SYMmetrically Dependent Models
ArisIng In Visual assessment Data

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SYMMETRICALLY DEPENDENT MODELS ARISING IN VISUAL ASSESSMENT DATA

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ABSTRACT. Given the data from bilateral visual assessments at k different instances, we define the contralateral correlations (C) between fellow eyes, and the lateral correlations (L) among k different assessments of the same eye. Under permutation symmetric dependence structure between observations from fellow eyes and among observations from the same eye, and obtain maximum likelihood estimates of L, C, and L-C. Large-sample estimates of the corresponding covariance structures are also obtained.

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1. Introduction

Observations between related measurements, such as with eyes, ears, siblings, etc. possess intrinsic symmetries. The motivation in the present article is with the assessment of human intra-ocular pressure (IOP). In particular, suppose that \((Y_{t1}, Y_{t2})\) represent IOP measurements in the left \((Y_{t1})\) and right \((Y_{t2})\) eye at pre and post treatment conditions (e.g., a topical beta blocker therapy). Here \(t = 1, 2\) for pre and post times; however, a more general context permits measurements at \(k\) time-points, so that \(t = 1, 2, \ldots, k\).

The two \(k\)-variate vectors \((Y_{11}, Y_{21}, \ldots, Y_{k1})\) and \((Y_{12}, Y_{22}, \ldots, Y_{k2})\) representing left and right eye observations are called lateral observations whereas the \(k\) bivariate vectors \((Y_{t1}, Y_{t2}), t = 1, \ldots, k\) are called contra-lateral observations. The concern is with the correlational inferences for these two types of observations. We assume an underlying normal distribution in which the contra-lateral correlations between any two time-points \(t\) and \(u\) are vision-symmetric, that is,

\[
\text{corr}(Y_{t1}, Y_{u2}) = \text{corr}(Y_{t2}, Y_{u1}) = \gamma_{tu},
\]

whereas the lateral correlations between any two time points \(t\) and \(u\) depend only on \(t\) and \(u\), that is,

\[
\text{corr}(Y_{t1}, Y_{u1}) = \text{corr}(Y_{t2}, Y_{u2}) = \lambda_{tu}.
\]

Further, there is symmetry within each time-point, \(\text{var}(Y_{t1}) = \text{var}(Y_{t2}) = \sigma_t^2\).

Let \(\Sigma\) indicate the covariance matrix of \(Y\). The \(2 \times 2\) diagonal and off-diagonal blocks have the form

\[
\Sigma_{tt} = \sigma_t^2 \begin{bmatrix} 1 & \gamma_{tt} \\ \gamma_{tt} & 1 \end{bmatrix}, \quad \Sigma_{tu} = \sigma_t \sigma_u \begin{bmatrix} \lambda_{tu} & \gamma_{tu} \\ \gamma_{tu} & \lambda_{tu} \end{bmatrix}, \quad t \neq u.
\]
For example, when $k=3$, the joint covariance matrix becomes

$$\Sigma = \begin{bmatrix}
\sigma_1^2 & \gamma_1 & 1 \\
\gamma_1 & \sigma_2 & \gamma_2 \\
1 & \gamma_2 & \sigma_3 \\
\end{bmatrix} \begin{bmatrix}
\lambda_1 \\
\lambda_2 \\
\lambda_3 \\
\end{bmatrix} \begin{bmatrix}
\gamma_1 \\
\gamma_2 \\
\gamma_3 \\
\end{bmatrix}. \tag{1.1}
$$

The matrix, $C$, of contralateral correlations pick out the relations between left and right eyes, whereas the matrix, $L$, of lateral correlations pick out the relations of a left eye (or right eye) at different time points. The matrix of standard deviations is represented by $\Delta$. In summary,

$$C = \begin{bmatrix}
\gamma_1 \\
\gamma_2 \\
\gamma_3 \\
\end{bmatrix}, \quad L = \begin{bmatrix}
1 & \lambda_1 \\
\lambda_1 & 1 \\
\lambda_1 & \lambda_1 \\
\end{bmatrix}, \quad \Delta = \begin{bmatrix}
\sigma_1 & 0 & 0 \\
0 & \sigma_2 & 0 \\
0 & 0 & \sigma_3 \\
\end{bmatrix}. \tag{1.2}
$$

In each case, the matrices $C$ and $L$ are symmetric.

Because our example dealt with two eyes, the vectors $Y_t$ were bivariate. However, in other contexts, $Y_t$ might be $p$-variate, rather than bivariate. For example, consider $k$ weekly insecticide spray treatments on the petals of a flower. Here we have $p$ petals instead of left and right eyes. For completeness we treat the more general case in which $Y_t = (Y_{t1}, \ldots, Y_{tp})$ and $t = 1, \ldots, k$. The symmetries in the visual outcome context are extended to the more general case.

2. Related History

In another ophthalmic assessment, a single joint measure $(Y_1, Y_2)$ of visual acuity is obtained from left and right eyes. Of concern are the extreme measurements of visual acuity, namely, the “best” acuity $Y_{(1)} = \min\{Y_1, Y_2\}$ and the “worst” acuity $Y_{(2)} = \max\{Y_1, Y_2\}$. This leads to an analysis of the covariance structure of the order statistics, and was studied by Olkin and Viana (1995) [see also (David 1996), Viana (1998)].

The model in which we have a $p$-dimensional normal vector with equal variances and equal correlations was introduced by Wilks (1946) in an educational context to determine whether $p$ examinations were “equivalent”. This model was extended by Votaw (1947) to a block covariance matrix in which each block was permutation symmetric. The structure was termed “compound symmetry”. The covariance hypotheses
studied were equality of variances and covariances. Subsequently, there have been a number of extensions of this model, in particular to the case in which the elements of a block are themselves matrices (e.g., Olkin (1974)). The important point to note is that although the compound symmetry structure is maintained, the hypotheses of interest in the context of visual outcomes are different.

In general, patterns in the population covariance are not reflected in the sample. For example, if \( \sigma_{12} = \sigma_{13} \) in the population, the estimate of the common value in \( \text{not} \) obtained as the average of the sample covariances. However, in the case of compound symmetry, the intuitive estimates are the maximum likelihood estimates (e.g., Olkin and Pratt (1958)).

If \( S \) indicates the sample covariance matrix of \( Y \) and \( S_{tu} \) indicates the submatrix of \( S \) corresponding to time-points \( t \) and \( u \), then the maximum likelihood estimates of the parameters are

\[
\hat{\sigma}_{tt}^2 = \frac{1}{p} \text{tr} S_{tt}, \tag{2.1a}
\]

\[
\hat{\gamma}_{tt} = \frac{e' S_{tt} e - \text{tr} S_{tt}}{(p - 1) \text{tr} S_{tt}}, \quad \hat{\gamma}_{tu} = \frac{e' S_{tu} e - \text{tr} S_{tu}}{(p - 1) \sqrt{\text{tr} S_{tt} \text{tr} S_{uu}}}, \quad t \neq u, \tag{2.1b}
\]

\[
\hat{\lambda}_{tu} = \frac{\text{tr} S_{tu}}{\sqrt{\text{tr} S_{tt} \text{tr} S_{uu}}}, \quad t \neq u. \tag{2.1c}
\]

3. Hypotheses

In the general case,

\[
\Sigma_{tt} = \sigma^2_t [\gamma_{tt} e e' + (1 - \gamma_{tt}) I], \quad \Sigma_{tu} = \sigma_t \sigma_u [\gamma_{tu} e e' + (\lambda_{tu} - \gamma_{tu}) I], \quad t \neq u, \tag{3.1}
\]

where \( e' = (1, \ldots, 1) \), and each block is of dimension \( p \). For example, when \( k = 2 \) and \( p = 3 \), the joint covariance structure is represented by

\[
\Sigma = \begin{bmatrix}
\Sigma_{11} & \Sigma_{12} \\
\Sigma_{21} & \Sigma_{22}
\end{bmatrix} = \begin{bmatrix}
\sigma^2_t & \gamma_{11} & \gamma_{11} \\
\gamma_{11} & 1 & \gamma_{11} \\
\gamma_{11} & \gamma_{11} & 1
\end{bmatrix} \begin{bmatrix}
\lambda_{12} & \gamma_{12} & \gamma_{12} \\
\gamma_{12} & \lambda_{12} & \gamma_{12} \\
\gamma_{12} & \gamma_{12} & \lambda_{12}
\end{bmatrix} + \begin{bmatrix}
\lambda_{12} & \gamma_{12} & \gamma_{12} \\
\gamma_{12} & \lambda_{12} & \gamma_{12} \\
\gamma_{12} & \gamma_{12} & \lambda_{12}
\end{bmatrix} \begin{bmatrix}
\sigma^2_t & \gamma_{21} & \gamma_{21} \\
\gamma_{21} & 1 & \gamma_{21} \\
\gamma_{21} & \gamma_{21} & 1
\end{bmatrix}.
\]

Correspondingly,

\[
C = \begin{bmatrix}
\gamma_{11} & \gamma_{12} \\
\gamma_{12} & \gamma_{22}
\end{bmatrix}, \quad L = \begin{bmatrix}
1 & \lambda_{12} \\
\lambda_{12} & 1
\end{bmatrix}, \quad \Delta = \begin{bmatrix}
\sigma_1 & 0 \\
0 & \sigma_2
\end{bmatrix}.
\]
We are interested in assessing two main hypotheses and obtain approximate confidence intervals under the assumption that $\Sigma_{tt}$ and $\Sigma_{tu}$ have the permutation symmetry structure indicated by (3.1). These hypothesis are:

$$H_1 : \gamma_{11} = \cdots = \gamma_{kk} = \gamma,$$
$$H_2 : \gamma_{tu} = \lambda_{tu}, \ t \neq u.$$ 

Note that $H_2$ is equivalent to

$$G \equiv L - C = \text{diag} \ (1 - \gamma_{11}, \ldots, 1 - \gamma_{kk}),$$

whereas $H_1 \cap H_2$ is equivalent to $L - C = (1 - \gamma)L$. In addition, we are interested in estimating $L$ and $C$ under these hypotheses. Alternative hypotheses in the context of this model are (i) $H_1$ vs $\overline{H}_1$, (ii) $H_2$ vs $\overline{H}_2$, (iii) $H_{12} \equiv H_2 \cap H_1$ vs $\overline{H}_{12}$, (iv) $H_{12}$ vs $H_2$, (v) $H_{12}$ vs $H_1$.

In the next section we provide estimates of the parameters under different models.

4. ESTIMATES AND LARGE-SAMPLE DISTRIBUTIONS

The unrestricted ML estimates $\hat{L}$, $\hat{C}$ and $\hat{\Delta}$ of $L$, $C$ and $\Delta$ follow from the canonical decomposition of the sample covariance matrix $S$. More specifically (as detailed in Appendix A), under the assumption of a multivariate normal distribution, the distribution of $nS$ decomposes into two orthogonal Wishart components $nE$ and $nF$, with respective parameters $E = \Delta(L + (p - 1)C)\Delta$ and $F = \Delta(L - C)\Delta$, so that $\hat{E} = E/n$ and $\hat{F} = F/n$ are the ML estimates of $E$ and $F$, respectively, based on the data $E, F$. The functional relation between $E, E$ and $E, F$ leads to the ML estimates of $C, L$ and $\Delta$. The connection with the original data comes from the fact that

$$pE_{tu} = e'S_{tu}e, \ p \ tr \ S_{tu} - e'S_{tu}e = pF_{tu},$$

where $E_{tu}$ indicates the $tu$ entry of $E$ and $S_{tu}$ indicates the corresponding block submatrix of $S$. This leads to (2.1a), (2.1b) and (2.1c). Note that the ML estimates depend on the data only through the maximal invariants under block-permutation symmetry, namely, the sum of the entries $(e'S_{tu}e)$ and the trace of the block sample covariance matrices.

We recall the notation $G \equiv L - C$. In order to carry out our analyses we require the large-sample distribution of $\hat{G}$ when $k=2$ and $p \geq 2$. The proof of the corresponding result is based on the standard delta method (see also Proposition 21.5.1 in Viana and Olkin (1997)). Because this development leads to lengthy expressions, we provide the details in Appendix B.
5. Intra-ocular Pressure Data

Viana and Olkin (1997) considered the study described by Sonty, Sonty and Viana (1996), in which intra-ocular pressure (IOP) measurements at pre-treatment ($Y_1$) and post treatment ($Y_2$) conditions were obtained from fellow glaucomatous eyes of $N = 15$ subjects on topical beta blocker therapy. The observed covariance matrices for IOP between fellow eyes are:

$$S_{11} = \begin{bmatrix} 12.410 & 7.019 \\ 7.019 & 12.924 \end{bmatrix}, \quad S_{22} = \begin{bmatrix} 17.029 & 15.371 \\ 15.371 & 17.352 \end{bmatrix},$$

whereas the sample cross-covariance matrix is

$$S_{12} = \begin{bmatrix} 11.671 & 9.348 \\ 8.200 & 10.076 \end{bmatrix}.$$

From Section 2, the estimated variances are

$$\hat{\sigma}_1^2 = \frac{\text{tr}S_{11}}{p} = 12.667, \quad \hat{\sigma}_2^2 = \frac{\text{tr}S_{22}}{p} = 17.190,$$

whereas the estimated correlations are

$$\hat{\gamma}_{11} = \frac{e'S_{11}e - \text{tr}S_{11}}{(p-1)\text{tr}S_{11}} = 0.553, \quad \hat{\gamma}_{22} = \frac{e'S_{22}e - \text{tr}S_{22}}{(p-1)\text{tr}S_{22}} = 0.894,$$

$$\hat{\lambda}_{12} = \frac{\text{tr}(S_{12})}{\sqrt{\text{tr}S_{22}\text{tr}S_{11}}} = 0.736, \quad \hat{\gamma}_{12} = \frac{e'S_{12}e - \text{tr}(S_{21})}{(p-1)\sqrt{\text{tr}S_{22}\text{tr}S_{11}}} = 0.594.$$

From Proposition B.1, the large-sample covariance matrix for $\hat{G}$ is

$$\text{Acov}(\hat{G}) = \begin{bmatrix} 0.48190 & 0.02143 & 0.071600 \\ 0.28413 & 0.08109 & 0.072276 \\ 0.071600 & 0.020836 & 0.040306 \end{bmatrix}. \quad (5.1)$$

To assess the hypothesis $H_1 : \gamma_{11} = \gamma_{22}$, we follow the steps of Example B.1. let

$$M = \begin{bmatrix} -1 & 0 & 1 \end{bmatrix},$$

so that (writing $G$ in vector form) $MG = 0$ represents $H_1$. Then, asymptotically,

$$Q = N(\hat{G}')M'[\text{Acov}(\hat{G}) M']^{-1}M(\hat{G}) \sim \chi^2_1,$$

and can be used to assess $H_1$. In the present case, we obtain $Q = 4.57$, with 1 degrees of freedom, thus suggesting a difference in association between left and right eye response during the experimental period, that is, the beta blocker therapy did alter the association between the IOP responses from fellow eyes.
To assess the lateral-contralateral homogeneity hypotheses \( H_2 \), let
\[
M = \begin{bmatrix}
0 & 1 & 0 \\
\end{bmatrix},
\]
so that, similarly, \( MG = 0 \) represents \( H_2 \) and \( Q \sim \chi^2_2 \). We obtain \( Q = 3.71 \), with 1 degrees of freedom, thus suggesting a heterogeneous (not proportional to \( ee' \)) covariance structure \( \Sigma_{12} \) between pre-treatment and post-treatment.

Similarly, to assess \( H_{12} \) we define
\[
M = \begin{bmatrix}
-1 & 0 & 1 \\
0 & 1 & 0 \\
\end{bmatrix},
\]
so that, similarly, \( M(G) = 0 \) represents \( H_{12} \) and \( Q \sim \chi^2_3 \). We obtain \( Q = 3.29 \), with 3 degrees of freedom, in support of the hypothesis that beta blocker therapy did alter the association between the IOP responses from fellow eyes and that the covariance structure \( \Sigma_{12} \) between pre-treatment and post-treatment is heterogeneous.

6. Discussion and Comments

We have observed that the relevant assumption of symmetry imposes that each block covariance sub-matrix \( \Phi \) of \( \Sigma \) is permutation symmetric, that is, \( \Phi \) has the form \( a_0 ee' + a_1 I \), for scalars \( a_0 \) and \( a_1 \). This form characterizes the permutation symmetry in the sense that for all permutation matrices \( g \) the conjugate covariance structure \( g\Phi g' \) remains constant if and only if \( \Phi \) has the form \( a_0 ee' + a_1 I \). In this case we say that \( \Phi \) has the symmetry of the permutation matrices. Equivalently, a matrix \( \Phi \) with real (or complex) entries commutes with every permutation matrix (of corresponding matrix dimension) if and only if \( \Phi \) has the form \( a_0 ee' + a_1 I \). Covariance structure with circular symmetry have been suggested, for example, in Viana, Olkin and McMahon (1993), and are characterized by the symmetry of the nullipotent permutation matrices.

To obtain the ML estimates of interest under the hypothesis \( H_1 \), note that the MLE of \( \gamma_{ht} \) in equation (2.1b) is equivalently obtained from the relation
\[
\frac{1 + (p - 1)\hat{\gamma}_{ht}}{(p - 1)\hat{\gamma}_{ht}} = \frac{E_{ht}}{F_{ht}}.
\]
(6.1)

It then follows that, under \( H_1 \), the ML estimate of the common correlation \( \gamma \) is obtained from the relation
\[
\frac{1 + (p - 1)\hat{\gamma}}{(p - 1)\hat{\gamma}} = \frac{\text{tr}E}{\text{tr}F}.
\]
(6.2)

Under the hypothesis \( H_2 \) of lateral-contralateral homogeneity, we obtain
\[
\hat{\gamma}_{tu} = \hat{\lambda}_{tu} = \frac{1}{p} \frac{e'S_{tt}e}{\sqrt{\text{tr}S_{tt} \text{tr}S_{uu}}}, \quad t \neq u,
\]
(6.3)
which is a weighted linear combination of the estimates (2.1b) and (2.1c) with weights \((p - 1)/p\) and \(1/p\).

Another example in which the determination of the symmetry between sets of variables has a particular interest is in the study of concomitants of order statistics under symmetrically dependent structure. Given two vectors \(Y\) and \(Z\) of same dimension, the ordered version \(Y\) of \(Y\) induces a corresponding permutation \(Z\) of the components of \(Z\), called the concomitant order statistic. When the joint covariance structure of \(Y\) and \(Z\) is permutation-symmetric given by (3.1), Lee (1998) has shown that the joint covariance structure of \((Y, Z)\) is in great part simplified by the assumption of full contralateral symmetry (\(\lambda_{tu} = \gamma_{tu}, t \neq u\)). For example, under the multivariate normal assumption,

\[
\text{Cov}(Z) = \Sigma_{22} + \frac{\sigma^2(\lambda_{12} - \gamma_{12})^2}{1 - \gamma_{11}}(C - I),
\]

where \(C\) is the covariance matrix of ordered independent standard normal observations. In this case, the interpretation is important: under the fully symmetric contralateral covariance structure, the covariance matrices of \(Z\) and \(Z\) coincide.
APPENDIX A.Canonical Forms and Elementary Estimates

The covariance matrix $\Sigma$ of $\mathbf{Y}$ is expressed as

$$
\Sigma = (\Delta \otimes I_p)(C \otimes e e' + (L - C) \otimes I_p)(\Delta \otimes I_p),
$$

where $I_p$ is the identity matrix of dimension $p$ and $e' = (1, \ldots , 1)$ has dimension $p$. To obtain a canonical form for $\Sigma$, let $\Gamma = I_k \otimes Q$, where $Q$ of dimension $p \times p$ is orthogonal with first row equal to $e'/\sqrt{p}$. It then follows that

$$
\Gamma \Sigma \Gamma' = (\Delta C \Delta) \otimes \text{diag}(p, 0, \ldots , 0) + (\Delta (L - C) \Delta) \otimes I_k.
$$

Moreover, there is a permutation $g$ (conjugating $\Delta \otimes I_p$ and $I_p \otimes \Delta$) such that

$$
g \Gamma \Sigma \Gamma' g' = \text{diag}(p, 0, \ldots , 0) \otimes (\Delta C \Delta) + I_k \otimes (\Delta (L - C) \Delta).
$$

The resulting canonical form is

$$
\text{diag}(E, F, \ldots , F),
$$

where $E = \Delta (L + (p - 1)C) \Delta$ and $F = \Delta (L - C) \Delta$. Both $E$ and $F$ have dimension $k \times k$ and the block $F$ is repeated $p$-1 times. In the simplest case ($p = k = 2$) the canonical form is $\text{diag}(E, F)$, with

$$
E = \begin{bmatrix}
\sigma_1^2(1 + \gamma_{11}) & \sigma_1 \sigma_2(\lambda_{12} + \gamma_{12}) \\
\sigma_1 \sigma_2(\lambda_{21} + \gamma_{21}) & \sigma_2^2(1 + \gamma_{22})
\end{bmatrix},
$$

$$
F = \begin{bmatrix}
\sigma_1^2(1 - \gamma_{11}) & \sigma_1 \sigma_2(\lambda_{12} - \gamma_{12}) \\
\sigma_1 \sigma_2(\lambda_{21} - \gamma_{21}) & \sigma_2^2(1 - \gamma_{22})
\end{bmatrix}.
$$

Because $E + (p - 1)F = p \Delta L \Delta$ and $E - F = p \Delta C \Delta$, it then follows that

$$
\frac{1}{p}(E + (p - 1)F)_{tt} = \sigma_t^2, \quad \frac{(E - F)_{tt}}{(E + (p - 1)F)_{tt}} = \gamma_{tt}, \quad t = 1, \ldots , k, \quad (A.1)
$$

$$
\frac{(E - F)_{tu}^2}{(E + (p - 1)F)_{tu}(E + (p - 1)F)_{uu}} = \gamma_{tu}^2, \quad t \neq u, \quad (A.2)
$$

$$
\frac{(E + (p - 1)F)_{tu}^2}{(E + (p - 1)F)_{tt}(E + (p - 1)F)_{uu}} = \lambda_{tu}^2, \quad t \neq u. \quad (A.3)
$$

In addition, for $\gamma_{tu} \neq 0$,

$$
\frac{(E + (p - 1)F)_{tu}}{(E - F)_{tu}} = \frac{\lambda_{tu}}{\gamma_{tu}} \quad t \neq u. \quad (A.4)
$$
Let $S$ indicate the sample covariance matrix associated with $\Sigma$ and $S_{tu}$ the blocks of $S$ corresponding to the blocks $\Sigma_{tu}$ of $\Sigma$. Let $\mathcal{E}$ and $\mathcal{F}_1, \ldots, \mathcal{F}_p$ indicate the diagonal blocks of $g\Gamma \Sigma \Gamma' g'$ corresponding to the diagonal blocks of $g\Gamma \Sigma \Gamma' g'$. Direct computation shows that

$$p\mathcal{E}_{tu} = e' S_{tu} e, \text{ the sum of the entries of } S_{tu}, \quad (A.5)$$
$$p\mathcal{F}_{tu} = p tr S_{tu} - e' S_{tu} e, \quad \text{where } \mathcal{F} = \sum_{i=1}^{p-1} \mathcal{F}_i. \quad (A.6)$$

Note, in the above expressions, that $\mathcal{E}_{tu}$ refers to the entry $tu$ of $\mathcal{E}$, whereas $S_{tu}$ indicates the $tu$ block submatrix of $S$.

When the distribution of $Y$ is multivariate normal then the distribution of $n\mathcal{E}$ is Wishart $W(\mathcal{E}, n)$, the distribution of $n\mathcal{F}$ is Wishart $W(\mathcal{F}, n(p-1))$, and $\mathcal{E}$ is independent of $\mathcal{F}$ ($n=\infty$). From the MLEs (based on the canonical transformation of sample covariance matrix) $\mathcal{E}$ and $\mathcal{F}$ of $E$ and $F$ and equations (A.1),(A.2),(A.3) and (A.4) we obtain the corresponding MLEs indicated by equations (2.1a),(2.1b) and (2.1c). For example,

$$\hat{\gamma}_{tt} = \frac{(\hat{E} - \hat{F})_{tt}}{(\hat{E} + (p-1)\hat{F})_{tt}} = \frac{(p-1)\mathcal{E} - \mathcal{F}}{(p-1)(\mathcal{E} + \mathcal{F})},$$

thus leading to (2.1b).

Examples B.2 and B.1 in Appendix B illustrate the computational steps in the canonical decomposition and the corresponding estimates.
Here we provide the necessary details required to obtain approximate confidence intervals for the parameters of interest. The following proposition gives the large-sample distribution of \( \hat{G} \) when \( k=2 \) and \( p \geq 2 \). The proof is based on the standard delta method. We interpret \( G \), in vector form, as \( (1 - \gamma_{11}, \lambda_{12} - \gamma_{12}, 1 - \gamma_{22}) \).

**Proposition B.1.** If the joint distribution of \((Y_t, Y_u)\) is multivariate normal with block-covariance structure \((k=2, p \geq 2)\) given by equation (3.1), then the asymptotic joint distribution of \( \sqrt{N}[\hat{G} - G] \) is bivariate normal with mean zero, variances

\[
[A\text{Cov}(\hat{G})]_{11} = 2 \frac{(-1 + \gamma_{11})^2 (1 + \gamma_{11} p - \gamma_{11})^2}{p(p-1)},
\]

\[
[A\text{Cov}(\hat{G})]_{22} = \frac{1}{2} \left[ 2p + \lambda_{12}^2 p + 2 \lambda_{12}^2 \gamma_{12}^2 p^2 + 2 \lambda_{12}^2 \gamma_{12}^2 p^2 - 4 \gamma_{12} \lambda_{12}^3 p + 4 \lambda_{12}^2 \gamma_{22} p + \\
+ \frac{4}{2} \lambda_{12}^2 \gamma_{11} p - 2 \gamma_{12} \lambda_{12} \gamma_{11}^2 p^2 + \lambda_{12}^2 \gamma_{22}^2 p^2 - 4 \gamma_{12}^2 p - 2 \lambda_{12} \gamma_{12}^3 p^2 \\
- 4 \lambda_{12}^2 \gamma_{11} - 4 \lambda_{12}^2 \gamma_{22} + 2 \lambda_{12}^4 p - 2 \gamma_{22} p - 2 \gamma_{11} p + 6 \lambda_{12}^2 - 2 \gamma_{12} \gamma_{12} + 6 \gamma_{12}^2 \\
- 8 \lambda_{12} \gamma_{12} p \gamma_{22} + 8 \lambda_{12} \gamma_{12} \gamma_{22} + 2 \gamma_{11} \gamma_{12} + 8 \gamma_{12} \gamma_{12} \gamma_{11} - 4 \gamma_{12} \lambda_{12} \gamma_{11}^2 p \\
- 8 \lambda_{12} \gamma_{11} \gamma_{12} - 4 \gamma_{12} \gamma_{12}^2 + 2 \gamma_{12} \lambda_{12} \gamma_{22}^2 + 4 \gamma_{11} \gamma_{12} \gamma_{12}^2 - 2 \gamma_{12} \lambda_{12} \gamma_{11}^2 \\
+ 8 \lambda_{12} \gamma_{12} \gamma_{11} p + 4 \gamma_{12} \gamma_{12} p \gamma_{22} + 4 \gamma_{12} \lambda_{12} \gamma_{22}^2 p - 2 \lambda_{12}^2 \gamma_{22} p - 2 \lambda_{12}^4 \\
- 2 \lambda_{12}^2 \gamma_{11}^2 p - 2 \lambda_{12}^2 \gamma_{12}^2 p + \lambda_{12}^2 \gamma_{11} + \lambda_{12}^2 \gamma_{22} - 4 \lambda_{12} \lambda_{12}^3 + 4 \gamma_{12} \lambda_{12} - 4 \lambda_{12}^2 \gamma_{22} \\
+ 8 \lambda_{12} \gamma_{12} p - 4 \lambda_{12}^2 p - 2 \lambda_{12} \lambda_{12} \gamma_{22}^2 p + \gamma_{12}^2 \gamma_{11} + 2 \gamma_{12}^4 p - 4 \gamma_{12}^4 p \\
- 2 \gamma_{12}^2 \gamma_{11}^2 p + \gamma_{12}^2 \gamma_{11}^2 p^2 - 2 \gamma_{12}^2 \gamma_{22}^2 p + \gamma_{12}^2 \gamma_{22}^2 p^2 + 2 \gamma_{12}^4 + \gamma_{12} \gamma_{12}^2 )/(p(p-1)),
\]

and covariances

\[
[A\text{Cov}(\hat{G})]_{12} = (-8 \gamma_{11}^2 p \lambda_{12} + 4 \gamma_{11} \gamma_{11}^3 p + \gamma_{12} \gamma_{11}^3 p^2 + \gamma_{12} \gamma_{11}^3 p \\
+ \gamma_{12} \gamma_{11}^3 p^2 + 3 \gamma_{12} \gamma_{11} p - 2 \gamma_{12} \gamma_{11} p^2 - \gamma_{12} \lambda_{12} \gamma_{11} - 2 \gamma_{11} \gamma_{12}^3 + \gamma_{11} \gamma_{12}^3 p^2 \\
+ \gamma_{12} \lambda_{12} \gamma_{11} p + \gamma_{11} \gamma_{12}^3 + 6 \gamma_{11}^2 \lambda_{12} + \gamma_{12} \gamma_{11} - 2 \gamma_{12} \gamma_{11}^2 + \gamma_{12} \gamma_{11}^3 \\
- 6 \gamma_{11} \lambda_{12} + 2 \lambda_{12} - 2 \lambda_{12} \gamma_{11}^3 + 6 \lambda_{12} \gamma_{11}^2 - \gamma_{12}^3 p^2 + 3 \gamma_{12}^3 p - 2 \gamma_{12}^3 - 8 \lambda_{12} \gamma_{12}^2 p \\
- 2 \gamma_{12} + 2 \lambda_{12} \gamma_{11} - 2 \lambda_{12} \gamma_{11} \gamma_{12}^2 + 4 \lambda_{12} \gamma_{11} \gamma_{12}^2 p - 2 \lambda_{12} \gamma_{11}^3 p \\
+ 4 \lambda_{12} \gamma_{11}^3 p + 4 \lambda_{12}^2 \gamma_{12} p - 2 \lambda_{12} \gamma_{11} \gamma_{12}^2 p + 2 \gamma_{12} \gamma_{12}^2 p^2 + 2 \gamma_{12}^2 p^2 \lambda_{12} \\
- 2 \lambda_{12} \gamma_{11} \gamma_{12}^2 \gamma_{12}^2 - 4 \lambda_{12}^2 \gamma_{12} )/(p(p-1)).
\]
\[ [\text{ACov}(\mathcal{G})]_{13} = 2 \left[ -2 \lambda_{12} \gamma_{12} p \gamma_{22} + \gamma_{11} \gamma_{12}^2 \gamma_{22} - 2 \gamma_{11} p \gamma_{12}^2 \gamma_{22} + \gamma_{12}^2 p^2 + \lambda_{12}^2 - \lambda_{12}^2 \gamma_{11} \gamma_{22} \\
+ 3 \gamma_{12}^2 + \gamma_{11} \gamma_{12}^2 p^2 \gamma_{22} - 2 \lambda_{12} \gamma_{11} p \gamma_{12} - 3 \gamma_{12}^2 p + \lambda_{12}^2 \gamma_{11} \gamma_{22} p - \gamma_{11} \gamma_{12}^2 p^2 \\
+ 2 \lambda_{12} \gamma_{12} p - \gamma_{12}^2 p^2 \gamma_{22} + 3 \gamma_{12}^2 p \gamma_{22} + 3 \gamma_{11} p \gamma_{12}^2 + 2 \lambda_{12} \gamma_{12} \gamma_{22} \\
+ 2 \lambda_{12} \gamma_{11} \gamma_{12} - 2 \gamma_{11} \gamma_{12}^2 - 2 \gamma_{12}^2 \gamma_{22} - 4 \lambda_{12} \gamma_{12}]/(p(p-1)) \right] \]

\[ [\text{ACov}(\mathcal{G})]_{23} = [ -9 \gamma_{12} p \gamma_{22} - 4 \gamma_{12} + 2 \lambda_{12} + 2 \lambda_{12} \gamma_{12}^2 - \gamma_{12}^3 p^2 + 3 \gamma_{12}^3 p - 4 \lambda_{12} \gamma_{22} + 9 \gamma_{12} \gamma_{22} \\
+ \gamma_{12} \gamma_{22}^2 - 2 \gamma_{12}^2 + \gamma_{12} \lambda_{12}^2 \gamma_{22} p + \gamma_{12}^3 \gamma_{22} + 2 \lambda_{12} \gamma_{22}^2 - 6 \gamma_{12} \gamma_{22}^2 \\
+ 9 \gamma_{12} \gamma_{22}^2 p - 3 \gamma_{12} \gamma_{22}^2 p^2 + 2 \lambda_{12} \gamma_{22} p + 2 \lambda_{12} p \gamma_{22} - 2 \lambda_{12} \gamma_{12}^2 p + 2 \gamma_{12} p - 2 \lambda_{12} \gamma_{22}^2 p \\
- 2 \gamma_{12}^3 p \gamma_{22} + \gamma_{12}^3 p^2 \gamma_{22} - \gamma_{12} \lambda_{12}^2 \gamma_{22} - 2 \gamma_{12} \gamma_{22}^3 p + \gamma_{12} \gamma_{22}^3 p^2]/(p(p-1)) \]

\[ [\text{ACov}(\mathcal{G})]_{33} = 2 \left( -1 + \gamma_{22} \right)^2 \left( 1 + \gamma_{22} p - \gamma_{22} \right) \]

\[ p(p-1) \]

**Proposition B.2.** If the joint distribution of \((Y_t, Y_u)\) is multivariate normal with block-covariance structure \((k=2, p \geq 2)\) given by equation (3.1), then the asymptotic joint distribution of \(\sqrt{N}[\Delta^2 - \Delta^2]\) is multivariate normal with mean zero and covariance matrix

\[
\text{ACov}(\Delta) = \frac{1}{p} \left[ \begin{array}{cc}
2 \sigma_1^4 (\gamma_{12}^2 p - \gamma_{12}^2 + 1) & 2 \sigma_1^2 \sigma_2^2 (\gamma_{12}^2 p - \gamma_{12}^2 + \lambda_{12}^2) \\
2 \sigma_1^2 \sigma_2^2 (\gamma_{12}^2 p - \gamma_{12}^2 + \lambda_{12}^2) & 2 \sigma_2^4 (\gamma_{22}^2 p - \gamma_{22}^2 + 1) 
\end{array} \right].
\]

**Proposition B.3.** If the joint distribution of \((Y_t, Y_u)\) is multivariate normal with block-covariance structure \((k=2, p \geq 2)\) given by equation (3.1), then the asymptotic joint distribution of \(\sqrt{N}[\bar{C} - C]\) is multivariate normal with mean zero, variances

\[ [\text{ACov}(\bar{C})]_{11} = 2 \left( -1 + \gamma_{11} \right)^2 \left( 1 + \gamma_{11} p - \gamma_{11} \right)^2]/(p(p-1)), \]

\[ [\text{ACov}(\bar{C})]_{22} = 1/2 \left[ 2 + 12 \gamma_{12}^2 \gamma_{11} p + 12 \gamma_{12}^2 p \gamma_{22} - 4 \gamma_{11} - 4 \gamma_{22} - 6 \gamma_{11} p \gamma_{22} + 2 \gamma_{11} p^2 \gamma_{22} + 4 \gamma_{12} \gamma_{11} \lambda_{12} \\
+ 4 \gamma_{12} \lambda_{12} \gamma_{22} + 4 \lambda_{12} \gamma_{12} p - 4 \gamma_{11} p^2 \gamma_{12}^2 - 4 \gamma_{12}^2 p^2 \gamma_{22} + 6 \gamma_{11} \gamma_{22} + 2 \lambda_{12}^2 - 8 \lambda_{12} \gamma_{12} \\
+ 2 \gamma_{12}^2 p^2 - 12 \gamma_{12}^2 p + 12 \gamma_{12}^2 + 2 \gamma_{11} p + 2 \gamma_{22} p - 4 \gamma_{12} \gamma_{11} \lambda_{12} - 4 \gamma_{12} \lambda_{12} \gamma_{22} p \\
- 8 \gamma_{12}^2 \gamma_{11} - 8 \gamma_{12} \gamma_{22} + 12 \gamma_{12}^2 \gamma_{11} + 2 \gamma_{12}^2 \gamma_{12} + \gamma_{12}^2 \gamma_{11} p + \gamma_{12}^2 \gamma_{11} p^2 + 2 \gamma_{12}^2 p \lambda_{12}^2 \\
+ \gamma_{12}^2 \gamma_{22} p^2 - 2 \gamma_{12}^2 \gamma_{22} p - 2 \gamma_{12}^2 \lambda_{12}^2 + 2 \gamma_{12}^4 + 2 \gamma_{12}^4 p^2 - 4 \gamma_{12}^4 p + \gamma_{12}^2 \gamma_{22}^2]/(p(p-1)), \]

\[ [\text{ACov}(\bar{C})]_{33} = 2 \left( -1 + \gamma_{22} \right)^2 \left( 1 + \gamma_{22} p - \gamma_{22} \right)^2]/(p(p-1)), \]
and covariances

\[
[A\text{Cov}(\hat{C})]_{12} = \begin{bmatrix} \gamma_{12} \gamma_{11} & \lambda_{12} & -2 \gamma_{12}^2 \gamma_{11} & p + \gamma_{11} p \lambda_{12} \\ -9 \gamma_{12} \gamma_{11} & 2 \lambda_{12} \gamma_{11} & -2 \lambda_{12}^2 \gamma_{11} p + 2 \gamma_{11} p \lambda_{12} & -2 \lambda_{12} \gamma_{11}^2 p + \gamma_{11} \lambda_{12}^2 p \\
2 \gamma_{12} \gamma_{11} & -3 \gamma_{12} \gamma_{11}^2 p - 4 \gamma_{12} & -2 \lambda_{12} \gamma_{12}^2 p + \gamma_{12} \gamma_{11} \lambda_{12} p \\
2 \gamma_{12} & -2 \gamma_{12} \gamma_{11} p - \gamma_{12}^2 p - 6 \gamma_{12} \gamma_{11} & -9 \gamma_{12} \gamma_{11} & -4 \gamma_{11} \lambda_{12} + 2 \lambda_{12} \gamma_{12} \\
3 \gamma_{12}^2 p - 2 \gamma_{12} & 9 \gamma_{12} \gamma_{11} \lambda_{12} p + 2 \lambda_{12} \gamma_{11}^2 p - 2 \lambda_{12}^2 \gamma_{11} & 1/(p (p - 1)) 
\end{bmatrix}
\]

\[
[A\text{Cov}(\hat{C})]_{13} = \begin{bmatrix} -2 \gamma_{12}^2 \gamma_{11} p - 2 \gamma_{12} \gamma_{12} p - 4 \lambda_{12} \gamma_{12} & 3 \gamma_{12}^2 \gamma_{11} & 2 \lambda_{12} \gamma_{12} p + 2 \gamma_{12} \gamma_{12} p \lambda_{12} & -3 \gamma_{12} \gamma_{11} \gamma_{12} p \\
+3 \gamma_{12}^2 \gamma_{11} & -2 \gamma_{12} \gamma_{11} \gamma_{12} p - 2 \gamma_{12} \gamma_{11} \gamma_{12} p \lambda_{12} & -2 \gamma_{12} \gamma_{12} \gamma_{11} p - 2 \gamma_{12} \gamma_{12} \gamma_{11} p \lambda_{12} & 2 \lambda_{12} \gamma_{12} \gamma_{11} \gamma_{12} p \\
+3 \gamma_{12}^2 p - 2 \gamma_{12} \gamma_{11} & -2 \gamma_{12} \gamma_{12} p - 2 \gamma_{12} \gamma_{11} \gamma_{12} p \lambda_{12} & 2 \lambda_{12} \gamma_{11} \gamma_{12} p - 2 \gamma_{12} \gamma_{12} \gamma_{11} \gamma_{12} p & 1/(p (p - 1)) 
\end{bmatrix}
\]

\[
[A\text{Cov}(\hat{C})]_{23} = \begin{bmatrix} -9 \gamma_{12} \gamma_{12} p \gamma_{12} & 2 \gamma_{12} \gamma_{12} p - 2 \gamma_{12} \gamma_{11} \gamma_{12} p \gamma_{12} & -3 \gamma_{12} \gamma_{12} p \gamma_{12} & -2 \gamma_{12} \gamma_{12} p \gamma_{12} \\
-2 \gamma_{12} \gamma_{12} & 9 \gamma_{12} \gamma_{12} b + 2 \gamma_{12} \gamma_{12} p \gamma_{12} & -2 \gamma_{12} \gamma_{12} \gamma_{12} p \gamma_{12} & 2 \lambda_{12} \gamma_{12} \gamma_{12} p \gamma_{12} \\
-4 \gamma_{12} \gamma_{12} & -2 \gamma_{12} \gamma_{12} p \gamma_{12} & -2 \gamma_{12} \gamma_{12} \gamma_{12} p \gamma_{12} & 2 \gamma_{12} \gamma_{12} \gamma_{12} p \gamma_{12} \\
-2 \gamma_{12} \gamma_{12} & 2 \gamma_{12} \gamma_{12} \gamma_{12} p \gamma_{12} & -2 \gamma_{12} \gamma_{12} \gamma_{12} p \gamma_{12} & 1/(p (p - 1)) 
\end{bmatrix}
\]

\textbf{Proposition B.4.} If the joint distribution of \((Y_1, Y_2)\) is multivariate normal with block-covariance structure \((k=2, p \geq 2)\) given by equation (3.1), then the asymptotic joint distribution of \(\sqrt{N} [\hat{T} - T]\) is normal with mean zero and variance

\[
ACov(\hat{T}) = 1/2 [ -2 \gamma_{12} \lambda_{12}^2 + 2 + 2 \gamma_{11} \gamma_{12} \gamma_{12} p - 2 \gamma_{11} \gamma_{12} + 2 \gamma_{12} \gamma_{12} p - 2 \gamma_{12} \\
+ 4 \gamma_{12} \gamma_{11} \lambda_{12} + 4 \gamma_{12} \lambda_{12} \gamma_{12} + 2 \lambda_{12}^4 + \lambda_{12}^2 \gamma_{11} \gamma_{12} \\
- \lambda_{12} \gamma_{11} \gamma_{12} + 2 \gamma_{12} \lambda_{12} \gamma_{12} p - \lambda_{12} \gamma_{12} \gamma_{12} \\
- 4 \gamma_{12} \gamma_{11} \lambda_{12} - 4 \gamma_{12} \lambda_{12} \gamma_{12} p - 4 \lambda_{12}^2 ]/p
\]
Example B.1. The following numerical example is for \(k=2\) and \(p=3\). Suppose

\[
\Sigma = \begin{bmatrix}
25 & 4 & 4 & 1 & 0.5 & 0.5 \\
4 & 25 & 4 & 0.5 & 1 & 0.5 \\
4 & 4 & 25 & 0.5 & 0.5 & 1 \\
1 & 0.5 & 0.5 & 9 & 2 & 2 \\
0.5 & 1 & 0.5 & 2 & 9 & 2 \\
0.5 & 0.5 & 1 & 2 & 2 & 9
\end{bmatrix}
\]

(B.1)

Correspondingly,

\[
C = \begin{bmatrix}
0.16 & 0.03 \\
0.03 & 0.22
\end{bmatrix}, \quad L = \begin{bmatrix}
1 & 0.06 \\
0.06 & 1
\end{bmatrix}, \quad \Delta = \begin{bmatrix}
5 & 0 \\
0 & 3
\end{bmatrix}.
\]

To obtain the canonical form of \(\Sigma\), let

\[
\Gamma = \begin{bmatrix}
1/3 \sqrt{3} & 1/3 \sqrt{3} & 1/3 \sqrt{3} & 0 & 0 & 0 \\
1/6 \sqrt{6} & 1/6 \sqrt{6} & -1/3 \sqrt{6} & 0 & 0 & 0 \\
1/2 \sqrt{2} & -1/2 \sqrt{2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1/3 \sqrt{3} & 1/3 \sqrt{3} & 1/3 \sqrt{3} \\
0 & 0 & 0 & 1/6 \sqrt{6} & 1/6 \sqrt{6} & -1/3 \sqrt{6} \\
0 & 0 & 0 & 1/2 \sqrt{2} & -1/2 \sqrt{2} & 0
\end{bmatrix},
\]

and let

\[
g = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]
indicate in matrix form the permutation that conjugates \( \Delta \otimes I_p \) and \( I_p \otimes \Delta \). It then follows that

\[
\begin{bmatrix}
33 & 2.0 & 0 & 0 & 0 & 0 \\
2.0 & 13 & 0 & 0 & 0 & 0 \\
0 & 0 & 21 & 0.50001 & 0 & 0 \\
0 & 0 & 0.50001 & 7 & 0 & 0 \\
0 & 0 & 0 & 0 & 21 & 0.50000 \\
0 & 0 & 0 & 0 & 0.50000 & 7 \\
\end{bmatrix}
\]

so that

\[
E = \begin{bmatrix} 33 & 2.0 \\ 2.0 & 13 \end{bmatrix}, \quad F = \begin{bmatrix} 21 & 0.5 \\ 0.5 & 7 \end{bmatrix},
\]

\[
E + (p - 1)F = \begin{bmatrix} 75 & 3 \\ 3 & 27 \end{bmatrix}, \quad E - F = \begin{bmatrix} 12 & 1.5 \\ 1.5 & 6 \end{bmatrix}.
\]

The following matrix \( S \) is a sample covariance matrix based on a pseudo-sample of size 50 from a multivariate normal with mean zero and covariance matrix \( \Sigma \) as in (B.1),

\[
S = \begin{bmatrix}
19.396 & 2.31 & 7.904 & 0.336 & -1.688 & -2.289 \\
2.31 & 23.738 & 2.021 & 0.407 & 2.065 & -1.038 \\
7.904 & 2.021 & 28.806 & 1.378 & -1.101 & -0.679 \\
0.336 & 0.407 & 1.378 & 6.826 & 1.729 & -1.49 \\
-1.688 & 2.065 & -1.101 & 1.729 & 8.134 & 1.194 \\
-2.289 & -1.038 & -0.679 & -1.49 & 1.194 & 6.808 \\
\end{bmatrix}
\]
We obtain

\[
g' \Sigma g' = \begin{bmatrix}
32.137 & -0.86963 & -4.6639 & 2.2176 & 0.62903 & 1.1615 \\
-0.86971 & 8.2110 & -0.33068 & 1.2016 & -2.0719 & -1.6297 \\
-4.6630 & -0.33067 & 20.547 & 0.75068 & -4.6500 & -1.3256 \\
2.2177 & 1.2019 & 0.75070 & 7.8057 & -0.38162 & 1.1721 \\
0.62928 & -2.0719 & -4.6503 & -0.38165 & 19.257 & 1.8410 \\
1.1615 & -1.6297 & -1.3256 & 1.1720 & 1.8410 & 5.7512
\end{bmatrix},
\]

so that

\[
\mathcal{E} = \begin{bmatrix}
32.137 & -0.869 \\
-0.869 & 8.211
\end{bmatrix}, \quad \mathcal{F} = \mathcal{F}_1 + \mathcal{F}_2 = \begin{bmatrix}
39.804 & 2.591 \\
2.591 & 13.557
\end{bmatrix}.
\]

Maximum likelihood estimation: To estimate \( \Delta^2 \), from (A.1), we use the corresponding diagonal entries of

\[
\frac{1}{p} (\mathcal{E} + \mathcal{F}) = \begin{bmatrix}
23.980 & 0.574 \\
0.574 & 7.256
\end{bmatrix}.
\]

However, because (see (A.5) and (A.6)),

\[
e' S_{11} e = 96.410 = p \mathcal{E}_{11}, \quad ptrS_{11} = e' S_{11} e = 119.41 = p \mathcal{F}_{11},
\]

\[
e' S_{12} e = -2.609 = p \mathcal{E}_{11}, \quad ptrS_{12} = e' S_{12} e = 7.775 = p \mathcal{F}_{12},
\]

\[
e' S_{22} e = 24.63 = p \mathcal{E}_{22}, \quad ptrS_{22} = e' S_{22} e = 40.67 = p \mathcal{F}_{22},
\]

these are MLEs of \( \Delta^2 \) based on the initial sample covariance matrix. Similarly, the MLE of \( \gamma_{11} \), from (A.1), is the corresponding diagonal entry of the matrix (understanding the division in the component-by-component sense),

\[
\frac{\mathcal{E} - \mathcal{F}/(p - 1)}{\mathcal{E} + \mathcal{F}} = \begin{bmatrix}
0.172 & -0.113 \\
-0.113 & 0.0683
\end{bmatrix}.
\]

These estimates coincide, based on the initial covariance matrix \( S \), with (2.1b). The off-diagonal entries of the above matrix are MLE of the ratios \( \gamma_{1u}/\lambda_{1u} \).
The MLE of $\gamma_{tu}$, from (A.2), is the corresponding off diagonal entry of the matrix

$$
\begin{bmatrix}
0.90090 & 0.39416 & 0.56058 \\
0.39414 & 0.17968 & 0.031772 \\
0.56059 & 0.031771 & 0.40100
\end{bmatrix},
$$

with entries defined by

$$
\frac{(E - F)/(p - 1)}{\sqrt{(E + F)_{tt}(E + F)_{uu}}},
$$

(B.2)

These estimates coincide with the estimates expressed as (2.1b).

The MLE of $\lambda_{tu}$, from (A.3), is $\hat{\lambda}_{tu} = 0.00109$. In general,

$$
\hat{\lambda}_{tu} = \frac{(E - F)_{tu}}{\sqrt{(E + F)_{tt}(E + F)_{uu}}}, \quad t \neq u.
$$

These estimates coincide with (2.1c), when based on $S$. These leads to

$$
\hat{G} = \begin{bmatrix}
0.828 & 0.00247 \\
0.00247 & 0.932
\end{bmatrix}.
$$

From Proposition B.3 we obtain

$$
\text{Acov}(\hat{G}) = \begin{bmatrix}
0.41280 & -0.00001953 & 0.0000015132 \\
-0.00001953 & 0.21252 & -0.000069240 \\
0.0000015132 & -0.000069240 & 0.37367
\end{bmatrix},
$$

whereas, from Proposition B.4,

$$
\text{Acov}(\hat{L}) = 0.34113.
$$

Moreover, from Proposition B.2,

$$
\text{Acov}(\hat{\Delta^2}) = \begin{bmatrix}
406.03 & 0.00057964 \\
0.00057964 & 35.425
\end{bmatrix},
$$

whereas, from Proposition B.1,

$$
\text{Acov}(\hat{G}) = \begin{bmatrix}
0.41280 & 0.0021671 & 0.0000015132 \\
-0.00001953 & 0.38585 & -0.00006923 \\
0.0000015132 & 0.0019312 & 0.37367
\end{bmatrix}.
$$

To assess the homogeneity hypotheses ($H_1$), let

$$
M = \begin{bmatrix}
-1 & 0 & 1
\end{bmatrix},
$$
so that (writing L-C=G in vector form) MG = 0 represents \( H_1 \). Then, asymptotically,
\[
Q = N(\bar{G}'M'[M \text{ Acov}(\bar{G}) M']^{-1}M(\bar{G}) \sim \chi^2_{k-1},
\]
and can be used to assess \( H_1 \). In the present case, we obtain \( Q=0.69 \), with 1 degrees of freedom. To assess the lateral-contralateral homogeneity hypotheses \( (H_2) \), let
\[
M = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix},
\]
so that, similarly, MG = 0 represents \( H_2 \) and \( Q \sim \chi^2_{k(k-1)/2} \). We obtain \( Q=0.00078 \), with 1 degrees of freedom. Similarly, to assess \( H_{12} \) we define
\[
M = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},
\]
so that, similarly, MG = 0 represents \( H_{12} \) and \( Q \sim \chi^2_{(k+2)(k-1)/2} \). We obtain \( Q=0.693 \), with 2 degrees of freedom.
The following proposition gives the large-sample distribution of $\hat{G}$ when $k=3$ and $p=2$. The proof is based on the standard delta method (see also Proposition 21.5.1 in Viana and Olkin (1997)). For that purpose, we interpret $\hat{G}$ in vector form with 6 components, as

$$(1 - \gamma_{11}, \lambda_{12} - \gamma_{12}, \lambda_{13} - \gamma_{13}, 1 - \lambda_{22}, \lambda_{23} - \gamma_{23}, 1 - \gamma_{33}).$$

**Proposition B.5.** If the joint distribution of $(Y_t, Y_u)$ is multivariate normal with block-covariance structure $(k=3, p=2)$ given by equation (3.1), then the asymptotic joint distribution of $\sqrt{N}[\hat{G} - G]$ is bivariate normal with mean zero, variances

$$[\text{ACov}(\hat{G})]_{11} = (-1 + \gamma_{11})^2 (1 + \gamma_{11})^2,$$

$$[\text{ACov}(\hat{G})]_{22} = 1 - \lambda_{12} \gamma_{12}^3 - 2 \lambda_{12} \gamma_{12} \gamma_{22} - 2 \lambda_{12} \gamma_{22} \gamma_{12} + \gamma_{12}^2 \gamma_{22} - \gamma_{22} - \gamma_{11}$$

$$- \frac{1}{2} \lambda_{12} \lambda_{12} \gamma_{11}^2 - \frac{1}{2} \gamma_{12} \lambda_{12} \gamma_{22}^2 + \frac{1}{2} \lambda_{12}^2 + \frac{1}{4} \lambda_{12}^2 \gamma_{11}^2 + \frac{1}{4} \lambda_{12}^2 \gamma_{22}^2 + \frac{1}{4} \lambda_{12}^2 \gamma_{11}^2 + \lambda_{12}^2 \gamma_{11}$$

$$+ \lambda_{12}^2 \gamma_{22} + \gamma_{11} \gamma_{22} - \frac{1}{2} \lambda_{12}^2 + \lambda_{12} \gamma_{12} - \frac{1}{2} \gamma_{12}^2 + \gamma_{11} \gamma_{12}^2 + \frac{1}{2} \gamma_{12}^4$$

$$+ \lambda_{12}^2 \gamma_{12} + \frac{1}{4} \gamma_{12}^2 \gamma_{11}^2 - \lambda_{12} \gamma_{12}^3 + \frac{1}{4} \gamma_{12}^2 \gamma_{22}^2,$$

$$[\text{ACov}(\hat{G})]_{33} = 1 - 2 \lambda_{13} \gamma_{13} \gamma_{33} - 2 \lambda_{13} \gamma_{13} \gamma_{13} - \gamma_{33} - \gamma_{11} - \frac{1}{2} \gamma_{13} \lambda_{13} \gamma_{11}^2 - \frac{1}{2} \gamma_{13} \lambda_{13} \gamma_{33}^2$$

$$+ \lambda_{13}^2 \gamma_{11} + \lambda_{13}^2 \gamma_{33} + \frac{1}{2} \lambda_{13}^4 - \lambda_{13} \gamma_{13}^3 - \gamma_{13} \lambda_{13}^3 + \gamma_{13}^2 \gamma_{33} + \gamma_{11} \gamma_{13}^2$$

$$+ \gamma_{13} \gamma_{33} - \frac{1}{2} \lambda_{13}^2 + \lambda_{13} \gamma_{13} - \frac{1}{2} \gamma_{13}^2 + \frac{1}{4} \lambda_{13}^2 \gamma_{11}^2 + \frac{1}{4} \gamma_{33}^2 \lambda_{13}^2 + \frac{1}{2} \gamma_{13}^4$$

$$+ \frac{1}{4} \gamma_{13}^2 \gamma_{11}^2 + \gamma_{13}^2 \lambda_{13}^2 + \frac{1}{4} \gamma_{33}^2 \gamma_{13}^2,$$

$$[\text{ACov}(\hat{G})]_{44} = (-1 + \gamma_{22})^2 (1 + \gamma_{22})^2,$$

$$[\text{ACov}(\hat{G})]_{55} = 1 - \gamma_{33} - \gamma_{22} + \gamma_{22} \lambda_{23}^2 + \gamma_{22} \gamma_{23}^2 + \lambda_{23}^2 \gamma_{33} - 2 \lambda_{23} \gamma_{23} \gamma_{22} + \gamma_{23}^2 \gamma_{33}$$

$$+ \gamma_{22} \gamma_{33} - \frac{1}{2} \lambda_{23}^2 + \lambda_{23} \gamma_{23} - \frac{1}{2} \gamma_{23}^2 + \frac{1}{2} \gamma_{23}^4 + \frac{1}{4} \gamma_{23}^2 \gamma_{22}^2$$

$$+ \frac{1}{4} \gamma_{23}^2 \gamma_{33}^2 + \frac{1}{4} \lambda_{23}^2 \gamma_{33}^2 + \frac{1}{2} \lambda_{23}^4 + \frac{1}{4} \gamma_{22}^2 \lambda_{23}^2 + \lambda_{23}^2 \gamma_{23}^2$$

$$- 2 \lambda_{23} \gamma_{23} \gamma_{33} - \gamma_{23} \lambda_{23}^3 - \lambda_{23} \gamma_{23}^3 - \frac{1}{2} \gamma_{23} \gamma_{23}^2 \lambda_{23} - \frac{1}{2} \gamma_{23} \lambda_{23} \gamma_{33}^2,$$
\[ [A\text{Cov}(\hat{G})]_{66} = (-1 + \gamma_{33})^2 (1 + \gamma_{33})^2, \]

and covariances

\[ [A\text{Cov}(\hat{G})]_{12} = \lambda_{12} - \lambda_{12} \gamma_{12}^2 + \frac{1}{2} \gamma_{12} \lambda_{12}^2 \gamma_{11} + \frac{1}{2} \gamma_{11} \gamma_{12}^2 \\
+ \frac{1}{2} \gamma_{12} \gamma_{11}^3 - \frac{1}{2} \gamma_{11} \gamma_{12} - \lambda_{12} \gamma_{11}^2 + 2 \lambda_{12}^2 \gamma_{12} + \lambda_{12} \gamma_{11} - 2 \gamma_{12} + 2 \gamma_{12} \gamma_{11}^2 \\
- \lambda_{12} \gamma_{11}^3 - \lambda_{12} \gamma_{11} \gamma_{12}^2 - \lambda_{12} \gamma_{11}^3 \gamma_{11} \]

\[ [A\text{Cov}(\hat{G})]_{13} = -\lambda_{13} \gamma_{13}^2 + \lambda_{13} + \frac{1}{2} \gamma_{13} \lambda_{13}^2 \gamma_{11} + \frac{1}{2} \gamma_{11} \gamma_{13}^3 + \frac{1}{2} \gamma_{13} \gamma_{11}^3 \\
- \lambda_{13} \gamma_{11}^2 - \frac{1}{2} \gamma_{11} \gamma_{13} - 2 \gamma_{13} + \lambda_{13} \gamma_{11} + 2 \gamma_{13} \lambda_{13}^2 - \lambda_{13} \gamma_{13}^2 \gamma_{11} \\
- \lambda_{13} \gamma_{11}^3 + 2 \gamma_{11}^2 \gamma_{13} - \lambda_{13} \gamma_{11}^3 \gamma_{11} \]

\[ [A\text{Cov}(\hat{G})]_{14} = \gamma_{12}^2 - 2 \lambda_{12} \gamma_{12} \gamma_{22} + \gamma_{11} \gamma_{12}^2 \gamma_{22} + \lambda_{12}^2 \\
+ \lambda_{12}^2 \gamma_{11} \gamma_{22} - 2 \lambda_{12} \gamma_{11} \gamma_{12} \]

\[ [A\text{Cov}(\hat{G})]_{15} = -\gamma_{11} \lambda_{12} \gamma_{13} + \frac{1}{2} \gamma_{23} \gamma_{11} \lambda_{12}^2 - \gamma_{23} \lambda_{12} \gamma_{12} - \gamma_{11} \gamma_{12} \lambda_{13} \\
+ \frac{1}{2} \gamma_{23} \gamma_{11} \gamma_{12}^2 - \gamma_{23} \lambda_{13} \gamma_{13} + \lambda_{12} \lambda_{13} + \gamma_{12} \gamma_{13} \\
+ \frac{1}{2} \gamma_{23} \gamma_{11} \lambda_{13}^2 + \frac{1}{2} \gamma_{23} \gamma_{11} \gamma_{13}^2 - 2 \gamma_{12} \lambda_{13} - 2 \lambda_{12} \gamma_{13} + 2 \lambda_{23} \lambda_{12} \gamma_{12} \\
- \lambda_{23} \gamma_{11} \gamma_{12}^2 + 2 \gamma_{11} \lambda_{12} \lambda_{13} - \lambda_{23} \gamma_{11} \lambda_{12}^2 - \lambda_{23} \gamma_{11} \gamma_{13}^2 \\
+ 2 \lambda_{23} \lambda_{13} \gamma_{13} + 2 \gamma_{11} \gamma_{12} \gamma_{13} - \lambda_{23} \lambda_{13} \gamma_{11} \gamma_{11} \]

\[ [A\text{Cov}(\hat{G})]_{16} = \gamma_{13}^2 + \lambda_{13}^2 \gamma_{11} \gamma_{33} + \gamma_{11} \gamma_{13}^2 \gamma_{33} - 2 \lambda_{13} \gamma_{13} \gamma_{33} - 2 \lambda_{13} \gamma_{11} \gamma_{13} + \lambda_{13}^2 \]
\[ [\text{ACov}(\bar{G})]_{23} = -\gamma_{11} \lambda_{12} \gamma_{13} - \gamma_{23} \lambda_{12} \gamma_{12} - \gamma_{23} - \gamma_{11} \gamma_{12} \lambda_{13} + \gamma_{11} \gamma_{12} \gamma_{13} \\
+ \lambda_{23} \lambda_{12} \gamma_{12} - \frac{1}{2} \gamma_{12}^2 \lambda_{23} + \gamma_{11} \lambda_{12} \lambda_{13} + \lambda_{23} + \frac{1}{4} \gamma_{12} \lambda_{12}^2 \gamma_{12} \\
+ \frac{1}{4} \gamma_{13} \lambda_{12} \gamma_{11}^2 + \lambda_{23} \lambda_{13} \gamma_{13} + \frac{1}{4} \lambda_{12}^3 \lambda_{13} + \frac{1}{4} \gamma_{13} \lambda_{12} \lambda_{23}^2 \\
+ \frac{1}{4} \gamma_{13} \lambda_{12} \lambda_{13}^2 + \frac{1}{4} \gamma_{13} \lambda_{12} \gamma_{23}^2 - \frac{1}{2} \lambda_{12}^2 \lambda_{23} - \gamma_{23} \lambda_{13} \gamma_{13} - \gamma_{11} \lambda_{23} \\
+ \gamma_{11} \gamma_{23} + \frac{1}{4} \gamma_{12} \lambda_{13} - \lambda_{12} \gamma_{13} + \frac{1}{2} \gamma_{12} \lambda_{13} + \frac{1}{4} \gamma_{12} \gamma_{13} + \frac{1}{4} \gamma_{12} \gamma_{13}^3 \\
- \frac{1}{2} \gamma_{13}^2 \lambda_{23} + \gamma_{23} \gamma_{12}^2 + \frac{1}{4} \lambda_{12} \lambda_{13}^3 + \frac{1}{4} \lambda_{12} \lambda_{13} \gamma_{11}^2 + \frac{1}{4} \lambda_{12} \lambda_{13} \gamma_{13}^2 \\
+ \frac{1}{4} \lambda_{12} \lambda_{13} \lambda_{23}^2 + \frac{1}{4} \lambda_{12} \lambda_{13} \gamma_{23}^2 + \frac{1}{4} \lambda_{12} \gamma_{12} \lambda_{13} - \frac{1}{2} \lambda_{13}^2 \lambda_{23} \\
- \frac{1}{2} \lambda_{13} \lambda_{12} \gamma_{11}^2 - \frac{1}{2} \lambda_{13} \lambda_{12} \gamma_{12} - \frac{1}{2} \lambda_{13} \lambda_{12} \gamma_{13}^2 - \frac{1}{2} \lambda_{13} \lambda_{12} \lambda_{23}^2 \\
- \frac{1}{2} \lambda_{13} \lambda_{12} \gamma_{23}^2 - \frac{1}{2} \lambda_{13} \gamma_{11}^3 - \frac{1}{2} \gamma_{12} \lambda_{13}^3 + \lambda_{13}^2 \gamma_{23} + \frac{1}{4} \gamma_{13} \lambda_{13}^3 \\
\] 

\[ [\text{ACov}(\bar{G})]_{24} = \lambda_{12} + \frac{1}{2} \gamma_{12} \gamma_{22}^3 - \lambda_{12} \gamma_{22}^2 - \frac{1}{2} \gamma_{12} \gamma_{22} + \frac{1}{2} \lambda_{12} \gamma_{12} \gamma_{22} - \lambda_{12} \gamma_{12}^2 + \frac{1}{2} \gamma_{12} \gamma_{22}^3 \gamma_{22} \] 

\[ [\text{ACov}(\bar{G})]_{25} = \frac{1}{4} \gamma_{23} \gamma_{12} \gamma_{13}^2 - \gamma_{13} + \gamma_{13} \gamma_{12}^2 + \frac{1}{4} \gamma_{23} \gamma_{12} \lambda_{23}^2 + \frac{1}{4} \gamma_{12} \gamma_{23}^3 - \frac{1}{2} \lambda_{13} \gamma_{23}^2 \\
+ \lambda_{13} + \frac{1}{4} \gamma_{23} \gamma_{12} \lambda_{13}^2 + \frac{1}{4} \gamma_{23} \gamma_{12} \gamma_{12}^2 + \frac{1}{4} \gamma_{23} \lambda_{12} \gamma_{12} + \frac{1}{4} \gamma_{23} \gamma_{12} \gamma_{13}^2 + \frac{1}{2} \gamma_{12}^2 \lambda_{13} \\
+ \frac{1}{4} \lambda_{12} \lambda_{23} - \lambda_{12} \gamma_{23} + \frac{1}{4} \gamma_{12} \gamma_{23} - \lambda_{13} \gamma_{22} + \gamma_{13} \gamma_{22} \\
- \gamma_{13} \lambda_{23} \gamma_{23} - \lambda_{12} \gamma_{22} \gamma_{23} - \gamma_{12} \gamma_{22} \lambda_{23} + \gamma_{12} \gamma_{22} \gamma_{23} - \gamma_{13} \lambda_{12} \gamma_{12} \\
- \frac{1}{2} \lambda_{23} \gamma_{12} \lambda_{13}^2 - \frac{1}{2} \lambda_{23} \gamma_{12} \gamma_{13}^2 + \lambda_{23} \lambda_{13} \gamma_{13} - \frac{1}{2} \lambda_{23} \gamma_{12} \gamma_{23}^2 - \frac{1}{2} \lambda_{23} \gamma_{12} \gamma_{23}^2 \\
+ \frac{1}{4} \lambda_{23} \lambda_{12} \gamma_{12}^2 - \frac{1}{2} \lambda_{23} \lambda_{12} \gamma_{12} - \frac{1}{2} \lambda_{23} \lambda_{12} \gamma_{13}^2 + \gamma_{13} \lambda_{23}^2 - \frac{1}{2} \lambda_{23} \gamma_{12}^2 - \frac{1}{2} \lambda_{12} \lambda_{23}^3 \\
+ \gamma_{12} \lambda_{12} \lambda_{13} - \frac{1}{2} \lambda_{12} \lambda_{13}^2 + \frac{1}{4} \lambda_{23} \lambda_{12} \lambda_{13} - \frac{1}{2} \lambda_{13} \lambda_{23}^2 + \frac{1}{4} \lambda_{23} \lambda_{12} \lambda_{23}^3 \\
+ \frac{1}{4} \lambda_{23} \lambda_{12} \lambda_{13}^2 + \frac{1}{4} \lambda_{23} \lambda_{12} \gamma_{13}^2 + \frac{1}{4} \lambda_{23} \lambda_{12} \gamma_{12}^2 + \frac{1}{4} \lambda_{23} \lambda_{12} \gamma_{23}^2 \] 

\[ [\text{ACov}(\bar{G})]_{26} = -\lambda_{13} \gamma_{23} \gamma_{33} - \lambda_{13} \gamma_{12} \gamma_{13} - \lambda_{23} \gamma_{12} \gamma_{23} - \gamma_{13} \lambda_{23} \gamma_{33} + \lambda_{13} \lambda_{23} + \gamma_{13} \gamma_{23} \\
+ \frac{1}{2} \gamma_{12} \lambda_{13} \gamma_{33} + \frac{1}{2} \gamma_{12} \lambda_{13} \gamma_{23}^2 + \frac{1}{2} \gamma_{12} \lambda_{23}^2 \gamma_{33} + \frac{1}{2} \gamma_{12} \lambda_{23}^2 \gamma_{23}^2 \]
$$\begin{align*}
\text{[ACov(\tilde{G})]}_{34} &= \lambda_{12} \gamma_{23} + \gamma_{12} \gamma_{23} + \frac{1}{2} \gamma_{13} \gamma_{22} \lambda_{23}^2 + \frac{1}{2} \gamma_{13} \gamma_{22} \gamma_{23}^2 - \gamma_{13} \lambda_{23} \gamma_{23} - \lambda_{12} \gamma_{22} \gamma_{23} \\
&- \gamma_{12} \gamma_{22} \lambda_{23} + \frac{1}{2} \gamma_{13} \gamma_{22} \lambda_{12}^2 + \frac{1}{2} \gamma_{13} \gamma_{22} \gamma_{12}^2 - \gamma_{13} \lambda_{12} \gamma_{12} \\
\text{[ACov(\tilde{G})]}_{35} &= -\gamma_{12} - \lambda_{13} \gamma_{23} \gamma_{33} - \lambda_{13} \gamma_{12} \gamma_{13} - \lambda_{23} \gamma_{12} \gamma_{23} + \lambda_{12}^2 + \lambda_{12} \gamma_{13}^2 \\
&+ \lambda_{13} \lambda_{12} \lambda_{13} - \lambda_{13} \lambda_{23} \gamma_{33} + \lambda_{13} \gamma_{23} \gamma_{33} - \frac{1}{2} \lambda_{23} \gamma_{13} \gamma_{23}^2 + \frac{1}{2} \lambda_{23} \lambda_{13} \gamma_{33} - \frac{1}{2} \lambda_{23} \gamma_{13} \gamma_{33}^2 \\
&- \frac{1}{2} \lambda_{23} \lambda_{13}^3 - \lambda_{12} \lambda_{23} \gamma_{33} + \lambda_{13} \gamma_{23} \gamma_{33} + \frac{1}{2} \lambda_{13} \lambda_{23} - \lambda_{13} \lambda_{23} - \frac{1}{2} \lambda_{13} \lambda_{23} + \frac{1}{4} \gamma_{13} \gamma_{23} \\
&+ \frac{1}{2} \lambda_{13}^2 - \frac{1}{2} \gamma_{23} \lambda_{13}^2 - \frac{1}{2} \gamma_{23} \lambda_{13} \lambda_{23}^2 + \frac{1}{4} \gamma_{23} \gamma_{23} \lambda_{23}^2 + \frac{1}{4} \gamma_{13} \gamma_{23}^3 \\
&+ \frac{1}{4} \gamma_{23} \gamma_{13} \gamma_{23}^2 + \frac{1}{4} \gamma_{23} \lambda_{13} \lambda_{23} + \frac{1}{4} \gamma_{23} \gamma_{23} \lambda_{23}^2 + \frac{1}{4} \gamma_{13} \gamma_{23} \\
&+ \frac{1}{4} \lambda_{23} \gamma_{13}^2 + \frac{1}{4} \lambda_{23} \gamma_{23} \gamma_{23}^2 \lambda_{13} - \frac{1}{2} \lambda_{12} \lambda_{23}^2 + \frac{1}{4} \lambda_{23} \lambda_{13} \gamma_{23}^2 \\
&+ \frac{1}{4} \lambda_{23} \lambda_{13}^3 + \frac{1}{4} \lambda_{23} \lambda_{12}^2 \lambda_{13} + \frac{1}{4} \lambda_{23} \gamma_{23}^2 \lambda_{13}^2 \\
\text{[ACov(\tilde{G})]}_{36} &= \frac{1}{2} \gamma_{13} \lambda_{13}^2 \gamma_{33} + \frac{1}{2} \gamma_{13}^3 \gamma_{33} - \frac{1}{2} \gamma_{33} \gamma_{13} - \gamma_{33} \lambda_{13}^2 - \gamma_{13} \gamma_{13} \lambda_{13} + \frac{1}{2} \gamma_{13} \gamma_{33}^3 + \lambda_{13} \\
\text{[ACov(\tilde{G})]}_{45} &= \frac{1}{2} \gamma_{23} \lambda_{23}^2 \gamma_{22} + \lambda_{23} - \lambda_{23} \gamma_{23}^2 + \frac{1}{2} \gamma_{23} \gamma_{22}^2 - \frac{1}{2} \gamma_{23} \gamma_{22}^2 \lambda_{23} + \frac{1}{2} \gamma_{23}^3 \gamma_{22} \\
&- \gamma_{23} \lambda_{23}^2 + 2 \lambda_{23} \gamma_{23} + 2 \gamma_{23} \gamma_{23}^2 + \gamma_{22} \lambda_{23} - \lambda_{23} \gamma_{23}^2 \gamma_{22} - \lambda_{23} \gamma_{23}^2 \gamma_{22} - 2 \gamma_{23} \\
\text{[ACov(\tilde{G})]}_{46} &= \gamma_{23}^2 + \gamma_{22} \gamma_{23}^2 \gamma_{33} + \gamma_{22} \lambda_{23}^2 \gamma_{33} + \lambda_{23}^2 - 2 \lambda_{23} \gamma_{23} \gamma_{22} - 2 \lambda_{23} \gamma_{23} \gamma_{33} \\
\text{[ACov(\tilde{G})]}_{56} &= \lambda_{23} - \lambda_{23} \gamma_{23}^2 - \lambda_{23} \gamma_{33}^2 + \frac{1}{2} \gamma_{23} \gamma_{33}^3 + \frac{1}{2} \gamma_{23}^3 \gamma_{33} + \frac{1}{2} \gamma_{23} \lambda_{23}^2 \gamma_{33} - \frac{1}{2} \gamma_{23} \gamma_{33}^3 
\end{align*}$$
Example B.2. The following numerical example for $k=3$ and $p=2$ may help sort out the various computations. Suppose

$$
\Sigma = \begin{bmatrix}
25 & 22.5 & 1.5 & 3.0 & 3.0 \\
22.5 & 25 & 3.0 & 1.5 & 3.0 \\
1.5 & 3.0 & 9 & 0.9 & 0.9 \\
3.0 & 1.5 & 0.9 & 9 & 0 \\
3.0 & 3.0 & 0.9 & 0 & 1 \\
3.0 & 3.0 & 0 & 0.9 & 0.5 \\
\end{bmatrix}.
$$

(B.3)

Correspondingly,

$$
C = \begin{bmatrix}
0.9 & 0.2 & 0.6 \\
0.2 & 0.1 & 0 \\
0.6 & 0 & 0.5 \\
\end{bmatrix}, \quad L = \begin{bmatrix}
1 & 0.1 & 0.6 \\
0.1 & 1 & 0.3 \\
0.6 & 0.3 & 1 \\
\end{bmatrix}, \quad \Delta = \begin{bmatrix}
5 & 0 & 0 \\
0 & 3 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}.
$$

To obtain the canonical form of $\Sigma$, let

$$
\Gamma = \begin{bmatrix}
1/2\sqrt{2} & 1/2\sqrt{2} & 0 & 0 & 0 \\
1/2\sqrt{2} & -1/2\sqrt{2} & 0 & 0 & 0 \\
0 & 0 & 1/2\sqrt{2} & 1/2\sqrt{2} & 0 & 0 \\
0 & 0 & 1/2\sqrt{2} & -1/2\sqrt{2} & 0 & 0 \\
0 & 0 & 0 & 0 & 1/2\sqrt{2} & 1/2\sqrt{2} \\
0 & 0 & 0 & 0 & 1/2\sqrt{2} & -1/2\sqrt{2} \\
\end{bmatrix},
$$

and $g$ indicate in matrix form the permutation that conjugates $\Delta \otimes I_p$ and $I_p \otimes \Delta$. It then follows that

$$
g\Gamma \Sigma \Gamma'g' = \begin{bmatrix}
47.6 & 4.52 & 6.0 & 0 & 0 & 0 \\
4.52 & 9.92 & 0.900 & 0 & 0 & 0 \\
6.0 & 0.900 & 1.50 & 0 & 0 & 0 \\
0 & 0 & 0 & 2.40 & -1.50 & 0 \\
0 & 0 & 0 & -1.50 & 8.12 & 0.900 \\
0 & 0 & 0 & 0 & 0.900 & 0.500 \\
\end{bmatrix}.
$$
so that

\[
E = \begin{bmatrix}
47.5 & 4.5 & 6.0 \\
4.5 & 9.9 & 0.9 \\
6.0 & 0.9 & 1.5
\end{bmatrix},
F = \begin{bmatrix}
2.5 & -1.5 & 0 \\
-1.5 & 8.1 & 0.9 \\
0 & 0.9 & 0.5
\end{bmatrix},
\]

\[
E + (p - 1)F = \begin{bmatrix}
50.0 & 3.0 & 6.0 \\
3.0 & 18.0 & 1.8 \\
6.0 & 1.8 & 2.0
\end{bmatrix},
E - F = \begin{bmatrix}
45.0 & 6.0 & 6.0 \\
6.0 & 1.8 & 0 \\
6.0 & 0 & 1.0
\end{bmatrix}.
\]

The following matrix \( S \) is a sample covariance matrix based on a pseudo-sample of size 50 from a multivariate normal with mean zero and covariance matrix \( \Sigma \) as in (B.3),

\[
S = \begin{bmatrix}
24.517 & 22.807 & 5.840 & 5.686 & 2.856 & 2.531 \\
5.840 & 6.704 & 7.695 & 1.753 & 1.406 & 0.537 \\
5.686 & 2.964 & 1.753 & 11.818 & -0.340 & 1.025 \\
2.856 & 3.068 & 1.406 & -0.340 & 0.941 & 0.395 \\
2.531 & 3.188 & 0.537 & 1.025 & 0.395 & 1.029
\end{bmatrix}.
\]

We obtain

\[
g' \Sigma g = \begin{bmatrix}
48.126 & 10.597 & 5.8216 & -0.798 & 1.9470 & 0.1024 \\
10.597 & 11.510 & 1.3140 & 0.9290 & -2.0616 & -0.24800 \\
5.8216 & 1.3140 & 1.3800 & -0.434 & 0.62900 & -0.04400 \\
-0.79800 & 0.92900 & -0.43450 & 2.5080 & -1.7930 & 0.22250 \\
1.9470 & -2.0616 & 0.62900 & -1.7930 & 8.0036 & 1.1170 \\
0.10250 & -0.24800 & -0.04400 & 0.22250 & 1.1170 & 0.59000
\end{bmatrix},
\]

so that

\[
\mathcal{E} = \begin{bmatrix}
48.126 & 10.597 & 5.8216 \\
10.597 & 11.510 & 1.3140 \\
5.8216 & 1.3140 & 1.3800
\end{bmatrix},
\mathcal{F} = \begin{bmatrix}
2.5080 & -1.7930 & 0.22250 \\
-1.7930 & 8.0036 & 1.1170 \\
0.22250 & 1.1170 & 0.59000
\end{bmatrix}.
\]
Maximum likelihood estimation: To estimate $\sigma_1^2$, from (A.1), we use the corresponding diagonal entries of

$$\frac{1}{p} \left( \mathcal{E} + \mathcal{F} \right) = \begin{bmatrix} 25.317 & 4.4020 & 3.0221 \\ 4.4020 & 9.7568 & 1.2155 \\ 3.0221 & 1.2155 & 0.98500 \end{bmatrix}.$$ 

However, because (see (A.5) and (A.6)),

$$e'S_{11}e = 96.245 = p \varepsilon_{11}, \quad ptrS_{11} - e'S_{11}e = 5.017 = p(p - 1)\mathcal{F}_{11},$$

these are MLEs of $\sigma^2$ based on the initial sample covariance matrix. Similarly, the MLE of $\gamma_{tt}$, from (A.1), is the corresponding diagonal entry of the matrix

$$\begin{bmatrix} 0.90094 & 1.4073 & 0.92637 \\ 1.4073 & 0.17969 & 0.081037 \\ 0.92637 & 0.081037 & 0.40102 \end{bmatrix},$$

with entries defined by $[(\mathcal{E} - \mathcal{F}/(p - 1))_{tt}]/[(\mathcal{E} + \mathcal{F})_{tt}]$. These estimates coincide, based on the initial covariance matrix $S$, with (2.1b). The off-diagonal entries of the above matrix are MLE of the ratios $\gamma_{tu}/\lambda_{tu}$. The MLE of $\gamma_{tu}$, from (A.2), is the corresponding off-diagonal entry of the matrix

$$\begin{bmatrix} 0.90090 & 0.39416 & 0.56058 \\ 0.39414 & 0.17968 & 0.031772 \\ 0.56059 & 0.031771 & 0.40100 \end{bmatrix},$$

with entries defined by

$$\frac{(\mathcal{E} - \mathcal{F}/(p - 1))_{tu}}{\sqrt{(\mathcal{E} + \mathcal{F})_{tt}(\mathcal{E} + \mathcal{F})_{uu}}}. \quad \text{(B.4)}$$

These estimates coincide with the estimates expressed as (2.1b).

The MLE of $\lambda_{tu}$, from (A.3), is the corresponding off-diagonal entry of

$$\begin{bmatrix} 0.99996 & 0.28006 & 0.60514 \\ 0.28008 & 0.99997 & 0.39207 \\ 0.60514 & 0.39207 & 0.99992 \end{bmatrix}.$$
These estimates coincide with (2.1c), when based on S. These leads, in summary, to

$$
\hat{G} = \begin{bmatrix}
0.1000 & -0.11400 & 0.040000 \\
-0.11400 & 0.82100 & 0.36200 \\
0.040000 & 0.36200 & 0.59900
\end{bmatrix}.
$$

From Proposition B.5 we obtain

$$
\text{Acov}(\hat{G}) = \begin{bmatrix}
0.036100 & -0.081469 & -0.023031 & 0.033210 & -0.023380 & 0.042440 \\
0.017469 & 0.09388 & 0.05049 & 0.20166 & 0.13486 & 0.085162 \\
0.047709 & 0.00189 & 0.06211 & 0.043520 & 0.06365 & 0.29677 \\
0.033210 & -0.46250 & -0.13384 & 0.93692 & 0.38512 & 0.15201 \\
0.033620 & -0.13290 & -0.15735 & 0.37690 & 0.51876 & 0.32449 \\
0.042440 & -0.086008 & -0.20009 & 0.15201 & 0.39098 & 0.70425
\end{bmatrix}.
$$

To assess the homogeneity hypotheses ($H_1$), let

$$M = \begin{bmatrix}
-1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 1
\end{bmatrix},$$

so that (writing $G$ in vector form) $MG = 0$ represents $H_1$. Then, asymptotically,

$$Q = N(\hat{G}')M'[\text{Acov}(\hat{G}) M']^{-1}M(\hat{G}) \sim \chi^2_{-1},$$

and can be used to assess $H_1$. In the present case, we obtain $Q=41.64$, with 2 degrees of freedom. To assess the lateral-contralateral homogeneity hypotheses ($H_2$), let

$$M = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}.$$
so that, similarly, $MG = 0$ represents $H_2$ and $Q \sim \chi^2_{k(k-1)/2}$. We obtain $Q=17.32$, with 3 degrees of freedom. Similarly, to assess $H_{12}$ we define

$$
M = \begin{bmatrix}
-1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},
$$

so that, similarly, $MG = 0$ represents $H_{12}$ and $Q \sim \chi^2_{(k+2)(k-1)/2}$. We obtain $Q=24.25$, with 5 degrees of freedom.
REFERENCES


