CONDITIONED DIFFUSIONS

by

HALINA FYRDMAN
Stern School of Business
New York University

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Department of Statistics
Sequoia Hall
STANFORD UNIVERSITY
Stanford, California 94305-4065

http://www-stat.stanford.edu
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HALINA FRYDMAN*
New York University

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ABSTRACT. This work considers a diffusion process $X(s), s \geq 0$, subject to killing at a rate which depends on time and current value of a diffusion. A conditioned process is defined, on $[0,t], t > 0$, by restricting the sample space of $X(s), s \geq 0$, to the sample paths which survived up to time $t$. It is shown that conditioned process is a diffusion and its infinitesimal parameters are derived in terms of the infinitesimal parameters of $X(s), s \geq 0$, and in terms of the survival function of $X(s), s \geq 0$. This result is applied to the problem of identification of a parametric form of a killing-rate function such that an unconditioned and an associated conditioned diffusion belong to the same family of diffusions. The specific results are obtained for Gaussian and for branching diffusions, a class of continuous space branching processes. For these diffusions the required killing-rate functions are identified and the properties of associated diffusions are discussed. In particular formulas for marginal survival functions of a Gaussian diffusion, and of a branching diffusion are obtained. A general formula for the survival function of a diffusion when the values of a diffusion are known at discrete times is also discussed. The applications of this work to survival analysis and possible extensions are indicated.

KEY WORDS: conditioned diffusion, killing-rate function, Gaussian and branching diffusions, survival function, stochastic covariate process.

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1. Introduction and Summary

Let $X(s), s \geq 0$, be a vector-valued diffusion process, and let $T$ be a nonnegative random variable with the property

\begin{equation}
P(T > t|X(s), s \geq 0) = P(T > t|X(s), 0 \leq s \leq t)
\end{equation}

for all $t > 0$. Assuming further that the conditional distribution of $T$ has smooth density function we define the conditional hazard function of $T$ as

*Postal address: Stern School of Business, 44 West 4th Street, New York, NY 10012, USA
(2) \[
q(t, X(\cdot)) = -\frac{(d/dt)P(T > t|X(s), 0 \leq s \leq t)}{P(T > t|X(s), 0 \leq s \leq t)}.
\]
It follows from the definition of \( q \) that

(3) \[
P(T > t|X(s), 0 \leq s \leq t) = \exp \left( -\int_0^t q(s, X(\cdot))ds \right).
\]
Suppose that \( X(s), s \geq 0 \), is a diffusion process with the conditional hazard-rate function depending on \( X(s) \) only through its current value, that is, \( q(s, X(\cdot)) = q(s, X(s)) \), where \( q \) is a function of \( s \) and \( X(s) \). In this case, as is customary in literature, we will refer to \( q \) as the killing-rate function, and to \( T \) as the killing time of the process \( X(s) \). Let \( X^*(s; t) \) be a process defined on \( 0 \leq s \leq t \) by

(4) \[
\mathcal{L}(X^*(\cdot; t)) = \mathcal{L}(X(\cdot)|T > t)
\]
where \( \mathcal{L}(Y) \) denotes the distribution law of a random variable \( Y \). Then \( X^*(s; t) \) is the conditioned process.

The main objective of this work is to develop some basic results for conditioned processes defined by (3) and (4), and consider their applications. Although the literature does not contain a comprehensive study of such processes, Karlin and Taylor (1981) consider some examples of conditioned diffusions.

Let

(5) \[
V(t|x; s) \equiv P(T > t|T > s, X(s) = x),
\]
be the marginal survival function of a diffusion \( X(s), s \geq 0 \). Our basic result (Theorem 1) states that if \( X(s), s \geq 0 \) is a diffusion with drift function \( \mu(s, x) \), and diffusion matrix \( D(s, x) \), undergoing killing at the rate \( q(s, x) \), then \( X^*(s; t), 0 \leq s \leq t \), is a diffusion with drift function

(6) \[
\mu^*(s, x) = \mu(s, x) + D(s, x)(\partial \ln V(t|x; s)/\partial x),
\]
and diffusion matrix \( D^*(s, x) = D(s, x) \). This result is established by showing that the transition density function of \( X^*(\tau; t), 0 \leq \tau \leq t \), satisfies an appropriate backward partial differential equation.

We then address the following question: what should be the parametric form of a killing-rate function so that an unconditioned diffusion and an associated conditioned diffusion belong to the same family of diffusions? We consider this question for Gaussian diffusions and for branching diffusions, a class of non-Gaussian diffusions, defined
in (58). After identifying the required \( q(s, x) \) we study the properties of an associated \( X^*(s; t), 0 \leq s \leq t \), and derive an expression for the marginal survival function in (5).

We also consider a more general survival function of a diffusion \( X(s), s \geq 0 \), than in (5), when the values of the process are known at discrete times, \( 0 \leq s = s_0 < s_1 < \ldots < s_j < t < \infty \):

\[
P(T > t| T > s, X(s_k) = x_k, 0 \leq k \leq j).
\]

We show that (7) can be expressed in terms of marginal survival distributions in (5), transition density of an unconditioned process \( X(s), s > 0 \), and transition densities of appropriate conditioned diffusions. This result is formulated for diffusions, but is more generally valid for Markov processes. The distribution in (7) is important for applications to survival analysis.

More specifically, for a Gaussian diffusion we show (Theorem 2) that \( X^*(s; t), 0 \leq s \leq t \), is also a Gaussian diffusion, if and only if the killing-rate function \( q(s, x) \) is a quadratic function of \( x \). Given a quadratic killing-rate function the function \( \partial \ln V(t|x; s)/\partial x \) is a linear function of \( x \). We derive the coefficients of this function as solutions of the Ricatti differential equation and the linear differential equation of a first degree. This establishes the drift function of \( X^*(s; t), 0 \leq s \leq t \). We then provide an explicit expression for \( V(t|x; s) \) in (5) in terms of the parameters of \( X^*(s; t), 0 \leq s \leq t \), and also discuss computation of (7). We illustrate our results with a number of examples. In special cases we obtain closed form expressions for the drift term of \( X^*(s; t), 0 \leq s \leq t \), for the mean and variance function of \( X^*(s; t), 0 \leq s \leq t \), and for survival functions. We also consider the distribution of the process \( X^*(t; t) \) as \( t \to \infty \).

We now relate these results to the existing literature. Theorem 2 was motivated by the result in Yashin (1985) who showed that for a Gaussian diffusion with a quadratic killing-rate function the distribution \( P(X(t) \leq y| T > t, X(s) = x), t > s \), is also Gaussian or, in our terminology, that the random variable \( X^*(t; t) \) has also Gaussian distribution. Theorem 2 extends this result by asserting that the process \( X^*(s; t), 0 \leq s \leq t \) is Gaussian. For a Gaussian diffusion with a quadratic \( q(s, x) \), Yashin (1985), Yashin, Manton and Stallard (1986a,b), and Yashin and Manton (1997) obtained the formula for the survival function (7) and in particular for (5). Their formula is based on the general expression for survival function with partial information, and Gaussian property of \( X^*(t; t) \). Our approach leads to a different formula for evaluation of (7). It would be of interest to compare the two formulas in terms of
their computational efficiency. Our formula for (5) generalizes the result in Myers (1981), which in turn subsumes the formula of Cameron and Martin (1944), to an arbitrary Gaussian diffusion with a quadratic $q(s, x)$. Myers (1981) uses stochastic calculus methods suggested by the Lipster and Shiryayev (1977) proof of Cameron and Martin (1944) formula. Our approach is different; it uses (6) and the backward differential equation for (5) to derive the parameters of $X^*(s; t), 0 \leq s \leq t$, and consequently the formulas for (5) and (7).

We apply our methods to branching diffusions. This class of diffusions plays an important role in the theory of branching processes (see Overbeck (1998)), and has been extensively used in financial modelling (see, e.g., Cox, Ingersoll and Ross (1985) or Duffie (1992)). For a branching diffusion we show (Theorem 3) that an associated conditioned diffusion is also a branching diffusion if and only if $q(s, x)$ is a linear function of $x$. Similarly as for Gaussian diffusions, we establish the parameters of the drift term of $X^*(s; t), 0 \leq s \leq t$, derive explicit expressions for the mean and variance function of $X^*(s; t), 0 \leq s \leq t$, and for the survival function in (5). Specializing to the branching diffusion with constant coefficients we obtain closed form expressions for all these functions.

The application of our approach to the issue of the identification of a killing-rate function such that an unconditioned and an associated conditioned diffusion belong to the same family of diffusions for other than Gaussian and square-root diffusions is currently under investigation. The methods developed here seem also applicable to the study of more general questions such as identification of a parametric form of $q(s, x)$ such that an unconditioned and conditioned diffusion belong to two different specified families of diffusions, e.g., their drift functions could be polynomials of different degrees in $x$. We leave this topic for future investigation.

This work has applications to survival analysis: if the conditional hazard-rate of failure $q$ is a function of a current value of a "marker", a time-varying covariate represented by a stochastic process $X(s), s \geq 0$, and time $s$, i.e., $q = q(s, X(s))$, then $X^*(s; t), 0 \leq s \leq t$, represents the evolution of a marker in a population that has survived up to time $t$. It is useful in this context to be able to identify the parametric form of $q$ which when combined with an underlying diffusion model would result in a conditioned diffusion with tractable properties. The development of methods for the computation of survival function from different observational plans of $X(s), s \geq 0$ is important for predicting probabilities of survival and for the estimation of unknown parameters of a hazard-rate function. The results obtained here for Gaussian processes can be used to provide an alternative form of the likelihood function for the
estimation problem considered in Yashin and Manton (1997). The results for branching diffusions can be similarly explored for application purposes. Marker models have been widely applied in medicine (see, e.g. Jewell and Kalbfleisch (1992), and Yashin and Manton (1997)), and more recently in finance (see, e.g., Lando (1996)). A different application of conditioned diffusions in finance is presented in Brown, Goetzmann and Ross (1995). Some results for continuous time Markov chains with killing are discussed in Berman and Frydman (1996).

The general results for conditioned diffusions are presented in in Section 2. The results for Gaussian diffusions are developed in Section 3, and for branching diffusions in Section 4.

2. Conditioned diffusions

Let \( X(s), s \geq 0 \), be a \( k \)-dimensional diffusion with the generator \( G \) defined by

\[
G := \frac{1}{2} \sum_{i,j=1}^{k} d_{ij}(s,x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{k} \mu_i(s,x) \frac{\partial}{\partial x_i},
\]

where \( \mu(s,x) = (\mu_1(s,x), ..., \mu_k(s,x)) \) is a drift vector and \( D(s,x) = (d_{ij}(s,x)) \) is a diffusion matrix. Let \( T \) be the killing time of the process \( X(s), s \geq 0 \). We consider the following distributions

\[
U \equiv U(\tau,y;t|x;s) = P(T > t, X(\tau) \leq y | T > s, X(s) = x),
\]

and

\[
R \equiv R(y;t|x;s) \equiv U(t,y;t|x;s),
\]

where \( 0 \leq s \leq \tau \leq t < \infty \). Then by (3) \( R \), and \( V \) given by (5), have the representations

\[
R(y;t|x;s) = E \left[ 1(X(t) \leq y) \exp \left\{ - \int_{s}^{t} q(u,X(u))du \right\} | X(s) = x \right],
\]

and

\[
V(t|x;s) = E \left[ \exp \left\{ - \int_{s}^{t} q(u,X(u))du \right\} | X(s) = x \right],
\]
respectively, and, as Feynman-Kac functionals, they satisfy, see, e.g., Rogers and Williams (1994)

\[
\frac{\partial R}{\partial s} = GR - q(s, x)R,
\]
and

\[
\frac{\partial V}{\partial s} = GV - q(s, x)V,
\]
with boundary conditions

\[R(y; t|x; t) = 1(x \leq y),\]
and

\[V(t|x; t) = 1,\]
respectively. Let \(r(y; t|x; s) = (d/dy)R(y; t|x; s)\). Here and in what follows we assume existence and smoothness of relevant functions. Then by (12) we also have

\[
\frac{\partial r}{\partial s} = Gr - q(s, x)r,
\]
with boundary condition

\[r(y; t|x; t) = \delta_{xy},\]
where \(\delta_{xy}\) is a Dirac delta function. Finally let

\[
h(y, \tau|x, s) = (d/dy)P(X(\tau) \leq y|X(s) = x),
\]
and

\[
h^*(y, \tau|x, s; t) = (d/dy)P(X(\tau) \leq y|T > t, X(s) = x),
\]
where \(0 \leq s \leq \tau \leq t < \infty\). We note that \(h(y, \tau|x, s)\) and \(h^*(y, \tau|x, s; t)\) are the transition density of an unconditioned and conditioned process, respectively.

**Theorem 1.** Let \(X(s), s \geq 0\), be a diffusion with the generator \(G\) defined in (8), and smooth transition density function \(h(y, \tau|x, s), 0 \leq s \leq \tau < \infty\). Then the conditioned process \(X^*(s; t), s \leq t\) is a diffusion with the generator

\[
G^* = \frac{1}{2} \sum_{i,j=1}^{k} d_{ij}(s, x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{k} \mu_i(s, x) + \sum_{j=1}^{k} d_{ij}(s, x)(\partial \ln V/\partial x_j) \frac{\partial}{\partial x_i},
\]
where \(V = V(t|x; s)\).
**Proof:** By definition the conditioned process has continuous sample and is a Markov process, and therefore it is a diffusion. It remains to show that $G^*$ is its generator. From the definition of $U$ in (9) we have
\[
u \equiv u(\tau, y; t|x; s) = \frac{dU(\tau, y; t|x; s)}{dy} =
\]
\[
P(T > t|T > \tau, X(\tau) = y)\frac{d}{dy}P(T > \tau, X(\tau) \leq y|T > s, X(s) = x) = V(t|y; \tau)r(y; \tau|x; s),
\]
and thus by (14)
\[
(17) \quad -\frac{\partial u}{\partial s} = Gu - qu.
\]
where $u = u(\tau, y; t|x; s)$ and $q = q(s, x)$. But, clearly, we can also express $u(\tau, y; t|x; s)$ as
\[
(18) \quad u(\tau, y; t|x; s) = h^*(y, \tau|x, s; t)V(t|x; s).
\]
To simplify notation in the rest of the proof we will suppress the dependence of the functions on their arguments. Now by (18), (13) and (17) we have
\[
\frac{\partial u}{\partial s} = \frac{\partial h^*V}{\partial s} = V \frac{\partial h^*}{\partial s} + h^* \frac{\partial V}{\partial s} =
\]
\[
V \frac{\partial h^*}{\partial s} + h^*(-GV + qV) = -Gh^*V + qh^*V,
\]
which implies that
\[
\frac{V \partial h^*}{\partial s} = -Gh^*V + h^*GV
\]
\[
= -\sum_{i=1}^{k} \mu_i \frac{\partial h^*V}{\partial x_i} - \frac{1}{2} \sum_{i,j=1}^{k} d_{ij} \frac{\partial^2 h^*V}{\partial x_i \partial x_j} + h^* \sum_{i=1}^{k} \mu_i \frac{\partial V}{\partial x_i} + \frac{1}{2} h^* \sum_{i,j=1}^{k} d_{ij} \frac{\partial^2 V}{\partial x_i \partial x_j}
\]
\[
= -\sum_{i=1}^{k} \mu_i V \frac{\partial h^*}{\partial x_i} - \frac{1}{2} \sum_{i,j=1}^{k} d_{ij} \frac{\partial (\partial h^*V)}{\partial x_j} + \frac{1}{2} h^* \sum_{i,j=1}^{k} d_{ij} \frac{\partial^2 V}{\partial x_i \partial x_j}
\]
\[
= -V \sum_{i=1}^{k} \mu_i \frac{\partial h^*}{\partial x_i} - \frac{1}{2} \sum_{i,j=1}^{k} d_{ij} \frac{\partial h^*V}{\partial x_i, \partial x_j} + \frac{\partial V}{\partial x_i} \frac{\partial h^*}{\partial x_j} + V \frac{\partial^2 h^*}{\partial x_i \partial x_j},
\]
or equivalently, that $h^*$ satisfies the backward equation

$$\frac{\partial h^*}{\partial s} = \sum_{i=1}^{k} \mu_i + \sum_{j=1}^{k} d_{ij} V^{-1}_{ij} \frac{\partial V}{\partial x_i} \frac{\partial h^*}{\partial x_j} + \frac{1}{2} \sum_{i,j=1}^{k} d_{ij} \frac{\partial^2 h^*}{\partial x_i \partial x_j},$$

which completes the proof. □

The next result is concerned with the general formula for survival function of a Markov process with killing when this process is observed at a sequence of time points. We formulate the result for diffusions, but similar result is valid for an arbitrary Markov process.

**Proposition 1.** Let $X(s), s \geq 0$ be a diffusion with transition density $h(y, \tau|x, s), s < \tau$, undergoing killing at a rate $q(s, x)$. Let $0 \leq s = s_0 < s_1 < \ldots < s_{j-1} < s_j = t < \infty$, be a sequence of times. Then

$$P(T > t|T > s, X(s_k) = x_k, 0 \leq k \leq j - 1)$$

$$= V(t|x_{j-1}; s_{j-1}) \prod_{k=1}^{j-1} h^*_k V(s_k|x_k-1; s_{k-1}),$$

(19)

where $h_k = h(x_k, s_k|x_{k-1}, s_{k-1}), h^*_k = h^*(x_k, s_k|x_{k-1}, s_{k-1}; s_k)$, and $V(s_k|x_k-1; s_{k-1})$ is given by (5).

**Proof:** By (3)

$$P(T > t|T > s, X(s_k) = x_k, 0 \leq k \leq j - 1) =$$

$$E \left\{ \exp \left( - \int_{s}^{t} q(u, X(u)) du \right) \left| X(s_k) = x_k, 0 \leq k \leq j - 1 \right. \right\} =$$

$$E \left[ \prod_{k=1}^{j} \exp \left( - \int_{s_k}^{s_{k-1}} q(u, X(u)) du \right) \left| X(s_k) = x_k, 0 \leq k \leq j - 1 \right. \right].$$

(20)

The Markov property implies that the subprocesses $X(s), s_{k-1} \leq s \leq s_k$, are conditionally independent, given the values $X(s_k), k = 1, \ldots, j - 1$, and that the conditional distribution of $X(s), s_{k-1} \leq s \leq s_k$, depends only on $X(s_{k-1})$ and $X(s_k)$. Thus (20) takes the form
\[
\prod_{k=1}^{j-1} E \left\{ \exp \left( - \int_{s_{k-1}}^{s_k} q(u, X(u)) du \right) | X(s_{k-1}) = x_{k-1}, X(s_k) = x_k \right\} 
\times E \left[ \exp \left( - \int_{s_{j-1}}^{t} q(u, X(u)) du \right) | X(s_{j-1}) = x_{j-1} \right]. 
\]

But, for \(1 \leq k \leq j-1\),

\[
E \left\{ \exp \left( - \int_{s_{k-1}}^{s_k} q(u, X(u)) du \right) | X(s_{k-1}) = x_{k-1}, X(s_k) = x_k \right\} = \frac{h^*_k}{h_k} V(s_k|x_{k-1}; s_{k-1}),
\]

which completes the proof. ■

It is seen from (19) that the survival function (7) can be expressed in terms of marginal survival distributions \(V(t|x; s)\) and transition densities \(h\) and \(h^*\).

For our development it is useful to rewrite the backward equation in (13) as follows

\[
- \frac{\partial \ln V}{\partial s} = \mu \frac{\partial \ln V}{\partial x} + \frac{1}{2} \left( \frac{\partial \ln V}{\partial x} \right)' D \left( \frac{\partial \ln V}{\partial x} \right) \\
+ \frac{1}{2} \sum_{i,j=1}^{k} d_{ij} \frac{\partial}{\partial x_i} \left( \frac{\partial \ln V}{\partial x_j} \right) - q,
\]

(21)

where \(z'\) denotes the transpose of a vector \(z\), and we suppressed the dependence of the functions on their arguments.

3. Gaussian Diffusions

Let \(X(s), s \geq 0\) be a \(k\)-dimensional diffusion with drift vector \(\mu(s, x) = a(s) + A(s)x\), and state-independent diffusion matrix \(D(s) = ((d_{ij}(s)))\), where \(a(s)\) is a continuous, \(k \times 1\) vector-valued function, and \(A(s)\) and \(D(s)\) are continuous, \(k \times k\) matrix-valued functions. We further assume that \(X(0) = x_0\), where \(x_0\) is a constant vector. Then \(X(s), s \geq 0\), is a Gaussian diffusion satisfying

\[
dX(s) = [a(s) + A(s)X(s)]ds + \sigma(s)dW(s), X(0) = x_0,
\]

(22)
where $\sigma(s)$ is such that $\sigma\sigma' = D$ and $W(s)$ is a $k$-dimensional standard Brownian motion. Let

\begin{equation}
q(s, x) = x'\theta_1(s)x + x'\theta_2(s) + \theta_3(s),
\end{equation}

where $\theta_1(s)$ is a $k \times k$ matrix-valued function, $\theta_2(s)$ is a $k \times 1$ vector-valued function, $\theta_3(s)$ is a scalar function, and $\theta_1, \theta_2$ and $\theta_3$ are such that $q(s, x) \geq 0$. The following theorem shows that conditioned diffusion corresponding to a Gaussian diffusion with a quadratic killing-rate function is also Gaussian.

**Theorem 2.** Let $X(s), s \geq 0$ be a Gaussian diffusion defined in (22). Then the conditioned process $X^*(s; t), s \leq t$, is also a Gaussian diffusion if and only if $q(s, x)$ is a quadratic function of $x$. If $q(s, x)$ is given by (23) then $\mu^*(s; t)$, the drift vector of $X^*(s; t), s \leq t$, is

\begin{equation}
\mu^*(s; t) = a(s) + A(s)x + D(s)[b^*(s; t) + A^*(s; t)x],
\end{equation}

where $A^*(s; t)$ is a $k \times k$ symmetric matrix which is a solution of the Riccati equation

\begin{equation}
\frac{\partial A^*(s; t)}{\partial s} = -A^*(s; t)D(s)A^*(s; t) - 2A'(s)A^*(s; t) + 2\theta_1(s),
\end{equation}

with boundary condition

\begin{equation}
A^*(t; t) = 0,
\end{equation}

and $b^*(s; t)$ is a $k \times 1$ vector which is a solution of the linear equation of first degree

\begin{equation}
\frac{\partial b^*(s; t)}{\partial s} = -[A'(s) + A^*(s; t)D(s)]b^*(s; t) - A^*(s; t)a(s) + \theta_2(s),
\end{equation}

with boundary condition

\begin{equation}
b^*(t; t) = 0.
\end{equation}

**Proof:** Suppose that conditioned diffusion is Gaussian. Then, by Theorem 1, we must have

\begin{equation}
\partial \ln V(t|x; s)/\partial x = A^*(s; t)x + b^*(s; t),
\end{equation}

for some $k \times k$ matrix-valued function $A^*(s; t)$, and $k \times 1$ vector-valued function $b^*(s; t)$. Equation (29) implies that
(30) \[ \ln V(t|x; s) = \frac{1}{2} x' A^*(s; t)x + x'b^*(s; t) + c(s; t), \]

for some function \( c^*(s; t) \), and also that \( A^*(s; t) \) is a symmetric matrix. Substituting (29) and (30) into (21) we obtain the following equation:

\[
\begin{align*}
-\frac{1}{2} x' \frac{\partial A^*}{\partial s} x - x' \frac{\partial b^*}{\partial s} - \frac{\partial c^*}{\partial s} \\
= (Ax + a)'(A^*x + b^*) + \frac{1}{2} (A^*x + b^*)' D(A^*x + b^*) + \\
\end{align*}
\]

(31) \[ \frac{1}{2} tr(DA^*) - q \]

where \( A^* = A^*(s; t), b^* = b^*(s, t), c^* = c^*(s; t), A = A(s), a = a(s), D = D(s), \) and \( q = q(s, x) \). It now follows immediately from inspection of (31) that \( q \) must indeed be a quadratic function of \( x \). We now show that if \( q \) is given by (23) then we can choose \( A^* \), \( b^* \) and \( c^* \) so that (31) holds. To this end we rewrite (31) as follows

\[
\begin{align*}
-\frac{1}{2} x' \frac{\partial A^*}{\partial s} x - x' \frac{\partial b^*}{\partial s} - \frac{\partial c^*}{\partial s} \\
= x' A' A^* x + \frac{1}{2} x' A'' D A^* x - x' \theta_1 x' \\
+ x' A'' a + x' A'b^* + x' A'' D b^* \\
- x' \theta_2 + a'b^* + \frac{1}{2} b'' D b^* \\
+ \frac{1}{2} tr(DA^*) - \theta_3, \\
\end{align*}
\]

(32)

where \( \theta_i = \theta_i(s) \) for \( i = 1, 2, 3 \). Equating the matrices of the quadratic forms in \( x \) on the left and right hand side of (32) gives (25). Similarly equating coefficient vectors of \( x' \) gives (27). Now by (30) and (11)

\[
\ln V(t|x; s) = \ln E\left\{ \exp \left[ - \int_{s}^{t} (q(u, X(u))du \right] \right| X(s) = x \}
\]

(33) \[ = \frac{1}{2} x' A^*(s; t)x + x'b^*(s; t) + c^*(s; t), \]

which implies stated boundary conditions (26) and (28), and completes the proof. \( \square \)

The methods for solving the Ricatti equation are discussed in Bellman and Kalaba (1965). The immediate consequence of Theorem 2 is the following formula for the marginal survival function.
Proposition 2. Let $X(s), s \geq 0$, be defined by (22) and $q(s, x)$ by (23) then
\[ V(t|x; s) = \exp \left[ \frac{1}{2} x'^* A^*(s; t)x + x'b^*(s; t) + c^*(s; t) \right], \]
where $A^*(s; t)$ satisfies (25) and (26), $b^*(s; t)$ satisfies (27) and (28), and $c^*(s; t)$ satisfies
\begin{equation}
\frac{\partial c^*(s; t)}{\partial s} = -\frac{1}{2} b'^* Db^* - \frac{1}{2} tr(DA^*) - a'b^* + \theta_3.
\end{equation}
with boundary condition $c^*(t, t) = 0$.

Proof: Follows from the proof of Theorem 2. The equation in (34) is obtained by equating constant terms with respect to $x$ in (32), and the condition $c^*(t, t) = 0$ follows from (33). □

Proposition 2 generalizes the result in Myers (1981) to an arbitrary Gaussian diffusion with a quadratic killing-rate function. We next consider the mean and variance function of the conditioned diffusion.

Corollary 1. Under the same assumptions as in Proposition 2, the mean and variance function of $X^*(\tau; t), 0 \leq s \leq \tau \leq t$, conditional on $X^*(s; t) = x$, a constant vector, are given by
\begin{equation}
EX^*(\tau; t) = \Phi(\tau) \left\{ x + \int_s^\tau \Phi^{-1}(r)(a(r) + D(r)b^*(r; t))dr \right\},
\end{equation}
and
\begin{equation}
V(X^*(\tau; t)) = \Phi(\tau) \left[ \int_s^\tau \Phi^{-1}(r)D(r)(\Phi^{-1}(r))'dr \right] \Phi(\tau)',
\end{equation}
respectively, where $\Phi(\tau) = \exp \left[ \int_s^\tau (A(u) + D(u)A^*(u; t))du \right]$.

Proof. Since, by Theorem 2, $X^*(s; t), s \leq t$, is a Gaussian diffusion, the expressions (35) and (36) follow immediately from the results for Gaussian diffusions, see, e.g., Arnold(1992). □

Above results can be used to compute survival function in (19) under assumptions of Proposition 2. It follows from Theorem 2 and Corollary 1 that $h_*^s$ is the density function of a normal random variable with mean given by (35) and variance given by (36) with $x = s_{k-1}$ and $\tau = t = s_k$. The marginal distribution $V(s_k|x_{k-1}; s_{k-1})$ can be computed using Proposition 2.

We next apply Theorem 2 and Proposition 2 to the model considered by Myers (1981).
Corollary 2. Let \( a(s) = -A(s)\gamma \) in (22), where \( \gamma \) is a \( k \times 1 \) constant vector, so that (22) becomes

\[
dX(s) = A(s)(X(s) - \gamma)ds + \sigma(s)dW(t), X(0) = x_0,
\]

and let

\[
q(s, x) = (x - \gamma)'\theta(s)(x - \gamma) + \eta(s),
\]

where \( \theta(s) \) is a nonnegative-definite, symmetric, \( k \times k \) matrix-valued function, and \( \eta(s) \) is a non-negative function. Then

\[
\mu^*(s, x) = (A(s) + D(s)A^*(s; t))(x - \gamma),
\]

and (Myers (1981))

\[
V(t|x; s) = \exp \left[ \frac{1}{2}(x - \gamma)'A^*(s; t)(x - \gamma) + \frac{1}{2} \int_s^t \text{tr} (D(u)A^*(u; t)) du \right] \times \exp \left[ -\int_s^t \eta(u)du \right],
\]

where \( A^*(s; t) \) satisfies

\[
\frac{\partial A^*(s; t)}{\partial s} = -A^*(s; t)D(s)A^*(s; t) - 2A'(s)A^*(s; t) + 2\theta(s),
\]

with

\[
A^*(t; t) = 0.
\]

Proof: Under stated assumptions equations in (25) and (27) become

\[
\frac{\partial A^*(s; t)}{\partial s} = -A^*(s; t)D(s)A^*(s; t) - 2A'(s)A^*(s; t) + 2\theta(s),
\]

and

\[
\frac{\partial b^*(s; t)}{\partial s} = -[A'(s) + A^*(s; t)D(s)]b^*(s; t) + A''(s; t)A(s)\gamma - 2\theta(s)\gamma.
\]

It is easily seen that the solution to (43) is of the form

\[
b^*(s; t) = -A^*(s; t)\gamma
\]
where \( A^*(s; t) \) satisfies (42) and thus (39) follows by Theorem 2. To compute survival function we consider equation (34) which, by (44), takes the form

\[
\frac{\partial c^*(s; t)}{\partial s} = -\frac{1}{2} \gamma' A''(s; t) D(s) A^*(s; t) \gamma - \frac{1}{2} \text{tr} \left( D(s) A^*(s; t) \right) \\
\gamma' A(s; t) \gamma + 2 \gamma' \dot{\theta}(s) \gamma + \eta(s) \\
= \frac{1}{2} \gamma' \frac{\partial A^*(s; t)}{\partial s} \gamma - \frac{1}{2} \text{tr} \left( D(s) A^*(s; t) \right) + \eta(s).
\]

(45)

Integrating (45) and taking into account boundary condition gives

\[
c^*(s; t) = \frac{1}{2} \gamma' A^*(s; t) \gamma + \frac{1}{2} \int_s^t \text{tr} \left( D(u) A^*(u; t) \right) du - \int_s^t \eta(u) du,
\]

so that by Proposition 2, (44) and (46) the survival function is indeed given by (40).

If \( X(s), s \geq 0 \) in (37) is a one-dimensional diffusion with constant coefficients: \( A(s) = A \) and \( D(s) = \sigma^2 \), and \( \theta(s) \equiv \theta \), where \( \theta \) is a positive constant, then, see Myers (1981), (42) with (26) has a closed form solution given by

\[
A^*(u; t) = -2\theta \frac{\exp[d(t - u)] - \exp[-g(t - u)]}{d \exp[-g(t - u)] + g \exp[d(t - u)]}, 0 \leq u \leq t,
\]

(47)

where \( f = \sqrt{A^2 + 2\sigma^2 \theta} \), \( d = f + A \) and \( g = f - A \). We note that \( A^*(u; t) < 0 \). Thus if also \( A < 0 \), then, by (39), \( X^*(s; t), s \leq t \), reverts to \( \gamma \) faster than \( X(s), s \geq 0 \). Since

\[
\sigma^2 \int_s^t A^*(u; t) du = \ln \{ d \exp[-g(t - u)] + g \exp[d(t - u)] \} |_s^t,
\]

(48)

it is seen from (40), (47) and (48), that survival function (5) can be evaluated in closed form. Similarly survival function (7) can be evaluated in closed form since, as shown below, \( \text{EX}^* (\tau; t) \) and \( V(X^*(\tau; t)) \) admit closed forms. By Corollary 1 and (48)

\[
\text{EX}^*(\tau; t) = \exp[(\tau - s)A] \frac{d \exp[-g(t - \tau)] + g \exp[d(t - \tau)]}{d \exp[-g(t - s)] + g \exp[d(t - s)]} (x - \gamma) + \gamma.
\]

To evaluate \( V(X^*(\tau; t)) \), let

\[
B(r) = \frac{\exp[-g(t - r)] - \exp[d(t - r)]}{d \exp[-g(t - r)] + g \exp[d(t - r)]},
\]
and note that
\[
\int_s^r \frac{\exp[2A(t-r)]}{d \exp[-g(t-r)] + g \exp[d(t-r)]} dr = (g + d)^{-2}B(r)|_s^r,
\]
so that by Corollary 1
\[
V(X^*(\tau; t)) = \sigma^2 \exp[-2A(t-\tau)] (g + d)^{-2} \times \\
\{d \exp[-g(t-\tau)] + g \exp[d(t-\tau)]\}^2 (B(\tau) - B(s)).
\]

In particular
\[
(49) \quad E X^*(t; t) = \frac{d + g}{d \exp[-f(t-s)] + g \exp[f(t-s)]} (x - \gamma) + \gamma,
\]
and
\[
(50) \quad V(X^*(t; t)) = \sigma^2 \left\{ \frac{\exp[f(t-s)] - \exp[-f(t-s)]}{d \exp[-f(t-s)] + g \exp[f(t-s)]} \right\}.
\]

We can now obtain the asymptotic distribution of the process \(X^*(t; t)\), as \(t \to \infty\). By Theorem 2, (49) and (50), this distribution is normal with mean equal to \(\gamma\) and variance equal to \(\sigma^2/g\). When \(\theta = 0\), i.e., when the killing-rate function is state-independent, and \(A < 0\), the asymptotic distribution of \(X^*(t; t)\) reduces to the asymptotic distribution of the unconditioned diffusion, namely, it is normal with mean equal to \(\gamma\) and variance equal to \(\sigma^2/(-2A)\).

The following special case of Corollary 2 is of historical interest as it leads to the first result concerning survival function of a diffusion with killing.

**Example 1.**

Let \(A(s) \equiv 0\), \(\sigma(s) = I\), an identity matrix, and \(x_0 = 0\) in (37) so that \(X(t), t \geq 0\) is a \(k\)-dimensional standard Brownian motion:
\[
(51) \quad dX(t) = dW(t), X(0) = 0,
\]
and let \(\gamma = 0\) and \(\eta(s) \equiv 0\) in (38) so that
\[
(52) \quad q(s, x) = x' \theta(s)x.
\]
Then
\[
(53) \quad dX^*(s; t) = A^*(s; t)X^*(s; t)ds + dW(s), X^*(0; t) = 0,
\]
and (Cameron and Martin (1944))

\begin{equation}
V(t|0; 0) = \exp \left[ \frac{1}{2} \int_0^t \tr A^*(u; t) du \right],
\end{equation}

where \( A^*(u; t) \) is a symmetric, nonpositive-definite matrix which satisfies

\begin{equation}
\frac{\partial A^*(s; t)}{\partial s} = -A^*(s; t) A^*(s; t) + 2 \theta(s),
\end{equation}

with boundary condition \( A^*(t; t) = 0 \).

When \( W(s) \) is a one-dimensional standard Brownian motion, and \( \theta(s) \equiv \theta \), then by (47) the solution to (55) is

\begin{equation}
A^*(s; t) = -\sqrt{2\theta} \tanh \left[ \sqrt{2\theta}(t - s) \right],
\end{equation}

and thus in this case (54) becomes

\begin{equation}
V(t|0; 0) = \left[ \cosh \sqrt{2\theta} t \right]^{-1/2}.
\end{equation}

In our final example we show that for one-dimensional Brownian motion with nonzero drift and killing-rate function given by (38) with \( \theta(s) \equiv \theta \), one again obtains closed form expressions for the drift function of the conditioned diffusion and for the survival function. We also discuss the asymptotic distribution of \( X^*(t; t) \) as \( t \to \infty \). We note that this example is not a special case of Corollary 2.

Let \( X(t), t \geq 0 \) be a one-dimensional Brownian motion:

\[ dX(t) = \alpha dt + \sigma dW(t), X(0) = x_0, \]

with the killing-rate function given by

\[ q(s, x) = (x - \gamma)' \theta(x - \gamma) + \eta(s), \]

where \( \theta \) is a positive constant, \( \eta(s) \) is a non-negative function, and \( \gamma \) is a constant. Then equations in (25) and (27) take the form

\[ \frac{\partial A^*(s; t)}{\partial s} = -\sigma^2 A^2(s; t) + 2 \theta, \]

and

\[ \frac{\partial b^*(s; t)}{\partial s} = -\sigma^2 A^*(s; t) b^*(s; t) - A^*(s; t) a - 2 \theta \gamma, \]
respectively. The solution to the above equations satisfying (26) and (28) is given by

\[ A^*(s; t) = -\sqrt{2\theta} \tanh \left( \sqrt{2\theta} \sigma(t - s) \right) / \sigma, \]

and

\[ b^*(s; t) = \sigma^{-2} a \left\{ \cosh^{-1} \left( \sqrt{2\theta} \sigma(t - s) \right) - 1 \right\} - \gamma A^*(s; t), \]

so that by (24), \( \mu^* = a \cosh^{-1} \left[ \sqrt{2\theta} \sigma(t - s) \right] - \sqrt{2\theta} \sigma(x - \gamma) \tanh \left[ \sqrt{2\theta} \sigma(t - s) \right] \).

Furthermore, a straightforward computation based on Corollary 1 with \( \tau = t \), gives

\[ EX^*(t; t) = x \cosh^{-1} \left[ \sqrt{2\theta} \sigma(t - s) \right] + \frac{a}{\sigma \sqrt{2\theta}} \tanh \left[ \sqrt{2\theta} \sigma(t - s) \right] + \gamma \left\{ 1 - \cosh^{-1} \left[ \sqrt{2\theta} \sigma(t - s) \right] \right\}, \]

and

\[ V(X^*(t; t)) = \frac{\sigma}{\sqrt{2\theta}} \tanh \left[ \sqrt{2\theta} \sigma(t - s) \right]. \]

It is seen from the above statements that survival functions (5) and (7) are available in closed forms. The asymptotic distribution of the process \( X^*(t; t), t \geq 0 \), as \( t \to \infty \), is normal with mean \( \gamma + a / \sqrt{2\theta} \sigma \) and variance \( \sigma / \sqrt{2\theta} \).

4. Branching diffusions

We consider a one-dimensional diffusion which satisfies

\[ dX(s) = [a(s) + A(s)X(s)] ds + \sigma_1(s) X(s)^{1/2} dW(s), X(0) = x_0, \tag{58} \]

where \( x_0 \geq 0 \) is a constant, and \( \sigma_1(s) > 0, a(s) > 0, A(s) \) are smooth functions. The condition \( a(s) > 0 \) guarantees that the state space of this diffusion is nonnegative real line, and that 0 is not an absorbing boundary. The processes generated by (58), when all coefficient functions are assumed to be constants, have been studied since Feller (1951); for the discussion of qualitative behavior of these processes and other references, see Overbeck (1998). These processes arise as limits of discrete space branching processes with immigration and thus we refer to them, and more generally to the processes generated by (58), as branching diffusions. In financial literature these processes are often called square-root diffusions, see, e.g., Duffie (1992). The theorem below specifies the form of a killing-rate function for (58) so that the drift function of the associated conditioned diffusion is also a linear function of \( x \).
Theorem 3. Let $X(t), t \geq 0$, be a diffusion satisfying (58). Then the conditioned process $X^*(s; t), s \leq t$, is a diffusion with a drift function

$$
\mu^*(s; t) = a(s) + [A(s) + \sigma_1^2(s)A^*(s; t)]x
$$

for some function $A^*(s; t)$ if and only if $q(s, x)$ is a linear function of $x$. If

$$
q(s, x) = \theta_1(s)x + \theta_2(s),
$$

where $\theta_1(s), \theta_2(s)$ are continuous functions for which $q(s, x) \geq 0$, then $A^*(s; t)$ satisfies

$$
\frac{\partial A^*(s; t)}{\partial s} = \frac{1}{2}[-\sigma_2^2(s)A^2(s; t) - 2A(s)A^*(s; t) + 2\theta_1(s)], s \leq t,
$$

with boundary condition $A^*(t; t) = 0$.

Proof: Suppose that the drift function of the conditioned diffusion is given by (59) for some function $A^*(s; t)$. Then Theorem 1 implies

$$
\frac{\partial \ln V(t|x; s)}{\partial x} = A^*(s; t),
$$

or equivalently

$$
\ln V(t|x; s) = A^*(s; t)x + b^*(s; t),
$$

for some function $b^*(s; t)$. Now by (21)

$$
-\frac{\partial \ln V(t|x; s)}{\partial s} = \frac{\partial A^*(s; t)}{\partial s}x - \frac{\partial b^*(s; t)}{\partial s} = \frac{1}{2}\sigma_1^2(s)A^2(s; t)x + (a(s) + A(s)x)A^*(s; t) - q(s, x)
$$

(62)

Inspection of (62) immediately implies that $q$ must be of the form given in (60). On the other hand if $q$ is given by (60) then by equating the coefficients of $x$ on the left and right side of (62) we obtain a differential equation (61) for the function $A^*(s; t)$. Boundary condition $A^*(t; t) = 0$ follows from

$$
\ln V(t|x; s) = \ln E\{\exp \left[-\int_s^t(q(u, X(u))du \right] | X(s) = x \}
$$

(63)

$$
= A^*(s; t)x + b^*(s; t),
$$

which completes the proof. ■
Proposition 3. Let \( X(t), t \geq 0 \) be defined by (58) and \( q(s, x) \) by (60). Then the associated survival function is given by

\[
V(t|x; s) = P(T > t|T > s, X(s) = x) = \exp \left[ A^*(s; t)x + \int_s^t a(u)A^*(u; t)du - \int_s^t \theta_2(u)du \right],
\]

where \( A^*(s; t) \) satisfies (61) with \( A^*(t; t) = 0 \).

Proof: By Theorem 3 we have \( V(t|x; s) = \exp [A^*(s; t)x + b^*(s; t)] \) where \( A^*(s; t) \) satisfies (61) with \( A^*(t; t) = 0 \). By equating constant terms in (62), we obtain differential equation for \( b^*(s; t) \):

\[
\frac{\partial b^*(s; t)}{\partial s} = -a(s)A^*(s; t) + \theta_2(s), s \leq t,
\]

for which boundary condition: \( b^*(t; t) = 0 \) follows from (63). The solution to (65) with \( b^*(t; t) = 0 \) is \( b^*(s; t) = \int_s^t a(u)A^*(u; t)du - \int_s^t \theta_2(u)du \).

Proposition 4. Under the same assumptions as in Proposition 3, the mean and variance function of \( X^*(\tau; t), 0 \leq s \leq \tau \leq t \), conditional on \( X^*(s; t) = x \), are given by

\[
EX^*(\tau; t) = \Phi(\tau) \left\{ x + \int_s^\tau a(r)\Phi^{-1}(r)dr \right\},
\]

and

\[
V(X^*(\tau; t)) = \Phi^2(\tau) \times \left\{ \int_s^\tau (2a(r) + \sigma_1^2(r))m(r)\Phi^{-2}(r)dr - \left[ \int_s^\tau a(r)\Phi^{-1}(r)dr \right]^2 \right\},
\]

respectively, where \( \Phi(\tau) = \exp \left[ \int_s^\tau (A(u) + \sigma_1^2(u))A^*(u; t)du \right] \) and \( m(\tau) = EX^*(\tau; t) \).

Proof: The proof extends the standard method for computing the mean and the variance of the Gaussian diffusion to the present case. By integrating (59) from \( s \) to \( \tau \), then taking expectation of both sides, and finally differentiating with respect to \( \tau \) we obtain an ordinary differential equation for the mean function \( m(\tau) \)

\[
m'(\tau) = a + [A + \sigma_1^2EX^*(\tau; t)]m(\tau), m(s) = x
\]
whose solution is (66). To obtain (67) we have by Ito’s rule
\[
\begin{align*}
    dX^*^2(r; t) &= \left\{ (2a(r) + \sigma^2_1(r))X^*(r; t) + 2(A(r) + \sigma^2_1(r))A^*(r; t)X^*^2(r; t) \right\} \, dr \\
    &\quad + 2\sigma_1(r)X^*^3/2(r; t) \, dW(r).
\end{align*}
\]
Now integrating the above equation from $s$ to $\tau$, then taking expectation of both sides, and differentiating with respect to $\tau$ gives an ordinary differential equation for $p(\tau) = EX^*^2(r; t)$:
\[
p'(\tau) = (2a + \sigma^2_1)\mu(\tau) + 2(A + \sigma^2_1 A^*(r; t)p(\tau), p(s) = x^2,
\]
whose solution is
\[
EX^*^2(r; t) = \Phi^2(\tau) \times \left\{ x^2 + \int_s^\tau (2a(r) + \sigma^2_1(r))\mu(r)\Phi^{-2}(r)\, dr \right\},
\]
which together with (66) implies (67). □

If we assume that (58) has constant coefficients and $\theta_1(s) \equiv \theta_1$ in (60), then equation (61) has a closed form solution given by
\[
A^*(u; t) = 2\theta_1 \exp \left[ -0.5g(t - u) \right] - \exp \left[ 0.5d(t - u) \right],
\]
where $f = \sqrt{A^2 + 2\sigma^2_1\theta_1}, d = f + A$, and $g = f - A$. Let
\[
H(u) = d \exp \left[ -0.5g(t - u) \right] + g \exp \left[ 0.5d(t - u) \right],
\]
Since
\[
\int_s^\tau A^*(u; t) \, du = \left( 2/\sigma^2_1 \right) \ln \left[ H(u) \right] \bigg|_s^\tau = \left( 2/\sigma^2_1 \right) \ln \left\{ (d + g)/H(s) \right\},
\]
it follows from Proposition 3 that marginal survival function (5) admits closed form. To evaluate the formula for the survival function in (19) we also require the transition density of the unconditioned and conditioned diffusion. The transition density of the diffusion in (58), when all coefficient functions are constant, is the noncentral chi-square density, (see Feller (1951) or Cox, Ingersoll and Ross (1985)), but numerical methods seem to be needed to compute the transition density of the conditioned
diffusion. However, as we now show we can obtain the mean and the variance of this density in closed form. A lengthy calculation based on (69), and on noting that

$$
\int_0^r \{\exp[-A(r - s)]/H^2(r)\}dr = 2(g + d)^{-2}\exp[-A(t - u)]F(r)|_s^t,
$$
gives

$$
EX^*(\tau; t) = x \exp[A(\tau - s)]H^2(\tau)/H^2(s) + 2a(g + d)^{-2}H^2(\tau)\exp[-A(t - \tau)][F(\tau) - F(s)],
$$
(70)

where

$$
F(u) = \{\exp[-.5g(t - u)] - \exp[.5d(t - u)]\}/H(u), 0 \leq u \leq t.
$$

Similarly a tedious but straightforward calculation, which uses the fact:

$$
\int_0^r \exp[-A(r - s)]F'(r)/H^2(r)dr
$$

$$
= \exp[-A(t - s)] \times \int_0^r \exp[A(t - r)]F(r)/H^2(r)dr
$$

$$
= \exp[-A(t - s)] \times (g + d)^{-2}H^2(r)|_s^t,
$$
gives the following expression for the variance of $X^*(\tau; t)$

$$
V(X^*(\tau; t)) = 2x\sigma_1^2(g + d)^{-2}\exp[A(2\tau - s - t)] \times [F(\tau) - F(s)]H^4(\tau)/H^2(s) + 2a\sigma_1^2(g + d)^{-4}\exp[-2A(t - \tau)]H^4(\tau)[F(\tau) - F(s)]^2.
$$
(71)

The expressions (70) and (71) simplify considerably when we set $\tau = t$. In this case

$$
EX^*(t; t) = x \exp[A(t - s)][(d + g)/H(s)]^2 - 2aF(s),
$$
(72)

and

$$
V(X^*(t; t)) = -2x\sigma_1^2F(s)\exp[A(t - s)][(g + d)/H(s)]^2 + 2a\sigma_1^2F^2(s).
$$
(73)

The expressions (72) and (73) are the mean and variance, respectively, of the density $h^*(y, t|x, s; t)$. As $t \to \infty$, $EX^*(t; t) \to 2a/g$, and $V(X^*(t; t)) \to 2a\sigma_1^2/g^2$. If $\theta_1 = 0$ and $A < 0$, then $2a/g = a/(-A)$, and $2a\sigma_1^2/g^2 = \alpha \sigma_1^2/2A^2$, which are the mean and
variance, respectively, of the limiting gamma distribution of the branching diffusion without killing. However, it is interesting to note that condition $A<0$ is not needed to obtain finite mean and variance of the asymptotic distribution of $X^*(t; t)$.

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