APPORTIONMENT OF SEATS IN
PROPORTIONAL REPRESENTATION SYSTEMS:
A MAJORIZATION COMPARISON OF DIVISOR METHODS

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Summary: Proportional representation is applied to such problems as the apportionment of a number of seats to each party proportionally to the number of votes received, or the apportionment of a number of seats to each constituency proportionally to its population. From the inception of the proportional representation movement it has been an issue whether larger parties are favored at the cost of smaller parties in one apportionment of seats as compared to another apportionment. A number of methods have been proposed and used in countries with a proportional representation system. These methods exhibit regularity of order that captures the preferential treatment of larger versus smaller parties. This order, namely majorization, permits the comparison of seat allocation in two apportionments. For divisor methods, we show that one method is majorized by another method if and only if their signpost ratios are increasing. This criterion is satisfied for the divisor methods with power mean rounding, and the divisor methods with stationary rounding. Majorization places the five traditional apportionment methods in the order as they are known to favor larger parties over smaller parties: Adams, Dean, Hill, Webster, and Jefferson.


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1. Introduction

Proportional representation is applied to such problems as the apportionment of a number of seats to each party proportionally to the number of votes received, or the apportionment of a number of seats to each constituency proportionally to its population. Alternative apportionment methods have been proposed dating from the Constitutional Congress in 1787 in the United States, and from the proportional representation movement in Europe that came into existence before 1900. The fundamental issue is how to allocate fractions to different parties. There are many papers dealing with this problem; we single out the seminal monograph by Balinski/Young (1982) as providing an excellent history as well as a mathematical foundation.

From the very beginning there has been the issue whether one apportionment method favors larger parties at the cost of smaller parties, as compared to another method. Arguments by well-known early founders in the United States, such as Jefferson, Hamilton, Adams, Webster, each proposed a method of apportionment. A perusal of the outcomes of these methods shows a regularity or order that captures the preferential treatment of larger versus smaller parties. It is this order, called majorization, that we here study.

The majorization ordering compares the subtotal of seats for the groups of the $k$ largest parties in one apportionment with the corresponding subtotal in another apportionment, where $k$ in turn is taken to be 1, 2, ... up to including all parties. One method is said to be majorized by another method if, whatever $k$, these subtotals do not decrease. That any group of larger parties is no worse off under one method as compared to another, describes in a quantitative manner what is otherwise paraphrased in a qualitative language.

The paper is organized as follows. In Section 2 we recall the well-established notion of vector majorization, and apply it to comparing two apportionment results obtained from different methods. The majorization ordering has a long history, with a special root in the social sciences, and we include some remarks on the history of the subject. For a review and mathematical discussions see Marshall/Olkin (1979). An earlier, influential forerunner was the book on inequalities by Hardy/Littlewood/Pólya (1934).

In Section 3 we extend the notion of majorization to apply to two methods of apportionment. For the particular subclass of apportionment methods that are divisor methods, Proposition 1 provides a determination of when majorization holds. In Section 4 this check is brought to bear on divisor methods with power mean rounding. Our most important result is that majorization puts the five traditional methods into the order as they are known to favor larger parties over smaller parties: Adams, Dean, Hill, Webster, and Jefferson.
Section 5 singles out another family of divisor methods, those with stationary rounding. This family permits a simpler technical handling than other families, and hence in many respects serves as a prototype for the theory. Some commentary on other methods constitutes Section 6. Section 7 appends biographical details of some of the individuals whose names persist in the literature on apportionment methods. All proofs are collected in Section 8.

2. Majorization of two apportionment results

Assume that, at the end of an election, parties \(1, 2, \ldots, \ell\), obtain vote counts \(v_1, v_2, \ldots, v_\ell\), respectively. It is convenient to take the parties ordered from largest to smallest in terms of their vote counts, so that \(v_1 \geq v_2 \geq \cdots \geq v_\ell\). Let \(M\) be the district magnitude, that is, the total number of seats to be apportioned among the \(\ell\) parties. The number of seats apportioned to party \(i\) is denoted by \(m_i\).

Let two apportionment outcomes \(m = (m_1, m_2, \ldots, m_\ell)\) and \(m' = (m'_1, m'_2, \ldots, m'_\ell)\) be given. Because parties with more votes cannot receive fewer seats than parties with fewer votes, we may assume that \(m_1 \geq m_2 \geq \cdots \geq m_\ell\), and \(m'_1 \geq m'_2 \geq \cdots \geq m'_\ell\). We say that the apportionment \(m\) is majorized by \(m'\) provided

\[
\begin{align*}
m_1 &\leq m'_1, \\
m_1 + m_2 &\leq m'_1 + m'_2, \\
& \vdots \\
m_1 + \cdots + m_{\ell-1} &\leq m'_1 + \cdots + m'_{\ell-1}, \\
m_1 + \cdots + m_{\ell} &= M = m'_1 + \cdots + m'_\ell.
\end{align*}
\]

Comparing the apportionment \(m\) and the apportionment \(m'\), the \(k\) largest parties thus gain in seats, or maintain their number, for all \(k = 1, 2, \ldots, \ell - 1\). The last equation means that in both cases the sum of all seats is equal to the number \(M\) of available seats. The subtotal comparisons in (1) are an appealing way of turning the idea of favoring larger parties at the cost of smaller parties into an operational, quantitative comparison.

It seems worthwhile to emphasize the descriptive power of the concept. For instance, Nohlen (1986, page 64) compares the Webster apportionment and the Jefferson apportionment. He makes a point that a transition from the Webster apportionment to the Jefferson apportionment may result in an additional seat for the largest party, as well as for the second-smallest party. Conversely, the smallest party could lose a seat, as could
the second-largest party. Put this way, there is left an impression that the transfers of a seat occur in a random fashion, and do not exhibit a systematic structure. This is not so, and majorization provides the appropriate language to capture the structural properties. The total number of seats of the \( k \) largest parties in the Jefferson apportionment is never smaller than the corresponding total in the Webster apportionment. Equivalently, the total number of seats of the \( k \) smallest parties in the Jefferson apportionment will be less than or equal to what those small parties total in the Webster apportionment.

In the electoral literature *Raschauer* (1971) and *Pennisi* (1998) are the only papers we know of that mention the notion of majorization, though only in passing. *Balinski/Young* (1982, page 118) introduce the following, closely related relation. An apportionment \( m \) is said to give up to another apportionment \( m' \) if, in every pairwise comparison of a larger party \( i \) with a smaller party \( j \), it cannot happen that the larger party \( i \) loses seats and the smaller party \( j \) gains seats. Unfortunately, this relation fails to be transitive. To clarify the notion of transitivity in this context, consider three apportionments:

\[
\begin{array}{ccc}
  m & m' & m'' \\
 1 & 10 & 11 & 11 \\
 2 & 6 & 5 & 5 \\
 3 & 3 & 3 & 4 \\
 4 & 2 & 2 & 1 \\
\end{array}
\] (2)

Going from \( m \) to \( m' \), party 2 gives up one seat to party 1. From \( m' \) to \( m'' \), party 4 gives up one seat to party 3. But comparing \( m \) with \( m'' \), party 2 loses a seat while party 3 gains a seat, thus violating the *Balinski/Young* relation. In contrast, majorization is a transitive relation so that, if \( m \) is majorized by \( m' \) and \( m' \) is majorized by \( m'' \), a *fortiori* \( m \) is majorized by \( m'' \).

The majorization ordering has been a helpful tool in many fields of science, including mathematics, statistics, chemistry, physics, and others. It should be emphasized, though, that the ordering has an independent and early origin in the social sciences. The political science and economics approach dates back to *Dalton* (1920, 1925) who studied the inequality of income. *Dalton’s* starting point was the simple idea that if a portion of income is transferred from a poor person to a rich person, then inequality is increased. Thus, in this ordering, the case where each person has the average is the most equal, and the case where one person has all the wealth is the most unequal. A vector \( m' \) of incomes can be reached from the vector \( m \) through a series of such transfers from poor to rich if and only if \( m \) is majorized by \( m' \). Then \( m' \) represents higher income inequality than does \( m \). For
example, when 3 units of a good must be shared by three individuals, then the apportionment \( m = (1, 1, 1) \) is less unequal than the apportionment \( m' = (2, 1, 0) \) which, in turn, is less unequal than \( m'' = (3, 0, 0) \). This type of comparison is afforded by the majorization ordering.

3. Majorization of two apportionment methods

In proportional representation electoral systems the apportionment \( m = (m_1, m_2, \ldots, m_\ell) \) is calculated from the vote counts \( v_1, v_2, \ldots, v_\ell \) and the district magnitude \( M \). A procedure that governs these calculations is called an apportionment method, or an electoral formula. Let \( A \) be the apportionment method that is stated in the applicable electoral law. The apportionment result then consists, practically almost always, of a single apportionment \( m = (m_1, m_2, \ldots, m_\ell) \). However, a general method must also accommodate tied situations, for instance when \( \ell \) parties with identical vote counts share \( \ell + 1 \) seats. Balinski/Young (1982, page 96) discuss such ties in detail. Our objective is to compare, not just two specific apportionment results, but two general apportionment methods.

**Definition.** Given the apportionment methods \( A \) and \( A' \), we say that \( A \) is majorized by \( A' \), denoted by \( A \prec A' \), if either they are equal, or for every number \( \ell \) of participating parties and for all proportions of votes \( v_1, v_2, \ldots, v_\ell \) and district magnitudes \( M \), every apportionment \( m \) of method \( A \) is majorized by every apportionment \( m' \) of method \( A' \).

In the set of all apportionment methods, this relation is a partial ordering. That is, it is reflexive (\( A \prec A \)), transitive (\( A \prec A' \) and \( A' \prec A'' \) implies \( A \prec A'' \)), and antisymmetric (\( A \prec A' \) and \( A' \prec A \) implies \( A = A' \)). Naturally, there is no necessity that any two arbitrary apportionment methods \( A \) and \( A' \) be comparable in the majorization ordering. The main result of the present paper is to establish a necessary and sufficient condition for determining majorization, under the assumption that the two apportionment methods are divisor methods. A divisor method \( A \) is defined through a sequence of numbers \( s(k) \) in the interval \([k, k + 1]\), for all \( k = 0, 1, 2, \ldots \). Balinski/Young (1982, page 64) picture \( s(k) \) as a "signpost", dividing the interval \([k, k + 1]\) into a left part where numbers are rounded down to \( k \), and a right part where numbers are rounded up to \( k + 1 \).

The numbers rounded this way are the quotients of the vote counts and a divisor, \( v_1/d, v_2/d, \ldots, v_\ell/d \), for some choice of divisor \( d > 0 \). If party \( i \) gets \( m_i \) seats, then necessarily \( s(m_i - 1) \leq v_i/d \leq s(m_i) \). The divisor \( d \) is adjusted so that the sum of all seats becomes equal to the district magnitude, \( m_1 + m_2 + \cdots + m_\ell = M \). The following result characterizes majorization through the signpost sequences defining the methods.
Proposition 1. Let $A$ and $A'$ be two distinct divisor methods with signpost sequences $s(0), s(1), s(2), \ldots$ and $s'(0), s'(1), s'(2), \ldots$, respectively. Then method $A$ is majorized by method $A'$ if and only if the sequence of ratios $s(k)/s'(k)$ is strictly increasing in $k$.

The proof of Proposition 1 is deferred to the Appendix. It is always the case that, as $k$ tends to infinity, the signpost ratios $s(k)/s'(k)$ can be estimated from below and from above by $k/(k+1) \leq s(k)/s'(k) \leq (k+1)/k$. Hence the signpost ratios converge to the limit one, as $k$ tends to infinity. If the condition of Proposition 1 holds, then convergence is monotonic and $s(k) < s'(k)$ for all $k$, meaning that a transition from method $A$ to $A'$ moves all signposts to larger values. Only $k = 0$ is an exception; when $s(0) = s'(0) = 0$ we define the quotient $0/0$ to be $0$.

To illustrate these ideas, let $A$ and $A'$ be two divisor methods with initial signposts $s(0) = 0.5$, $s(1) = 1.4$, and $s'(0) = 0.5$, $s'(1) = 1.6$. If two parties have vote counts $v_1 = 75$ and $v_2 = 25$, then two seats are apportioned according to $m = (2, 0)$, and $m' = (1, 1)$, respectively. With divisors $d = 51$, and $d' = 49$, this is easily checked:

$$\frac{v_1}{d} = \frac{75}{51} = 1.47 > 1.4 = s(1) \Rightarrow m_1 = 2, \quad \frac{v_2}{d} = \frac{25}{51} = 0.49 < 0.5 = s(0) \Rightarrow m_2 = 0;$$

$$\frac{v_1}{d'} = \frac{75}{49} = 1.53 < 1.6 = s'(1) \Rightarrow m'_1 = 1, \quad \frac{v_2}{d'} = \frac{25}{49} = 0.51 > 0.5 = s'(0) \Rightarrow m'_2 = 1.$$

The growth of the second signpost from 1.4 to 1.6 makes it increasingly difficult for party 1 to secure as many seats as before. Technically, the apportionment $m$ fails to be majorized by $m'$, and the signpost ratios $s(0)/s'(0) = 1 > s(1)/s'(1) = 0.875$ fail to be increasing. The virtue of Proposition 1 lies in the fact that it can be successfully applied to two practically important families of divisor methods, those with power mean rounding (Section 4) and those with stationary rounding (Section 5).

4. Divisor methods with power mean rounding

The divisor method with power mean rounding of order $p$ is given by the signpost sequence

$$s_p(k) = \left( \frac{k^p + (k+1)^p}{2} \right)^{1/p}, \quad (3)$$

with power parameter $p \in [-\infty, \infty]$. That is, $s_p(k)$ is the mean of order $p$ of the integers $k$ and $k+1$. The limiting case $p = -\infty$ has $s_{-\infty}(k) = k$, the other extreme is $s_{\infty}(k) = k+1$. When $p$ tends to zero we obtain the geometric mean, $s_0(k) = \sqrt{k(k+1)}$. The cases $p = 1$ and $p = -1$ give the arithmetic mean and harmonic mean, respectively.
Particular members of this family are the five traditional apportionment methods that figure prominently in the seminal monograph of Balinski/Young (1982). These methods carry the names of the various individuals who proposed them or fought for them; see Section 7 for some biographical details. For \( p = -\infty, -1, 0, 1, \infty \) we obtain, respectively, the Adams method (divisor method with rounding up), the Dean method (divisor method with harmonic rounding), the Hill method (divisor method with geometric rounding, method of equal proportions), the Webster method (divisor method with standard rounding, method of Sainte-Laguë), and the Jefferson method (divisor method with rounding down, method d'Hondt, method of Hagenbach-Bischoff). The majorization relationship in this family reduces to a simple comparison of the parameters.

**Proposition 2.** The divisor method with power mean rounding of order \( p \) is majorized by the divisor method with power mean rounding of order \( p' \) if and only if \( p \leq p' \).

The proof is given in the Appendix. In particular, majorization puts the five traditional divisor methods into the sequence

\[
\text{Adams} \prec \text{Dean} \prec \text{Hill} \prec \text{Webster} \prec \text{Jefferson}. \tag{4}
\]

The fact that the traditional apportionment methods (4) are increasingly favorable to larger parties at the cost of smaller parties is worked out from various perspectives by Balinski/Young (1982). Yet another viewpoint is now added by the majorization ordering (1). That the apportionment results of the traditional methods are ordered by majorization is plainly visible in the tables provided in Balinski/Young (1982, pages 158–176).

More generally, when the vote counts \( v_1, v_2, \ldots, v_\ell \) are given, it is instructive to investigate the sequence of apportionments as \( p \) increases. Because of successive majorization the apportionments stay constant until, at some order \( p_0 \), a seat is transferred from a smaller party to a larger party. A transfer to party \( i \) from party \( j \) at \( p(i, j) \), say, necessitates a tie,

\[
\frac{v_i}{d} = s_{p(i,j)}(m_i), \quad \frac{v_j}{d} = s_{p(i,j)}(m_j - 1). \tag{5}
\]

That is, parties \( i \) and \( j \) have \( m_i \) and \( m_j \) seats immediately before the transfer, and \( m_i + 1 \) and \( m_j - 1 \) immediately after the transfer. Solving for \( d \) in (5) eliminates \( d \), and determines \( p(i, j) \) as a zero of the equation

\[
\frac{v_i}{v_j} - \frac{s_{p(i,j)}(m_i)}{s_{p(i,j)}(m_j - 1)} = 0. \tag{6}
\]
The solution, if it exists, can easily be found numerically; if there is none we set \( p(i, j) = \infty \). Between the \( \ell(\ell-1)/2 \) possible two party pairings, let \( p_0 \) denote the smallest of the numbers \( p(i, j) \). In other words, among all ties that are possible, the one at \( p_0 \) is first to surface.

We illustrate the procedure with Examples 1 and 2 in Exhibit 1. Example 1, adapted from Pólya (1919c, page 301), demonstrates the diversity of results in a three party system. The proportions of votes are assumed to be 59.5, 25.32, 15.18 percent, respectively, and \( M = 10 \) seats are to be apportioned. The Adams apportionment \((5,3,2)\) provides the starting point at \( p = -\infty \); here the largest party gets 50 percent of all seats, the second-largest party 30 percent, and the third-largest party 20 percent. By computer calculation, we find that the smallest solution of equation (6) occurs at \( i = 1 \) and \( j = 2 \), with value \( p_0 = -3.290884 \). At this \( p_0 \), the second-largest party gives up a seat to the largest party. The apportionment \((6,2,2)\) remains in effect long enough to also include \( p = -1 \) (Dean) and \( p = 0 \) (Hill). At \( p_0 = 0.978385 \), the third-largest party transfers a seat to the second-largest party, leading to the Webster apportionment \((6,3,1)\). Finally, at \( p_0 = 14.315221 \) the apportionment \((7,2,1)\) is reached and stays constant until \( p = \infty \) (Jefferson); now the largest party gets 70 percent of the seats, the second-largest party 20 percent, and the third-largest party 10 percent.

Example 2 is taken from the Balinski/Young (1982, page 96) monograph, showing that there are vote counts for which the five traditional divisor methods of Adams, Dean, Hill, Webster, and Jefferson produce different apportionments. We see that, between Webster and Jefferson, a sixth apportionment fits in the sequence.

For large numbers, the divisor methods with power mean rounding essentially reduce to three methods only, the Adams method \((p = -\infty)\), the Webster method \((p = 1)\), and the Jefferson method \((p = \infty)\). This is due to the limiting relationship

\[
\lim_{k \to \infty} \left( s_p(k) - k \right) = \begin{cases} 
1 & \text{for } p = \infty, \\
1/2 & \text{for } p \in (-\infty, \infty), \\
0 & \text{for } p = -\infty.
\end{cases}
\]

(The limit is obtained using l'Hospital's rule, as \( z = 1/k \) tends to zero in \( s_p(k) - k = [\{(1 + (1 + z)^p)/2\}^{1/p} - 1]/z. \) Thus, in the intervals \([k, k + 1]\) with \( k \) large, the signposts \( s(k) \) move to the left endpoints \( k \), the midpoints \( k + 1/2 \), or the right endpoints \( k + 1 \). In contrast, a family maintaining its richness also for large values of \( k \) is the following.
Exhibit 1: Divisor methods with power mean rounding, or with stationary rounding, I
Two examples of identical apportionment sequences

<table>
<thead>
<tr>
<th>Example 1 (adapted from Pólya, 1919c, page 301)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Order $p$ from $-\infty$ to $-3.291$ (0.979, 14.316)</td>
</tr>
<tr>
<td>Shift $q$ from 0 to $0.222$ (0.498, 0.964)</td>
</tr>
<tr>
<td>Named methods Adams, Dean, Hill, Webster, Jefferson</td>
</tr>
<tr>
<td>Vote counts</td>
</tr>
<tr>
<td>1</td>
</tr>
<tr>
<td>2</td>
</tr>
<tr>
<td>3</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Example 2 (from Balinski/Young, 1982, page 96)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Order $p$ from $-\infty$ to $-3.364$ (0.262, 5.597, 15.599)</td>
</tr>
<tr>
<td>Shift $q$ from 0 to $0.109$ (0.471, 0.634, 0.812)</td>
</tr>
<tr>
<td>Named methods Adams, Dean, Hill, Webster, Jefferson</td>
</tr>
<tr>
<td>Vote counts</td>
</tr>
<tr>
<td>1</td>
</tr>
<tr>
<td>2</td>
</tr>
<tr>
<td>3</td>
</tr>
<tr>
<td>4</td>
</tr>
<tr>
<td>5</td>
</tr>
<tr>
<td>6</td>
</tr>
</tbody>
</table>

The first example apports 10 seats among 3 parties, the second, 36 seats among 6 parties. Any apportionment column is majorized by its successor, in that a smaller party gives a seat up to a larger party, as is indicated by arrows. By transitivity, any apportionment column to the left is majorized by any other apportionment column to the right. This can also be verified from the defining partial sum comparison (1); the $k$ largest parties together keep their number of seats or gain, for all $k = 1, \ldots, \ell$. The apportionment sequences happen to be identical for the divisor methods with power mean rounding, and for the divisor methods with stationary rounding. The ranges of the order $p$ and the shift $q$ leading to the apportionments are given in the first four lines of each table.
5. Divisor methods with stationary rounding

The divisor method with stationary rounding of shift $q$ is given by the signpost sequence

$$s_q(k) = (1 - q)k + q(k + 1) = k + q,$$  (8)

where $q \in [0, 1]$. Here, the point of discontinuity in each interval $[k, k + 1]$ has the same location relative to the lower and upper endpoint, and thus is shift invariant. This is what motivates use of the attribute "stationary", as in Balinski/Racev (1997, page 13).

The family of divisor methods with stationary rounding contains three of the five traditional apportionment methods, the Adams method ($q = 0$), the Webster method ($q = 1/2$), and the Jefferson method ($q = 1$). Again the parameters immediately determine the majorization ordering.

**Proposition 3.** The divisor method with stationary rounding of shift $q$ is majorized by the divisor method with stationary rounding of shift $q'$ if and only if $q \leq q'$.

The proof in the Appendix shows that the distinct advantage of this family is that it much easier to handle technically. For a set of vote counts $v_1, v_2, \ldots, v_n$, we may now list the sequence of apportionments evolving under the divisor methods with stationary rounding. Equation (6), with $q(i,j)$ in place of $p(i,j)$, is easily solved to give

$$q(i,j) = \frac{m_i v_j - (m_j - 1)v_i}{v_i - v_j}.$$  (9)

Let $q_0$ be the smallest of these values $q(i,j)$. When $q$ passes through $q_0$, the apportionment $m$ changes by transferring a seat from a smaller party to a larger party, as is dictated by majorization.

Examples of the resulting sequence of apportionments are included in Exhibit 1, where in Examples 1 and 2 the sequences of apportionments under the stationary methods and under the power mean methods are the same. In other cases they may differ, as is demonstrated by Example 3 in Exhibit 2.

6. Commentary

From the point of view of proportional representation, we find it impossible to recommend one apportionment sequence as better justified, more self-explanatory, or otherwise superior to another. As Balinski/Racev (1997, page 14) put it, "one should not assume
**Exhibit 2:** Divisor methods with power mean rounding, or with stationary rounding, II

An example of different apportionment sequences

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**Example 3 (from Balinski/Rachev, 1997, page 14)**

<table>
<thead>
<tr>
<th>Order $p$ from $-\infty$ to $-27.336$</th>
<th>$0.333$ to $0.405$</th>
<th>$0.781$ to $16.824$</th>
<th>$19.714$ to $71.650$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Vote counts</strong></td>
<td><strong>Apportionment of 100 seats (Power mean rounding)</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>42919</td>
<td>41</td>
<td>42</td>
</tr>
<tr>
<td>2</td>
<td>13048</td>
<td>13</td>
<td>13</td>
</tr>
<tr>
<td>3</td>
<td>10879</td>
<td>11</td>
<td>11</td>
</tr>
<tr>
<td>4</td>
<td>10581</td>
<td>10</td>
<td>10</td>
</tr>
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<td>5</td>
<td>9547</td>
<td>10</td>
<td>9</td>
</tr>
<tr>
<td>6</td>
<td>5708</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>7</td>
<td>2502</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>8</td>
<td>1898</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>9</td>
<td>1461</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>10</td>
<td>1457</td>
<td>2</td>
<td>2</td>
</tr>
</tbody>
</table>

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**Shift $q$ from 0 to 0.154**

<table>
<thead>
<tr>
<th>Vote counts</th>
<th><strong>Apportionment of 100 seats (Stationary rounding)</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>42919</td>
</tr>
<tr>
<td>2</td>
<td>13048</td>
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<td>3</td>
<td>10879</td>
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Along the family of divisor methods with power mean rounding (top), or the family of divisor methods with stationary rounding (bottom), the apportionment sequence always starts with the Adams apportionment ($p = -\infty$, $q = 0$), and terminates with the Jefferson apportionment ($p = \infty$, $q = 1$). In the above example, the sequences differ in-between.
that the stationary methods yield every ‘reasonable’ rounding”, nor—we would like to add—does any other family of methods depending on a single parameter.

The two signpost sequences (3) and (8) generate “compromises” between \( k \) and \( k + 1 \). However, other compromises or averages can be constructed that satisfy the condition that \( s(k)/s'(k) \) be strictly increasing in \( k \). Indeed, for any two signpost sequences \( s(k) \) and \( s'(k) \) with strictly increasing ratios define \( s_q(k) = (1 - q)s(k) + qs'(k) \) for \( q \in [0, 1] \). This sequence satisfies Proposition 1. Note that for \( s(k) = k \) and \( s'(k) = k + 1 \), this reproduces the stationary signposts (8).

Second, Dorfleritner/Klein (1999, page 151) propose the family \( s_r(k) = k^{1-r}(k+1)^r \) for \( r \in [0, 1] \), which includes the Adams method \( (r = 0) \), the Hill method \( (r = 1/2) \), and the Jefferson method \( (r = 1) \). Third, the signposts \( s_t(k) = \log\left([(e^{tk} + e^{(t+1)k})/2]^{1/t}\right) \) produce yet another family, beginning with Adams \((t = -\infty)\), passing through Webster \((t = 0)\), and ending with Jefferson \((t = \infty)\).

### 7. Some biographical details

Balinski/Young (1982) aptly review the various contributions to the apportionment problem by US-American politicians and scientists, in particular John Quincy Adams, James Dean, Josef A. Hill, Daniel Webster, and Thomas Jefferson. We add a few biographical details of the proportional representation protagonists on the European continent; for additional information see, for example, Kopfermann (1991).

Victor d'Hondt (* 20 November 1841, † 30 May 1901) was professor of tax law and civil rights at the University of Ghent (Beatse, 1913). D'Hondt, an activist in the Association Réformiste Belge, proposed the apportionment method that, in Europe, was named after him and published widely on this method; see, for instance, d'Hondt (1885). No photo of d'Hondt seems to be available in the literature.

Eduard Hagenbach-Bischoff (* 20 February 1833, † 23 December 1910) was a physics professor at the University of Basel (Huber, 1960). As a member of the Canton legislature he became a proponent of the method d'Hondt, and simplified the calculations to obtain its apportionments. Of his many publications on the subject we mention the booklet (1905).

André Sainte-Laguë (* 20 April 1882, † 18 January 1950) was a professor of applied mathematics at the Conservatoire national des arts et métiers in Paris (Chastenet, 1994). Early in his career, while teaching at the Lycée in Douai, he published two papers (1910a,b) analyzing the optimality properties of apportionment methods.
George Pólya (* 13 December 1887, † 7 September 1985) was one of the eminent mathematicians of this century (Taylor/Taylor, 1993). His collected papers comprise four volumes of more than 2000 pages; the Pólya Picture Album (1987) is a fascinating document of the scientific history of this century. Pólya authored five papers (1918, 1919a–d) scrutinizing the various apportionment methods then in use in Switzerland.

Appendix: Proofs

Proof of Proposition 1. For the direct part, let A and A' be two distinct divisor methods satisfying $A < A'$. We need to show that $s(k)/s'(k) < s(k+1)/s'(k+1)$ for all $k$. Our proof is indirect, assuming the contrary,

$$\frac{s(k+1)}{s'(k+1)} \leq \frac{s(k)}{s'(k)} \quad \text{for some integer } k \geq 0. \quad (10)$$

The left hand side of (10) is bounded from below by $(k + 1)/(k + 2) > 0$. Consequently $s(k) > 0$, and $a = s(k+1)/s(k) \geq 1$. Now the interval

$$I = \left[\frac{s(k+1)}{s(k)}, \frac{s'(k+1)}{s'(k)}\right] \quad (11)$$

is nonempty, by (10), and its left endpoint $a$ satisfies $a \in [1, \infty)$. If the interval is nondegenerate, we can choose two integers $v_1$ and $v_2$ such that $v_1/v_2$ lies in the interior of $I$; because of $v_1/v_2 \geq a \geq 1$, we have $v_1 \geq v_2$. If the interval degenerates, $I = \{a\}$, we can still define two proportions of votes $v_1 = a/(1 + a) > 0$ and $v_2 = 1/(1 + a) > 0$ with $v_1/v_2 = a \in I$. This construction provides us with a two party situation, with vote counts (or proportions of votes) $v_1$ and $v_2$. We choose a district magnitude of $M = 2k + 2$.

We claim that $m = (k + 2, k)$ is an apportionment under method $A$. We establish our claim by verifying the max-min inequality of Balinski/Young (1982, page 100), according to which $m$ is an apportionment under method $A$ if and only if

$$\max \left\{ \frac{v_1}{s(k+2)}, \frac{v_2}{s(k)} \right\} \leq \min \left\{ \frac{v_1}{s(k+1)}, \frac{v_2}{s(k-1)} \right\}. \quad (12)$$

That is, we need to check four inequalities,

$$\frac{v_1}{s(k+2)} \leq \frac{v_1}{s(k+1)}, \quad (12a)$$
$$\frac{v_1}{s(k+2)} \leq \frac{v_2}{s(k-1)}, \quad (12b)$$
$$\frac{v_2}{s(k)} \leq \frac{v_1}{s(k+1)}, \quad (12c)$$
$$\frac{v_2}{s(k)} \leq \frac{v_2}{s(k-1)}. \quad (12d)$$
But (12a) follows from \( s(k + 1) \leq s(k + 2) \), (12b) from \( v_1/v_2 \leq s'(k + 1)/s'(k) \leq s(k + 2)/s(k - 1) \), (12c) from \( s(k + 1)/s(k) \leq v_1/v_2 \), and (12d) from \( s(k - 1) \leq s(k) \).

We also claim that \( m' = (k + 1, k + 1) \) is an apportionment under method \( A' \). For this to hold true the max-min inequality takes the form

\[
\max \left\{ \frac{v_1}{s'(k + 1)}, \frac{v_2}{s'(k + 1)} \right\} \leq \min \left\{ \frac{v_1}{s'(k)}, \frac{v_2}{s'(k)} \right\}. \tag{13}
\]

Again we need to check four inequalities,

\[
\frac{v_1}{s'(k + 1)} \leq \frac{v_1}{s'(k)}, \tag{13a}
\]

\[
\frac{v_1}{s'(k + 1)} \leq \frac{v_2}{s'(k)}, \tag{13b}
\]

\[
\frac{v_2}{s'(k + 1)} \leq \frac{v_1}{s'(k)}, \tag{13c}
\]

\[
\frac{v_2}{s'(k + 1)} \leq \frac{v_2}{s'(k)}. \tag{13d}
\]

Now (13a) follows from \( s'(k) \leq s'(k + 1) \), (13b) from \( v_1/v_2 \leq s'(k + 1)/s'(k) \), (13c) from \( s'(k)/s'(k + 1) \leq s(k + 1)/s(k) \leq v_1/v_2 \), and (13d) from \( s'(k) \leq s'(k + 1) \).

In summary, the methods \( A \) and \( A' \) produce the apportionments \( m = (k + 2, k) \) and \( m' = (k, k) \) where, evidently, \( m \) is not majorized by \( m' \). This contradicts the assumption \( A \prec A' \), thus invalidating (10).

For the converse part, we follow the lines of argument in Balinski/Young (1982, page 118), and Balinski/Rachecv (1997, page 15). Let the signpost ratios be strictly increasing. For some vote counts \( v_1, v_2, \ldots, v_t \) and district magnitude \( M \), let \( m \) be an apportionment under method \( A \) and \( m' \) an apportionment under \( A' \). We prove, for all \( v_i > v_j \), that \( m_i \leq m'_i \) or \( m_j \geq m'_j \); this forces \( m \) to be majorized by \( m' \), see the Lemma below. Otherwise, there exist two vote counts \( v_i > v_j \) satisfying

\[
m_i > m'_i \quad \text{and} \quad m_j < m'_j. \tag{14}
\]

As a consequence, there are divisors \( d \) for \( A \) and \( d' \) for \( A' \) such that

\[
\frac{v_i}{d} \geq s(m_i - 1), \quad \frac{v_j}{d} \leq s(m_j); \quad \frac{v_i}{d'} \leq s'(m'_i), \quad \frac{v_j}{d'} \geq s'(m'_j - 1). \tag{15}
\]

This leads to the first and last inequalities in

\[
\frac{v_i}{v_j} \leq \frac{s'(m'_i)}{s'(m'_j - 1)} \leq \frac{s'(m_i - 1)}{s'(m_j)} \leq \frac{s(m_i - 1)}{s(m_j)} \leq \frac{v_i}{v_j}. \tag{16}
\]
The second inequality follows from (14), while the strict inequality holds by assumption on the monotonicity of the signpost ratios. But (16) is a contradiction, whence (14) cannot hold true. The proof is complete.

The following Lemma is of intrinsic interest in the theory of majorization.

**Lemma.** If \( m_i \leq m'_i \) or \( m_j \geq m'_j \) for all \( i < j \), then \( m \) is majorized by \( m' \). The converse is not generally true.

**Proof.** The proof is indirect. Suppose that \( m \) is not majorized by \( m' \), then for some \( i \) we have

\[
\begin{align*}
m_1 & \leq m'_1, \\
m_1 + m_2 & \leq m'_1 + m'_2, \\
& \vdots \\
m_1 + \cdots + m_{i-1} & \leq m'_1 + \cdots + m'_{i-1}, \\
m_1 + \cdots + m_{i-1} + m_i & > m'_1 + \cdots + m'_{i-1} + m'_i.
\end{align*}
\]

Consequently, we must have \( m_i > m'_i \). However, the total sums are equal, so that it must be that \( m_j < m'_j \) for some \( j > i \). For the converse part, we refer to (2) where, although \( m \) is majorized by \( m'' \), we have \( m_2 = 6 > 5 = m'_2 \) and \( m_3 = 3 < 4 = m'_3 \). The proof of the Lemma is complete.

**Proof of Proposition 2.** Let \( p < r \) define the power mean signposts \( s_p(k) \) and \( s_r(k) \) as in (3). We wish to establish monotonicity of the signpost ratios \( s_p(k)/s_r(k) \). This is the same as showing, for all \( k \), that

\[
g(r) = \frac{s_r(k+1)}{s_r(k)} < \frac{s_p(k+1)}{s_p(k)} = g(p).
\]

In other words, the function \( g(r) \) in (18) is strictly decreasing in \( r \). Upon setting \( x_1 = k+2, \ x_2 = k+1 \) and \( y_1 = k+1, y_2 = k \), we may rewrite \( g(r) \) in the form

\[
g(r) = \left(\frac{(k+1)^r + (k+2)^r}{(k^r + (k+1)^r)}\right)^{1/r} = \left(\frac{\sum_{i=1}^{2} x_i^r}{\sum_{j=1}^{2} y_j^r}\right)^{1/r}.
\]

Since \( x_1 > x_2 > 0 \) and \( y_1 > y_2 \geq 0 \) and \( y_1/x_1 > y_2/x_2 \), Proposition 5.B.3 in Marshall/Olkin (1979, page 130) applies and states that \( g(r) \) is decreasing in \( r \). Moreover, the function \( g \) is analytic, whence if it is constant on some open interval then it is constant on the whole real line. This is not the case, as it falls from \( g(-\infty) = (k+1)/k \) down to \( g(\infty) = (k+2)/(k+1) \). Hence \( g \) is strictly decreasing, and the proof is complete.
Proof of Proposition 3. Let \( s_q(k) \) and \( s_{q'}(k) \) be stationary signposts as given by (8). Straightforward calculation gives

\[
s_q(k+1)s_{q'}(k) - s_q(k)s_{q'}(k+1) = q' - q.
\]

Hence if \( q < q' \), then \( s_q(k)/s_{q'}(k) \) is strictly increasing in \( k \). The proof is complete.

References


