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WITH ARBITRARY REAL DILATIONS

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Orthonormal Wavelets and Tight Frames
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Abstract. The objective of this paper is to establish a complete characterization of tight frames, and particularly of orthonormal wavelets, for an arbitrary dilation factor \( a > 1 \), that are generated by a family of finitely many functions in \( L^2 := L^2(\mathbb{R}) \). This is a generalization of the fundamental work of G. Weiss and his colleagues who considered only integer dilations. As an application, we give an example of tight frames generated by one single \( L^2 \) function for an arbitrary dilation \( a > 1 \) that possess "good" time-frequency localization. As another application, we also show that there does not exist an orthonormal wavelet with "good" time-frequency localization when the dilation factor \( a > 1 \) is irrational such that \( a^j \) remains to be irrational for any positive integer \( j \). This answers a question in Daubechies' Ten Lectures book for almost all irrational dilation factors. Other applications include a generalization of the notion of \( s \)-elementary wavelets of Dai and Larson to \( s \)-elementary wavelet families with arbitrary dilation factors \( a > 1 \). Generalization to dual frames is also discussed in this paper.

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1. Introduction and results

Let $1 < a < \infty$ and $b > 0$. A finite collection $\Psi_L = \{\psi_1, \ldots, \psi_L\}$ of functions in $L^2 := L^2(\mathbb{R})$ is said to generate an affine frame

$$\mathcal{F}_a := \{\psi_{\ell,j,k}(x) := a^{j/2}\psi_\ell(a^j x - kb): j, k \in \mathbb{Z}, \ \ell = 1, \ldots, L\}$$

of $L^2$, if there exist positive constants $A$ and $B$, with $0 < A \leq B < \infty$, such that

$$A\|f\|^2 \leq \sum_{\ell=1}^{L} \sum_{j,k \in \mathbb{Z}} |\langle f, \psi_{\ell,j,k} \rangle|^2 \leq B\|f\|^2, \quad \text{all } f \in L^2. \tag{1.2}$$

Here and throughout, $\langle , , \rangle$ and $\|\| \text{ denote the inner product and norm, respectively, for } L^2$, and $a$ is called the dilation factor of $\mathcal{F}_a$. Also, in (1.2) $A$ and $B$ are called upper and lower frame bounds of $\mathcal{F}_a$. If $A$ and $B$ can be chosen to be the same constant, then $\mathcal{F}_a$ is called a tight frame of $L^2$. For tight frames, the frame bound $A = B$ can be assumed to be $1$, simply by dividing each frame generator $\psi_\ell \in \Psi_L$ by $\sqrt{A}$. In other words, when we discuss tight frames, we can always use the definition

$$\sum_{\ell=1}^{L} \sum_{j,k \in \mathbb{Z}} |\langle f, \psi_{\ell,j,k} \rangle|^2 = \|f\|^2, \quad \text{all } f \in L^2, \tag{1.3}$$

assuming that $\Psi_L$ has been properly normalized.

It is clear that a frame is complete in $L^2$. Hence, it follows from the definition (1.3) that a tight frame $\mathcal{F}_a$ is an orthonormal basis of $L^2$, if and only if

$$\|\psi_1\| = \cdots = \|\psi_L\| = 1. \tag{1.4}$$

This can be easily seen by choosing any function in $\mathcal{F}_a$ as $f$ in (1.3) and noting that $\|\psi_{\ell,j,k}\| = \|\psi_\ell\|$. Therefore, any characterization of tight frames always translates to a characterization of orthonormal wavelets, when the restriction in (1.4) is taken into consideration.

In this paper, we give a characterization of tight frames (and hence, of orthonormal wavelets) in terms of the Fourier transform, defined by

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx.$$
Our investigation of this problem was motivated by the fundamental work of G. Weiss and his colleagues described in details in Hernández and Weiss [11; see the Preface, pp. 105–112, and pp. 332–347] as well as the matrix dilation generalization by Calogero [4, 5] and Bownik [3]. We note, however, that when restricted to one dimension, the results in [3, 4, 5] become generalization of [11] from dilation factor $a = 2$ to only integers $a \geq 2$. The main result in our paper is a complete generalization from dilation factor $a \in \mathbb{Z}, a \geq 2$, to an arbitrary dilation factor $a > 1$. To facilitate the statement of this result, we need the following notation. An $\alpha \in \mathbb{R}$ of the form

$$\alpha = \frac{m}{a^j}, \quad m, j \in \mathbb{Z},$$

is called an "$a$-adic number." For any $a$-adic number $\alpha$, set

$$I_a(\alpha) = \{(j, m) \in \mathbb{Z}^2 : \alpha = \frac{m}{a^j}\}. \quad (1.6)$$

We have the following result.

**Theorem 1.** Let $a > 1$ and $\Psi_L \subset L^2$. Then $\mathcal{F}_a$ as in (1.1) is a tight frame of $L^2$, as defined by (1.3), if and only if $\Psi_L$ satisfies

$$\frac{1}{b} \sum_{\ell=1}^{L} \sum_{(j, m) \in I_a(\alpha)} \overline{\psi}_\ell(a^j \omega) \hat{\psi}_\ell \left(a^j \omega + \frac{2m\pi}{b}\right) = \delta_{\alpha, 0}, \text{ a.e.}, \quad (1.7)$$

for all $a$-adic numbers $\alpha$.

Here and throughout, $\delta_{\alpha, 0}$ denotes the (generalized) Kronecker symbol

$$\delta_{\alpha, 0} := \begin{cases} 1, & \text{if } \alpha = 0, \\ 0, & \text{if } \alpha \in \mathbb{R}\setminus\{0\}. \end{cases} \quad (1.8)$$

To better understand the formulation (1.7), we divide the set of all dilation factors $a$ into three disjoint sets, namely,

$$(1, \infty) = E_1 \cup E_2 \cup E_3,$$

where

$$E_1 = \{a > 1: a^j \in \mathbb{Z} \text{ for some integer } j > 0\}, \quad (1.9)$$

$$E_2 = \{a > 1: a^j \text{ is noninteger rational for some integer } j > 0\}, \quad (1.10)$$

$$E_3 = \{a > 1: a^j \text{ is irrational for all integers } j > 0\}. \quad (1.11)$$
Since \( E_1 \) is already a generalization of the result in Hernández and Weiss [11] as mentioned above, as well as any integer dilation [3, 4, 5], let us first discuss this case.

Let \( a \in E_1 \) and \( \gamma > 0 \) be the smallest integer such that
\[
 n_a := a^\gamma \in \mathbb{Z}.
\] (1.12)

For each non-zero \( a \)-adic number \( \alpha = m_0/a^{j_0} \) with fixed \((j_0, m_0) \in \mathbb{Z}^2\), write \( m_0 = n_a^t r \), where \( 0 \leq t \in \mathbb{Z} \) and \( n_a \not| r \). Let \((j, m) \in I_a(\alpha)\). Since \( m/m_0 = a^{j-j_0} \) is rational, it is clear that \( j-j_0 = k \gamma \) for some \( k \in \mathbb{Z} \). Hence, we have
\[
 m = m_0 a^{j-j_0} = n_a^t r a^{k \gamma} = n_a^{(t+k)} r, \ n_a \not| r.
\]

So, for \((j, m)\) to run over \( I_a(\alpha) \), \( k \) runs from \(-t\) to \( \infty \), and (1.7) becomes
\[
\frac{1}{b} \sum_{\ell = 1}^{L} \sum_{k = -t}^{\infty} \hat{\psi}(a^\gamma \cdot \omega)^{\bar{\ell}} \left( a^{j_0 + k \gamma} \omega + \frac{2n_a^{k+t} r \pi}{b} \right)
\]
\[
= \frac{1}{b} \sum_{\ell = 1}^{L} \sum_{k = -t}^{\infty} \hat{\psi}(a^{j_0 - t \gamma} n_a^{k+t} \omega)^{\bar{\ell}} \left( n_a^{k+t} \left( a^{j_0 - t \gamma} \omega + \frac{2r \pi}{b} \right) \right) = \delta_{\alpha,0}.
\]

That is, absorbing \( a^{j_0 - t \gamma} \) by \( \omega \) gives
\[
\frac{1}{b} \sum_{\ell = 1}^{L} \sum_{j = 0}^{\infty} \hat{\psi}(n_a^{j} \omega)^{\bar{\ell}} \left( n_a^{j} \left( \omega + \frac{2r \pi}{b} \right) \right) = \delta_{\alpha,0}
\]

for all \( r \) with \( n_a \not| r \). This can be summarized as follows.

**Corollary 1.** Let \( 1 < a \in E_1 \) and \( n_a = a^\gamma \in \mathbb{Z} \) as in (1.12), where \( \gamma > 0 \) is the smallest integer. Then \( F_a \) is a tight frame of \( L^2 \), if and only if
\[
\frac{1}{b} \sum_{\ell = 1}^{L} \sum_{j = -\infty}^{\infty} |\hat{\psi}(a^j \omega)|^2 = 1, \ \text{a.e.}
\] (1.13)

and
\[
\left\{ \begin{array}{l}
\frac{1}{b} \sum_{\ell = 1}^{L} \sum_{j = 0}^{\infty} \hat{\psi}(n_a^{j} \omega)^{\bar{\ell}} \left( n_a^{j} \left( \omega + \frac{(sn_a + d)2 \pi}{b} \right) \right) = 0, \ \text{a.e.,}
\end{array} \right.
\]

for \( d = 1, \ldots, n_a - 1 \) and \( s \in \mathbb{Z} \).

Observe that when \( a = 2 \) and \( b = 1 \), we have \( n_a = a = 2 \), so that (1.14) becomes
\[
\frac{1}{b} \sum_{\ell = 1}^{L} \sum_{j = 0}^{\infty} \hat{\psi}(2^j \omega)^{\bar{\ell}} \left( 2^j \left( \omega + \frac{(2s + 1)2 \pi}{b} \right) \right) = 0, \ \text{a.e.,}
\]

for all \( s \in \mathbb{Z} \), as in Hernández and Weiss [11].

For \( a \in E_2 \), Theorem 1 becomes the following.
Corollary 2. Let \(1 < a \in E_2\) and \(r_a := a^\gamma = \frac{p}{q}\), where \(p > q \geq 2\) with \((p, q) = 1\) and \(0 < \gamma\) is the smallest integer for which \(a^\gamma\) is rational. Then \(F_a\) is a tight frame of \(L^2\), if and only if \(\Psi_L\) satisfies both (1.13) and

\[
\begin{align*}
\frac{1}{b} \sum_{\ell=1}^{L} \sum_{j=0}^{s} \hat{\psi}_\ell(r_a^j \omega) \hat{\psi}_\ell \left( r_a^j \left( \omega + \frac{2q^st\pi}{b} \right) \right) &= 0, \ a.e. \quad (1.15) \\
& \text{for } s = 0, 1, \ldots \text{ and all } 0 \neq t \in \mathbb{Z}, \text{ with } p, q \nmid t.
\end{align*}
\]

The proof of (1.15) is similar to that of (1.14), except that in the factorization of \(m_0\), we have

\[m_0 = p^u q^v t,\]

where \(u\) and \(v\) are nonnegative integers and \(p, q \nmid t\). Hence,

\[m = p^u q^v t a^{k}\gamma = p^u q^v \left( \frac{p}{q} \right)^k t = p^{u+k} q^{v-k} t;\]

and this restricts the range of \(k\) to \(-u \leq k \leq v\), or \(0 \leq j \leq s\) when \(j := k + u\) and \(s := v + u\) in (1.15).

Finally, for \(1 < a \in E_3\), it is clear that \(I_a(\alpha)\) is a singleton, so that for \(\alpha \neq 0\), (1.7) becomes

\[
\frac{1}{b} \sum_{\ell=1}^{L} \hat{\psi}_\ell(a^j \omega) \hat{\psi}_\ell \left( a^j \omega + \frac{2m\pi}{b} \right) = 0, \ a.e.,
\]

or equivalently,

\[
\frac{1}{b} \sum_{\ell=1}^{L} \hat{\psi}_\ell(\omega) \hat{\psi}_\ell \left( \omega + \frac{2m\pi}{b} \right) = 0, \ a.e., 0 \neq m \in \mathbb{Z}. \quad (1.16)
\]

Observe that if \(\Psi_L\) satisfies (1.16), it also satisfies (1.14) in Corollary 1 and (1.15) in Corollary 2. Hence, we have the following.

Corollary 3. Let \(1 < a < \infty\) and \(\Psi_L \subset L^2\).

(a) If \(\Psi_L\) satisfies both (1.13) and (1.16), then \(F_a\) is a tight frame of \(L^2\).

(b) If \(1 < a \in E_3\) and \(F_a\) is a tight frame of \(L^2\), then \(\psi_L\) must satisfy both (1.13) and (1.16).

As a direct consequence of Theorem 1, we also have the following.
Corollary 4. Let \( 1 < a < \infty \) be an arbitrary dilation factor, and \( \Psi_L \subset L^2 \) such that \( \hat{\psi}_\ell(\omega) \geq 0 \) a.e. for \( \ell = 1, \ldots, L \). Then \( \mathcal{F}_a \) is a tight frame of \( L^2 \), if and only if \( \Psi_L \) satisfies both (1.13) and

\[
\hat{\psi}_\ell(\omega)\hat{\psi}_\ell \left( \omega + \frac{2m\pi}{b} \right) = 0, \text{ a.e. for all } 0 \neq m \in \mathbb{Z} \text{ and } \ell = 1, \ldots, L. \tag{1.17}
\]

Recall from our earlier discussion that under the assumption in Corollary 4, the wavelet family \( \mathcal{F}_a \) is an orthonormal basis of \( L^2 \), if and only if (1.13), (1.17), and

\[
\|\hat{\psi}_1\| = \cdots = \|\hat{\psi}_L\| = 2\pi
\]

are satisfied. This trivial observation is useful for studying MSF (minimally supported frequency) wavelets (see [11, p. 349 and p. 394]).

Let us now focus on irrational dilation factors. As dilation factors, these irrational numbers fall into two significantly different categories, as follows.

(i) For \( a \in E_3 \) (i.e. \( \alpha^j \) remains to be an irrational for any positive integer \( j \)), we will see in Section 3.2 that there does not exist any orthonormal wavelet, with dilation factor \( a \), that gives “good” time-frequency localization. This answers a question in Daubechies [8, p. 16], for \( a \in E_3 \).

(ii) For \( a \in E_1 \cup E_2 \) (i.e. \( \alpha^j \) becomes an integer, or at least a rational, for some positive integer \( j \)), however, these irrational numbers, \( a \), can be treated as integer or rational dilation factors, as shown in the following Proposition 1. This points out an essential departure from the non-existence result of Auscher [1], within the frame work of multiresolution analyses, as mentioned in Daubechies [8, p. 323]. For rational dilation factors, on the other hand, see Auscher [2].

Proposition 1. Let \( 1 < a \in E_1 \cup E_2 \) and \( 1 \leq \gamma = \gamma_a \in \mathbb{Z} \) be the smallest integer for which \( r = r_\alpha := a^\gamma \) is a rational number (and particularly an integer, if \( a \in E_1 \)). Also, for \( \psi_\ell \in L^2 \), let \( \psi_\ell^{\#}(x) := \gamma^{-1/2}\psi_\ell(x) \).

(a) If the family

\[
\mathcal{F}_r := \{ \psi_{j,k,\ell}(x) := r^{j/2}\psi_\ell(r^j x - kb) : j, k \in \mathbb{Z}, \ell = 1, \ldots, L \} \tag{1.18}
\]
is a tight frame of $L^2$ in the sense of (1.3), then

$$F_a^\#: = \{\psi_{\ell,j,k}^\#(x) := a^{j/2}\psi_{\ell}^\#(a^j x - kb): j, k \in \mathbb{Z}, \ell = 1, \ldots, L\}$$

is also a tight frame of $L^2$ in the sense of (1.3).

(b) If $F_a^\#$ is a tight frame of $L^2$ in the sense of (1.3), then a necessary and sufficient condition for $F_r$ to be a tight frame of $L^2$ in the sense of (1.3) is that

$$\sum_{\ell=1}^L \sum_{j \in \mathbb{Z}} |\tilde{\psi}_{\ell}(r^\gamma \omega)|^2 = \text{constant} \neq 0 \text{ a.e.}$$

(1.20)

The proof of Theorem 1 and Proposition 1 will be given in Section 2. In Section 3, we construct a single function $\psi \in L^2$ (i.e. $\Psi_L$ is a singleton) that satisfies (1.13) and (1.16), where the dilation factor $a > 1$ is arbitrary. Hence, according to Corollary 3, there always exists a tight frame $F_a$, with arbitrary dilation factor $a > 1$, which is generated by one function in $L^2$. In fact, this generating function $\psi$ can be chosen to have "good" time-frequency localization. We will also apply Corollary 4 to generalize the notion of s-elementary wavelets of Dai and Larson [7] from dilation $a = 2$ to arbitrary $a > 1$.

In Section 4, we generalize Theorem 1 and its corollaries to dual frames, again for any dilation factor $a > 1$.

2. Proof of main results

To prove Theorem 1 as well as its generalization to dual frames, we first establish a unified lemma. For this purpose, we need the following notations and definitions.

In addition to $\Psi_L$ and $F_a$ defined in Section 1, let $\tilde{\Psi}_L = \{\tilde{\psi}_1, \ldots, \tilde{\psi}_L\} \subset L^2$ and

$$\tilde{F}_a := \{\tilde{\psi}_{\ell,j,k}(x) := a^{j/2}\tilde{\psi}_{\ell}(a^j x - kb): j, k \in \mathbb{Z}, \ell = 1, \ldots, L\}$$

(2.1)

generated by $\tilde{\Psi}_L$. For any $F_a$ and $\tilde{F}_a$, we consider the bi-linear functional

$$P(f, g) := \sum_{\ell=1}^L \sum_{j, k \in \mathbb{Z}} \langle f, \psi_{\ell,j,k} \rangle \langle \tilde{\psi}_{\ell,j,k}, g \rangle, \quad f, g \in L^2,$$

(2.2)

on $L^2 \times L^2$, where the weak convergence of the bi-infinite series is defined by

$$\lim_{j_1, j_2 \rightarrow \infty} \lim_{k_1, k_2 \rightarrow \infty} \sum_{\ell=1}^L \sum_{j=-j_1}^{j_2} \sum_{k=-k_1}^{k_2} \langle f, \psi_{\ell,j,k} \rangle \langle \tilde{\psi}_{\ell,j,k}, g \rangle.$$

(2.3)
We say that \((\mathcal{F}_a, \widetilde{\mathcal{F}}_a)\) is a dual pair if the finite series in (2.2) converges in the sense of (2.3) and satisfies
\[
P(f, g) = \langle f, g \rangle, \quad \text{all } f, g \in L^2.
\] (2.4)

In this regard, we recall the following notion of Bessel families (or sequences). For instance, the family \(\mathcal{F}_a\) in (1.1) is called a Bessel sequence if it satisfies the upper bound (but not necessarily the lower bound) condition in (1.2); i.e.
\[
\sum_{\ell=1}^{L} \sum_{j,k \in \mathbb{Z}} |\langle f, \psi_{\ell,j,k} \rangle|^2 \leq B \| f \|^2, \quad \text{all } f \in L^2,
\] (2.5)

and \(B\) is called a Bessel bound. Hence, if both \(\mathcal{F}_a\) and \(\widetilde{\mathcal{F}}_a\) are Bessel sequences, then it is clear that the series (2.2) is absolutely convergent. For \(\Psi := \Psi_L\) and \(\widetilde{\Psi} := \widetilde{\Psi}_L\), we also need the notation
\[
L_\Psi(\omega) := \sum_{\ell=1}^{L} \sum_{j \in \mathbb{Z}} |\widehat{\psi}_\ell(a^j \omega)|^2, \quad L_{\widetilde{\Psi}}(\omega) = \sum_{\ell=1}^{L} \sum_{j \in \mathbb{Z}} |\widehat{\psi}_\ell(a^j \omega)|^2.
\] (2.6)

In addition, we let
\[
B := \{ f \in L^\infty(\mathbb{R}) : f \text{ has compact support} \}
\] (2.7)

and
\[
\hat{B} := \{ g : \hat{g} \in B \}.
\] (2.8)

The main result in this section is the following.

**Lemma 1.** Let \(a > 1\) be an arbitrary dilation factor and \(\Psi = \Psi_L, \widetilde{\Psi} = \widetilde{\Psi}_L \subset L^2\) satisfy \(L_\Psi, L_{\widetilde{\Psi}} \in L^\infty\). Then for any \(f, g \in \hat{B}\), the series (2.2) converges in the sense of (2.3). Furthermore, if
\[
P(f, g) = \langle f, g \rangle, \quad \text{all } f, g \in \hat{B},
\] (2.9)

then \(\Psi_L\) and \(\widetilde{\Psi}_L\) satisfy
\[
\frac{1}{b} \sum_{\ell=1}^{L} \sum_{(j,m) \in I_a(\alpha)} \overline{\psi}_\ell(a^{j} \omega) \hat{\psi}_\ell \left( a^{j} \omega + \frac{2m \pi}{b} \right) = \delta_{a,0}, \text{ a.e.},
\] (2.10)
for all \( a \)-adic numbers \( \alpha \).

Let us first apply this lemma to the proof of Theorem 1.

**Proof of Theorem 1.** Let \( \tilde{\Psi}_L = \Psi_L \). If \( \mathcal{F}_a \) is a tight frame of \( L^2 \), then \( P(f, f) = \|f\|^2 \)
for all \( f \in L^2 \). Consequently, it follows that \( L_\Psi \in L^\infty \) and \( P(f, g) = (f, g) \) for all \( f, g \in L^2 \). Thus, by Lemma 1, (2.10) holds for \( \tilde{\Psi}_L = \Psi_L \); i.e. (1.7) holds.

To establish the converse, we first observe that

\[
P(f, f) = \frac{1}{2\pi b} \sum_{\ell=1}^L \sum_{j, s \in \mathbb{Z}} \int_{-\infty}^{\infty} \hat{f}(\omega) \hat{\psi}_\ell(a^{-j} \omega) \hat{f} \left( \omega + \frac{2\pi s a^j}{b} \right) \overline{\psi}_\ell \left( a^{-j} \omega + \frac{2\pi s}{b} \right) d\omega,
\]

where the series certainly converges for \( f \in \tilde{B} \), provided that \( L_\Psi \in L^\infty \). Indeed, for \( \text{supp } \hat{f} \subset [-H, H] \), there exists an \( J \in \mathbb{Z} \), such that

\[
|P(f, f)| = \frac{1}{2\pi b} \left| \sum_{\ell=1}^L \sum_{j \leq J} \sum_{|s| \leq S_j} \int_{-H}^{H} \hat{f}(\omega) \hat{\psi}_\ell(a^{-j} \omega) \hat{f} \left( \omega + \frac{2\pi s a^j}{b} \right) \overline{\psi}_\ell \left( a^{-j} \omega + \frac{2\pi s}{b} \right) d\omega \right|,
\]

\[
\leq \frac{1}{2\pi b} \sum_{\ell=1}^L \sum_{j \leq J} \sum_{|s| \leq S_j} \left( \int_{-H}^{H} \left| \hat{f}(\omega) \right|^2 d\omega \right)^{1/2} \left( \int_{-H}^{H} \left| \hat{\psi}_\ell(a^{-j} \omega) \right|^2 d\omega \right)^{1/2}
\]

\[
\times \left( \int_{-H}^{H} \left| \hat{f}(\omega) \hat{\psi}_\ell(a^{-j} \omega) \right|^2 d\omega \right)^{1/2}
\]

\[
\leq \text{const} \sum_{\ell=1}^L \sum_{j \leq J} a^{-j} ||\hat{f}||_\infty \left( \int_{-H}^{H} \left| \hat{\psi}_\ell(a^{-j} \omega) \right|^2 d\omega \right)^{1/2}
\]

\[
\leq \text{const} \||\hat{f}||_\infty \sum_{\ell=1}^L ||\hat{\psi}_\ell||,
\]

where \( S_j \leq \frac{bH}{\pi} a^{-j} \).

For \( a > 1 \), let \( \Lambda(a) \) denote the set of distinct \( a \)-adic numbers; i.e.

\[
\Lambda(a) = \left\{ \alpha \in \mathbb{R} : \text{there exist } j, m \in \mathbb{Z} \text{ such that } \alpha = \frac{m}{a^j} \right\}.
\]

Using this notation, we can re-write \( P(f, f) \) as

\[
P(f, f) = \frac{1}{2\pi b} \sum_{\ell=1}^L \sum_{\alpha \in \Lambda(a)} \int_{-\infty}^{\infty} \hat{f}(\omega) \hat{f} \left( \omega + \frac{2\pi \alpha}{a^j} \right) \left[ \sum_{(j, s) \in I_a(\alpha)} \hat{\psi}_\ell(a^j \omega) \overline{\psi}_\ell \left( a^j \omega + \frac{2\pi s}{b} \right) \right] d\omega.
\]

(2.11)
Now suppose that (1.7) is valid. Then on one hand, we have $L_{\psi}(\omega) = 1$ a.e., so that $L_{\psi} \in L^\infty$; and on the other hand, (2.11) can be simplified to be

$$P(f, f) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{f}(\omega) \tilde{f}(\omega) d\omega = \|f\|^2,$$  
(2.12)

for all $f \in \hat{B}$. Since $\hat{B}$ is dense in $L^2$, (2.12) holds for all $f \in L^2$. That is, $\mathcal{F}_a$ is a tight frame of $L^2$. This completes the proof of Theorem 1.

We now turn to the proof of the main lemma.

**Proof of Lemma 1.** Let $f, g \in \hat{B}$ and let $K > 0$ such that both $\hat{f}$ and $\hat{g}$ vanish outside $[-K, K]$. We first prove that (2.4) is convergent. Fix $j \in \mathbb{Z}$ and consider

$$G_j := \sum_{\ell=1}^{L} \sum_{k \in \mathbb{Z}} \langle f, \psi_{\ell,j,k} \rangle \langle \tilde{\psi}_{\ell,j,k}, g \rangle.$$  
(2.13)

Since $L_{\psi}$ and $L_{\tilde{\psi}}$ are in $L^\infty$, $\hat{\psi}_\ell$ and $\tilde{\psi}_\ell$, $\ell = 1, \ldots, L$, are also in $L^\infty$. Hence, the above series that defines $G_j$ is convergent, and by the Parseval identity for Fourier transforms, we have

$$G_j = \frac{1}{4\pi^2} \sum_{\ell=1}^{L} \sum_{k \in \mathbb{Z}} a_j \int_{-\infty}^{\infty} \hat{f}(a_j \xi) \tilde{\psi}_\ell(\xi) e^{ikb\xi} d\xi \int_{-\infty}^{\infty} \bar{g}(a_j \omega) \tilde{\psi}_\ell(\omega) e^{-ikb\omega} d\omega.$$  
(2.14)

For convenience, set

$$T := \frac{2\pi}{b},$$  
(2.15)

and consider the $T$-periodic functions

$$F_j(\omega) := \sum_{\ell=1}^{L} \sum_{s \in \mathbb{Z}} \hat{f}(a_j (\omega + sT)) \tilde{\psi}_\ell(\omega + sT).$$  
(2.16)

Clearly, $F_j$'s are locally square integrable and the Fourier series of $F_j$, which are convergent in $L^2(0, T)$, are given by

$$F_j(\omega) = \sum_{k \in \mathbb{Z}} \left( \frac{1}{T} \int_{0}^{T} F_j(\xi) e^{i k b \xi} d\xi \right) e^{-i k b \omega} = \frac{b}{2\pi} \sum_{\ell=1}^{L} \sum_{k \in \mathbb{Z}} \left( \int_{-\infty}^{\infty} \hat{f}(a_j \xi) \tilde{\psi}_\ell(\xi) e^{i k b \xi} d\xi \right) e^{-i k b \omega}.$$  
(2.17)
For the quantities $G_j$ in (2.13), set

$$P_{j_1,j_2} := \sum_{j=-j_1}^{j_2} G_j. \quad (2.18)$$

Then by (2.14), (2.16), and (2.17), we have

$$P_{j_1,j_2} = \frac{1}{2\pi b} \sum_{\ell=1}^{L} \sum_{j=-j_1}^{j_2} \int_{-\infty}^{\infty} \tilde{g}((\omega) \tilde{\psi}_\ell(a^{-j}\omega) \left[ \sum_{s \in \mathbb{Z}} \tilde{f}(\omega + a^j s T) \tilde{\psi}_\ell(a^{-j}\omega + s T) \right] d\omega. \quad (2.19)$$

Hence, since $\tilde{f}$ and $\tilde{g}$ vanish outside $[-K, K]$, we have, for $|a^j T| \geq 2K$ and $s \neq 0$,

$$\int_{-\infty}^{\infty} \tilde{g}(\omega) \tilde{\psi}_\ell(a^{-j}\omega) \tilde{f}(\omega + a^j s T) \tilde{\psi}_\ell(a^{-j}\omega + s T) d\omega = 0,$$

so that for $J \geq \log_a \frac{2K}{T}$,

$$\lim_{j_2 \to \infty} P_{-J,j_2} = \sum_{\ell=1}^{L} \sum_{j=J}^{\infty} \int_{-\infty}^{\infty} \tilde{g}(\omega) \tilde{\psi}_\ell(a^{-j}\omega) \left[ \sum_{s \in \mathbb{Z}} \tilde{f}(\omega + a^j s T) \tilde{\psi}_\ell(a^{-j}\omega + s T) \right] d\omega$$

$$= \sum_{\ell=1}^{L} \sum_{j=J}^{\infty} \int_{-\infty}^{\infty} \tilde{g}(\omega) \tilde{f}(\omega) \tilde{\psi}_\ell(a^{-j}\omega) \tilde{\psi}_\ell(a^{-j}\omega) d\omega. \quad (2.20)$$

For $j \leq J$, we see that

$$|G_j| = \left| \sum_{\ell=1}^{L} \int_{-K}^{K} \tilde{g}(\omega) \tilde{\psi}_\ell(a^{-j}\omega) \left[ \sum_{s \in \mathbb{Z}} \tilde{f}(\chi_{[-K-a^j s T, K-a^j s T]})(\omega + a^j s T) \tilde{\psi}_\ell(a^{-j}\omega + s T) \right] d\omega \right|$$

$$\leq \|\tilde{g}\|_\infty \|\tilde{f}\|_\infty \sum_{\ell=1}^{L} \sum_{|s| \leq \frac{2K}{T a^j}} \left( \int_{-K}^{K} |\tilde{\psi}_\ell(a^{-j}\omega)|^2 d\omega \right)^{1/2} \left( \int_{-2K}^{2K} |\tilde{\psi}_\ell(a^{-j}\omega)|^2 d\omega \right)^{1/2}$$

$$\leq \|\tilde{g}\|_\infty \|\tilde{f}\|_\infty \sum_{\ell=1}^{L} \left( \frac{4K}{T a^j} + 1 \right) \left( \int_{-K}^{K} |\tilde{\psi}_\ell(a^{-j}\omega)|^2 d\omega \right)^{1/2} \left( \int_{-2K}^{2K} |\tilde{\psi}_\ell(a^{-j}\omega)|^2 d\omega \right)^{1/2}.$$

Thus, we have

$$\sum_{j=-J_1}^{J-1} |G_j| \leq \text{const} \|\tilde{g}\|_\infty \|\tilde{f}\|_\infty$$

$$\times \left[ \left( \sum_{\ell=1}^{L} |\tilde{\psi}_\ell|^2 \right)^{1/2} \right] \left( \sum_{\ell=1}^{L} \|\tilde{\psi}_\ell\|^2 \right)^{1/2} + \|L_\Psi\|_\infty \|L_\Phi\|_\infty < \infty. \quad (2.21)$$
The above inequality (2.21) implies that the limit, \( \lim_{j_1 \to \infty} P_{j_1,J-1} \), exists, and hence by combining this with (2.20), we conclude that the series

\[
P(f, g) = \sum_{\ell = 1}^{\infty} \sum_{j,k} \langle f, \psi_{\ell,j,k} \rangle \langle \tilde{\psi}_{\ell,j,k}, g \rangle
\]

is convergent in the sense of (2.3).

From the above proof, it is also easy to see that the series

\[
P(f, g) = \frac{1}{2\pi b} \sum_{\ell=1}^{L} \sum_{j=-\infty}^{j} \sum_{s=-\infty}^{s} \int_{\mathbb{R}} \bar{g}(\omega) \tilde{\psi}_{\ell}(a^{-j} \omega) \tilde{f}(\omega + a^j s T) \bar{\psi}_{\ell}(a^{-j} \omega + sT) d\omega
\]

is absolutely convergent. Hence, we can rewrite \( P \) as

\[
P(f, g) = \frac{1}{2\pi b} \sum_{\ell=1}^{L} \sum_{\alpha \in \Lambda(\alpha)} \int_{\mathbb{R}} \bar{g}(\omega) \tilde{f}(\omega + \alpha T) \left[ \sum_{(j,s) \in I_\alpha(0)} \tilde{\psi}_{\ell}(a^j \omega) \bar{\psi}_{\ell}(a^j \omega + sT) \right] d\omega.
\]

Next, we decompose the above series into two parts:

\[
P(f, g) := M(f, g) + R(f, g),
\]

where

\[
M(f, g) = \frac{1}{2\pi b} \sum_{\ell=1}^{L} \int_{\mathbb{R}} \bar{g}(\omega) \tilde{f}(\omega) \left[ \sum_{(j,s) \in I_\alpha(0)} \tilde{\psi}_{\ell}(a^j \omega) \bar{\psi}_{\ell}(a^j \omega + sT) \right] d\omega
\]

and

\[
R(f, g) = \frac{1}{2\pi b e} \sum_{\ell=1}^{L} \sum_{\alpha \in \Lambda(\alpha) \setminus \{0\}} \int_{\mathbb{R}} \bar{g}(\omega) \tilde{f}(\omega + \alpha T) \left[ \sum_{(j,s) \in I_\alpha(0)} \tilde{\psi}_{\ell}(a^j \omega) \bar{\psi}_{\ell}(a^j \omega + sT) \right] d\omega.
\]

Since \( I_\alpha(0) = \mathbb{Z} \times \{0\} \), we see that (2.24) can be written as

\[
M(f, g) = \frac{1}{2\pi b} \int_{\mathbb{R}} \bar{g}(\omega) \tilde{f}(\omega) \left[ \sum_{\ell=1}^{L} \sum_{j \in \mathbb{Z}} \tilde{\psi}_{\ell}(a^j \omega) \bar{\psi}_{\ell}(a^j \omega) \right] d\omega.
\]

Let \( \omega_0 \in \mathbb{R} \setminus \{0\} \) and \( \varepsilon > 0 \) be given. Set

\[
H_\alpha(\varepsilon) := [\omega_0 - \alpha T - \varepsilon, \omega_0 - \alpha T + \varepsilon], \quad \alpha \in \Lambda(\alpha).
\]
Also, consider \( f = g = f_1 \), defined by
\[
\hat{f}_1(\omega) = \frac{1}{\sqrt{2\varepsilon}} \chi_{\mathcal{H}_0(\varepsilon)}(\omega).
\]
Then
\[
M(f_1, f_1) = \frac{1}{4\pi \varepsilon} \int_{\omega_0 - \varepsilon}^{\omega_0 + \varepsilon} \sum_{\ell=1}^{L} \sum_{j \in \mathbb{Z}} \tilde{\psi}_\ell(\alpha^j \omega) \bar{\psi}_\ell(\alpha^j \omega) d\omega
\]
and
\[
R(f_1, f_1) = \frac{1}{4\pi \varepsilon} \sum_{\ell=1}^{L} \sum_{\alpha \in \Lambda(\alpha) \setminus \{0\}} \int_{H_0(\varepsilon) \cap H_\alpha(\varepsilon)} \left[ \sum_{(j,s) \in I_\alpha(\alpha)} \tilde{\psi}_\ell(\alpha^j \omega) \bar{\psi}_\ell(\alpha^j \omega + sT) \right] d\omega.
\]
By the Cauchy Inequality, we see that
\[
|R(f_1, f_1)| \leq \frac{1}{4\pi \varepsilon} \left( \sum_{\ell=1}^{L} \sum_{\alpha \in \Lambda(\alpha) \setminus \{0\}} \sum_{(j,s) \in I_\alpha(\alpha)} \int_{H_0(\varepsilon) \cap H_\alpha(\varepsilon)} |\tilde{\psi}_\ell(\alpha^j \omega)|^2 d\omega \right)^{1/2}
\times \left( \sum_{\ell=1}^{L} \sum_{\alpha \in \Lambda(\alpha) \setminus \{0\}} \sum_{(j,s) \in I_\alpha(\alpha)} \int_{H_0(\varepsilon) \cap H_\alpha(\varepsilon)} |ar{\psi}_\ell(\alpha^j \omega + sT)|^2 d\omega \right)^{1/2}.
\]
Now it is clear that if \( |\alpha| > 2\varepsilon/T \), then
\[
\text{meas}[H_0(\varepsilon) \cap H_\alpha(\varepsilon)] = 0.
\]
On the other hand, if \( |\alpha| \leq 2\varepsilon/T \) and \( (j,s) \in I_\alpha(\alpha) \) with \( \alpha \neq 0 \), we have \( |s| \geq 1 \), and hence
\[
a^{-j} \leq \frac{2\varepsilon}{|s|T} \leq \frac{2\varepsilon}{T} \quad \text{and} \quad |a^{-j} sT| \leq 2\varepsilon.
\]
Thus, setting
\[
Q := \left| \log_a \frac{2\varepsilon}{T} \right| \quad \text{and} \quad q_j := 2\varepsilon a^j / T,
\]
we have
\[
|R(f_1, f_1)| \leq \frac{1}{4\pi \varepsilon} \left( \sum_{\ell=1}^{L} \sum_{j \geq Q \ |s| \leq q_j, s \neq 0} \int_{\omega_0 - \varepsilon}^{\omega_0 + \varepsilon} |\tilde{\psi}_\ell(\alpha^j \omega)|^2 d\omega \right)^{1/2}
\times \left( \sum_{\ell=1}^{L} \sum_{j \geq Q \ |s| \leq q_j, s \neq 0} \int_{\omega_0 - 3\varepsilon}^{\omega_0 + 3\varepsilon} |\bar{\psi}_\ell(\alpha^j \omega)|^2 d\omega \right)^{1/2}.
\]
Let \( \varepsilon \) be sufficiently small so that \(|\omega_0| - 3\varepsilon > 0\) and that the intervals \((a^j(|\omega_0| - 3\varepsilon), a^j(|\omega_0| + 3\varepsilon)), \ j \geq 0\), are mutually disjoint. Then we see that

\[
\sum_{\ell=1}^L \sum_{j \geq Q} \sum_{|s| \leq q_j, s \neq 0} \int_{\omega_0 - \varepsilon}^{\omega_0 + \varepsilon} \left| \hat{\psi}_\ell(a^j \omega) \right|^2 d\omega \\
\leq \frac{4\varepsilon}{T} \sum_{\ell=1}^L \sum_{j \geq Q} a^j \int_{\omega_0 - \varepsilon}^{\omega_0 + \varepsilon} \left| \hat{\psi}_\ell(a^j \omega) \right|^2 d\omega \\
\leq \frac{4\varepsilon}{T} \sum_{\ell=1}^L \int_{|\omega| \geq a^Q(|\omega_0| - \varepsilon)} \left| \hat{\psi}_\ell(\omega) \right|^2 d\omega,
\]

(2.30)

and

\[
\sum_{\ell=1}^L \sum_{j \geq Q} \sum_{|s| \leq q_j, s \neq 0} \int_{\omega_0 - 3\varepsilon}^{\omega_0 + 3\varepsilon} \left| \hat{\psi}_\ell(a^j \omega) \right|^2 d\omega \leq \frac{4\varepsilon}{T} \sum_{\ell=1}^L \int_{|\omega| \geq a^Q(|\omega_0| - 3\varepsilon)} \left| \hat{\psi}_\ell(\omega) \right|^2 d\omega.
\]

(2.31)

Combining (2.29)–(2.31), we obtain

\[
|R(f_1, f_1)| = \text{const} \left( \sum_{\ell=1}^L \int_{|\omega| \geq \frac{T}{2\varepsilon}(|\omega_0| - \varepsilon)} \left| \hat{\psi}_\ell(\omega) \right|^2 d\omega \right) \\
\times \left( \sum_{\ell=1}^L \int_{|\omega| \geq \frac{T}{2\varepsilon}(|\omega_0| - 3\varepsilon)} \left| \hat{\psi}_\ell(\omega) \right|^2 d\omega \right)^{1/2} \to 0,
\]

(2.32)

as \( \varepsilon \to +0 \). Then, it follows from (2.9), (2.23), (2.25), (2.28) and (2.32), that

\[
\frac{1}{4\pi b\varepsilon} \int_{\omega_0 - \varepsilon}^{\omega_0 + \varepsilon} \sum_{\ell=1}^L \sum_{j \in \mathbb{Z}} \hat{\psi}_\ell(a^j \omega) \overline{\hat{\psi}_\ell(a^j \omega)} d\omega + o(1) = \frac{1}{2\pi}.
\]

(2.33)

By taking the limit of (2.33) as \( \varepsilon \to +0 \), we arrive at

\[
\frac{1}{b} \sum_{\ell=1}^L \sum_{j \in \mathbb{Z}} \hat{\psi}_\ell(a^j \omega_0) \overline{\hat{\psi}_\ell(a^j \omega_0)} = 1,
\]

(2.34)

for almost all \( \omega_0 \in \mathbb{R} \), i.e. (2.10) holds for \( \alpha = 0 \).

Next we prove (2.10) with \( \alpha \neq 0 \). First, putting (2.34) into (2.25), and combining this result with (2.9) and (2.23), we get

\[
R(f, g) = 0, \quad f, g \in \hat{B}.
\]

(2.35)
Now, let $\alpha_0 = a^{-j_0}m_0 \in \Lambda(a) \setminus \{0\}$, and decompose $R(f, g)$ into two parts:

$$R(f, g) = R_1(f, g) + R_2(f, g),$$  \hspace{1cm} (2.36)

where

$$R_1(f, g) := \frac{1}{2\pi b} \sum_{\ell=1}^L \int_{\mathbb{R}} \tilde{g}(\omega) \tilde{f}(\omega + \alpha_0 T) \left[ \sum_{(j,s) \in I_n(\alpha_0)} \tilde{\psi}_\ell(a^j \omega) \overline{\psi}_\ell(a^j \omega + sT) \right] d\omega$$

and

$$R_2(f, g) := \frac{1}{2\pi b} \sum_{\ell=1}^L \sum_{\alpha \in \Lambda(a) \setminus \{0, \alpha_0\}} \int_{\mathbb{R}} \tilde{g}(\omega) \tilde{f}(\omega + \alpha T) \left[ \sum_{(j,s) \in I_n(\alpha)} \tilde{\psi}_\ell(a^j \omega) \overline{\psi}_\ell(a^j \omega + sT) \right] d\omega.$$

Assume that $\omega_0 \in \mathbb{R} \setminus \{0\}$, $0 < \varepsilon/T < 1/4$, and consider $f = f_2, g = g_2$ defined by

$$\widetilde{g}_2(\omega) := \frac{1}{\sqrt{2\varepsilon}} \chi_{[\omega_0 - \varepsilon, \omega_0 + \varepsilon]}(\omega)$$

and

$$\widetilde{f}_2(\omega + \alpha_0 T) := \frac{1}{\sqrt{2\varepsilon}} \chi_{[\omega_0 - \varepsilon, \omega_0 + \varepsilon]}(\omega).$$

It is clear that for almost all $\omega_0 \in \mathbb{R}$,

$$\lim_{\varepsilon \to 0} R_1(f_2, g_2) = \frac{1}{2\pi b} \sum_{\ell=1}^L \sum_{(j,s) \in I_n(\alpha_0)} \tilde{\psi}_\ell(a^j \omega) \overline{\psi}_\ell(a^j \omega_0 + sT).$$  \hspace{1cm} (2.37)

Next, to estimate $R_2(f_2, g_2)$, let $J_0$ be any natural number that satisfies $a^{-J_0} > 4|\alpha_0|$. If $s \neq 0$ and $j \leq -J_0$, then

$$|a^{-j} s - \alpha_0| \geq a^{-j} |s| - a^{-J_0} |\alpha_0| \geq \frac{3}{4} a^{-j} \geq \frac{2\varepsilon}{T}.$$ 

From the definition of $f_2$ and $g_2$, we see that if $\alpha = a^{-j} s$ satisfies $|\alpha T - \alpha_0 T| > 2\varepsilon$, then $\tilde{g}_2(\omega) \tilde{f}_2(\omega + \alpha T) = 0$. Thus, $R_2(f_2, g_2)$ can be rewritten as

$$R_2(f_2, g_2) = \frac{1}{2\pi b} \sum_{\ell=1}^L \sum_{j=J_1}^\infty \sum_{|a^{-j} s - \alpha_0| \leq 2\varepsilon/T, s \neq 0} \int_{\mathbb{R}} \tilde{g}_2(\omega) \tilde{f}_2(\omega + a^{-j} sT)$$

$$\times \tilde{\psi}_\ell(a^j \omega) \overline{\psi}_\ell(a^j \omega + sT) d\omega$$

$$+ \frac{1}{2\pi b} \sum_{\ell=1}^L \sum_{j=-J_0}^{J_1} \sum_{|a^{-j} s - \alpha_0| \leq 2\varepsilon/T, s \neq 0} \int_{\mathbb{R}} \tilde{g}_2(\omega) \tilde{f}_2(\omega + a^{-j} sT)$$

$$\times \tilde{\psi}_\ell(a^j \omega) \overline{\psi}_\ell(a^j \omega + sT) d\omega$$

$$=: R_{2,1} + R_{2,2},$$  \hspace{1cm} (2.38)
where $J_1$ is a sufficiently large natural number. Similar to (2.32), we can prove that for an arbitrarily given $\eta > 0$, there exists an integer $J_1$, such that

$$|R_{2,1}| = O(1) \left( \sum_{\ell=1}^{L} \int_{a^{j_1}(|\omega_0| - \varepsilon)}^{\infty} |\hat{\psi}_\ell(\omega)|^2 d\omega \right)^{1/2} \left( \sum_{\ell=1}^{L} \int_{a^{j_1}(|\omega_0| - 3\varepsilon)}^{\infty} |\hat{\psi}_\ell(\omega)|^2 d\omega \right)^{1/2} \leq \eta. \tag{2.39}$$

For any $j \in \mathbb{Z}$, the number of $s \in \mathbb{Z}\setminus\{0\}$ that satisfy $|a^{-j}s - \alpha_0| \leq 2\varepsilon/T$ is not more than $a^j4\varepsilon/T + 1$. Thus, the summation in $R_{2,2}$ is a finite sum. Hence, we have

$$\lim_{\varepsilon \to 0} R_{2,2} = 0. \tag{2.40}$$

Combining (2.35)–(2.40), we obtain

$$0 = \frac{1}{2\pi b} \sum_{\ell=1}^{L} \sum_{(j,s) \in I_s(a_0)} \hat{\psi}_\ell(a^j\omega_0)\overline{\hat{\psi}_\ell(a^j\omega_0 + sT)} + O(\eta). \tag{2.41}$$

Since $\eta$ is arbitrarily given, (2.41) implies (2.10) with $\alpha = a_0 \in \Lambda(a)\setminus\{0\}$. Since $a_0 \neq 0$ is arbitrary and (2.34) holds, we complete the proof of Lemma 1. \hfill \blacksquare

The proof of Proposition 1 depends on Corollaries 1 and 2, as follows.

**Proof of Proposition 1.**

(a) For each $j \in \mathbb{Z}$, write

$$j = q\gamma + p, \quad 0 \leq p \leq \gamma - 1.$$

Hence, for any $f \in L^2$, we have

$$\Phi(f) := \sum_{\ell=1}^{L} \sum_{j,k \in \mathbb{Z}} \left| \langle f, \psi_{\ell,j,k}^\# \rangle \right|^2$$

$$= \frac{1}{\gamma} \sum_{p=0}^{\gamma-1} \sum_{\ell=1}^{L} \sum_{q,k \in \mathbb{Z}} r^{q/2+p/2\gamma} \int_{-\infty}^{\infty} \overline{\psi}_\ell \left( r^q \left( r^{p/\gamma} x - kb \right) \right) f(x) dx^2$$

$$= \frac{1}{\gamma} \sum_{p=0}^{\gamma-1} \sum_{q \in \mathbb{Z}} \sum_{\ell=1}^{L} \sum_{k \in \mathbb{Z}} r^{q/2} \int_{-\infty}^{\infty} \overline{\psi}_\ell \left( r^q x - kb \right) f \left( r^{-p/\gamma} x \right) dx^2.$$

So, if $\mathcal{F}_\gamma$ in (1.18) is a tight frame of $L^2$ in the sense of (1.3), we have

$$\sum_{\ell=1}^{L} \sum_{q,k \in \mathbb{Z}} r^{q/2} \int_{-\infty}^{\infty} \overline{\psi}_\ell \left( r^q x - kb \right) f \left( r^{-p/\gamma} x \right) dx^2 = r^{p/\gamma} ||f||^2,$$

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and this gives
\[ \Phi(f) = \frac{1}{\gamma} \sum_{p=0}^{\gamma-1} r^{-p/\gamma} \| f \|^2 = \| f \|^2. \]

In other words, \( \mathcal{F}_{a}^{\#} \) is a tight frame of \( L^2 \) in the sense of (1.3).

(b) Conversely, suppose that \( \mathcal{F}_{a}^{\#} \) is a tight frame of \( L^2 \) in the sense of (1.3). Then by Corollaries 1 or 2 (depending on \( a \in E_1 \) or \( a \in E_2 \)), we have
\[ \frac{1}{b} \sum_{\ell=1}^{L} \sum_{j \in \mathbb{Z}} \left| \hat{\psi}_{\ell}^{\#}(a^j \omega) \right|^2 = 1 \text{ a.e.} \tag{2.42} \]
as well as (1.14) or (1.15). Now, assume that (1.20) holds. Then combining (2.42) with (1.20), we have
\[ \frac{1}{b} \sum_{\ell=1}^{L} \sum_{j \in \mathbb{Z}} \left| \hat{\psi}_{\ell}(r^j \omega) \right|^2 = 1 \text{ a.e.} \tag{2.43} \]
Thus, since (1.14) or (1.15) also holds, it follows from Corollary 1 or Corollary 2 that \( \mathcal{F}_{\tau} \) is a tight frame of \( L^2 \) in the sense of (1.3).

On the other hand, if \( \mathcal{F}_{\tau} \) is a tight frame of \( L^2 \), then (1.20) holds by Corollaries 1 or 2. This completes the proof of Proposition 1. \( \blacksquare \)

3. Applications

The first objective of this section is to show that the characterization in Theorem 1 is meaningful, in that for an arbitrary dilation factor \( a > 1 \), there always exist \( \Psi_L \)‘s that generate tight frames \( \mathcal{F}_{a} \) of \( L^2 \). In fact, even for \( L = 1 \) (i.e. \( \Psi_L = \Psi_1 \) is a singleton), tight frames with good time-frequency localization can be constructed by applying the theory developed in this paper. This will be discussed in Section 3.1. On the other hand, it is also shown, at least for \( 1 < a \in E_3 \), that \( \Psi_1 \) cannot generate an orthonormal basis of \( L^2 \) with good time-frequency localization. This observation, to be discussed in Section 3.2, partially answers a question in Daubechies’ Ten Lectures book [8, p. 16]. As another application of the theory in this paper, we study, in Section 3.3, minimally supported frequency (MSF) wavelets as introduced by Weiss and his colleagues [11, 9–10], and particularly s-elementary (SE) wavelets of Dai and Larson [7], for arbitrary dilation factors \( a > 1 \).
3.1. Tight frames with good time-frequency localization. Let $a > 1$, $b = 1$ and consider an arbitrary order of smoothness $m \geq 1$. Set $\xi = a^J$, where $J := \lfloor \log_a \pi - 2 \rfloor$. It is clear that $a^2 \xi \leq \pi$, and the function $\psi_0$, defined by

$$\hat{\psi}_0(\omega) := \chi_{[\xi, a\xi)[\cup[-a\xi, -\xi]}(\omega),$$

satisfies

$$\sum_{j \in \mathbb{Z}} |\hat{\psi}_0(a^j \omega)|^2 = 1, \quad \text{and} \quad \hat{\psi}_0(\omega)\hat{\psi}_0(\omega + k\pi) = 0, \quad k \in \mathbb{Z}.$$ 

Hence, by Theorem 1, we may conclude that $\{\psi_0, j, k(x) = a^{j/2}\psi_0(a^j x - k) : j, k \in \mathbb{Z}\}$ is a tight frame of $L^2$. This frame, however, does not have good time-frequency localization. So we need to modify $\hat{\psi}_0$ to $\hat{\psi} \in C^m(\mathbb{R})$. To do so, we set $n \in \mathbb{N}$ such that $2^n - 1 > m$ and define

$$r_0(x) = \begin{cases} 0, & \text{if } x \leq -1, \\ \sin \left[ \frac{\pi}{4} (1 + x) \right], & \text{if } |x| \leq 1, \\ 1, & \text{if } x \geq 1, \end{cases}$$

$$r_{\ell}(x) = r_{\ell-1} \left( \sin \frac{\pi x}{2} \right), \quad \ell = 1, 2, \ldots, n.$$ 

Then, the function $r_\ell(x) \in C^{2^\ell-1}([\mathbb{R})$ and hence, $r_n \in C^m(\mathbb{R})$. Let $\varepsilon$ be any real number with $0 < \varepsilon \leq (a - 1)\xi/(a + 1)$. Now, define $\hat{\psi}$ by

$$\hat{\psi}(\omega) := \begin{cases} r_n \left( \frac{\omega - \xi}{\varepsilon} \right), & \text{if } |\omega - \xi| \leq \varepsilon, \\ 1, & \text{if } \xi + \varepsilon \leq \omega \leq a\xi - a\varepsilon, \\ r_n \left( \frac{a\xi - \omega}{a\varepsilon} \right), & \text{if } |\omega - a\xi| \leq a\varepsilon, \\ 0, & \text{if } 0 \leq \omega < \xi - \xi \quad \text{or} \quad a\xi + a\varepsilon < \omega < \infty, \\ \psi(-\omega), & \text{if } \omega < 0. \end{cases}$$

It is clear that $\hat{\psi} \in C^m(\mathbb{R})$, with supp $\hat{\psi} \subset [-\pi, \pi]$. Thus, $\hat{\psi}$ satisfies $\hat{\psi}(\omega)\hat{\psi}(\omega + 2k\pi) = 0$, for any integer $k \neq 0$. On the other hand, by the definition of $r_n$, it is easy to prove that $\hat{\psi}$ also satisfies

$$\sum_{j \in \mathbb{Z}} |\hat{\psi}(a^j \omega)|^2 = 1, \quad \omega \in \mathbb{R}. \quad (3.1)$$

Indeed, if $1 \leq \omega < 1 + a^{-J}\varepsilon$, then

$$\hat{\psi}(a^j \omega) = 0, \quad \text{if } j \neq J \quad \text{or} \quad J + 1,$$
but

\[ |\hat{\psi}(a^J\omega)|^2 + |\hat{\psi}(a^{J+1}\omega)|^2 = r_n \left( \frac{a^J\omega - \xi}{\varepsilon} \right)^2 + r_n \left( \frac{a^{J+1}\omega - \xi}{a\varepsilon} \right)^2 = 1. \]

If \( a(1 - a^{-J}\varepsilon) \leq \omega \leq a, \) we can similarly prove that (4.1) holds. If \( 1 + a^J\varepsilon < \omega < a(1 - a^{-J}\varepsilon), \) then \( |\hat{\psi}(a^J\omega)|^2 = 1 \) and \( \hat{\psi}(a^J\omega) = 0, \) for \( j \neq J. \) Thus, (3.1) holds as well. Since \( \hat{\psi}(\omega) \) is an even function, the same equality holds for all \( \omega < 0. \)

3.2. A question of Daubechies. Next we address a problem posed in Daubechies [8, p. 16] namely: "It is an open question whether there exist orthonormal wavelet bases (necessarily not associated with a multiresolution analysis), with good time-frequency localization, and with irrational \( a\)." Here, we discuss this question only for the case where the irrational dilation factor \( a \in E_3. \) By Theorem 1, we see that if \( a \in E_3 \) and \( \{\psi_{j,k}(x) := a^{j/2}\psi(a^j x - k), j, k \in \mathbb{Z}\} \) is an orthonormal basis of \( L^2, \) then \( \psi \) must satisfy:

\[ \sum_{j \in \mathbb{Z}} |\hat{\psi}(a^j\omega)|^2 = 1, \quad \text{and} \quad \hat{\psi}(\omega)\hat{\psi}(\omega + 2k\pi) = 0, \quad k \in \mathbb{Z}\setminus\{0\}. \]

It is not hard to see that these conditions imply that \( |\hat{\psi}(\omega)| = \chi_E(\omega), \) where \( E \) is a measurable set with Lebesgue measure \( 2\pi. \) Thus, there does not exist an orthonormal wavelet with dilation factor \( a \in E_3, \) that has good time-frequency localization.

3.3. MSF Wavelets. It is easy to verify that if \( \mathcal{F}_a, \) defined as in (1.1) with \( b = 1, \) constitutes an orthonormal basis of \( L^2, \) then

\[ \sum_{k \in \mathbb{Z}} |\hat{\psi}_\ell(\omega + 2k\pi)|^2 = 1, \quad \text{a.e.,} \quad \ell = 1, \ldots, L. \]

(For \( L = 1, \) see [11]). These equalities imply that the measure of each of the sets

\[ G_\ell := \{\omega: \hat{\psi}_\ell(\omega) \neq 0\}, \quad \ell = 1, \ldots, L, \]

is at least \( 2\pi. \) Hence, following [11, p. 349], we will call \( \Psi_L = \{\psi_1, \ldots, \psi_L\} \) a minimally supported frequency (MSF) wavelet family, provided that \( \mathcal{F}_a \) is an orthonormal basis of \( L^2 \)

and that \( \text{meas} \ G_\ell = 2\pi, \ \ell = 1, \ldots, L. \)
In another development of wavelet theory, Dai and Larson [7] studied the so-called s-elementary (SE) wavelets. (Here, the prefix “s” stands for “set.”) To generalize this notion from dilation \( a = 2 \) to arbitrary \( a > 1 \), we call a collection \( \Psi_L = \{ \psi_1, \ldots, \psi_L \} \) of functions an SE wavelet family, provided that \( \mathcal{F}_a \) defined as in (1.1) with \( b = 1 \) is an orthonormal basis of \( L^2 \) and that \( \psi_\ell, \ell = 1, \ldots, L, \) are defined by

\[
\hat{\psi}_\ell = \chi_{S_\ell}, \quad \ell = 1, \ldots, L, \tag{3.2}
\]

for some measurable sets \( S_\ell \) in \( \mathbb{R} \).

Analogous to a result in Dai and Larson [7] for \( L = 1 \) and \( a = 2 \), we will show that an SE wavelet family can be characterized by the properties of “translation congruence” and “dilation congruence” of the sets as in [7]. From this characterization, it is again evident, as in [7], that an SE wavelet family must be an MSF wavelet family.

A pair of measurable sets \( E \) and \( F \) in \( \mathbb{R} \) is said to be translation congruent modulo \( 2\pi \) (or \( E \) is translation congruent to \( F \) modulo \( 2\pi \)) if there exists a measurable bijection \( \mu: E \to F \) such that

\[
(\mu(s) - s)/2\pi \in \mathbb{Z}, \quad \text{a.e. on } E;
\]

and is said to be dilation congruent modulo \( a \) (or \( E \) is dilation congruent to \( F \) modulo \( a \)), if there exists a measurable bijection \( \tau: E \to F \) such that

\[
\log_a(\tau(s)/s) \in \mathbb{Z}, \quad \text{a.e. on } E.
\]

As an application of Corollary 4, we establish the following result.

**Corollary 5.** Let \( \Psi_L \) be a collection of functions defined as in (3.2). Then, the family \( \mathcal{F}_a \) defined in (1.1) with \( b = 1 \) is an orthonormal basis of \( L^2 \), if and only if

i) for \( 1 \leq \ell \leq L \), each set \( S_\ell \) is translation congruent to \([0, 2\pi) \) modulo \( 2\pi \);

ii) \( \text{meas}(S_\ell \cap S_m) = 0 \), whenever \( \ell \neq m \); and

iii) the union \( S := \bigcup_{\ell=1}^L S_\ell \) is dilation congruent to the set \((-a, -1] \cup [1, a) \) modulo \( a \).

**Proof.** Suppose that the sets \( S_1, \ldots, S_L \) satisfy the conditions i), ii) and iii). Let \( \omega \) be arbitrarily chosen in \([1, a) \). Since \( \text{meas}(S_\ell \cap S_m) = 0 \), for \( \ell \neq m \), and \( S \) is dilation congruent
to \([-a, -1) \cup (1, a]\) modulo \(a\), there exists only one \(j \in \mathbb{Z}\) and one \(\ell \in \{1, \ldots, L\}\) such that \(a^j \omega \in S_\ell\), except for a null set. Thus, we have

\[
L_\psi(\omega) := \sum_{\ell=1}^{L} \sum_{j \in \mathbb{Z}} |\hat{\psi}_\ell(a^j \omega)|^2 = 1, \quad \text{a.e.}
\]  

(3.3)
on \([1, a]\). Similarly, (3.3) holds for \(\omega \in (-a, 1]\) a.e. Now, since \(L_\psi(\omega) = L_\psi(a \omega)\), (3.3) holds for almost all \(\omega \in \mathbb{R}\).

On the other hand, since each \(S_\ell\) is translation congruent to \([0, 2\pi]\) modulo \(2\pi\), we have

\[
\hat{\psi}_\ell(\omega) \hat{\psi}_\ell(\omega + 2k\pi) = 0, \quad \text{a.e.,}
\]

(3.4)
for all \(k \neq 0\) and all \(\ell\), and that meas \(S_\ell = 2\pi, \ell = 1, \ldots, L\). The last condition implies that \(||\psi_\ell||^2 = 1\). Thus, by Corollary 4, \(\mathcal{F}_a\) is an orthonormal basis of \(L^2\).

Conversely, if \(\mathcal{F}_a\) is an orthonormal basis of \(L^2\), then by Corollary 4, we see that (1.13) and (3.4) hold. Now, (3.4) implies that \(S_\ell\) is translation congruent to a subset \(\tilde{S}_\ell\) of \([0, 2\pi]\) modulo \(2\pi\). Since \(||\psi_\ell||^2 = 1\) must be 1, it follows that \(\tilde{S}_\ell = [0, 2\pi]\) except for a null set. On the other hand, (1.13) implies that \(S_\ell, \ell = 1, \ldots, L\), are mutually disjoint except for a null set, and the set \(S = \bigcup_{\ell=1}^{L} S_\ell\) is dilation congruent to \((-a, -1) \cup [1, a]\) modulo \(a\). This completes the proof of Corollary 5.

\[\blacksquare\]

In the following, we give two examples of SE wavelet families, one being a singleton, and the other consisting of two wavelets. We remark that by Corollary 5, any SE wavelet family is an MSF wavelet family also.

**Example 1.** Let \(L = 1\) and \(S = [\alpha, \beta) \cup (\gamma, \delta]\), where \(-\infty < \alpha < \beta < 0 < \gamma < \delta < \infty\).

Under these assumptions, the set \(S\) satisfies the conditions i), ii), and iii) of Corollary 5 if and only if

\[
\gamma = \beta + 2k\pi, \quad \text{for some } k \in \mathbb{N},
\]

(3.5)

\[
\delta = a\gamma,
\]

(3.6)

\[
\alpha = a\beta,
\]

(3.7)
and

\[(\beta - \alpha) + (\delta - \gamma) = 2\pi.\]  \hspace{1cm} (3.8)

By (3.6)--(3.8), we may conclude that

\[\gamma - \beta = 2\pi/(a - 1).\]  \hspace{1cm} (3.9)

Also, by (3.5) and (3.9), we conclude that \(a\) must be a rational number of the type

\[a = (k + 1)/k,\]

where \(k \in \mathbb{N}\). We remark that dilation factors of this type have been used to construct Meyer-typed wavelets by David (see [8, p. 323].)

If \(\beta = -\gamma,\ \alpha = -\delta\), then \(\hat{\psi}\) is an even function, and hence \(\psi\) is real-valued. In this case, we have

\[S = [-\pi, -k\pi) \cup (k\pi, \pi].\]

**Example 2.** Let \(L = 2\) and let \(a > 1\) be any dilation factor. Set

\[S_1 = [\alpha_1, \beta_1) \cup [\gamma_1, \delta_1)\]

and

\[S_2 = [\alpha_2, \beta_2) \cup [\gamma_2, \delta_2),\]

where

\[0 < \alpha_1 < \beta_1 < \gamma_1 < \delta_1 < \infty\]  \hspace{1cm} (3.10)

and

\[-\infty < \alpha_2 < \beta_2 < \gamma_2 < \delta_2 < 0.\]  \hspace{1cm} (3.11)

Define \(\Psi_2 = \{\psi_1, \psi_2\}\) by \(\hat{\psi}_1 = \chi_{s_1}\) and \(\hat{\psi}_2 = \chi_{s_2}\). By Corollary 5, \(\mathcal{F}_a\) defined as in (1.1) with \(b = 1\) is an orthonormal basis of \(L^2\), if and only if both \(S_1\) and \(S_2\) are translation congruent to \([0, 2\pi]\) modulo \(2\pi\) and the set \(S_1 \cup S_2\) is dilation congruent to \([-a, -1) \cup (1, a]\) modulo \(a\). It is easy to verify that these conditions are equivalent to the following equalities:

\[\gamma_1 = \beta_1 + 2k_1\pi, \hspace{1cm} k_1 \in \mathbb{N},\]  \hspace{1cm} (3.12)
\( \gamma_1 = a^{m_1} \beta_1, \quad m_1 \in \mathbb{N}, \) \hfill (3.13)

\( \delta_1 = a^{m_1 + 1} \alpha_1, \) \hfill (3.14)

\( \delta_1 = \alpha_1 + 2(k_1 + 1)\pi, \) \hfill (3.15)

\( \beta_2 = \gamma_2 - 2k_2\pi, \quad k_2 \in \mathbb{N}, \) \hfill (3.16)

\( \beta_2 = a^{m_2} \gamma_2, \quad m_2 \in \mathbb{N}, \) \hfill (3.17)

\( \alpha_2 = a^{m_2 + 1} \delta_2, \) \hfill (3.18)

\( \alpha_2 = \delta_2 - 2(k_2 + 1)\pi. \) \hfill (3.19)

The solution of (3.12)–(3.15) is given by

\[
\begin{align*}
\alpha_1 &= \frac{2(k_1 + 1)\pi}{a^{m_1 + 1} - 1}, \\
\beta_1 &= \frac{2k_1\pi}{a^{m_1} - 1}, \\
\gamma_1 &= \frac{2a^{m_1}k_1\pi}{a^{m_1} - 1}, \\
\delta_1 &= \frac{2a^{m_1 + 1}(k_1 + 1)\pi}{a^{m_1 + 1} - 1}.
\end{align*}
\] \hfill (3.20)

From this, we see that the condition (3.10) is equivalent to

\[
\frac{a^{m_1} - 1}{(a - 1)a^{m_1}} < k_1 < \frac{a}{a - 1}(a^{m_1} - 1).
\]

Since

\[
\lim_{m_1 \to \infty} \frac{a^{m_1} - 1}{(a - 1)a^{m_1}} = \frac{1}{a - 1}
\]

and

\[
\lim_{m_1 \to \infty} \frac{a}{a - 1}(a^{m_1} - 1) = \infty,
\]

for large \( m_1 \in \mathbb{N}, \) the set

\[
\mathbb{N} \cap \left( \frac{a^{m_1} - 1}{(a - 1)a^{m_1}}, \frac{a}{a - 1}(a^{m_1} - 1) \right)
\]

is non-empty. Thus, there exist \( m_1 \) and \( k_1, \) such that \( \alpha_1, \beta_1, \gamma_1, \delta_1 \) in (3.20) satisfy (3.10) and (3.12)–(3.15). Similarly, there exist \( \alpha_2, \beta_2, \gamma_2, \delta_2 \) such that the conditions (3.11) and (3.16)–(3.19) are satisfied. For these \( \alpha_1, \beta_1, \gamma_1, \delta_1 \) and \( \alpha_2, \beta_2, \gamma_2, \delta_2, \) the collection \( \Psi_2 \) is an SE wavelet family.

**4. Generalization to dual frames**

Theorem 1 and its corollaries can be generalized to the more general setting of dual frames. Precisely we have the following.
Theorem 2. Let \( a > 1 \), and let \( \Psi_L = \{\psi_1, \ldots, \psi_L\} \) and \( \tilde{\Psi}_L = \{\tilde{\psi}_1, \ldots, \tilde{\psi}_L\} \) be in \( L^2 \).
Suppose that both affine families \( \mathcal{F}_a \) and \( \tilde{\mathcal{F}}_a \) generated by \( \Psi_L \) and \( \tilde{\Psi}_L \), respectively, are Bessel sequences. Then \( (\mathcal{F}_a, \tilde{\mathcal{F}}_a) \) forms a dual pair of frames, if and only if
\[
\frac{1}{b} \sum_{\ell=1}^{L} \sum_{(j,m) \in I_a(\alpha)} \tilde{\psi}_\ell(a^j \omega) \tilde{\psi}_\ell \left( a^j \omega + \frac{2m\pi}{b} \right) = \delta_{\alpha,0}, \text{ a.e.} \tag{4.1}
\]
for all \( a \)-adic numbers \( \alpha \).

This theorem can be proved in a similar way as the proof of Theorem 1 by applying Lemma 1 together with the following well-known result.

Lemma 2. Let \( \{f_n\} \) and \( \{g_n\} \) be two sequences in \( L^2 \). Then the pair \( (\{f_n\}, \{g_n\}) \) forms a dual pair of frames in \( L^2 \) if and only if both \( \{f_n\} \) and \( \{g_n\} \) are Bessel sequences in \( L^2 \) and that
\[
\sum_n \langle f, f_n \rangle \langle g_n, g \rangle = \langle f, g \rangle, \quad f, g \in L^2.
\]

Of course, the formulation (4.1) can be simplified as in Corollaries 1, 2, and 3, according to the values of \( a \). More precisely, if \( a \in E_1 \), then (4.1) is equivalent to
\[
\frac{1}{b} \sum_{\ell=1}^{L} \sum_{j=-\infty}^{\infty} \tilde{\psi}_\ell(a^j \omega) \tilde{\psi}_\ell(a^j \omega) = 1, \quad \text{a.e.} \tag{4.2}
\]
and
\[
\left\{ \begin{array}{l}
\frac{1}{b} \sum_{\ell=1}^{L} \sum_{j=1}^{\infty} \tilde{\psi}_\ell(n_a^j \omega) \tilde{\psi}_\ell \left( n_a^j \left( \omega + \frac{2(sn_a + d)\pi}{b} \right) \right) = 0, \quad \text{a.e.,} \tag{4.3}
\end{array} \right.
\]

for \( d = 1, \ldots, n_a - 1 \) and \( s \in \mathbb{Z} \),

where \( n_a = a^\gamma \in \mathbb{Z} \) with the smallest \( \gamma \in \mathbb{Z} \) as in (1.12). If \( a \in E_2 \), then (4.1) is equivalent to the totality of (4.2) and
\[
\left\{ \begin{array}{l}
\frac{1}{b} \sum_{\ell=1}^{L} \sum_{j=0}^{s} \tilde{\psi}_\ell(r_a^j \omega) \tilde{\psi}_\ell \left( r_a^j \left( \omega + \frac{2qt\pi}{b} \right) \right) = 0, \quad \text{a.e.,} \tag{4.4}
\end{array} \right.
\]

for \( s = 0, 1, \ldots \) and all \( 0 \neq t \in \mathbb{Z} \) with \( p, q \mid t \),

where \( r_a = a^\gamma = \frac{p}{q}, (p, q) = 1 \), with the smallest \( \gamma \in \mathbb{Z} \) as defined in Corollary 2. Finally, for \( a \in E_3 \), the condition (4.1) is equivalent to the totality of (4.2) and
\[
\frac{1}{b} \sum_{\ell=1}^{L} \tilde{\psi}_\ell(\omega) \tilde{\psi}_\ell \left( \omega + \frac{2m\pi}{b} \right) = 0, \quad \text{a.e.,} \quad 0 \neq m \in \mathbb{Z}. \tag{4.5}
\]
Furthermore, it is clear that if $\Psi_L$ and $\tilde{\Psi}_L$ satisfy (4.5), they also satisfy (4.3) and (4.4). Thus, for any real number $a > 1$, (4.2) and (4.5) always imply that $(\mathcal{F}_a, \tilde{\mathcal{F}}_a)$ is a dual pair of frames of $L^2$, provided that $\mathcal{F}_a$ and $\tilde{\mathcal{F}}_a$ are Bessel sequences.

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**References**