EXERCISE BOUNDARIES AND EFFICIENT APPROXIMATIONS TO AMERICAN OPTION PRICES AND HEDGE PARAMETERS

by

Farid Aitsahlia
TzeLeung Lai

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Department of Statistics
STANFORD UNIVERSITY
Stanford, California
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FARID AITSAHLIA
Financial Engines
1804 Embarcadero Road
Palo Alto, CA

TZE LEUNG LAI
Department of Statistics
Stanford University

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Department of Statistics
Sequoia Hall
STANFORD UNIVERSITY
Stanford, California 94305-4065

http://www-stat.stanford.edu
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Farid AITSAHLIA*  Tze Leung LAI
Financial Engines  Department of Statistics
1804 Embarcadero Road  Stanford University
Palo Alto, CA 94303  Stanford, CA 94305

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presented. Address correspondence to Farid AitSahlia, Financial Engines, 1804 Embarcadero Road, Palo
Alto, CA 94303. Tel: (650) 565-4988, fax: (650) 565-4905, email:faitsahlia@FinancialEngines.com
Abstract

This paper presents a new numerical method to solve the integral equation defining the early exercise boundary of an American option. It is shown that the early exercise boundaries of standard American options are well approximated by linear splines with a few knots, implying that the new solution method can actually be carried out on a coarse grid of time points with reasonable accuracy. This leads to a fast and reasonably accurate method to compute the early exercise boundaries, values and hedge parameters of American options. In this connection, a brief survey of recent developments in approximations to American option prices and hedge parameters are also given.
1 INTRODUCTION

The explosive growth in the use of derivatives by investors and institutions (financial and otherwise) has fueled the need for their efficient and accurate valuation. The problem of efficient and accurate valuation of American options has a large literature, which can be broadly characterized by two directions. The first involves a discretization scheme to approximate the continuous-time pricing problem. This approach is used in the finite difference method of Brennan and Schwartz (1977), Courtadon (1982) and Wu and Kwok (1997) to solve the corresponding free boundary PDE, and in the binomial tree method of Cox et al. (1979) who approximate the underlying price process by a discrete process on a tree to which a dynamic programming algorithm can be applied.

To improve computational efficiency, a second direction has emerged whereby analytical characterizations or approximations are sought to circumvent the fine resolution required by direct discretization for accurate valuation. It can be traced back to Geske and Johnson (1984) who first characterized American options as compound European options and then used the Richardson extrapolation (with 3 or 4 points, typically) for the numerical approximation phase. The past few years witnessed the emergence of a new approach to approximate American option prices and hedge parameters, based on the integral representation formula of the difference between American and European option prices due to Kim (1990), Jacka (1991) and Carr et al. (1992). Although the representation formula is exact, it requires the determination of the early exercise boundary for its implementation. This integral representation formula also leads to an integral equation for the early exercise boundary. Huang et al. (1996) proposed to approximate the integrands in both
the integral representation formula and in the integral equation by piecewise constant functions, with \( n = 1, 2, 3 \) pieces, yielding three crude approximations \( P_1, P_2, \) and \( P_3 \) to the option price, and then to combine them via a three-point Richardson extrapolation scheme so that greater accuracy can be achieved. Instead of using piecewise constant approximations to the integrands, Ju (1998) subsequently proposed to approximate the boundary by a piecewise exponential function, for which the integral in the representation formula can be evaluated in closed form. This approach leads to greater accuracy than that of Huang et al. (1996), while maintaining the latter’s computational simplicity. Ju (1998) reported numerical studies showing that his method with \( n = 3 \) pieces substantially improves those of Geske and Johnson (1984) and of Huang et al. (1996) and other approximations in the literature. By using the finite difference method to solve for the free boundary PDE in one example, he computed an approximation to the early exercise boundary and found it to differ substantially from his piecewise exponential approximations with \( n = 1, 2, 3 \) pieces. This led him to conclude that “an accurate estimate of the early exercise boundary is not required for pricing an American option accurately.”

Ait-Sahalia and Lai (1999) recently carried out extensive computations of early exercise boundaries for a wide range of maturities, interest rates, dividend rates, volatilities and strike prices via reparametrization to reduce American option valuation to a single optimal stopping problem for standard Brownian motion, indexed by one parameter in the absence of dividends, and by two parameters otherwise. Their results show, however, that the early exercise boundary is actually very well approximated by a piecewise exponential boundary which uses a small number of pieces. This explains Ju’s finding that his method has superior performance over other approximation approaches. In this article, we make use of Ju’s
closed-form integration to develop (i) a new numerical method to solve the integral equation defining the early exercise boundary and (ii) an improvement of Ju's approximation to the boundary. It involves three basic ideas. The first is a change of variables under which the exercise boundary appears in the integrand in a simpler form. Moreover, under this change of variables, an exponential function is transformed into a linear function. The second idea uses a numerically stable root finding algorithm in solving for the boundary points at successive times $t_i$. The third idea is to use, instead of the usual step function approximation to the boundary as in Kim (1990), a linear interpolation for the boundary between two adjacent time points and to apply Ju's (1998) closed-form expression for the premium integral when the boundary is piecewise linear in the transformed coordinates (or equivalently, piecewise exponential in the original coordinates).

The last idea above is of particular importance in developing a good approximation to the boundary that involves only a few time points. Specifically, if the actual boundary for the transformed coordinates should be nearly linear over a wide interval with endpoints $t_i$ and $t_{i-1}$, then interpolating the boundary linearly between $t_i$ and $t_{i-1}$ and using Ju's analytic formula for the premium integral between $t_i$ and $t_{i-1}$ would yield an almost exact value of the premium integral even though $t_i$ and $t_{i-1}$ may be quite far apart. Solving the integral equation for the early exercise boundary at narrowly spaced time points provides a benchmark to compare approximations that solve the integral equation on a coarse grid consisting of only a few points, as is done in Section 3 which confirms that the boundary is indeed well approximated by a linear spline having only a few knots. Hence the three key ideas mentioned above also lead to an efficient approximation to the early exercise boundary that involves finding its values at a few time points and linearly interpolating
elsewhere (in the transformed coordinate system). This fast and accurate approximation to
the early exercise boundary yields an efficient method for the valuation of American options,
thanks to Ju's (1998) closed-form expression for the premium integral. Moreover, we also
develop here a closed-form expression for the hedge parameters using this piecewise linear
approximation (in the transformed coordinate system) for the early exercise boundary.

This article is organized as follows. Section 2 describes our method to solve the integral
equation defining the early exercise boundary. Numerical results and efficient linear
spline approximation to the boundary are given in Section 3. Section 4 gives closed-form
expressions for the approximate computation of hedge parameters. Section 5 compares our
approximation to American option prices and exercise boundaries to those of Ju (1998) and
Huang et al. (1996) and provides a deeper understanding of their methods and results via
the benchmark developed in Section 2. In this connection we also give a brief review of
other approximations in the literature. Section 6 summarizes and concludes the article.

2 A NUMERICAL METHOD TO SOLVE THE INTEGRAL
EQUATION FOR THE EARLY EXERCISE BOUNDARY

In the standard Black-Scholes environment with a riskless interest rate $r$ and an underlying
asset having volatility $\sigma$ and paying dividend at rate $\mu$, the price of an American option at
time $t$ is the optimum value in the optimal stopping problem

$$U(t, P) = \sup_{\tau \in \mathcal{T}_{a, b}} E \left\{ e^{-r(t-\tau)} f(P_\tau) | P_t = P \right\},$$

(1)

where $P_t = P_0 e^{(r-\mu-\frac{\sigma^2}{2})t + \sigma W_t}$ with initial security price $P_0$, $\{W_t\}$ is a standard Brownian
motion (so that the stochastic process $\{P_t\}$ is a geometric Brownian motion), and $\mathcal{T}_{a, b}$ is
the set of stopping times taking values between \( a \) and \( b \) with \( b > a \). Given a strike price \( K \), the payoff \( f(P) \) in (1) is \( (K - P)^+ \) for a put, and \( (P - K)^+ \) for a call. We shall focus on American puts, as American calls on dividend-paying securities can be evaluated similarly.

Kim (1990), Jacka (1991) and Carr et al. (1992) have obtained the following representation for an American put:

\[
U(t, P) = U_E(t, P) + \int_t^T \left\{ r Ke^{-r(t-t)} N(-d_2(P, B_t, \tau - t)) \\
-\mu Pe^{-\mu(\tau-t)} N(-d_1(P, B_t, \tau - t)) \right\} d\tau, \tag{2}
\]

in which \( N(.) \) denotes the standard normal distribution function, \( B_t \) is the early exercise boundary, \( U_E \) the corresponding European put price given by

\[
U_E(t, P) = Ke^{-(T-t)} N(-d_2(P, K, T - t)) - Pe^{-\mu(T-t)} N(-d_1(P, K, T - t)),
\]

where \( d_1(x, y, \tau) = (\ln(x/y) + (\tau - \mu + \sigma^2/2)\tau) / \sigma \sqrt{\tau} \) and \( d_2(x, y, \tau) = d_1(x, y, \tau) - \sigma \sqrt{\tau} \).

Since \( U(t, B_t) = K - B_t \), it follows from (2) that \( B_t \) satisfies the integral equation

\[
K - B_t = U_E(t, B_t) + \int_t^T \left\{ r Ke^{-r(\tau-t)} N(-d_2(B_t, B_\tau, \tau - t)) \\
-\mu B_t e^{-\mu(\tau-t)} N(-d_1(B_t, B_\tau, \tau - t)) \right\} d\tau. \tag{3}
\]

Kim (1990) proposed to solve (3) numerically by dividing the interval \([0, T]\) into \( n \) subintervals \([t_{i-1}, t_i]\), with \( t_0 = 0 \), \( t_n = T \) and \( t_i - t_{i-1} = T/n \), and replacing (3) by the system of nonlinear equations

\[
K - B_{t_i} = U_E(t_i, B_{t_i}) + \sum_{j=i+1}^n \left\{ r Ke^{-r(t_j-t_i)} N \left(-d_2(B_{t_i}, B_{t_j}, t_j - t_i)\right) \\
-\mu B_{t_i} e^{-\mu(t_j-t_i)} N \left(-d_1(B_{t_i}, B_{t_j}, t_j - t_i)\right) \right\} T/n, \tag{4}
\]
which is solved recursively backwards with $B_{t_n} = K$ if $\mu \leq r$ and $B_{t_n} = rK/\mu$ if $\mu > r$.

The basic idea behind (4) is to approximate the integrand in (3) by a step function, which gives good accuracy when $n$ is large.

2.1 Change of variables

The boundary $B_t$ appears in (3) (and therefore (4) also) in a complicated manner. Letting $\rho = r/\sigma^2$ and $\alpha = \mu/r$, we introduce the change of variables

$$s = \sigma^2(t - T), \quad z = \log(P/K) - (\rho - \alpha\rho - 1/2)s,$$

so that the boundary $B_t$ becomes $\overline{z}(s)$ in the new coordinate system, with $B_t = K e^{\overline{z}(s) + (\rho - \alpha\rho - 1/2)s}$.

Note that a piecewise exponential boundary $B_t$ is transformed to a piecewise linear function of $s$. Moreover, the integral equation (3) can be expressed in the simpler form

$$1 - e^{\overline{z}(s) + (\rho - \alpha\rho - 1/2)s} = e^{\rho s} \left\{ N( -\overline{z}(s)/\sqrt{s} ) - e^{\overline{z}(s) - s/2} N( -\overline{z}(s)/\sqrt{s} - \sqrt{s} ) \right\} +$$

$$\rho e^{\rho s} \int_0^s \left[ e^{-\rho u} N \left( \frac{\overline{z}(u) - \overline{z}(s)}{\sqrt{u} - s} \right) - \alpha e^{-(\rho\rho u + s/2 + \overline{z}(s))} N \left( \frac{\overline{z}(u) - \overline{z}(s)}{\sqrt{u} - s} - \sqrt{u} - s \right) \right] du.$$  \hspace{1cm} (6)

The transformation (5) was introduced in Ait-Sahalia and Lai (1999) to reduce the optimal stopping problem (1) to a canonical form that involves only one parameter, $\rho$, in the absence of dividends and two parameters, $\rho$ and $\alpha$, when dividends are paid. An important advantage of this transformation is that for values of $\sigma$ (between 0.1 and 0.4) that are of practical interest, the time horizon $\sigma^2 T$ in the canonical scale is only a small fraction of $T$.

2.2 Evaluation of integrals

To solve (6), instead of using a step function approximation to the entire integrand as in (4), we use a piecewise linear approximation to $\overline{z}(\cdot)$. This is clearly more accurate but requires
evaluation of the integral which has a closed-form expression when $\bar{z}(\cdot)$ is piecewise linear.

First note that

$$\rho e^{\alpha s} \int_0^s \left[ e^{-\rho u} N \left( \frac{\bar{z}(u) - z}{\sqrt{u-s}} \right) - \alpha e^{-(\alpha \rho u + s/2)} N \left( \frac{\bar{z}(u) - z}{\sqrt{u-s}} - \sqrt{u-s} \right) \right] du =$$

$$1 - e^{\rho s} - e^{z+(\rho-\alpha \rho-1/2)s} (1 - e^{\alpha \rho s}) - \int_0^{-s} \rho e^{-\alpha t} N \left( \frac{z - \bar{z}(s+t)}{\sqrt{t}} \right) dt$$

$$+ e^{z+(\rho-\alpha \rho-1/2)s} \int_0^{-s} \alpha e^{-\alpha \rho t} N \left( \frac{z - \bar{z}(s+t) + t}{\sqrt{t}} \right) dt.$$  \hspace{1cm} (7)

To evaluate the last two integrals in (7), suppose that $s = s_m < \ldots < s_0 = 0$ divide the interval $[s, 0]$ into $m$ subintervals such that

$$\bar{z}(u) = \beta_i u + \gamma_i \text{ for } s_i \leq u \leq s_{i-1} \quad (1 \leq i \leq m).$$  \hspace{1cm} (8)

Let $\tau_i = s_i - s_m$. Then $\bar{z}(t + s_m) - z = -(b_i t + c_i)$ for $\tau_i \leq t \leq \tau_{i-1}$, where $b_i = -\beta_i$, $c_i = z - \gamma_i - \beta_i s_m$, noting that $\tau_i + s_m = s_i$. Let $a_i = \sqrt{b_i^2 + 2\rho}$. Then for $1 \leq i \leq m$,

$$\int_{\tau_{i-1}}^{\tau_i} \rho e^{-\alpha t} N \left( \frac{z - \bar{z}(t + s_m)}{\sqrt{t}} \right) dt =$$

$$e^{-\rho \tau_i} N \left( b_i \tau_i^{1/2} + c_i \tau_i^{-1/2} \right) - e^{-\rho \tau_{i-1}} N \left( b_i \tau_{i-1}^{1/2} + c_i \tau_{i-1}^{-1/2} \right) +$$

$$\frac{1}{2} \left( \frac{b_i}{a_i} + 1 \right) e^{(a_i - b_i) c_i} \left\{ N \left( a_i \tau_i^{1/2} + c_i \tau_i^{-1/2} \right) - N \left( a_i \tau_{i-1}^{1/2} + c_i \tau_{i-1}^{-1/2} \right) \right\} +$$

$$\frac{1}{2} \left( \frac{b_i}{a_i} - 1 \right) e^{-(a_i + b_i) c_i} \left\{ N \left( a_i \tau_{i-1}^{1/2} - c_i \tau_{i-1}^{-1/2} \right) - N \left( a_i \tau_i^{1/2} - c_i \tau_i^{-1/2} \right) \right\},$$  \hspace{1cm} (9)

which corresponds to Eq. (5) of Ju (1998). In the case where $\tau_i = 0$, we replace $\tau_i$ by $t > 0$ and take the limit as $t \to 0$, which amounts to setting $c_i \tau_i^{-1/2} = 0$, or $\infty$, or $-\infty$ according as $c_i = 0$, or $c_i > 0$, or $c_i < 0$. Similarly, letting $\bar{b}_i = b_i + 1$, $\bar{a}_i = (\bar{b}_i^2 + 2\alpha \rho)^{1/2}$, we have

$$\int_{\tau_i}^{\tau_{i-1}} \alpha e^{-\alpha \rho t} N \left( \frac{z - \bar{z}(t + s_m) + \bar{b}_i}{\sqrt{t}} \right) dt =$$

RHS of (9) with $(b_i, a_i)$ replaced by $((\bar{b}_i, \bar{a}_i)$ and $\rho$ by $\alpha \rho$,

$$\hspace{1cm} \text{(10)}$$
which corresponds to Eq. (6) of Ju (1998). When either \((a_i - b_i)c_i\) or \(-(a_i + b_i)c_i\) exceeds some large number (say 400), its exponential in (9) becomes exceedingly large, causing numerical instability. In these situations, which typically do not occur unless \(\tau_{i-1} - \tau_i\) is very small, we evaluate the LHS of (9) by the midpoint rule

\[
\int_{\tau_i}^{\tau_{i-1}} e^{-\rho t} N \left( \frac{z - \bar{z}(t + s_m)}{\sqrt{t}} \right) dt \approx (e^{-\rho \tau_i} - e^{-\rho \tau_{i-1}}) \frac{N \left( \frac{z - \bar{z}(\tau_i^* + s_m)}{\sqrt{\tau_i^*}} \right)}{\sqrt{\tau_i^*}},
\]

where \(\tau_i^* = \frac{\tau_{i+1} - \tau_i}{2}\). We also evaluate the LHS of (10) by the midpoint rule if \((\bar{a}_i - \bar{b}_i)c_i\) or \(- (\bar{a}_i + \bar{b}_i)c_i\) is large. Since \(\int_0^{-s_m} = \sum_{i=1}^{m} \int_{\tau_i}^{\tau_{i-1}}\), the integral in the RHS of (7) can be expressed as a sum of terms in (9)–(10).

### 2.3 Solving for the boundary at successive time points

To solve the integral equation (6) recursively on a grid of time points \(s_0 = 0 > s_1 > \ldots > s_n = -\sigma^2 T\), we linearly interpolate the boundary between \(s_j\) and \(s_{j+1}\) and use (7) together with (9), (10) or (11), to evaluate the integrals in (6) in terms of simple sums. The recursion is initialized at \(s_0 = 0\) by \(\bar{z}(0) = 0\) if \(0 \leq \alpha \leq 1\) and \(\bar{z}(0) = -\ln \alpha\) if \(\alpha > 1\). For \(1 \leq m \leq n\) and \(1 \leq j \leq m\), let \(\bar{z}_j = \bar{z}(s_j)\) and \(\tau_j = s_j - s_m\). Suppose that \(\bar{z}_0, \ldots, \bar{z}_{m-1}\) have been determined. Then \(b_i = -\beta_i = (\bar{z}_i - \bar{z}_{i-1}) / (s_{i-1} - s_i)\), \(\gamma_i = \bar{z}_{i-1} - \beta_i s_{i-1}\), and \(c_i = \bar{z}_i - \bar{z}_{i-1} + \beta_i \tau_{i-1}\) for \(1 \leq i \leq m - 1\). To determine \(\bar{z}_m\), let \(z\) be a candidate value and let \(b(z) = (z - \bar{z}_{m-1}) / (s_{m-1} - s_m)\), \(a(z) = (b^2(z) + 2\rho)^{1/2}\), \(\bar{b}(z) = b(z) + 1\), \(\bar{a}(z) = \left(\bar{b}^2(z) + 2\alpha\rho\right)^{1/2}\). Noting that \(c(z) := z - \bar{z}_{m-1} - (s_{m-1} - s_m)b(z) = 0\) and \(\tau_m = 0\), we obtain from (6)–(10) the following equation (in the variable \(z\)) defining \(\bar{z}_m\):

\[
1 - e^{z + (\rho - \alpha \rho^{-1/2})s_m} = e^{\rho s_m} \left\{ N(-z/\sqrt{-s_m}) - e^{-s_m/2} N(-z/\sqrt{-s_m} - \sqrt{-s_m}) \right\}
\]
+1 - e^{\rho s} - e^{\epsilon + (\rho - \alpha - 1/2) s} (1 - e^{\alpha s}) + e^{-\rho \tau_{m-1}} N \left( \frac{b(z)}{a(z)} \tau_{m-1}^{1/2} \right) - 1/2
\]
\[
- \frac{b(z)}{a(z)} N \left( a(z) \tau_{m-1}^{1/2} - \frac{1}{2} \right) - \sum_{i=1}^{m-1} A_i(z) + e^{\epsilon + (\rho - \alpha - 1/2) s} \left( \frac{b(z)}{a(z)} \left( N \left( \frac{\tilde{a}(z)}{\tilde{a}(z)} \tau_{m-1}^{1/2} \right) - \frac{1}{2} \right) \right)
\]
\[
+ \frac{1}{2} - e^{\alpha \rho \tau_{m-1}} N \left( \frac{\tilde{b}(z)}{\tilde{a}(z)} \tau_{m-1}^{1/2} \right) + \sum_{i=1}^{m-1} \tilde{A}_i(z) \right),
\]
(12)

where \( A_i(z) \) is given by the RHS of (9) and \( \tilde{A}_i(z) \) by the RHS of (10).

The nonlinear equation (12) can be solved by a Van Wijngaarden–Dekker–Brent–type method; see Press et al. (1992). First, as shown in Ait-Sahalia and Lai (1999), the early exercise boundary \( \bar{z}(s) \) is bounded above by \( \bar{z}_u(s) \) and bounded below by \( \bar{z}_l(s) \), where

\[
\bar{z}_u(s) = - (\rho (1 - \alpha) - 1/2) s - (\ln \alpha)^+ ,
\]

\[
\bar{z}_l(s) = - (\rho (1 - \alpha) - 1/2) s - \ln(\theta/(\theta - 1)) ,
\]

\[
\theta = - \left( \rho (1 - \alpha) - \frac{1}{2} - \left( \rho (1 - \alpha) - \frac{1}{2} \right)^2 + 2 \theta \right)^{1/2} .
\]

Let \( D(z) \) denote the difference between the two sides of (12); specifically, \( D(z) = \text{LHS} - \text{RHS} \).

If \( D(\bar{z}_{m-1}) = 0 \), we have found the solution and can set \( \bar{z}_m = \bar{z}_{m-1} \). If \( D(\bar{z}_{m-1}) < 0 \), set \( z' = \bar{z}_{m-1} \) and \( z'' = \bar{z}_l(s_m) \) after checking that \( D(z'') > 0 \). In the unlikely event that the latter condition is violated, set \( \bar{z}_m = \bar{z}_{m-1} \). If \( D(\bar{z}_{m-1}) > 0 \), set \( z'' = \bar{z}_m = \bar{z}_{m-1} \) and \( z' = \bar{z}_u(s_m) \) after checking that \( D(z') < 0 \), whose unlikely violation results in setting \( \bar{z}_m = \bar{z}_{m-1} \). After bracketing the solution in this way between \( z' \) and \( z'' \) with \( D(z') < 0 \) and \( D(z'') > 0 \), we can use successive linear approximations that replace \( D(z) \) by the linear function \( D_1(z) \) with \( D_1(z') = D(z') \) and \( D_1(z'') = D(z'') \). Let \( z^* = z'' - D(z'')(z' - z'') / \{ D(z') - D(z'') \} \) be the solution of \( D_1(z) = 0 \). Note that \( z^* \) lies between \( z' \) and \( z'' \). If \( D(z^*) < 0 \), reset \( z' \) at \( z^* \).

If \( D(z^*) > 0 \), reset \( z'' \) at \( z^* \). Proceeding inductively in this way, the procedure terminates when a solution is reached or when \( |z' - z''| \) falls below some prescribed tolerance level.
3 NUMERICAL RESULTS AND AN EFFICIENT APPROXIMATION TO THE EARLY EXERCISE BOUNDARY

Using even spacing in the choice of grid points $s_0 = 0 > s_1 > \ldots > s_n = -\sigma^2 T$ in Section 2, i.e. $s_i - s_{i+1} = \delta := \sigma^2 T / n$ for $0 \leq i \leq n - 1$, Table 1 gives values of the early exercise boundary $\bar{z}(\cdot)$ obtained by using the method of Section 2 to solve the Volterra integral equation (6) with $\delta = 10^{-2}, 10^{-3}$ or $10^{-4}$. These results show convergence of the method with diminishing $\delta$; moreover they show that the integral equation approach is already quite accurate for $\delta = 10^{-2}$ and $-s \geq 0.05$. Note that for $\delta = 10^{-2}$, $s = -0.01$ corresponds to the first time point at which the integral equation method computes a boundary value to start the recursion and $s = -0.05$ is the fifth recursive stage.

INSERT TABLE 1 ABOUT HERE

For small $\delta$, a much faster method to compute $\bar{z}(\cdot)$ is the corrected Bernoulli walk method introduced by Chernoff and Petkau (1986). This method has recently been used to compute the benchmark values of early exercise boundaries of American options by Ait-Sahalia and Lai (1999). Table 1 also gives for comparison the values of $\bar{z}(\cdot)$ computed by the corrected Bernoulli walk method with time step size $\delta = 10^{-4}, 10^{-5}$ or $10^{-6}$. The results show that for $0.002 \leq -s < 0.005$ the corrected Bernoulli walk method needs $\delta = 10^{-6}$ to give boundary values comparable to those obtained by using the method of Section 2 to solve the integral equation (6). For this range of $-s$, using $\delta = 10^{-4}$ in the corrected Bernoulli walk method produces a substantial relative error for the boundary. For $-s \leq .001$, Table 1 shows that even smaller values of $\delta$ than $10^{-6}$ are needed for the corrected Bernoulli walk method.
Although the numerical procedure of Section 2 is computationally expensive for small values of $\delta$ because of the large number of computations and memory needed in evaluating $A_i(z)$ and $\tilde{A}_i(z)$ for $1 \leq i \leq m - 1$ and different values of $z$ in (12) when $m$ is large, it is very fast when $m(\leq n)$ is small. For $T = 1$ and $\sigma = .1$, $n = c^2T/\delta = 10$ if we choose $\delta = 10^{-2}$, which gives quite accurate results for the boundary values at $-s \geq 0.05$. On the other hand, the Bernoulli method with 10 time steps ($\delta = 10^{-2}$) is grossly inaccurate.

As noted in Ait-Sahalia and Lai (1999), the Chernoff-Petkau continuity correction to compute the optimal stopping boundary $\bar{z}(\cdot)$ at $s$ via the Bernoulli walk approximation to Brownian motion requires boundedness of the derivative $\bar{z}'(\cdot)$ in some neighborhood of $s$ and is not applicable around $s = 0$. Since the corrected Bernoulli walk method can be applied for $s \leq -0.005$, we can restrict to $0 > s > -0.005$ in applying the preceding numerical procedure to solve the integral equation that defines the optimal stopping boundary. In particular, with $\delta = 10^{-4}$, the range $0 > s > -0.005$ only involves $n = 50$ time steps. Let $s_0 = -0.005$ and $t_0 = T + s_0/\sigma^2$. Having computed $\bar{z}(\cdot)$ for $0 > s \geq s_0$ by using the preceding procedure to solve the integral equation defining $\bar{z}(\cdot)$, we can use (13) below together with (9) and (10) to evaluate the value function at $s_0$ as the terminal payoff of the corrected Bernoulli walk method that computes $\bar{z}(\cdot)$ by backward induction for $s \leq -s_0$. As shown by Lai, Yao and Ait-Sahalia (1999), this hybrid approach that combines the corrected Bernoulli walk method of Chernoff and Petkau (for $s \leq s_0$) with the integral equation method (for $s_0 \leq s < 0$) has $o(\sqrt{\delta})$ error in computing the boundary $\bar{z}(\cdot)$. Note the computational efficiency of this hybrid approach, which exploits the recursive nature of the corrected Bernoulli walk method and only involves $-s_0/\delta$ time steps for the integral equation method applied to the initial time segment $[-s_0, 0)$.
Figure 1 plots the graphs of \( \bar{\mathcal{E}}(s) \) for \(-0.3 \leq \delta \leq 0\), \( \rho = 0.5 \) and for different values of \( \alpha \), computed by the preceding approach with \( \delta = 10^{-4} \). It shows that \( \bar{\mathcal{E}}(.) \) is well approximated by a linear spline with knots at \( s = 0, -.005, -.025, -.05, -.1, -.15, -.3 \). For \( \sigma = .1 \), \( s = -.3 \) corresponds to a very long maturity \( T = 30 \) (years). The approximate piecewise linearity of \( \bar{\mathcal{E}}(.) \) shown in Figure 1 is also confirmed by extensive numerical computations over a wide range of \( \rho \) and \( \alpha \) values for puts and calls in Ait-Sahalia and Lai (1999). Hence, using a small number of time steps in the procedure of Section 2 results in a reasonably accurate and fast approximation to the early exercise boundary.

INSERT FIGURE 1 ABOUT HERE

4 APPLICATION TO VALUATION AND HEDGING

As in Section 2 we again focus on American puts since the treatment of American calls on dividend-paying securities is similar. After the change of variables (5), the value of (2) can be written as

\[
U(t, P) = Ke^{qs} \left\{ N(-z/\sqrt{-s}) - \frac{e^{z-s/2}N(-z/\sqrt{-s} - \sqrt{-s})}{\sqrt{-s}} \right\} \\
+ \rho Ke^{qs} \int_{s}^{0} \left[ e^{-\rho u} N\left( \frac{\bar{z}(u) - z}{\sqrt{u - s}} \right) - \alpha e^{-\alpha pu - s/2 + z} N\left( \frac{\bar{z}(u) - z}{\sqrt{u - s}} - \sqrt{u - s} \right) \right] du. \tag{13}
\]

Suppose \( \bar{\mathcal{E}}(.) \) can be approximated by a piecewise linear function of the form (8). Then in view of (7) and \( \int_{0}^{s} = \sum_{i=1}^{m} \int_{\tau_{i-1}}^{\tau_{i}} \), we can express the integral in (13) as a sum of the terms in (9) and (10) (in which \( c_{i} \tau_{i}^{1/2} \) is set to be equal to \( 0, \infty \) or \( -\infty \) according as \( c_{i} = 0 \), \( c_{i} > 0 \) or \( c_{i} < 0 \) when \( \tau_{i} = 0 \) or in (11) when \( (a_{i} - b_{i})c_{i} \) or \( -(a_{i} + b_{i})c_{i} \) is too large.

Formula (13) also leads to explicit expressions for the hedge parameters through differ-
entiation. In particular, the parameter delta can be expressed as

\[
\frac{\partial U}{\partial \tau}(t, P) = -e^{\alpha \rho s} N \left(-\frac{Z}{\sqrt{-s}} - \sqrt{-s} \right) - \rho e^{-z - \frac{\rho - \alpha \rho - 1/2}{2}} \int_0^{-s} e^{-\rho \tau} n \left(\frac{E(\tau + s) - Z}{\sqrt{\tau}}\right) d\tau
\]

\[
- \int_0^{-s} \alpha e^{-\alpha \rho \tau} N \left(\frac{E(\tau + s) - Z - \tau}{\sqrt{\tau}}\right) + \alpha \tau \int_0^{-s} e^{-\alpha \rho \tau} n \left(\frac{E(\tau + s) - Z - \tau}{\sqrt{\tau}}\right) d\tau, \tag{14}
\]

where \( n(x) = e^{-x^2/2}/\sqrt{2\pi} \) is the standard normal density function. Suppose \( E(s) \) can be approximated by a piecewise linear function of the form (8). Then the second integral in (14) can be written as

\[
\int_0^{-s} \alpha e^{-\alpha \rho \tau} N \left(\frac{E(\tau + s) - Z - \tau}{\sqrt{\tau}}\right) d\tau = 1 - e^{\alpha \rho s} - \int_0^{-s} \alpha e^{-\alpha \rho \tau} N \left(\frac{Z - E(\tau + s) + \tau}{\sqrt{\tau}}\right) d\tau,
\]

which can therefore be expressed as a sum of terms of the form (10), with \( s = s_m \). Since \( a_i^2 = b_i^2 + 2\rho \), the first integral can also be written as a sum of the terms

\[
\int_{\tau_i}^{\tau_i-1} e^{-\rho \tau} n \left(\frac{E(\tau + s_m) - Z}{\sqrt{\tau}}\right) d\tau = \int_{\tau_i}^{\tau_i-1} e^{-\rho \tau} n \left(b_i \tau^{1/2} + c_i \tau^{-1/2}\right) d\tau
\]

\[
= (2\pi)^{-1/2} e^{-b_i c_i} \int_{\tau_i}^{\tau_i-1} \tau^{-1/2} e^{-a_i^2 \tau + c_i^2 \tau^{-1}} d\tau
\]

\[
= a_i^{-1} e^{-(a_i + b_i)c_i} \left\{ N \left(a_i \tau_i^{1/2} + c_i \tau_i^{-1/2}\right) - N \left(a_i \tau_i^{1/2} + c_i \tau_i^{-1/2}\right) \right\}
\]

\[
+ a_i^{-1} e^{-(a_i + b_i)c_i} \left\{ N \left(a_i \tau_i^{1/2} - c_i \tau_i^{-1/2}\right) - N \left(a_i \tau_i^{1/2} - c_i \tau_i^{-1/2}\right) \right\}. \tag{15}
\]

The last equality above can be derived by using the change of variables

\[
x = a_i \tau^{1/2} + c_i \tau^{-1/2}, \quad y = a_i \tau^{1/2} - c_i \tau^{-1/2},
\]

so that \( dx = \frac{1}{2} \left(a_i \tau^{-1/2} - c_i \tau^{-3/2}\right) d\tau, \quad dy = \frac{1}{2} \left(a_i \tau^{-1/2} + c_i \tau^{-3/2}\right) d\tau, \) and by using the identity

\[
\tau^{-1/2} = \left\{ \frac{a_i \tau^{-1/2} - c_i \tau^{-3/2}}{a_i \tau^{-1/2} + c_i \tau^{-3/2}} \right\} / (2a_i).
\]

13
Similarly, letting \( \tilde{b}_i = b_i + 1 \) and \( \tilde{a}_i = \left( \tilde{b}_i^2 + 2\alpha \rho \right)^{1/2} \) as before, we can express the third integral as a sum of the terms

\[ \int_{\tau_i}^{\tau_{i-1}} \frac{e^{-\alpha \varrho \tau}}{\sqrt{\tau}} n \left( \frac{z(\tau + s_m) - z - \tau}{\sqrt{\tau}} \right) d\tau = \int_{\tau_i}^{\tau_{i-1}} \frac{e^{-\alpha \varrho \tau}}{\sqrt{\tau}} n \left( \tilde{b}_i \tau^{1/2} + c_i \tau^{-1/2} \right) d\tau \]

\[ = \tilde{a}_i^{-1} e^{(\tilde{a}_i - \tilde{b}_i)c_i} \left\{ N \left( \tilde{a}_i \tau_{i-1}^{1/2} + c_i \tau_{i-1}^{-1/2} \right) - N \left( \tilde{a}_i \tau_i^{1/2} + c_i \tau_i^{-1/2} \right) \right\} \]

\[ + \tilde{a}_i^{-1} e^{-(\tilde{a}_i + \tilde{b}_i)c_i} \left\{ N \left( \tilde{a}_i \tau_{i-1}^{1/2} - c_i \tau_{i-1}^{-1/2} \right) - N \left( \tilde{a}_i \tau_i^{1/2} - c_i \tau_i^{-1/2} \right) \right\}. \]  \hspace{1cm} (16)

When \( \tau_i = 0 \) in (10), (15) or (16), we take \( c_i \tau_i^{-1/2} = 0 \), or \( \infty \), or \( -\infty \) according as \( c_i = 0 \), or \( > 0 \), or \( < 0 \).

The closed-form valuation and hedge parameter formulas given by (13)–(16) together with (9) and (10) are based on the piecewise linear approximation (8) to the early exercise boundary after the change of variables (5). We have seen in Section 3 that \( \tilde{z}(\cdot) \) can be well approximated by a linear spline with a few knots that are determined by using the numerical method of Section 2. Table 2 gives the results for the option prices and deltas computed by using a 3-piece linear approximation (denoted by LSP3) of \( \tilde{z}(\cdot) \), which places the first knot at \( s = -0.005 \) and evenly spaces the remaining two knots and which limits to only 5 iterations in the solution of (12) to achieve greater speed than previously proposed approximations. These results are overwhelmingly within 0.1% of the corresponding benchmark values computed by (i) using (13) and \( \tilde{z}(\cdot) \) determined by the hybrid method in Section 3 with \( \delta = 10^{-4} \), and (ii) using the binomial tree method with 10,000 steps.

INSERT TABLE 2 ABOUT HERE
5 COMPARISON WITH OTHER APPROXIMATIONS

Shortly after the Geske-Johnson approximation which is reviewed in Section 1, McMillan (1986) and Barone-Adesi and Whaley (1987) developed an alternative analytic approximation of the option price. The basic idea is to approximate the PDE for the difference between American and European option prices by an ODE. Applying the boundary conditions to the ODE gives a nonlinear equation for a constant (time-invariant) approximation $P^*$ to the exercise boundary, and a closed-form expression involving $P^*$ yields an approximation to the American option price. Although the method is fast and gives an adequate approximation to the option price when the maturity is short or very long, it suffers from lack of accuracy for intermediate life-spans. Moreover, $P^*$ fails to capture the shape of the time-varying exercise boundary.

Bunch and Johnson (1992) introduced a modification of the Geske-Johnson method. Broadie and Detemple (1996) developed upper and lower bounds for an American option and used a convex combination of these bounds with empirically determined weights to approximate the option price. Carr (1998) discretized the time dimension of the PDE with a few points and used a randomization method to approximate the option price. Ju (1998) has carried out an extensive numerical study comparing these approximations with his and that of Huang et al. (1996), and has found that his piecewise exponential approximation to the early exercise boundary using 3 evenly spaced pieces gives the best performance in terms of both speed and accuracy. We shall therefore focus only on Ju's method but also consider briefly the closely related method of Huang et al. (1996).

Since a piecewise exponential exercise boundary $B_t$ is equivalent to a piecewise linear
after the change of variables (5), our method and Ju's basically use the same functional form to approximate the early exercise boundary. However, there are important differences. First, like the actual boundary, our approximation is continuous; it is a linear spline. On the other hand, Ju chooses the coefficients $Q_m$ and $q_m$ of the exponential function $Q_m e^{q_m t}$ separately for each of 3 pieces, resulting in discontinuities at the time points between successive intervals. Secondly, instead of even spacing between successive time points, we allow unevenly spaced knots to better fit the shape of the early exercise boundary as explained in Section 3, and the number of pieces in our piecewise linear approximation depends on $\sigma^2 T$ and may vary between 1 and 6. Thirdly, our algorithm in Section 2 to solve (10) for $\bar{z}(s_m)$ is simpler than Ju's method which has to determine the two (instead of one) parameters $Q_m$, $q_m$ recursively.

The differences between our method and Ju's are primarily due to major differences in the rationale behind the two approaches. The motivation behind our use of the linear spline approximation to $\bar{z}(.)$ with a few knots comes from an extensive numerical study in which we computed $\bar{z}(.)$ almost exactly for different values of $\rho$ and $\alpha$ and which shows that $\bar{z}(.)$ can indeed be well approximated by such linear splines. In contrast, Ju (1998) was motivated by his attempt to use the closed-form expression for the integral in (2) when $B_t$ is piecewise exponential to improve on the much cruder approximation of Huang et al. (1996) that treats the entire integrand as piecewise constant even though the interval $[0, T]$ is coarsely partitioned into 3 subintervals of equal width. Since he felt that the actual exercise boundary might differ substantially from a piecewise exponential curve with 3 or fewer pieces, he used two adjustable parameters $q_m, Q_m$ to fit an exponential function in each piece. The following example illustrates the adequacy of Ju's 3-piece approximation.
(with jump discontinuities) to the early exercise boundary and compares it with ours and that of Huang et al. (1996).

EXAMPLE 1. Ju (1998, p. 635) has considered the following American put with $K = 100$, $T = 3$ (years), $\sigma = .2$, $r = .08$ and $\mu = .12$, and has reported values of the coefficients $q_m^{(3)}, Q_m^{(3)}$ ($m = 1, 2, 3$) in the 3-piece exponential curve (denoted by EXP3) to approximate the early exercise boundary. Using these values, Figure 2 plots Ju's approximation and the "exact" (benchmark) boundary which is computed by the hybrid approach described in Section 3 with $\delta = 10^{-4}$. Also shown in Figure 2 is our linear spline approximation with 4 unevenly spaced pieces (denoted by LSP4), with the first knot at $s = -.005$ followed by three evenly spaced knots. The figure shows that our approximation is close to the early exercise boundary. Although the first piece of Ju's approximation is not quite close, it has little effect on the integrand in (13), as shown in Figure 3 which plots the integrands in (13) for $P = 100$ using the exact, Ju's and our exercise boundaries, showing close agreement between them. On the other hand, if we approximate the integrand by a piecewise constant function using 4 pieces as in Huang et al. (1996, p. 287), then its graph (denoted by HSY4), also shown in Figure 3, differs substantially from that of the integrand associated with the exact boundary, explaining why their method performs worse than Ju's as shown in Ju's Table 2. Also shown in Figure 2 is the step function approximation of Huang et al. (HSY4) that differs substantially from the exact boundary.

INSERT FIGURES 2 AND 3 ABOUT HERE

To solve for $(q_m^{(n)}, Q_m^{(n)})$ recursively in an $n$-piece exponential approximation to $B_t$, Ju needs two nonlinear equations $f_1(q_m^{(n)}, Q_m^{(n)}) = 0$ and $f_2(q_m^{(n)}, Q_m^{(n)}) = 0$, where $f_1$ comes directly from (3) by writing the integral in the RHS as $\sum_{i=1}^{n} f_{n_i}^{t_i-1}$, and $f_2$ comes from the
integral equation obtained by differentiating the RHS of (2) with respect to \( P \) and setting it equal to \(-1\) (= derivative of \( K - P \)) when \( P = B_t \). These two nonlinear equations are solved by Newton’s method that requires good starting values. In contrast, the method in Section 2 to solve (12) for \( \bar{z}(s_m) \) only involves a one-dimensional search, which can be carried out by the Van Wijngaarden-Dekker-Brent method that is numerically much more stable than Newton’s method.

As indicated by Ju (1998), his \( n \)-piece exponential approximation to the early exercise boundary involves besides the \( 2n \) parameters \( q_1^{(n)}, Q_1^{(n)}; \ldots; q_n^{(n)}, Q_n^{(n)} \) of the \( n \) pieces also the \( 2(n - 1) \) parameters \( q_1^{(1)}, Q_1^{(1)}; \ldots; q_{n-1}^{(n-1)}, Q_{n-1}^{(n-1)} \) of the 1-piece, \( \ldots, (n - 1) \)-piece approximations. The reason is that he uses \((q_{i-1}^{(j-1)}, Q_{i-1}^{(j-1)})\) as the starting value for solving the two nonlinear equations defining the parameters of the function \( Q_1^{(j)} \exp(q_1^{(j)}t) \) for \((j - 1)T/j < t \leq T\). In particular, to initialize the one-piece approximation, he can rely on the good time-invariant approximation \((0, P^*)\) to the early exercise boundary provided by MacMillan (1986) and Barone-Adesi and Whaley (1987) whose method is described at the beginning of this section. For \( i \geq 2 \), \((q_{i-1}^{(n)}, Q_{i-1}^{(n)})\) is used as the starting value for solving the nonlinear equations defining \((q_i^{(n)}, Q_i^{(n)})\). This “bottom-up” approach requires the determination of \( P^* \) and \( 2(n - 1) \) additional parameters simply to initialize \( q_1^{(n)}, Q_1^{(n)} \). In contrast, our method is more direct and involves \( n \) equations, one for each \( \bar{z}(s_i) \), in our \( n \)-piece linear approximation to \( \bar{z}(\cdot) \).

It is natural to try to bypass Ju’s bottom-up approach by initializing the solution for \((q_1^{(n)}, Q_1^{(n)})\) more directly without using \((q_1^{(n-1)}, Q_1^{(n-1)})\) which in turn involves \((q_1^{(n-2)}, Q_1^{(n-2)})\), \( \ldots, (q_1^{(1)}, Q_1^{(1)})\), as was done by Gao et al. (2000) in their extension of Ju’s approximation to American barrier options. An obvious way is to initialize with the \((0, P^*)\) of MacMillan,
Barone-Adesi and Whaley. Another way is to initialize with \((0, B_T)\), as suggested by Gao et al. (2000), in which \(B_T = K \min(1, r/\mu)\). A third way is to initialize with the first piece of our piecewise linear approximation for the transformed coordinates (5), which yields the starting value \((\bar{q}, \bar{Q})\) for \((q_1^{(n)}, Q_1^{(n)})\), where

\[
\bar{q} = (\bar{z}(0) - \bar{z}(s_1))n/T + r - \mu - \sigma^2/2, \quad \bar{Q} = B_T e^{-\bar{q}T},
\]  

(17)

noting that \(T/n\) is the width of each subinterval when \([0, T]\) is partitioned into \(n\) subintervals of equal width.

In his extensive numerical study comparing his method with other approximations in the literature, Ju (1998) uses 3 pieces and no more than 3 iterations in applying Newton’s procedure to solve the two nonlinear equations for the \((q, Q)\) of each piece, after initializing the procedure in the way described above. This small number of pieces and of iterations makes his method faster than many other approximations in the literature, as shown in Tables 1-4 of his paper. Since he builds the three-piece approximation “bottom-up” from the one- and two-piece approximations, it is natural for him to combine the option values obtained from the one-, two- and three-piece approximations by using the Richardson extrapolation to obtain a slightly more accurate approximation than that involving only the 3-piece approximation.

Table 3 compares Ju’s method involving a 3-piece exponential approximation (EXP3), whose results are taken from Table 2 of Ju (1998), with our linear spline approximation LSP4. To be comparable to Ju in speed, we have limited the number of iterations in our solution of (12) to 5 in Table 3 and Figure 2. LSP4 and EXP3 are in close agreement with the benchmark values computed by the binomial tree method with 10,000 steps. Table 3 also
considers the three simpler ways to initialize \((q_1^{(3)}, Q_1^{(3)})\) described above, namely, starting with \((0, P^*)\) or \((0, B_T)\), or with the \((\bar{q}, \bar{Q})\) defined in (17). It shows the superiority of Ju’s elaborate choice of the starting value over other equally plausible choices, which produce either less accurate or numerically unstable results. In particular, the entries marked by * in Table 3 denote results where the Newton’s iterative procedure to solve Ju’s simultaneous nonlinear equations is terminated because of a singular (or nearly singular) Jacobian matrix.

**INSERT TABLE 3 ABOUT HERE**

6 CONCLUSION

Ait-Sahalia and Lai (1999) used the corrected Bernoulli walk approach introduced by Chernoff and Petkau (1986) to compute the early exercise boundary of a standard American option. In this paper we use a different approach based on numerical solution of the integral equation defining the early exercise boundary. More importantly, by combining both approaches, we obtain a hybrid method that is an improvement over either approach in accuracy and speed. A new method is also developed to solve the integral equation numerically.

The approximately piecewise linear shape, with a few unevenly spaced pieces, of the early exercise boundary in the canonical scale, already noted in Ait-Sahalia and Lai (1999), suggests that our new method to solve the integral equation defining the early exercise boundary can be applied to a coarse grid with a few time points to yield a fast and reasonably accurate approximation. It also explains why Ju’s (1998) method that involves solving simultaneous nonlinear equations for the parameters of piecewise exponential approximations to the early exercise boundary (in the original coordinates) is both faster and
more accurate than previous approximations in the literature. We have noted in Section 5 that another ingredient for the success of Ju's method is his elaborate starting value for the iterative solution of these simultaneous nonlinear equations, and that other equally plausible starting values can result in singular or ill-conditioned Jacobian matrices in the Newton-type iterations used by Ju to solve the simultaneous nonlinear equations. In contrast, our method to solve for \( \bar{x}(t_i) \) involves a numerically stable one-dimensional search and has superior convergence properties.
REFERENCES


Courtadon, G.R. (1982). A more accurate finite difference approximation for the valuation


### TABLE 1. Early exercise boundary, in canonical scale (5), using the Volterra integral equation (6) or the corrected Bernoulli walk method for American put with $\rho = 0.5$ and $\alpha = 0$

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### Table 2

Accuracy of 3-piece Linear Spline Approximation (LSP3) relative to Binomial and Hybrid benchmarks

Case I: $T = 3, \sigma = 0.1, r = 0.06$

Case II: $T = 0.3, \sigma = 0.2, r = 0.06$

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Case II | .09 | 80 | 20.802 | 20.803 | 20.803 | -0.906 | -0.906 | -0.906 |
|        |     | 90 | 12.422 | 12.422 | 12.422 | -0.748 | -0.748 | -0.748 |
|        |     | 100| 6.183  | 6.183  | 6.183  | -0.491 | -0.491 | -0.491 |
|        |     | 110| 2.535  | 2.534  | 2.535  | -0.250 | -0.250 | -0.250 |
|        |     | 120| 0.865  | 0.865  | 0.865  | -0.100 | -0.100 | -0.100 |
|        | .06| 80 | 20.093 | 20.094 | 20.055 | -0.949 | -0.949 | -0.945 |
|        |     | 90 | 11.545 | 11.545 | 11.542 | -0.742 | -0.742 | -0.740 |
|        |     | 100| 5.504  | 5.504  | 5.510  | -0.462 | -0.462 | -0.462 |
|        |     | 110| 2.154  | 2.154  | 2.158  | -0.223 | -0.223 | -0.223 |
|        |     | 120| 0.701  | 0.701  | 0.702  | -0.085 | -0.085 | -0.085 |
|        | .03| 80 | 20.000 | 20.000 | 19.968 | -1.000 | -1.000 | -0.995 |
|        |     | 90 | 10.953 | 10.953 | 10.906 | -0.760 | -0.760 | -0.759 |
|        |     | 100| 4.961  | 4.961  | 4.940  | -0.443 | -0.443 | -0.440 |
|        |     | 110| 1.843  | 1.843  | 1.841  | -0.200 | -0.200 | -0.200 |
|        |     | 120| 0.570  | 0.570  | 0.571  | -0.072 | -0.072 | -0.072 |
|        | .00| 80 | 20.000 | 20.000 | 19.996 | -1.000 | -1.000 | -0.996 |
|        |     | 90 | 10.522 | 10.523 | 10.534 | -0.794 | -0.794 | -0.794 |
|        |     | 100| 4.493  | 4.492  | 4.493  | -0.427 | -0.427 | -0.427 |
|        |     | 110| 1.578  | 1.577  | 1.582  | -0.180 | -0.180 | -0.181 |
|        |     | 120| 0.462  | 0.462  | 0.465  | -0.061 | -0.061 | -0.062 |
TABLE 3. Comparison of benchmark values (Binomial) of American option prices $(K = 100, T = 3, \sigma = 0.2, r = 0.08)$ with 5 fast approximations: LSP4, EXP3, Ju$(0, P^*)$, Ju$(0, B_T)$, Ju$(\bar{q}, \bar{Q})$. The last three represent initializing Ju’s method at three alternative starting values.

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<th>EXP3</th>
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Figure 1 - Optimal Stopping Boundaries: Standard Put with $\rho = 0.5$
Figure 2 - Approximations to Early Exercise Boundary
Figure 3 - Approximations to Premium Integrand