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Farid AitSahlia
Lorenz Imhof
Tze Leung Lai

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Department of Statistics
STANFORD UNIVERSITY
Stanford, California
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FARID AITSAHLIA
Financial Engines

LORENS IMHOF
RWTH Aachen

TZE LEUNG LAI
Department of Statistics
Stanford University

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Department of Statistics
Sequoia Hall
STANFORD UNIVERSITY
Stanford, California 94305-4065

http://www-stat.stanford.edu
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Abstract

This article presents a new numerical method to compute the prices and early exercise boundaries of American barrier options. Numerical results obtained by this method show that after a change of variables, the exercise boundaries are well approximated by certain piecewise linear functions, leading to two fast and accurate methods to compute exercise boundaries, prices and hedge parameters. Comparison with other competing methods is also included.

*We thank Nengjiu Ju and Bin Gao for providing us with the codes of their procedures. Address correspondence to Tze Leung Lai, Department of Statistics, Sequoia Hall, Stanford University, Stanford, CA 94305. Tel: (650) 723-2622. Fax: (650) 725-8977. Email: lail@stat.stanford.edu
The explosive growth in the use of derivatives by investors and institutions has fueled the need for fast and accurate valuation. In the past two decades substantial progress has been made in this direction for standard American options. Geske and Johnson (1984) characterized American options as compound European options and then used the Richardson extrapolation (with 3 or 4 points typically) to approximate the option price. MacMillan (1986) and Barone-Adesi and Whaley (1987) approximated the partial differential equation (PDE) for the difference between American and European option prices by an ordinary differential equation (ODE) and thereby derived an approximate valuation formula. Broadie and Detemple (1996) developed upper and lower bounds for an American option and used a convex combination of these bounds with empirically determined weights to approximate the option price. Carr (1998) discretized the time dimension of the PDE with a few points and used a randomization method to approximate the option price. Huang et al. (1996), Ju (1998) and AitSahlia and Lai (1999) developed approximations to the early exercise boundary and used them to derive approximations to American option prices and hedge parameters.

In particular, Huang et al. approximated the early exercise boundary by a step function while Ju used a piecewise exponential function to approximate the boundary. Ju (1998) reported numerical studies showing that his method with \( n = 3 \) pieces substantially improves earlier approximations in the literature in speed and accuracy. AitSahlia and Lai (1999) carried out extensive computations of early exercise boundaries for a wide range of maturities, interest rates, dividend rates, volatilities and strike prices via reparametrization to reduce American option valuation to a single optimal stopping problem for standard Brownian motion, indexed by one parameter in the absence of dividends and by two parameters otherwise. Their results show that the early exercise boundary is well approximated by a piecewise exponential boundary which uses a small number of pieces, and explain why Ju's method has superior performance over previous approximation approaches.

The past decade also witnessed important developments in the valuation of American barrier options. Barrier options are widely used by institutional investors, banks and corporations in their risk management, and American-style options give their holders the additional flexibility of early exercise. Boyle and Lau (1994) pointed out that naive ap-
plication of the Cox-Ross-Rubinstein binomial tree method can result in significant errors even when a large number of time steps is used because the barrier typically lies between two adjacent layers of nodes in the lattice. They proposed to reduce the size of errors by refining the time partition so that the resulting lattice has layers as close as possible to the barrier. Ritchken (1995) introduced a trinomial tree method, Cheuk and Vorst (1996) used a time-dependent shift of the trinomial tree, while Figlewski and Gao (1999) developed an "adaptive mesh model" to address difficulties with lattice methods for barrier options. By using piecewise constant or piecewise exponential functions to approximate the early exercise boundaries, Gao et al. (2000) derived approximations to the values and hedge parameters for American barrier options, similar to those for standard options. They measured the accuracy of these approximations by the differences from corresponding values computed by Ritchken's method with a large number (between 10,027 and 21,385) of time steps. Because of the computational task involved, they conducted numerical studies on two relatively small samples of 48 contracts each. Moreover, in the absence of known convergence results for Ritchken's method, the adequacy of these "benchmark values" may be questionable.

In this article we introduce an alternative approach to compute benchmark values that are accurate up to an $O(n^{-1})$ error, where $n$ is the number of time steps. It is based on a modification of the corrected Bernoulli walk method of Chernoff and Petkau (1986) to solve a canonical form of optimal stopping problems for American barrier options. This modification addresses the difficulties with lattice methods in the presence of a barrier and computes not only the option prices but also the entire exercise boundary. Using this approach, we carry out extensive computations, which show that the exercise boundaries of American barrier options are well approximated by piecewise exponential functions with a small number of pieces. Using this result, we develop two fast and accurate approximations to the option prices and hedge parameters. They are compared with that of Gao et al. (2000) and with other approximations in terms of speed and accuracy (using benchmark values computed by both our and Ritchken's methods that are in close agreement, but with ours much faster than Ritchken's method).

This article is organized as follows. Section 1 presents new methods to compute bench-
mark values for the early exercise boundary, option prices and hedge parameters. Section 2 provides efficient approximations to these benchmark values by using a continuous piecewise exponential function, with a few unevenly spaced pieces, to approximate the exercise boundary. Section 3 gives an alternative approximation based on extending Ju's discontinuous piecewise exponential approximation of the exercise boundary to barrier options. In this connection we also extend the approximation of MacMillan (1986) and Barone-Adesi and Whaley (1987) that has been used by Ju to initialize Newton's method for solving the nonlinear equations for the two parameters of the exponential functions in each piece. Section 4 presents numerical results comparing our approximations with others and with benchmark values. While Sections 1-4 consider primarily "up-and-out" put options to fix the ideas, Section 5 studies "down-and-in" put options and summarizes how these methods can be extended to other American barrier options. Section 6 gives some concluding remarks.

1 Benchmark Values and Early Exercise Boundary

Under the usual assumptions of a riskless interest rate \( r \) and an underlying asset which follows a geometric Brownian motion (GBM) with volatility \( \sigma \) and which pays dividend at rate \( \mu \), the price of an American barrier option at time \( t \leq T \) (\( = \) expiration date) that entails no arbitrage opportunities before exercise is given by

\[
P(t, S) = \sup_{\tau \in \mathcal{T}_{a,b}} E\{e^{-r(t-t_0)}g(S_{\tau}; \tau, \tau_H)|S_t = S\},
\]

where \( S_t = S_0e^{(r-\mu-\sigma^2/2)t+\sigma W_t} \), with initial price \( S_0 \), \( \{W_t\} \) is standard Brownian motion, \( \tau_H \) is the first time that \( S_t \) crosses the barrier (to be defined precisely below), and \( \mathcal{T}_{a,b} \) is the set of stopping times taking values between \( a \) and \( b \) with \( b > a \); see Karatzas (1988). Given a strike price \( K \), the payoff \( g(S; \tau, \tau_H) \) in (1) is \( (K - S)^+1_{\{\tau < \tau_H\}} \) for a knock-out put and \( (S - K)^+1_{\{\tau < \tau_H\}} \) for a knock-out call, and \( 1_{\{\tau < \tau_H\}} \) is replaced by \( 1_{\{\tau \geq \tau_H\}} \) for the corresponding knock-in options. Here \( \tau_H = \inf\{t \leq T : S_t \geq H\} \) for an "up-and-out" or "up-and-in" barrier, and \( \tau_H = \inf\{t \leq T : S_t \leq H\} \) for a "down-and-out" or "down-and-in" barrier.
Introduce the change of variables

$$\rho = \frac{r}{\sigma^2}, \quad \alpha = \mu/r, \quad \gamma = \ln(H/K);$$

(2)

$$u = \sigma^2(t-T), \quad z = \ln(S/K).$$

(3)

To fix the ideas, consider an up-and-out put with barrier \(H > K\). In this case, (1) can be expressed as \(Ke^{\mu u}w(u, z)\), where

$$w(u, z) = \sup_{\tau \in \mathcal{T}_{u,0}} E\{e^{-\rho \tau}(1-e^{Z_{\tau}})+1_{\{\tau < \tau(\gamma)\}}|Z_u = z\},$$

(4)

\[Z_s = (\rho - \alpha \rho - \frac{1}{2})s + W_s, \ s \leq 0,\] is a Wiener process with drift and \(\tau(\gamma) = \inf\{s \leq 0 : Z_s \geq \gamma\}\).

Note that for the optimal stopping problem (4) the horizon is always 0, and therefore for a given set of parameters \((\rho, \alpha, \gamma)\), only one numerical program need be implemented for all expiration dates \(T\). Moreover, for values of \(\sigma\) (between .1 and .4) that are of practical interest, the time horizon \(\sigma^2 T\) in the canonical scale (3) is only a small fraction of \(T\). Let

$$\lambda = \rho - \alpha \rho - 1/2, \ u_0 = 0 \text{ and } u_k = u_{k-1} - \delta \text{ for } k \geq 1.$$  

To solve numerically the optimal stopping problem (4), we first approximate \(Z_{u_k}\) by a Bernoulli random walk \(\Sigma_{k=1}^k X_i\) on a lattice that contains \(\gamma\). Specifically let

$$L_\delta = \{\gamma - \sqrt{\delta} \ (1 + \delta \lambda^2)^{1/2} \cdot j : j = 0, 1, 2, \ldots\}.$$  

(5)

The Bernoulli random walk has absorbing barrier \(\gamma\), time increment \(\delta\) and space increment \(X_i\), such that the \(X_i\) are independent Bernoulli random variables with

$$P\{X_i = \pm \sqrt{\delta}(1 + \delta \lambda^2)^{1/2}\} = \frac{1}{2} \left(1 \pm \frac{\lambda \sqrt{\delta}}{\sqrt{1 + \delta \lambda^2}}\right).$$

(6)

Note that \(X_i\) has mean \(\lambda \delta = E(Z_{t+\delta} - Z_t)\) and variance \(\delta = \text{Var}(Z_{t+\delta} - Z_t)\).

Approximating \(\{Z_t\}\) by the above Bernoulli random walk that has the same absorbing barrier \(\gamma\) and incremental mean and variance as \(\{Z_t\}\), our basic idea is to approximate (4) by the backward recursion

$$w(u_{n+1}, z) = \max\{e^{-\rho u_n}(1-e^z)^+, \ Ew(u_n, z + X_n)\},$$

(7)
with boundary condition \( w(u_i, \gamma) = 0 \) for all \( i \). Since (7) only computes \( w(u_i, z) \) for \( z \in L_\delta \), the value of \( w(u_i, z) \) at \( z \notin L_\delta \) can be computed by interpolation, e.g. by using linear interpolation, or Lagrange's interpolation formula with a quadratic interpolation polynomial (see Press et al. (1992)). Similarly we can obtain the value of \( w(u, z) \) for \( u \notin \{u_0, u_1, \ldots \} \) by interpolation.

The stopping boundary of the discrete-time optimal stopping problem for the approximating Bernoulli walk is given by \( \tilde{z}_\delta(u_i) = \max\{z \in L_\delta: w(u_i, \gamma) = e^{-\rho u_i}(1 - e^z)^+\} \). We can use the continuity correction of Chernoff and Petkau (1986) to compute the continuous-time optimal stopping boundary \( \tilde{z}(\cdot) \) as follows: Let

\[
\begin{align*}
x_0^\delta(u_i) &= \tilde{z}_\delta(u_i) + \sqrt{\delta}, \quad x_1^\delta(u_i) = \tilde{z}_\delta(u_i) + 2\sqrt{\delta}, \\
D_j(u_i) &= e^{-\rho u_i} \{1 - \exp(x_j^\delta(u_i))^\}^+ - w(u_i, x_j^\delta(u_i)) \quad \text{for} \quad j = 0, 1, \\
\bar{z}(u_i) &= x_0^\delta(u_i) + \sqrt{\delta |D_1(u_i)/\{2D_1(u_i) - 4D_0(u_i)\}|.}
\end{align*}
\]

As noted by AitSahlia and Lai (2000), the derivative \( \bar{z}'(u) \) of the boundary of an American barrier put becomes infinite as \( u \) approaches 0 and the Chernoff-Petkau correction cannot be used for \( u \) near 0. We describe below a more accurate method to compute \( \bar{z}(u) \) and \( w(u, z) \) for \( u \) near 0.

To begin with, as shown in AitSahlia and Lai (2000), \( P(t, S) \) (\( = e^{\rho u} w(u, z) \) in the case \( K = 1 \)) has the following decomposition:

\[
e^{\rho u} w(u, z) = f(u, z) - e^{2\lambda(\gamma - z)} f(u, 2\gamma - z) \\
+ \rho \int_u^0 e^{-\rho(s-u)} \left\{ N\left(d(\lambda, s, u, z)\right) - e^{2\lambda(\gamma - z)} N\left(d(\lambda, s, u, 2\gamma - z)\right) \\
- \alpha e^z \left[ N\left(d(\lambda + 1, s, u, z)\right) - e^{2(\lambda+2)(\gamma - z)} N\left(d(\lambda + 1, s, u, 2\gamma - z)\right) \right] \right\} ds,
\]

where \( d(\xi, s, u, y) = \{\bar{z}(s) - y - \xi(s - u)\} / \sqrt{s - u} \) and

\[
f(u, z) = e^{\rho u} N\left(-\frac{z - \lambda u}{\sqrt{-u}}\right) - e^{z + \rho u} N\left(-\frac{z - (\lambda + 1) u}{\sqrt{-u}}\right).
\]

In the transformed coordinates (3), \( f(u, z) \) and \( f(u, z) - e^{2\lambda(\gamma - z)} f(u, 2\gamma - z) \) are, respectively, the European standard and barrier option prices. Let \( \psi(u, z) \) denote the RHS of (8). Setting \( z = \bar{z}(u) \) in (8) yields the following integral equation for the exercise boundary:

\[
1 - e^{\bar{z}(u)} = \psi(u, \bar{z}(u)).
\]
Instead of initializing the recursion (7) at \( n = 0 \) with \( w(u_0, z) = (1 - e^z)^+ \), we initialize (7) at \( n_0 \) and determine \( w(u_{n_0}, z) \) via (8), in which the integral can be expressed in closed form when \( \tilde{z}(u) \) for \( u_i \leq u \leq u_{i-1} \) is obtained by linearly interpolating \( \tilde{z}(u_i) \) and \( \tilde{z}(u_{i-1}) \). Using this closed form to compute the integral in (8), \( \tilde{z}(u_i) \) can be determined recursively from (9) for \( 0 \leq i \leq n_0 \), with \( \tilde{z}(0) = -(\log \alpha)^+ \). The details are given in Section 2. Theoretical analysis in AitSahalia and Lai (2000) shows that this "hybrid" method, which combines the decomposition approach for \( u \) near 0 with the Bernoulli walk method for \( u \leq u_{n_0} \), yields option values with \( O(\delta) \) error and the optimal stopping boundary \( \tilde{z}(.) \) with \( o(\sqrt{\delta}) \) error.

Differentiating the decomposition (8) for \( P(t, S) = e^{\alpha u} w(u, z) \) with respect to \( S = e^z \), we can express the hedge ratio \( \Delta \) in the canonical scale (3) as

\[
\frac{\partial P}{\partial S} = -e^{\alpha u} \left\{ -\frac{z - (\lambda + 1)u}{\sqrt{-u}} - e^{(2\lambda + 2)(\gamma-z)} \frac{2\gamma - z - (\lambda + 1)u}{\sqrt{-u}} \right.
\]

\[
+ 2\lambda e^{2\lambda(2\lambda+1)z} f(u, 2\gamma - z) + \rho \int_u^0 e^{-\rho(s-u)} \left\{ e^{-z} \left[ -\phi(\lambda, s, u, z) \right] \right. \]

\[
- e^{2\lambda(\gamma-z)} \left( \phi(\lambda + 1, s, u, 2\gamma - z) - 2\lambda N (d(\lambda + 1, s, u, 2\gamma - z)) \right) \]

\[
- \alpha \left[ \left( 1 - e^{(2\lambda+2)(\gamma-z)} \right) N (d(\lambda + 1, s, u, 2\gamma - z)) - \left( 1 + e^{(2\lambda+2)(\gamma-z)} \right) \right] \]

\[
\times \left( \phi(\lambda + 1, s, u, 2\gamma - z) - (2\lambda + 2) N(d(\lambda + 1, s, u, 2\gamma - z)) \right) \right\} ds, \quad (10)
\]

where \( \phi(\xi, s, u, y) = n(d(\xi, s, u, y)) / \sqrt{s - u} \), with \( n(.) \) being the standard normal density function. Since \( \tilde{z}(.) \) can be computed with \( o(\sqrt{\delta}) \) error, we can also compute (10) with \( o(\sqrt{\delta}) \) error by using the closed-form integration formula in Section 2 that linearly interpolates between \( \tilde{z}(u_i) \) and \( \tilde{z}(u_{i-1}) \). Other hedge parameters such as gamma, vega and rho can be computed similarly.

2 An Efficient Approximation

In this section we first consider computation of the early exercise premium and solution of the integral equation (9). Our numerical results show that \( \tilde{z}(.) \) is well approximated by a
linear spline with a few knots. We then make use of such approximations to develop a fast method to compute option values, hedge parameters and $\bar{z}(\cdot)$ approximately.

To evaluate the integral in (8) we use Ju's (1998) closed-form expression for the integral when $\bar{z}(\cdot)$ is piecewise linear. Note that in view of the transformation (3), a piecewise linear $\bar{z}(\cdot)$ corresponds to the piecewise exponential $B_t$ used by Ju to approximate the exercise boundaries of standard American options. Specifically, assuming that $u = u_m < \ldots < u_0 = 0$, partition $[u, 0]$ into $m$ subintervals (not necessarily of equal width) such that

$$\bar{z}(s) = b_is + \alpha_i \quad \text{for } u_i \leq s \leq u_{i-1} \quad (1 \leq i \leq m),$$

and letting $\tau_i = u_i - u$, $a_i = \sqrt{b_i^2 + 2m}$, $c_i = \alpha_i + b_iu - z + \lambda u$, Ju showed that for $1 \leq i \leq m$,

$$\int_{\tau_i}^{\tau_{i-1}} e^{-\rho t} N \left( \frac{\bar{z}(t + u) - z + \lambda u}{\sqrt{t}} \right) dt = \int_{\tau_i}^{\tau_{i-1}} e^{-\rho t} N \left( b_i \sqrt{t} + \frac{c_i}{\sqrt{t}} \right) dt$$

$$= e^{-\rho \tau_i} N \left( b_i \tau_i^{1/2} + c_i \tau_i^{-1/2} \right) - e^{-\rho \tau_{i-1}} N \left( b_i \tau_{i-1}^{1/2} + c_i \tau_{i-1}^{-1/2} \right)$$

$$+ \frac{1}{2} \left( \frac{b_i}{a_i} + 1 \right) e^{(a_i - b_i) \alpha_i} \left\{ N \left( a_i \tau_{i-1}^{1/2} + c_i \tau_{i-1}^{-1/2} \right) - N \left( a_i \tau_i^{1/2} + c_i \tau_i^{-1/2} \right) \right\}$$

$$+ \frac{1}{2} \left( \frac{b_i}{a_i} - 1 \right) e^{-(a_i + b_i) \alpha_i} \left\{ N \left( a_i \tau_i^{1/2} - c_i \tau_i^{-1/2} \right) - N \left( a_i \tau_{i-1}^{1/2} - c_i \tau_{i-1}^{-1/2} \right) \right\}.$$  

(12)

In the case $\tau_i = 0$, we replace $\tau_i$ by $t > 0$ and take the limit as $t \to 0$, which amounts to setting $c_i \tau_i^{-1/2} = 0$, or $\infty$, or $-\infty$ according as $c_i = 0$, or $c_i > 0$, or $c_i < 0$. The other integrands in (8) can be treated similarly.

To solve the integral equation (9) numerically, let $\bar{z}_j = \bar{z}(u_j)$. Suppose $\bar{z}_1, \ldots, \bar{z}_{m-1}$ have already been determined. Let $z$ be a candidate value for $\bar{z}_m$ and let $b(z) = (z - \bar{z}_{m-1})/(u_m - u_{m-1})$, $a(z) = z - b(z)u_m$. Linearly interpolating $\bar{z}(\cdot)$ between $u_j$ and $u_{j-1}$ for $1 \leq j \leq m$ gives the piecewise linear function (11) with $b_m = b(z)$ and $\alpha_m = a(z)$. In view of (9), using the preceding closed-form expression to evaluate the integral in (8) leads to the following nonlinear equation for $\bar{z}_m(= z)$:

$$1 - e^{z} = \psi(u_m, z),$$

(13)

which can be solved by a Van Wijngaarden-Dekker-Brent-type method; see Press et al. (1992).

This method to compute $\bar{z}(u_i)$ is fast when the number $m$ of time steps is small. However, because of the iterative procedure to solve (12) and the non-recursive computations required
to compute $\psi(u_m, z)$, it becomes computationally expensive for large $m$. This is the rationale behind the hybrid method in Section 1, which only uses (9) to compute the early exercise boundary for $u$ near 0 (where $\bar{z}'(u)$ is large) and to initialize the recursive Bernoulli walk method at $u_{n_0}$, where $u_{i-1} - u_i \equiv \delta$ with a small $\delta$.

Figure 1 plots the graphs of $\bar{z}(u)$ for $-0.3 \leq u \leq 0$, $\gamma = 1.2$, $\alpha = 0$ and different values of $\rho$, computed by the hybrid method with $\delta = 10^{-4}$. It shows that $\bar{z}(\cdot)$ is well approximated in each case by a linear spline with 6 knots. The approximate piecewise linearity of $\bar{z}(\cdot)$ suggests that we can in fact use a small number $n$ of time steps in the preceding procedure to solve (9) for $\bar{z}(\cdot)$. It often suffices to choose $n$ as small as 4, with unevenly spaced knots at $-\sigma^2T$, $(-.4)\sigma^2T$, $(-.08)\sigma^2T$ and $(-.005)\sigma^2T$. Figure 2 plots such an approximation to $\bar{z}(\cdot)$ in the case $\gamma = 1.1$, $\alpha = 1/3$, $\rho = 1.5$ and $\sigma^2T = .12$, showing that the approximation is quite good. Doubling the number of time steps, with 8 unevenly spaced knots at $-1, -0.8, -0.6, -0.4, -0.2, -0.1, -0.05$ and $-0.005$ multiplied by $\sigma^2T$, gives a closer approximation.

INSERT FIGURES 1 AND 2 ABOUT HERE

Once the linear spline approximation to $\bar{z}(\cdot)$ is determined, we can use (8) and (12) to compute option values. The hedge ratio $\Delta$ can be computed via (10), (12) and the following analogue of (12):

$$\int_{\tau_i}^{\tau_{i-1}} \frac{e^{-\rho t}}{\sqrt{t}} n \left( \frac{\bar{z}(t + u) - z + \lambda u}{\sqrt{t}} \right) dt = \int_{\tau_i}^{\tau_{i-1}} \frac{e^{-\rho t}}{\sqrt{t}} n \left( b_i \sqrt{t} + \frac{c_i}{\sqrt{t}} \right) dt$$

$$= (2\pi)^{-1/2} e^{-b_i c_i} \int_{\tau_i}^{\tau_{i-1}} t^{-1/2} e^{-(a_i^2 t + c_i^2 t^{-1})/2} dt$$

$$= a_i^{-1} e^{(a_i - b_i)c_i} \left\{ N \left( a_i \tau_{i-1}^{-1/2} + c_i \tau_{i-1}^{-1/2} \right) - N \left( a_i \tau_i^{-1/2} + c_i \tau_i^{-1/2} \right) \right\}$$

$$+ a_i^{-1} e^{-(a_i + b_i)c_i} \left\{ N \left( a_i \tau_{i-1}^{1/2} - c_i \tau_{i-1}^{-1/2} \right) - N \left( a_i \tau_i^{1/2} - c_i \tau_i^{-1/2} \right) \right\}. \quad (14)$$

The last equality above can be derived by using the change of variables

$$x = a_i t^{1/2} + c_i t^{-1/2}, \quad y = a_i t^{1/2} - c_i t^{-1/2},$$
so that \( dx = \frac{1}{2} \left( a_i t^{-1/2} - c_i t^{-3/2} \right) dt, \ dy = \frac{1}{2} \left( a_i t^{-1/2} + c_i t^{-3/2} \right) dt, \) and by the identity
\[
t^{-1/2} = \left\{ \left( a_i t^{-1/2} - c_i t^{-3/2} \right) + \left( a_i t^{-1/2} + c_i t^{-3/2} \right) \right\} / (2a_i).
\]

3 Extension of Ju’s Method

When there is no barrier, Ju (1998) used the following method to determine a piecewise exponential approximation to the original early exercise boundary \( B_t \). The interval \([0,T]\) is divided into a small number (3 or 4) of evenly spaced pieces, and on the \( i \)th piece \( B_t \) is approximated by an exponential function \( Q_i e^{\alpha_i t} \). Although piecewise exponential \( B_t \) is equivalent to piecewise linear \( \bar{z}(u) \), Ju’s method is different from that in Section 2 because unlike the linear spline, the piecewise exponential approximation is not assumed to be continuous, resulting in 2n parameters \( q_1^{(n)}, Q_1^{(n)}, \ldots, q_n^{(n)}, Q_n^{(n)} \) for an \( n \)-piece exponential approximation instead of the \( n \) parameters \( \bar{z}(u_1), \ldots, \bar{z}(u_n) \) for the \( n \)-piece linear spline. The \( (q_m^{(n)}, Q_m^{(n)}) \) are determined recursively by solving two nonlinear equations \( f_1(q_m^{(n)}, Q_m^{(n)}) = 0 \) and \( f_2(q_m^{(n)}, Q_m^{(n)}) = -1 \), where \( f_1 \) comes directly from the integral equation (9) which can be expressed in the original variables as
\[
K - B_t = p(t, B_t) + \int_t^T e^{-r(t-t)} \{ rK[N(-d_2(B_t, B_{\tau}, \tau - t))] - (H/B_t)^{2\lambda} N(-d_2(H^2/B_t, B_{\tau}, \tau - t)) - \mu B_t[N(-d_1(B_t, B_{\tau}, \tau - t))] - (H/B_t)^{2\lambda+2} N(-d_1(H^2/B_t, B_{\tau}, \tau - t))\} d\tau, \tag{15}
\]
in which \( p(t, S) \) is the European barrier put price and
\[
d_1(x, y, \tau) = \left\{ \ln(x/y) + (r - \mu + \sigma^2/2)\tau \right\} / (\sigma/\sqrt{\tau}), \tag{16}
\]
\[
d_2(x, y, \tau) = d_1(x, y, \tau) - \sigma \sqrt{\tau}.
\]

Differentiating both sides of (15) with respect to the critical price \( B_t \) yields the equation \( f_2 = -1 \).

Good starting values are needed to solve the nonlinear simultaneous equations \( f_1 = 0, f_2 = -1 \) for \( (q_m^{(n)}, Q_m^{(n)}) \) by Newton’s method. When there is no barrier, Ju (1998) uses \( (q^{(j-1)}_i, Q^{(j-1)}_1) \) to initialize \( (q^{(j)}_i, Q^{(j)}_1) \) for \( 2 \leq j \leq n \) and \( (q^{(n)}_{i-1}, Q^{(n)}_{i-1}) \) to initialize \( (q^{(n)}_i, Q^{(n)}_1) \).
for $2 \leq i \leq n$. To initialize $(q_i^{(1)}, Q_i^{(1)})$ for the one-piece approximation, he uses $(0, B^*)$ with $B^*$ given by the approximation to $B_0$ due to MacMillan (1986) and Barone-Adesi and Whaley (1987). To extend his approach to American barrier options, we shall develop the corresponding approximation $B^*$ when there is a barrier. After computing such $B^*$, we can also follow his procedure of replacing the $n$-piece exponential function by a step function (i.e., $q_i^{(n)} = 0$ for all $i$) if $|B^* - B_T|/B_T \leq .05$ (with $B_T = \min\{Kr/\mu, K\}$) to avoid nonconvergence of Newton’s method when the $n$-piece exponential function is relatively flat.

### 3.1 Extension of the MacMillan/Barone-Adesi/Whaley Method

Let $p(t, S)$ and $P(t, S)$ denote, respectively, the price of a European and an American barrier put when the underlying security has price $S$ at time $t$. Define $\rho$ and $\alpha$ by (2) and let $\rho^* = 2\rho, \alpha^* = 2\rho(1 - \alpha), \tau = T - t.$ In the continuation region, the early exercise premium $\pi(\tau, S)$, defined as $P(T - \tau, S) - p(T - \tau, S)$, satisfies the Black-Scholes PDE

$$
\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 \pi}{\partial S^2} + (\tau - \mu) S \frac{\partial \pi}{\partial S} - \frac{\partial \pi}{\partial \tau} = \tau \pi.
$$

Following MacMillan (1986) and Barone-Adesi and Whaley (1987), we first seek solutions of the form

$$
\pi(\tau, S) = g(\tau)h(\tau, S), \quad (17)
$$

where $g(\tau) = 1 - e^{-\tau \tau}$. In other words, $h$ must satisfy the PDE

$$
S^2 \frac{\partial^2 h}{\partial S^2} + \alpha^* S \frac{\partial h}{\partial S} - \frac{\rho^*}{g} h - (1 - g)\rho^* \frac{\partial h}{\partial g} = 0. \quad (18)
$$

We next drop the term $(1 - g)\rho^* \partial h/\partial g$ in (18) to obtain the ODE

$$
S^2 \frac{\partial^2 h}{\partial S^2} + \alpha^* S \frac{\partial h}{\partial S} - \frac{\rho^*}{g} h = 0,
$$

a general solution of which is

$$
h(\tau, S) = a_1(\tau) S^{b_1(\tau)} + a_2(\tau) S^{b_2(\tau)}, \quad (19)
$$

where

$$
b_1(\tau) = \left[-(\alpha^* - 1) - \sqrt{(\alpha^* - 1)^2 + 4\rho^*/g(\tau)} \right]/2,
$$

11
\[ b_2(\tau) = \left[ -(\alpha^* - 1) + \sqrt{(\alpha^* - 1)^2 + 4\rho^* / g(\tau)} \right] / 2. \]

To fully specify \( h \) in (17) we rely on the barrier and optimal exercise conditions to determine \( a_1(\tau) \) and \( a_2(\tau) \). Clearly \( h(\tau, H) = 0 \) and thus \( a_2(\tau) = -a_1(\tau)H^{b_1(\tau) - b_2(\tau)} \). Let \( \bar{S} \) be the critical price at \( t = T - \tau \). Then the optimal exercise conditions at \((t, \bar{S})\) can be expressed as

\[ K - \bar{S} = p(T - \tau, \bar{S}) + a_1g \left[ \bar{S}^{b_1} - H^{b_1} \left( \bar{S} / H \right)^{b_2} \right], \tag{20} \]

\[ -1 = \frac{\partial p}{\partial \bar{S}}(T - \tau, \bar{S}) + a_1g \left[ b_1 \bar{S}^{b_1 - 1} - q_2 H^{b_1 - 1} \left( \bar{S} / H \right)^{b_2 - 1} \right], \tag{21} \]

where to simplify notation we omit \( \tau \) from \( a_1(\tau), a_2(\tau), b_1(\tau), b_2(\tau) \) and \( g(\tau) \). From (21) we obtain

\[ a_1 = 1 + \left\{ \frac{\partial p}{\partial \bar{S}}(T - \tau, \bar{S}) \right\} / \left\{ g \left[ b_1 \bar{S}^{b_1 - 1} - b_2 H^{b_1 - 1} \left( \bar{S} / H \right)^{b_2 - 1} \right] \right\}. \tag{22} \]

Combining (22) with (20) yields

\[ K - \bar{S} = p(T - \tau, \bar{S}) - \left\{ 1 + \frac{\partial p}{\partial \bar{S}}(T - \tau, \bar{S}) \right\} \frac{\bar{S}^{b_1} - H^{b_1 - 1} \left( \bar{S} / H \right)^{b_2 - 1}}{b_1 \bar{S}^{b_1 - 1} - b_2 H^{b_1 - 1} \left( \bar{S} / H \right)^{b_2 - 1}}, \tag{23} \]

where \( \frac{\partial p}{\partial \bar{S}}(T - \tau, \bar{S}) \) is the hedge ratio \( \Delta \) for the corresponding European barrier put at \( t = T - \tau \) and \( S = \bar{S} \), while \( p(T - \tau, \bar{S}) \) is given by \( f(u, z) \) in (8) after the change of variables (3). We can solve the nonlinear equation (23) for \( \bar{S} \) by a Van Wijngaarden–Dekker–Brent–type method. In particular, \( \bar{S} \) determined in this way at \( t = 0 \) (or equivalently \( \tau = T \)) gives the value of \( B^* \) for initializing the one-piece exponential approximation to the exercise boundary in Ju’s approach.

### 3.2 Implementation Issues

As indicated above, Ju’s approach requires solving two simultaneous nonlinear equations for the parameters \( q_i, Q_i \) of each of the \( n \) exponential functions \( Q_i e^{a_i t} \) that are pieced together to approximate the exercise boundary. These equations are solved by Newton’s method that uses an elaborate initializing scheme, beginning with a one-piece, followed by a two- (and then three-, etc.) piece approximation. For standard American options, Ju recommends
choosing \( n = 3 \) and his computer program that implements this approach only uses 3 iterations in Newton's method to solve for each \((q_i, Q_i)\) plus 6 iterations to solve for the \(B^*\) that initializes the one-piece approximation. This ensures a fast procedure. Moreover, since the one- and two-piece approximations have to be computed before the three-piece approximation, it takes little extra work to compute the option price (or hedge ratio) by using a three-point Richardson extrapolation scheme to combine the prices (or hedge ratios) computed from the one-, two- and three-piece approximations to the exercise boundary.

The same implementation can be applied to American knock-out options, to which we have extended the method of MacMillan, Barone-Adesi and Whaley. However, there are many more terms to compute, especially in the dividend-paying case. Solving two simultaneous equations \(f_1(q_i^{(n)}, Q_i^{(n)}) = 0, f_2(q_i^{(n)}, Q_i^{(n)}) = -1\) by Newton's method requires computation of many terms that appear in the first and second partial derivatives of the right hand side of (15) with respect to the critical price, and this makes Ju's method for up-and-out puts substantially slower in the presence than in the absence of dividends.

In their extension of Ju's approximation to barrier options, Gao et al. (2000) bypass the preceding extension of MacMillan, Barone-Adesi and Whaley to initialize the one-piece exponential approximation. Instead of following Ju to initialize the solution for \((q_i^{(n)}, Q_i^{(n)})\), they initialize at \(B_T = K \min\{1, r/\mu\}\) but allow a larger upper bound on the number of iterations in applying Newton's method to solve the simultaneous nonlinear equations. Their procedure, which will be denoted by \(G^{EXP}\), will be compared in the next section with the preceding extension of Ju's procedure.

4 Numerical Results and Discussion

In this section we demonstrate, through a large sample study, the accuracy and speed of the spline approximation to \(\tilde{z}(u)\) in Section 2 based only on a few knots, and of our extension of Ju's method in Section 3 using a three-piece exponential approximation to \(B_t\). We also include in the study our extension of MacMillan, Barone-Adesi and Whaley, which is used to initialize Ju's method. Also included in the study is \(G^{EXP}\), which is the procedure of Gao et al. (2000) to approximate \(B_t\) with a 3-piece exponential function and which
we implemented by using a computer program supplied by Gao. Besides the piecewise exponential approximation, Gao et al. (2000) have also proposed to approximate $B_t$ by a piecewise constant function (step function), which we denote by $G^{\text{STEP}}$ in our numerical study. These approximations are compared with benchmark values computed by using the hybrid method in Section 1 with $\delta = 10^{-4}$. Also included for comparison with benchmark values is Ritchken’s (1995) method with 10,000 and 800 time steps. All computations were performed on a shared Silicon Graphics Challenge workstation with four 64-bit Mips processors.

Table 1 reports the results from 1200 American up-and-out options where the parameters are independently drawn from the following distributions: Time to maturity $T$ is uniform between $1/24$ and 3 years, the barrier $H$ has probability $1/3$ of assuming each value in \{110, 150, 200\}, the current stock price $S$ is uniform between 80 and $0.9 \times H$, the volatility $\sigma$ is uniformly distributed on $[0.1, 0.6]$ and the riskless interest rate $r$ is uniformly distributed on $[0.02, 0.15]$. In addition, $K = 100$ and $\mu = 0$. The prices are in dollars and the entries of Table 1 represent the differences in cents between the prices generated by the hybrid method of Section 1 (benchmark) and the alternative methods.

In Table 1, the columns labeled R10000 and R800 represent Ritchken’s method with the number of time steps fixed at 10,000 and 800, respectively. The columns labeled SP4 and SP8 correspond to the linear spline approximation in Section 2, where $\tilde{z}(\cdot)$ is approximated by, respectively, 4- and 8-point splines at $-1, -0.8, -0.6, -0.4, -0.2, -0.1, -0.05, -0.005 \times \sigma^2 T$, and at $-1, -0.8, -0.6, -0.4, -0.2, -0.1, -0.05, -0.005 \times \sigma^2 T$. The column labeled Ju corresponds to our extension of Ju’s method in Section 3 using a 3-piece approximation to $B_t$. The column labeled MBW corresponds to our extension of the method of MacMillan (1986) and Barone-Adesi and Whaley (1987) in Section 3.1. The columns labeled $G^{\text{EXP}}$ and $G^{\text{STEP}}$ correspond to the methods of Gao et al. (2000) that approximate $B_t$ with a 3-piece exponential and a step function, respectively. Table 1 shows that both SP8 and Ju have the best performance in terms of accuracy and speed, and Ju has smaller maximum error. SP4 and $G^{\text{STEP}}$ are faster but less accurate procedures, with SP4 having a smaller root mean squared error (RMSE) than $G^{\text{STEP}}$. The $G^{\text{EXP}}$ method only converged in 851 out of the 1200 cases; the entries marked by asterisks (*) in Table 1 are actually for 849 of these 851 contracts. The
inclusion of the remaining two contracts for which there was convergence would have raised the maximum absolute error and the RMSE considerably (to 378 and 13 cents, and to 102 and 3.5 cents if only the largest is excluded).

Although both Ju and $G^{\text{EXP}}$ are based on piecewise exponential approximations to $B_t$ with 3 evenly spaced pieces, they differ in their choice of starting values for solving the simultaneous nonlinear equations defining the parameters of the exponential functions, as explained in Section 3.2. It is well known that two-dimensional root finding algorithms are sensitive to starting values; see Press et al. (1992). Somehow the meticulous initialization scheme used in Ju has resulted in a numerically stable solution, while the simpler and more direct initialization scheme used in $G^{\text{EXP}}$ has resulted in nonconvergence for about 30% of the cases in Table 1. In fact Newton’s procedure to solve the simultaneous equations was terminated in each of these cases because of a singular (or nearly singular) Jacobian matrix.

In situations where $G^{\text{EXP}}$ gives numerically stable results, the prices computed by $G^{\text{EXP}}$ and Ju are typically close to each other. This is illustrated on a set of 24 contracts in Table 2, where benchmark values computed by the hybrid method in Section 1 are also given together with the option prices (in dollars) obtained by other methods. Ritchken’s method with 800 time steps, R800, cannot be implemented when the current stock price $S$ is too close to the barrier $H(S = 100, H = 101$ in Table 2); in this case a larger number of steps is needed. Note that $R10000$ is within .09 cents of the benchmark values for all 24 contracts, consistent with the results of the large sample study in Table 1.

Table 3 gives the values of the hedge ratios $\Delta$ of these 24 contracts computed by different methods. The benchmark values are computed via (10), (12) and (14), in which the boundary $\tilde{z}(\cdot)$ is determined by the hybrid method of Section 1 with $\delta = 10^{-4}$. The approximations SP4, SP8, Ju, $G^{\text{EXP}}$ and $G^{\text{STEP}}$ also compute $\Delta$ via (10), (12) and (14), but use linear spline approximations with 4 or 8 knots to approximate $\tilde{z}(\cdot)$ or piecewise exponential/constant functions to approximate $B_t$ with 3 evenly spaced pieces. The tree methods R10000 and R800 use numerical differentiation to determine $\Delta$.

INSERT TABLES 1-3 ABOUT HERE
5 American Down-and-In Puts and Other Barrier Options

Generally investors consider put options as insurance against possible drops in the value of, say, a stock they are holding. A down-and-in put enables its holder to reduce the cost of such insurance by requiring that it be effective only if the stock price hits a barrier \( H \) below the current price \( S \). The price of a European down-and-in put option with strike price \( K \) at expiration date \( T \) in the setting of Section 1 is given by

\[
p_i(S, t) = -Se^{-\mu(T-t)}N(-d_1(S, H, T - t)) + Ke^{-\tau(T-t)}N(-d_2(S, H, T - t))
+ Se^{-\mu(T-t)}(H/S)^{2\lambda+2}\{N(d_1(H^2, SK, T - t)) - N(d_1(H, S, T - t))\}
- Ke^{-\tau(T-t)}(H/S)^{2\lambda}\{N(d_2(H^2, SK, T - t)) - N(d_2(H, S, T - t))\},
\]

where \( d_1(x, y, \tau) \) and \( d_2(x, y, \tau) \) are defined in (16).

An American down-and-in put option gives its holder the additional flexibility of early exercise. Ait-Sahalia and Lai (2000) have shown that its price can be decomposed as the European option price \( p_i(S, t) \) plus an integral that involves a standard American put:

\[
P_i(S, t) = p_i(S, t) + \int_t^T h(\sigma^2(\tau - t); z)\pi(T - \tau, H)d\tau,
\]  
(24)

where \( z = \ln(S/K) \), \( \pi(T - \tau, H) \) is the early exercise premium of a standard American put with maturity \( T - \tau \), strike price \( K \) and initial stock price \( H \), and

\[
g(u; z) = u^{-3/2}\gamma - z|n((\gamma - z - \lambda u)/\sqrt{u}) \quad (g(0; z) = 0) \]  
(25)

is the density function of the first time when the Wiener process \( z + \lambda t + W_t \) hits the barrier \( \gamma = \ln(H/K) \). Using the change of variables \( u = \sigma^2(\tau - t) \), the integral in (24) can be expressed as

\[
\int_0^{\sigma^2(T-t)} g(u; z)\pi(T - t - \sigma^{-2}u, H)du.
\]  
(26)

We now describe a fast and accurate approximation to (26). First we use Ju's (1998) method to approximate the early exercise premium \( \pi \) of a standard American option. Multiplying it by \( g(u; z) \) then gives the value of the integrand. For fast valuation, the integral has to be approximated by a sum of a few terms. To do this we make use of the special
feature of the density function (25), which is skewed towards and concentrated around its maximum at $u^* = (-3 + [9 + 4\lambda^2(\gamma - z)^2])/2(2\lambda^2)$. In the case $u^* \geq \sigma^2(T - t)$, evaluate (26) by the trapezoidal rule that divides $[0, \sigma^2(T - t)]$ into $n$ evenly spaced subintervals and evaluates the integrand at the endpoints of the subintervals.

Suppose $u^* < \sigma^2(T - t)$. Take $0 < p < 1/2$ and let $u' = (1 - p)u^*$, $u'' = u^* + p\{\sigma^2(T - t) - u^*\}$. Decompose the integral (26) as $\int_0^u u' + \int_{u'}^{u^*} u'' + \int_{u^*}^{\sigma^2(T-t)} u''$. Divide $[u', u^*]$ into 2 subintervals of equal width $pu^*/2$ and compute $\int_{u'}^{u^*}$ by Simpson's rule that involves evaluation of the integrand at $u', u^*$ and $(u' + u^*)/2$. Similarly divide $[u^*, u'']$ into 2 subintervals of equal width and compute $\int_{u^*}^{u''}$ by using Simpson's rule. The integrals $\int_0^{u'}$ and $\int_{u'}^{\sigma^2(T-t)}$ are computed by using the trapezoidal rule with $n - 4$ subintervals for $[0, u'] \cup [u'', \sigma^2(T - t)]$, keeping the ratio of the number of subintervals for $[0, u']$ to that for $[u'', \sigma^2(T - t)]$ roughly equal to $u'/\{\sigma^2(T - t) - u''\}$. We may not be able to do this if $u'$ or $\sigma^2(T - t) - u''$ is too small, in which case we allocate one interval to $[0, u']$ (or to $[u'', \sigma^2(T - t)]$) and $n - 5$ subintervals to the other interval.

Table 4 illustrates the accuracy of this procedure with $n = 8$ and $p = .2$. The benchmark values were obtained by applying Simpson's rule with subinterval width .0025 to compute the integral (26) and using a binomial tree with 10,000 time steps to compute the standard American option price whose difference from the standard European option price given by the Black-Scholes formula yields the early exercise premium $\pi(T - \sigma^{-2}u, H)$; here $t = 0$. The table shows that our fast procedure that uses 8 unevenly spaced time points together with Ju's approximation to standard American options gives results within penny accuracy. To confirm that our choice of time points to perform numerical integration adapts well to the density function (25), we also used the same method to compute the integral $\int_0^{\sigma^2T} g(u; z)du$, which is the probability of hitting the barrier during the life of the contract. Table 4 shows that our fast numerical integration scheme also gives reasonably accurate results for this down-crossing probability.

INSERT TABLE 4 ABOUT HERE

We have so far considered American up-and-out and down-and-in put options. The same methods apply to calls, and there is a "put-call symmetry" between an up-and-out put and a down-and-out call, as shown by Gao et al. (2000). In the case $H \leq K$ for up-and-out
puts, Gao et al. (2000) have shown that it is optimal to exercise the option immediately when there is no dividend, while Ait-Sahalia and Lai (2000) have shown that if the dividend rate \( \mu \) is larger than \( rK/H \) then there is an early exercise boundary for which a modified version of the decomposition formula (8) still holds, although (8) itself fails to hold in the case \( H \leq K \) (or equivalently \( \gamma \leq 0 \)). The hybrid method of Section 1 is still applicable to compute the early exercise boundary in this case.

6 Concluding Remarks

In Section 1 we have introduced the hybrid method to compute for American knock-out options both the prices and the exercise boundaries with known convergence properties, giving benchmark values for the prices that are accurate up to an \( O(n^{-1}) \) error and for the exercise boundary that is accurate up to an \( o(n^{-1/2}) \) error. It is considerably faster than previous modifications of tree methods for American barrier options (such as Ritchken's method), whose convergence properties have not been established.

By using the hybrid method and the reparameterization (2)-(3) to perform extensive computations of the exercise boundaries over a wide range of maturities, interest rates, dividend rates, volatilities and barrier/strike prices, we have found that the exercise boundaries of American knock-out options can be well approximated by continuous piecewise exponential functions that use a small number of pieces, or equivalently for the transformed coordinates (3), by linear splines with a few knots. Note in this connection that the time horizon \( \sigma^2 T \) under the transformation (3) is only a small fraction of the maturity \( T \). These findings have led us to two approximations of the option prices and hedge parameters, whose accuracy and speed relative to the benchmark values derived from the hybrid method and also to other competing methods have been assessed in a large sample study.

The first approximation, considered in Section 2, uses a linear spline with a few unevenly spaced knots to approximate the exercise boundary \( \tilde{x}(\cdot) \) in the coordinate system (3). The uneven spacing of the knots has the advantage of following the actual boundary more closely. A simple rule is devised for knot placement in the 4-knot and 8-knot schemes of Sections 2 and 4. The slope of each linear piece of the spline can be found by solving a
one-dimensional nonlinear equation, as the intercept is fixed by continuity of the spline. An important advantage of this piecewise linear approximation (or piecewise exponential approximation in the original coordinates), first noted by Ju (1999), is that it leads to a closed-form expression for the integral defining the early exercise premium; moreover, there are similar closed-form expressions for the hedge parameters, as shown in Section 2.

The second approximation, considered in Section 3, removes the continuity requirement in the piecewise exponential function that approximates $B_t$. This has the advantage that each piece is approximated separately, without relying on the previously fitted pieces. Of course the previously fitted pieces still have an impact on the accuracy of the current piece since they appear in the integral defining the early exercise premium, but they are not used directly as parameter(s) (such as the intercept of $\tilde{\xi}(\cdot)$ in the spline approximation) of the current piece. It is, therefore, less important to choose the endpoints of each piece to match closely the actual boundary, and the simple choice of evenly spaced endpoints proposed by Ju (1998) has been found to perform well. The disadvantage of this flexibility is that we now have two (instead of one) parameters to determine for each piece of $B_t$, resulting in a system of two nonlinear equations that require good starting values. Ju (1998) has developed an elaborate scheme to initialize Newton’s method for solving these equations and has found that it works well in the case of standard American options. In the course of generalizing it to barrier options, we have also extended the classical method of MacMillan (1986) and Barone-Adesi and Whaley (1987).

Our numerical study has demonstrated the sensitivity of two-dimensional root-finding algorithms to starting values. Using other equally plausible starting values (such as that of Gao et al. (2000)) to solve these simultaneous nonlinear equations can result in singular or ill-conditioned Jacobian matrices in the Newton-type iterations. In contrast, the spline approximations to $\tilde{\xi}(\cdot)$ in Section 2 only involve the slope parameter of each piece, which can be determined by a numerically stable one-dimensional search that, unlike the two-dimensional search in Ju (1998) and Gao et al. (2000), does not require computing the first and second partial derivatives of the right hand side of (15) with respect to the critical price.

We have also considered in Section 5 American knock-in options, for which the integral
defining the early exercise premium is very different from that of an American knock-out option. By making use of the special feature of the first passage density function (25), we have developed a fast and accurate valuation method for these options.
REFERENCES


Table 1: Summary of price deviation (in cents) from hybrid benchmark for 1200 randomly generated puts in no-dividend case

<table>
<thead>
<tr>
<th></th>
<th>R10000</th>
<th>R800</th>
<th>SP4</th>
<th>SP8</th>
<th>MBW</th>
<th>Ju</th>
<th>GSTEP</th>
<th>GEXP</th>
</tr>
</thead>
<tbody>
<tr>
<td>Max abs. error (cents)</td>
<td>0.46</td>
<td>1.15</td>
<td>4.84</td>
<td>2.21</td>
<td>27.63</td>
<td>0.81</td>
<td>6.38</td>
<td>6.25*</td>
</tr>
<tr>
<td>Max rel. error (%)</td>
<td>0.12</td>
<td>0.28</td>
<td>2.65</td>
<td>0.71</td>
<td>15.21</td>
<td>0.59</td>
<td>0.43</td>
<td>0.53*</td>
</tr>
<tr>
<td>RMSE (cents)</td>
<td>0.07</td>
<td>0.13</td>
<td>0.66</td>
<td>0.18</td>
<td>6.18</td>
<td>0.16</td>
<td>1.56</td>
<td>0.48*</td>
</tr>
<tr>
<td>RMSRE (%)</td>
<td>0.01</td>
<td>0.03</td>
<td>0.13</td>
<td>0.07</td>
<td>1.99</td>
<td>0.04</td>
<td>0.13</td>
<td>0.05*</td>
</tr>
<tr>
<td># abs. err. &gt; 1 cent</td>
<td>0</td>
<td>1</td>
<td>143</td>
<td>6</td>
<td>882</td>
<td>0</td>
<td>409</td>
<td>21*</td>
</tr>
<tr>
<td># rel. err. &gt; 1%</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>249</td>
<td>0</td>
<td>0</td>
<td>0*</td>
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<tr>
<td>CPU time (sec)</td>
<td>15799</td>
<td>118</td>
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<td>6.81</td>
<td>0.68</td>
<td>7.30</td>
<td>1.60</td>
<td>37.67</td>
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</tbody>
</table>

Starting with R10000, the columns represent, respectively, from left to right the method of Ritchken with 10,000 steps and 800 steps, spline approximations with 4 and 8 pieces, our extensions of MBW (McMillan, Barone-Adesi and Whaley) and of Ju (3-piece exponential approximation), and the piecewise constant and exponential approximations of Gao et al. The asterisk * indicates that the entries apply to only 849 out of the original 1200 puts; see text. The CPU time for generating the benchmark option prices by the hybrid method was 504.5 seconds, much faster than R10000.
Table 2: Option prices for American up-and-out puts ($r = 0.04$, $\sigma = 0.2$, $T = 3$, $K = 100$, $\mu = 0$)

<table>
<thead>
<tr>
<th>S</th>
<th>H</th>
<th>Benchmark</th>
<th>R10000</th>
<th>R8000</th>
<th>SP4</th>
<th>SP8</th>
<th>MBW</th>
<th>Ju</th>
<th>G^{STEP}</th>
<th>G^{EXP}</th>
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</thead>
<tbody>
<tr>
<td>80.00</td>
<td>101.00</td>
<td>20.0000</td>
<td>20.0000</td>
<td>20.0000</td>
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<tr>
<td>95.00</td>
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Benchmark values are generated by the hybrid method described in Section 1 with $\delta = 10^{-4}$. Starting with R10000, the columns represent, respectively, from left to right the method of Ritchken with 10,000 steps and 800 steps, spline approximations with 4 and 8 pieces, our extensions of MBW (MacMillan, Barone-Adesi and Whaley) and of Ju (3-piece exponential approximation), and the piecewise constant and exponential approximations of Gao et al. For the entry NA, see text.
Table 3: Hedge ratios for American up-and-out puts ($r = 0.04, \sigma = 0.2, T = 3, K = 100, \mu = 0$)

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Benchmark values are generated by the hybrid method described in Section 1 with $\delta = 10^{-4}$. Starting with R10000, the columns represent, respectively, from left to right the method of Ritchken with 10,000 steps and 800 steps, spline approximations with 4 and 8 pieces, our extensions of Ju (3-piece exponential approximation), and the piecewise constant and exponential approximations of Gao et al. For the entry NA, see text.
Table 4: Down-and-in put option prices

Case I: $r = 0.06$, $\mu = 0.09$, $\sigma = 0.2$, $T = 3$, $K = 100$

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FIGURE 1. Optimal Stopping Boundaries: Up-and-Out Put

\[ \alpha = 0 \quad \gamma = 1.2 \]

\[ \rho = 2.0 \quad \rho = 1.5 \quad \rho = 1.0 \quad \rho = 0.5 \quad \rho = 0.4 \quad \rho = 0.3 \quad \rho = 0.2 \quad \rho = 0.1 \]
FIGURE 2. Optimal Stopping Boundaries: Up-and-Out Put

\[ \rho = 1.5 \]
\[ \alpha = 1/3 \]
\[ \gamma = 1.1 \]