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Technical Report No. 2001-16
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This research was supported in part by National Science Foundation
grant DMS-00-72523

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Abstract

This paper presents an efficient method to compute the values and early exercise boundaries of American lookback options. A key idea underlying the method is the reduction of option valuation to a single optimal stopping problem for standard Brownian motion and an associated path-dependent functional, indexed by one parameter in the absence of dividends and by two parameters in the presence of a dividend rate. Numerical results obtained by this method show that, after a space-time transformation, the stopping boundaries are well approximated by certain piecewise linear functions with a few pieces, leading to fast and accurate approximations for American lookback option values. A decomposition formula for American lookback options is derived and applied not only to the development of these approximations but also to the asymptotic analysis of the early exercise boundary near the expiration date.

KEY WORDS: American options, lookback options, optimal stopping problem, space-time transformation.

*Preliminary versions of this paper were presented at the PACAP/FMA Conference (July 1999) and the World Congress of Nonlinear Analysts (July 2000). The authors thank the conference participants for their stimulating comments. The first author gratefully acknowledges research support by the National Science Foundation and the Center for Advanced Study in the Behavioral Sciences.
1. INTRODUCTION

There are two aspects to an American option problem, whose complete solution should provide both the option value and an optimal exercise strategy. In some cases, analytic solutions are available: for example, McKean (1965), Shepp and Shiryaev (1993), and Duffie and Harrison (1993) give formulas for perpetual contracts, while Merton (1973) establish that standard American calls written on non-dividend paying stocks reduce to European calls. More often, a numerical approach is needed to price an American option, using binomial trees, finite difference methods, Monte Carlo simulation and analytic approximations. The binomial tree approach, introduced by Cox, Ross, and Rubinstein (1979), approximates the underlying geometric Brownian motion price process by a discrete-time process on a tree to which a dynamic programming algorithm can be applied to solve the valuation problem. For lookback and Asian options, Hull and White (1993) and Barraquand and Pudet (1996) propose modifications of the conventional binomial tree to compute option prices. As an alternative, finite difference methods solve the free-boundary partial differential equation defining the function that expresses the option value in terms of time to maturity and the stock price; see Brennan and Schwartz (1977). The use of Monte Carlo simulation for American option valuation has been more recent; see Boyle, Broadie, and Glasserman (1997) for a review. To reduce the computational task, various analytic approximations have been developed; see Ingersoll (1998), Ju (1998), and Ait-Sahalia and Lai (1999) for recent reviews of these approximations in the case of American vanilla options, and Gao, Huang, and Subrahmanyam (2000) in the case of American barrier options.

In this paper, we propose a new approach to compute both the price and the optimal exercise boundary, which is a two-dimensional surface in $\mathbb{R}^3$, of an American lookback option. A key idea in this approach is to use a space-time transformation that reduces the valuation of a family of American lookback options to a single canonical optimal stopping problem for standard Brownian motion, indexed by one (resp. two) parameter(s) in the absence (resp. presence) of dividends. Denoting the volatility by $\sigma$, calendar time is scaled by $\sigma^2$ when transformed into “canonical” time so that for values of $\sigma$ (between 0.1 and 0.4) that commonly arise, the canonical time horizon $\sigma^2 T$ is only a small fraction of $T$. The transformation also removes the dependence on a prescribed root node when a Bernoulli walk is used to approximate Brownian motion. We can thereby compute not only the option price but also the entire early exercise boundary by backward induction.

Another contribution of the paper is a decomposition formula for an American lookback option as the sum of the corresponding European value and an “early exercise premium” that is given
by a double integral whose integrand is an explicit function of the early exercise boundary. We first make use of this result to derive the asymptotic behavior of the early exercise boundary near expiration. We then use it to develop fast approximate methods to compute option prices, similar to those developed by Ju (1998) and Ait-Sahalia and Lai (1999) for American vanilla options. Specifically, extensive computations of early exercise boundaries using the Bernoulli walk approach show that these boundaries in the “canonical” coordinates are well approximated by piecewise linear functions with a few pieces, which in turn provide fast and accurate approximations to the early exercise premium. It is, therefore, almost as easy to compute these approximations to American lookback prices as their vanilla counterparts in Ju (1998) and Ait-Sahalia and Lai (1999).

The paper is organized as follows. Section 2 describes the change of variables leading to the canonical optimal stopping problem. Section 3 describes the Bernoulli walk approach to compute the optimal stopping boundary and presents numerical results obtained by applying this approach to American lookback puts on non-dividend paying stocks, along with certain analytical properties of the boundaries. In Section 4, we derive a decomposition formula for an American lookback option as the sum of the corresponding European value and an early exercise premium. Two fast approximate methods for computing the option prices are presented in Section 5. One method is based on a tabulation approach while the other is based on an integral equation defining the early exercise boundary. In Section 6, we provide numerical comparisons of our approximation with other methods. Section 7 summarizes and concludes the paper.

2. A CANONICAL OPTIMAL STOPPING PROBLEM

In the standard Black-Scholes environment, the price of a security (under the risk-neutral measure $Q$) is represented by a geometric Brownian motion

$$S_t = S_0 \exp \{(r - q - \sigma^2/2)t + \sigma B_t\},$$

(2.1)

where $S_0$ is the initial security price and $\{B_t, \ t \geq 0\}$ is a standard Brownian motion with $B_0 = 0$. Here, $r$ is the riskless rate of return, $q$ stands for the dividend rate paid by the underlying security and $\sigma$ is the standard deviation of the security's return (i.e., the volatility). As shown by Karatzas (1988), the American lookback option price $P(t, S_t, S^*_t)$ at time $t \in [0, T]$ that entails no arbitrage opportunities before exercise is given by

$$P(t, S, S^*) = \sup_{\tau \in T_{t,T}} \mathbb{E} \left\{ e^{-r(T-t)} f(S_\tau, S^*_\tau) \middle| S_t = S, S^*_t = S^* \right\},$$

(2.2)
where $\mathcal{T}_{a,b}$ is the set of stopping times taking values between $a$ and $b(>a)$, the payoff function for the put (resp. call) with strike price $K$ is given by $f(S,S^*) = f(S^*) = (K - S^*)^+$ (resp. $(S^* - K)^+$), and $S^*_r = \min_{u \in [0,r]} S_u$ (resp. $\max_{u \in [0,r]} S_u$).

**Proposition 2.1.** The value function $P(t,S,S^*)$ in (2.2) is continuous. Moreover, it is non-increasing in $(S,S^*)$ for the put and nondecreasing in $(S,S^*)$ for the call.

**Proof.** First consider the put and define the flow

$$S^*_u(t,S,S^*) = \min \left\{ S^*, \min_{t \leq r \leq u} S^e^{(r-q-\sigma^2/2)(r-t)+\sigma B_{r-t}} \right\}, \quad u \geq t, \quad S \geq S^*.$$

Following Jailett, Lamberton, and Lapeyre (1990, proof of Proposition 2.2),

$$|P(t_2,S_2,S_2^*) - P(t_1,S_1,S_1^*)|$$

$$\leq E \left[ \sup_{t_2 \leq u \leq T} \left| e^{-r(u-t_2)} (K - S^*_u(t_2,S_2,S_2^*))^+ - e^{-r(u-t_1)} (K - S^*_u(t_1,S_1,S_1^*))^+ \right| \right]$$

$$+ E \left[ \sup_{t_1 \leq u \leq t_2} \left| e^{-r(u-t_1)} (K - S^*_u(t_1,S_1,S_1^*))^+ - e^{-r(u-t_1)} (K - S^*_u(t_1,S_1,S_1^*))^+ \right| \right].$$

Note that the last two terms are uniformly integrable, since $(K-\cdot)^+$ is bounded above by $K$.

Continuity and monotonicity of the value function $P$ follow from continuity of the flow and from monotonicity of $(K - S^*_u(t,S,S^*))^+$ in $S$ and $S^*$. For the call, the flow $S^*_u(t,S,S^*)$ is associated with the maximum asset price in the interval $[t,u]$. Accordingly, uniform integrability of both terms can be derived from the following inequalities with $\nu > 1$:

$$E[(\text{first term})^\nu] \leq E(S^*_T(t_2,S_2,S_2^*))^\nu + E(S^*_T(t_1,S_1,S_1^*))^\nu < C_1,$$

$$E[(\text{second term})^\nu] \leq 2E(S^*_T(t_1,S_1,S_1^*))^\nu < C_2,$$

for all $0 \leq t_1 < t_2 \leq T$, where $C_1$ and $C_2$ are universal constants independent of $t_1$ and $t_2$. \qed

We can reduce the number of parameters $K$, $T$, $r$, $q$, $\sigma$ in the optimal stopping problem (2.2) by certain space-time transformations. First, by dividing all prices by $K$, we can reduce the problem to the case $K = 1$ which we shall assume throughout the sequel. Next introduce the change of variables

$$s = \sigma^2(t-T), \quad z = \log(S/K) - (\rho - \gamma \rho - 1/2)s, \quad z^* = \log(S^*/K),$$

(2.3)

$$p(s,z,z^*) = e^{r(T-t(s))} P(t(s),S(s,z),S^*(z^*)) / K,$$

where $\rho = r/\sigma^2$ and $\gamma = q/r$. Then, using the Markovian structure of the problem and the scaling property of Brownian motion, we have the following Brownian optimal stopping problem for American lookback options.
Proposition 2.2. Let \( \rho = r/\sigma^2 \) and \( \gamma = q/r \). The value of an American lookback option can be obtained from the value function

\[
p(s, z, z^*) = \sup_{T \in T_{s,0}} E\{g(\tau, W^*_\tau(s, z^*))\}, \quad \text{with } W_s = z, \quad W^*_s = z^*, \quad s \in [-\sigma^2 T, 0],
\]

where \( \{W_u, s \leq u \leq 0\} \) is a standard Brownian motion, \( g(u, w) = e^{-\rho u}(1 - e^u)^+ \) for a put, \( g(u, w) = e^{-\rho u}(e^w - 1)^+ \) for a call, and

\[
W^*_u(s, z^*) = \min \left\{ z^*, \min_{s \leq u \leq t} (\rho - \gamma \rho - 1/2)t + W_t \right\}
\]

for the put option, with \( \min \) in (2.5) replaced by \( \max \) for the call option.

Note that for problem (2.4) the horizon is always 0. By virtue of (2.5), for a given set of parameters \( (\rho, \gamma) \), only one numerical program need be implemented for all expiration dates \( T \). Also, from (2.3), \( s = -\sigma^2 T \) at time \( t = 0 \). Therefore, for values of \( \sigma \) (between 0.1 and 0.4) and \( T \) (between 0.08 and 1.5) that commonly arise in practice, \( \sigma^2 T \) is typically small (not exceeding 0.3). Note that \( p \) is defined on \( \mathcal{A} = \{(s, x, y) \in (-\infty, 0] \times \mathbb{R}^2 : y \leq x + (\rho - \gamma \rho - 1/2)s\} \) for the put, while the domain becomes \( \mathcal{A} = \{(s, x, y) \in (-\infty, 0] \times \mathbb{R}^2 : y \geq x + (\rho - \gamma \rho - 1/2)s\} \) for the call. In view of Proposition 2.1 and the transformations (2.3), the value function \( p(s, z, z^*) \) given by (2.4) is continuous. Moreover, it is nonincreasing in \( (z, z^*) \) for the put and nondecreasing in \( (z, z^*) \) for the call. Thus, the optimal stopping boundary \( \overline{z}(s, z^*) \) for the put is the smallest \( z \geq z^* - (\rho - \gamma \rho - 1/2)s \) such that \( p(s, z, z^*) = g(s, z^*) \), with \( s \leq 0 \) and \( z^* < 0 \) (since it is not optimal to stop when \( e^{z^*} \geq 1 \), or equivalently, when \( z^* \geq 0 \)). Similarly the boundary for the call is the largest \( z \leq z^* - (\rho - \gamma \rho - 1/2)s \) such that \( p(s, z, z^*) = g(s, z^*) \), with \( s \leq 0 \) and \( z^* > 0 \).

The stopping region \( \mathcal{S} \) is the collection of all points in \( \mathcal{A} \) such that \( z \geq \overline{z}(s, z^*) \) for the put and \( z \leq \overline{z}(s, z^*) \) for the call. The continuation region is \( \mathcal{K} := \mathcal{A} \setminus \mathcal{S} \). We can retrieve the solution \((P, \overline{S})\) to the original pricing problem (2.2) by mapping back as follows: \( P(t, S, S^*) = Ke^{pS}p(s, z, z^*) \) and \( \overline{S}(t, S^*) = Ke^{\overline{z}(s, z^*) + (\rho - \gamma \rho - 1/2)s} \).

Since \( p(s, z, z^*) \) and \( g(s, z^*) \) are continuous functions, it follows that \( \overline{z}(s, z^*) \) is continuous. Moreover, by applying backward induction to the corresponding discrete-time optimal stopping problem for a normal random walk with increments having mean 0 and variance \( \delta \) (with time \( s \) restricted to the set \( \{0, -\delta, -2\delta, \ldots\} \)) and taking the limit as \( \delta \to 0 \), it can be shown that \( \overline{z}(s, z^*) \) is nondecreasing in \( z^* \) and approaches \( \infty \) as \( z^* \to 0 \) in the case of a put, and that \( \overline{z}(s, z^*) \) is nonincreasing in \( z^* \) and approaches \( -\infty \) as \( z^* \to 0 \) in the case of a call. Moreover, \( \overline{z}(s, z^*) - z^* \to -(\rho - \gamma \rho - 1/2)s \) as \( z^* \to -\infty \) for a put, or as \( z^* \to \infty \) for a call. Furthermore, \( \overline{z}(0, z^*) = z^* \).

A schematic illustration of the stopping boundary is given in Figure 1. Note that unlike standard
American options for which \( \bar{z}(0) \) depends on \( \gamma \) (and in particular \( \bar{z}(0) = \infty \) for calls with \( \gamma = 0 \) that are exercised only at maturity), American lookback puts and calls always satisfy \( \bar{z}(0, z^*) = z^* \), irrespective of the value of \( \gamma \).

**INSERT FIGURE 1 ABOUT HERE**

### 3. A NUMERICAL METHOD USING BERNOUlli WALKS

In principle, the classical binomial tree method can be directly extended to deal with path-dependent options; see, for example, Hull and White (1993). A binomial tree is grown, with root node \( S_0 \) at time 0, to approximate the geometric Brownian motion (2.1). At each node a set of possible (or representative) values of the path-dependent quantity \( S_t^j \) is included so that (2.2) can be computed via a backward induction algorithm to solve the corresponding optimal stopping problem for the approximating binomial tree. The change of variables (2.3) transforms (2.2) into the optimal stopping problem (2.4) for standard Brownian motion (with \( E(dW_s) = 0 \) and \( \text{Var}(dW_s) = ds \)). In view of the functional central limit theorem, a standard Brownian motion can be approximated by a symmetric Bernoulli random walk, so (2.4) can likewise be computed via the following backward induction algorithm applied to the approximating random walk.

Specifically, choose a small \( \delta > 0 \) and discretize time and space as follows. Let \( s_0 = 0 \) and \( s_j = s_{j-1} - \delta \) for \( j \geq 1 \). Let

\[
Z_\delta = \left\{ n\sqrt{\delta} : n \text{ is an integer} \right\} = \left\{ 0, \pm \sqrt{\delta}, \pm 2\sqrt{\delta}, \ldots \right\}.
\]

We will approximate Brownian motion by a symmetric Bernoulli random walk with time increment \( \delta \) and space increment \( X_i \), where the \( X_i \) are independent Bernoulli variables with \( \Pr(X_i = \sqrt{\delta}) = 1/2 = \Pr(X_i = -\sqrt{\delta}) \). Then, for the lookback put option, we deduce from (2.5) that the states \((z, z^*)\) evolve from time \( s_i \) to \( s_{i-1} = s_i + \delta \) as follows:

\[
\begin{align*}
z &\rightarrow z_{i\pm} = z \pm \sqrt{\delta}, \\
z^* &\rightarrow z^*_{i\pm} = \min \left\{ z^*, (\rho - \gamma \rho - 1/2)s_{i-1} + (z \pm \sqrt{\delta}) \right\}.
\end{align*}
\]

For the lookback call option, we replace \( \min \) in (3.1) with \( \max \). Therefore, the optimal stopping problem (2.4) can be approximated by the backward induction

\[
p(s_i, z, z^*) = \max \{ g(s_i, z^*), [p(s_{i-1}, z^*_+ + z^*_-) + p(s_{i-1}, z^*_-, z^*_+)]/2 \}. \quad (3.2)
\]

Each point \((z, z^*) \in Z_\delta \times Z_\delta \) with \( z \geq z^* - (\rho - \gamma \rho - 1/2)s_i \) (admissible region) can be determined to be a stopping or continuation point at time \( s_i \) according to whether \( p(s_i, z, z^*) = g(s_i, z^*) \) or
\( p(s_i, z, z^*) > g(s_i, z^*) \). We use a "continuity correction," proposed by Chernoff and Petkau (1986), to obtain the optimal stopping boundary for Brownian motion from the discrete-time and discrete-state boundary associated with (3.2). Let

\[
\begin{align*}
    z_0^0(s_i, z^*) &= \bar{z}_d(s_i, z^*) - \sqrt{\delta}, \\
    z_0^1(s_i, z^*) &= \bar{z}_d(s_i, z^*) - 2\sqrt{\delta}, \\
    z_1^0(s_i, z^*) &= \bar{z}_d(s_i, z^*) + \sqrt{\delta}, \\
    z_1^1(s_i, z^*) &= \bar{z}_d(s_i, z^*) + 2\sqrt{\delta},
\end{align*}
\]

for the put,

\[
D_j(s_i, z^*) = g(s_i, z^*) - p(s_i, z_j^j(s_i, z^*), z^*), \quad \text{for } j = 0, 1,
\]

where \( \bar{z}_d(s_i, z^*) \) is the last (i.e, smallest for put and largest for call) point in \( Z_\delta \) for which \( p(s_i, z, z^*) = g(s_i, z^*) \). The early exercise boundary at \( (s_i, z^*) \) can be computed from \( \bar{z}_d(s_i, z^*) \) via the Chernoff-Petkau continuity correction as follows:

\[
\bar{z}(s_i, z^*) = z_0^0(s_i, z^*) \pm \sqrt{\delta} |D_1(s_i, z^*)/(2D_1(s_i, z^*) - 4D_0(s_i, z^*))|,
\]

where the + and − signs apply to the put and the call, respectively. It can be shown under certain conditions that the continuity correction can approximate the continuous-time boundary with \( o(\sqrt{\delta}) \) error; see Lai, Yao, and Ait-Sahalia (2001). One such condition is boundedness of the derivative \( (\partial/\partial s)\bar{z}(s, z^*) \) in some neighborhood of \( s_i \). Numerical results show that the derivative is large only in a very small neighborhood of \( s = 0 \), so the Chernoff-Petkau correction can be applied when \( s_i \leq -0.005 \). For \( 0 > s > -0.005 \), \( \bar{z}(s, z^*) \) is close to \( z^* \) and the uncorrected \( \bar{z}_d \) typically suffices to approximate this small portion of the early exercise boundary, although a more refined approximation via an integral equation approach is available; see the last paragraph of Section 5.

As an illustration, the backward recursion (3.2) is implemented on non-dividend paying puts (so \( \gamma = 0 \)) for \( \rho = 0.1, 0.25, 0.5, 1.0, 2.0 \) with \( \delta = 10^{-3} \). Our choice of \( \rho = r/\sigma^2 \) is based on the ranges of interest rates 0.01 \( \leq r \leq 0.1 \) and volatilities 0.1 \( \leq \sigma \leq 0.4 \), commonly encountered in practice. The optimal stopping boundary for \( \rho = 0.5 \) is shown in Figure 2. A side-by-side cross-sectional plot of the boundaries associated with the five values of \( \rho \) we examine is presented in Figure 3. In each case, we see that, for a fixed value of \( z^* \), the optimal stopping boundary in the canonical scale can be well approximated by linear splines with a few knots. We tabulate one such approximation for \( \rho = 0.5 \) in Table 1. This approximation uses knots at eight canonical time points \( (s = -0.3, -0.2, -0.15, -0.1, -0.05, -0.05, -0.01, -0.005) \) for six levels of the path-dependent variable \( (z^* = -2.0, -1.0, -0.5, -0.25, -0.1, -0.05) \).

INSERT FIGURES 2, 3 AND TABLE 1 ABOUT HERE

Continuity corrections can also be applied to improve the accuracy of option prices computed by the binomial tree or Bernoulli walk method. In Section 6 we describe two such continuity
corrections and use them to compute accurate benchmark values for comparison with two fast approximations developed in Section 5.

4. A DECOMPOSITION FORMULA

We first derive an integral representation of the American lookback put value and use it to deduce properties of the exercise boundary near expiration. We will indicate, at the end of this section, corresponding results for the lookback call option.

4.1 Decomposition of an American Lookback Put

We first give an explicit formula for the value of a European lookback put option in terms of the canonical parameters $\rho = \tau / \sigma^2$ and $\gamma = q / \rho$. We then express an American lookback put as the sum of a European lookback put and an early exercise premium which can be expressed as an integral. An explicit expression of the integrand is then derived from a formula for the conditional distribution of the state variables in Lemma 4.3.

**Proposition 4.1.** If $\gamma \neq 1$, the value of a European lookback put option is given by

\begin{equation}
P_{E}(t, S, S^*) = e^{-\tau \rho} \left[ \frac{S}{2 \rho (1 - \gamma)} \left( \frac{S}{K^*} \right)^{-2 \rho (1 - \gamma)} N(-\lambda_3) - S e^{-q \tau} \left[ 1 + \frac{1}{2 \rho (1 - \gamma)} \right] N(-\lambda_1) \right],
\end{equation}

where $K^* = \min\{K, S^*\}$, $\tau = T - t$,

\[
\lambda_1 = \frac{\log(S/K^*) + (\tau - q + \sigma^2/2) \tau}{\sigma \sqrt{\tau}}, \quad \lambda_2 = \lambda_1 - \sigma \sqrt{\tau} \quad \text{and} \quad \lambda_3 = \lambda_1 - \frac{2(\tau - q)}{\sigma} \sqrt{\tau}.
\]

If $\gamma = 1$, then $\lambda_1 = \lambda_3 = (\sigma \sqrt{\tau})^{-1} \log(S/K^*) + \sigma \sqrt{\tau}/2$ and the European value is given by

\begin{equation}
P_{E}(t, S, S^*) = e^{-\tau \rho} \left[ K - K^* N(\lambda_2) - S N(-\lambda_1) - S e^{-\tau \rho} \sigma \sqrt{\tau} [\lambda_3 N(-\lambda_3) - n(-\lambda_3)] \right].
\end{equation}

**Proof.** The European value given in (4.1a) can be obtained from the dividend-free pricing formulas (17)–(18) of Conze and Viswanathan (1991) as follows: replace $T$ by $\tau$ and replace every occurrence of $r$ by $r - q$, but still use $e^{-\tau \rho}$ as the discount factor. The remaining case $\gamma = 1$ can be obtained via direct evaluation of the usual risk-neutral expectation or as the limiting case of (4.1a) as $\gamma \to 1$. Taking the latter approach, we examine the Taylor series approximation of the third and
fifth terms when $\gamma$ is close to 1. Expanding up to $1 - \gamma$ yields

$$
\frac{Se^{-\tau}}{2\rho(1 - \gamma)} \left( \frac{S}{K^*} \right)^{-2\rho(1-\gamma)} N(-\lambda_3) - \frac{Se^{-\tau}}{2\rho(1 - \gamma)} N(-\lambda_1)
$$

$$
\pm \frac{Se^{-\tau}}{2\rho(1 - \gamma)} \left\{ \left[ 1 - 2\rho(1 - \gamma) \log \left( \frac{S}{K^*} \right) \right] N(-\lambda^*) + \frac{r(1 - \gamma)\tau}{\sigma \sqrt{\tau}} n(-\lambda^*) \right\}
$$

$$
- \left[ 1 + r(1 - \gamma)\tau \right] \left[ N(-\lambda^*) - \frac{r(1 - \gamma)\tau}{\sigma \sqrt{\tau}} n(-\lambda^*) \right]
$$

$$
= Se^{-\tau} \left\{ - \left[ \log \left( \frac{S}{K^*} \right) + \frac{\sigma^2 \tau}{2} \right] N(-\lambda^*) + \sigma \sqrt{\tau} n(-\lambda^*) \right\},
$$

where $\lambda^* = (\sigma \sqrt{\tau})^{-1} \log (S/K^*) + \sigma \sqrt{\tau}/2$. Equation (4.1b) follows by letting $\gamma \to 1$. \hfill \Box

**Proposition 4.2.** The value of an American lookback put has the decomposition

$$
P(t, S, S^*) = P_E(t, S, S^*) + r \int_t^T e^{-r(\tau - t)} E \left[ (K - S_t^*) 1_{\{S_t^* \notin C_t \}} \right] dt,
$$

where $C_t$ denotes the continuation region at time $\tau$.

**Proof.** As shown by Wilmott, Dewynne, and Howison (1993, pages 205–206), $P$ satisfies the following PDE in the continuation region $C$:

$$
LP := -rP + \frac{\partial P}{\partial t} + (r - q)S \frac{\partial P}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} = 0,
$$

subject to

$$
\lim_{S \uparrow S(t, S^*)} P(t, S, S^*) = K - S^* \quad \text{and} \quad \lim_{S \uparrow S(t, S^*)} \frac{\partial P(t, S, S^*)}{\partial S} = 0.
$$

The second condition in (4.3) corresponds to the condition of smooth fit at the optimal stopping boundary. This smooth fit condition and the fact that $\{S^* \geq K\}$ belongs to the continuation region imply that $P$ is continuously differentiable at all $(t, S, S^*)$ with $0 \leq t < T$. Moreover, $P$ is twice continuously differentiable in $S$ except along the curve $S = \overline{S}(t, S^*)$. Therefore, application of the generalized Ito’s formula (cf. Krylov (1980)) to $e^{-rt}P$ yields

$$
e^{-r(T-t)} (K - S_{T_t}^*)^+ = P(t, S_t, S_t^*) + \int_t^T e^{-r(\tau - t)} LP d\tau + \int_t^T e^{-r(\tau - t)} \sigma S_{\tau} \frac{\partial P}{\partial S} dW_\tau,
$$

where $P(t, S_t, S_t^*)$ is the American value at time $t$. Since $P(t, S_t, S_t^*) = P(t, S_t, S_t^*) 1_{\{S_t \notin C_t \}} + (K - S_t^*) 1_{\{S_t^* \notin C_t \}}$, it then follows from (4.2) that

$$
e^{-r(T-t)} (K - S_{T_t}^*)^+
$$

$$
= P(t, S_t, S_t^*) - \int_t^T e^{-r(\tau - t)} \left[ r(K - S_t^*) 1_{\{S_t, S_t^* \notin C_t \}} \right] d\tau + \int_t^T e^{-r(\tau - t)} \sigma S_{\tau} \frac{\partial P}{\partial S} dW_\tau.
$$
Taking expectations yields the desired conclusion since \( P_E(t, S_t, S_t') = E_t[e^{-r(T-t)}(K - S_t')]^+ \) and the integral with respect to Brownian motion is a martingale null at zero.

\[
\text{Lemma 4.3. For } \bar{x} > 0 \text{ and } y \leq \min\{\bar{x}, S^*\},
\]

\[
\Pr(S_T \geq \bar{x}, S_T' \in dy \mid S_t = S, S_t' = S^*) = \frac{\sqrt{2(\rho-\gamma-1)}}{S^2(1-\gamma)^{-1}} F(S, \bar{x}, y, \tau - t) dy,
\]

and for \( \bar{x} \geq S^* \),

\[
\Pr(S_T \geq \bar{x}, S_T' = S^* \mid S_t = S, S_t' = S^*) = G(S, \bar{x}, S^*, \tau - t),
\]

where

\[
F(x, \bar{x}, y, \tau) = (2\rho(1 - \gamma) - 1) N(-d(x, \bar{x}, y, \tau)) + \frac{2n(-d(x, \bar{x}, y, \tau))}{\sigma \sqrt{\tau}},
\]

\[
G(x, \bar{x}, y, \tau) = N(-d(x, \bar{x}, x, \tau)) - \left(\frac{y}{x}\right)^{2\rho(1 - \gamma) - 1} N(-d(x, \bar{x}, y, \tau)),
\]

\[
d(x, \bar{x}, y, \tau) = \frac{\log(x \bar{x}/y^2) - (r - q - \sigma^2/2)\tau}{\sigma \sqrt{\tau}}.
\]

\[
\text{Proof. Let } X_T = \log(S_T/S_0) \text{ and } Y_T = \log(S_T'/S_0), \text{ where } S_0 \text{ is the initial asset price. Then}
\]

\[
X_T = \log(S_T/S_t) + X_t \quad \text{and} \quad Y_T = \log(S_T'/S_t) + X_t.
\]

From the joint distribution of \((\log(S_T-t/S_0), \log(S_T'-t/S_0))\) given by Lemma 2 of Conze and Viswanathan (1991), we deduce that for \(-\infty < x < \infty \) and \( y \leq \min\{x, Y_t\} \),

\[
\Pr(X_T \geq x, Y_T \geq y \mid X_t, Y_t)
\]

\[
= N\left(\frac{-x + X_t + \mu(\tau-t)}{\sigma \sqrt{\tau-t}}\right) - e^{2\mu(y-X_t)/\sigma^2} N\left(\frac{-x + 2y - X_t + \mu(\tau-t)}{\sigma \sqrt{\tau-t}}\right),
\]

where \( \mu = r - q - \sigma^2/2 \). Moreover, for \( x \geq Y_t \), \( \lim_{y \uparrow Y_t} \Pr(X_T \geq x, Y_T > y \mid X_t, Y_t) > 0 \) whenever \( Y_t \neq X_t \). Thus, \( \Pr(X_T \geq x, Y_T > y \mid X_t, Y_t) \) exists and is given by this positive limit. Applying the transformation \( S_T = S_0 e^{X_T}, S_T' = S_0 e^{Y_T} \) yields (4.4a)-(4.4b).

\[
\text{Theorem 4.4. The value of an American lookback put can be expressed as}
\]

\[
P(t, S, S^*) = P_E(t, S, S^*) + r(K - S^*)^+ \int_t^T e^{-r(\tau-t)} G(S, \bar{S}(\tau, S^*), S^*, \tau - t) d\tau
\]

\[
+ \frac{r}{S^2(1-\gamma)^{-1}} \int_0^{K^*} H(y) \int_t^T e^{-r(\tau-t)} F(S, \bar{S}(\tau, y), y, \tau - t) d\tau dy,
\]

where \( K^* = \min\{K, S^*\} \) and \( H(y) = (K - y)^{2(\rho-\gamma-1)} \).

\[
\text{Proof. Noting that } C_T = \{(\bar{x}, y) : \bar{x} > \bar{S}(\tau, y), y < K \} \text{ and that } \bar{S}(\tau, y) \geq y, \text{ we can apply (4.4a)}
\]

and (4.4b) to evaluate the integral in Proposition 4.2, yielding (4.5).
4.2 Exercise Boundary Near Expiration

The first application of decomposition formula (4.5) is to derive the asymptotic behavior of the exercise boundary near expiration. In particular, we use (4.5) to prove the following.

**Lemma 4.5.** For $S^* \leq K$, $P(t, S, S^*) = (1 + r\tau)P_E(t, S, S^*) + o(\tau)$ as $\tau = T - t \to 0$.

**Proof.** Let $\lambda_i$ $(i = 1, 2, 3)$ be as defined in Proposition 4.1 with $K^* = S^*$. We have the following approximations for the second and third terms on the right hand side of (4.5) for $S^* \leq K$ as $\tau \to 0$:

Second term $= r\tau (K - S^*) e^{-rT} G(S, \bar{S}(T, S^*), S^*, \tau) + o(\tau)$

$= r\tau (K - S^*) e^{-rT} G(S, S^*, S^*, \tau) + o(\tau)$

$= r\tau e^{-rT} (K - S^*) \left[ N(\lambda_2) - \left(\frac{S^*}{\bar{S}}\right)^{2\rho(1-\gamma) - 1} N(-\lambda_3) \right] + o(\tau).$

Third term $= \frac{r}{S^{2\rho(1-\gamma) - 1}} \int_0^{S^*} H(y) r e^{-rT} F(S, \bar{S}(T, y), y, \tau) dy + o(\tau)$

$= r\tau e^{-rT} \int_0^{S^*} (K - y) \frac{y^{2(\rho - \gamma \rho - 1)}}{S^{2\rho(1-\gamma) - 1}} F(S, y, y, \tau) dy + o(\tau).$

Now note that

$$\int_0^{S^*} \frac{y^\alpha}{S^{2\rho(1-\gamma) - 1}} F(S, y, y, \tau) dy = \frac{\alpha + 1}{2\rho(1 - \gamma) - 1} \frac{y^{\alpha + 1}}{S^{2\rho(1-\gamma) - 1}} N(-d(S, y, y, \tau)) \bigg|_0^{S^*}$$

$$+ \left[ 2\rho(1 - \gamma) - 1 \right] \int_0^{S^*} \frac{y^\alpha}{S^{2\rho(1-\gamma) - 1}} n(-d(S, y, y, \tau)) \frac{1}{\sigma \sqrt{\tau}} dy.$$

Setting $\alpha$ (and then $\alpha - 1$) equal to $2(\rho - \gamma \rho - 1)$, we can apply the change of variables $z = (\sigma \sqrt{\tau})^{-1} \log(y/S)$ to evaluate the last integral above and obtain the following:

Third term $= r\tau e^{-rT} \left\{ \left[ K - S^* + \frac{S^*}{2\rho(1 - \gamma)} \right] \left(\frac{S^*}{\bar{S}}\right)^{2\rho(1-\gamma) - 1} N(-\lambda_3) + KN(-\lambda_2) \right\}$

$$- r\tau e^{-rT} S \left[ 1 + \frac{1}{2\rho(1 - \gamma)} \right] N(-\lambda_1) + o(\tau).$$

Note that the second and third terms add up to $r\tau P_E(t, S, S^*) + o(\tau)$. \hfill \Box

**Theorem 4.6.** For $S^* \leq K$, as $t \uparrow T$,

$$\frac{\bar{S}(t, S^*) - S^*}{S^*} \sim \sigma \sqrt{(T - t)} \log(T - t).$$
Proof. For notational simplicity write $S$ for $S(t, S^*)$. Let $\tau = T - t$ and $y = (\sigma \sqrt{\tau})^{-1} \log(S/S^*)$, so $S/S^* = e^{\sigma y \sqrt{\tau}}$. Multiplying the equation $K - S^* = (1 + \rho \tau)P_E(t, S, S^*) + o(\tau)$ through by $e^{\rho \tau}/S^*$ yields

\begin{equation}
\frac{e^{\rho \tau} - 1}{1 + \rho \tau} \left( \frac{K}{S^*} - 1 \right) = \left[ N_2 + \frac{e^{(1 - 2\rho(1 - \gamma))\sigma y \sqrt{\tau}}}{2\rho(1 - \gamma)} N_3 \right] - e^{\sigma y \sqrt{\tau} + (\tau - y)\tau} \left[ 1 + \frac{1}{2\rho(1 - \gamma)} \right] N_1 + o(\tau),
\end{equation}

where $N_i = N(-\lambda_i)$, $i = 1, 2, 3$, and the $\lambda_i$'s are given in Proposition 4.1 with $K^* = S^*$. Since $\overline{S}(t, S^*) \rightarrow S^*$ as $\tau \rightarrow 0$ and $\overline{S}(t, S^*)$ is continuous in $t$, $y \rightarrow \infty$ as $\tau \rightarrow 0$ and $e^{\sigma y \sqrt{\tau} + b\tau} = 1 + a\sigma y \sqrt{\tau} + o(\sigma y \sqrt{\tau})$, $N_1 = N(-y) - \frac{1}{2}\{2\rho(1 - \gamma) + 1\} \sigma \sqrt{\tau} n(-y) + o(\tau)$, $N_2 = N(-y) - \frac{1}{2}\{2\rho(1 - \gamma) + 1\} \sigma \sqrt{\tau} n(-y) + o(\tau)$, $N_3 = N(-y) + \frac{1}{2}\{2\rho(1 - \gamma) + 1\} \sigma \sqrt{\tau} n(-y) + o(\tau)$. Substituting these approximations into the right hand side of (4.7) yields

\begin{equation}
N(-y)[-2\sigma y \sqrt{\tau} + o(\sigma y \sqrt{\tau})] + n(-y)[2\sigma \sqrt{\tau} + 2\sigma^2 y \tau + o(y \tau)] + o(\tau).
\end{equation}

Assume for the moment that $y^n \sqrt{\tau} = o(1)$ for all $n \in \mathbb{N}$. Using the identity $N(-y) = n(-y)(1/y - 1/y^3 + O(1/y^5))$ simplifies (4.8) to $2\sigma \sqrt{\tau} n(-y)/y^2 + o(\tau)$, from which it follows that

$$
\frac{2\sigma \sqrt{\tau} e^{-y^2/2}}{\sqrt{2\pi} y^2} \sim \tau \frac{K}{S^*} - 1 \text{ as } \tau \rightarrow 0,
$$

since $(e^{\rho \tau} - 1)(1 + \rho \tau)^{-1} = \tau \rho + o(\tau)$. Taking logarithms yields $y \sim \sqrt{\log \tau}$, which is consistent with our earlier assumption that $y^n \sqrt{\tau} = o(1)$ for all $n \in \mathbb{N}$. This is equivalent to $\log(S/S^*) \sim \sigma \sqrt{\tau} \log \tau$, which is in turn equivalent to (4.6).

4.3 American Lookback Calls

Unlike the case of standard American calls which are exercised only at maturity for $\gamma = 0$, early exercise boundaries exist for American lookback calls for all parameter values $(\rho, \gamma)$. One can solve for this boundary using the backward induction procedure described in Section 3. Adopting the same notation in (4.1a)-(4.1b), except with $K^* = \max\{K, S^*\}$ and $S^*$ representing the maximum asset price here, the value of a European lookback call is

\begin{align}
\text{(4.9a) } P_E(t, S, S^*) &= S e^{-\rho \tau} \left[ 1 + \frac{1}{2\rho(1 - \gamma)} \right] N(\lambda_1) \\
&\quad - e^{\tau \rho} \left[ K - K^* N(-\lambda_2) + \frac{S}{2\rho(1 - \gamma)} \left( \frac{S}{K^*} \right)^{-2\rho(1 - \gamma)} N(\lambda_3) \right], \text{ if } \gamma \neq 1,
\end{align}

\begin{align}
\text{(4.9b) } P_E(t, S, S^*) &= e^{-\tau \rho} [SN(\lambda_1) + K^* N(-\lambda_2) - K] + S e^{-\tau \rho} \sigma \sqrt{\tau} [\lambda_3 N(\lambda_3) + n(\lambda_3)], \text{ if } \gamma = 1.
\end{align}
Accordingly, with \( d(x, \bar{x}, y, \tau) \) defined as in Theorem 4.4, the value of an American lookback call is given by

\[
P(t, S, S^*) = P_E(t, S, S^*) + r(S^* - K)^+ \int_t^T e^{-r(t-\tau)} G(S, \bar{S}(\tau, S^*), S^*, \tau - t) \, d\tau + \frac{r}{S^{2\rho(1-\gamma)-1}} \int_{K^*}^\infty H(y) \int_t^T e^{-r(t-\tau)} F(S, \bar{S}(\tau, y), y, \tau - t) \, d\tau \, dy,
\]

where \( H(y) = (y - K)^{2(\rho-\gamma\rho-1)} \),

\[
F(x, \bar{x}, y, \tau) = \frac{2n(d(x, \bar{x}, y, \tau))}{\sigma \sqrt{\tau}} - (2\rho(1-\gamma) - 1)N(d(x, \bar{x}, y, \tau)),
\]

\[
G(x, \bar{x}, y, \tau) = N(d(x, \bar{x}, x, \tau)) - \left(\frac{y}{x}\right)^{2\rho(1-\gamma)-1} N(d(x, \bar{x}, y, \tau)).
\]

Moreover, as in Theorem 4.6, we have

\[
\frac{S^* - \bar{S}(t, S^*)}{S^*} \sim \sigma \sqrt{(T-t)|\log(T-t)|}
\]

as \( t \uparrow T \), where \( \bar{S}(t, S^*) \) is the early exercise boundary of the American lookback call.

5. TWO FAST APPROXIMATE VALUATION METHODS

In this section we show how a piecewise linear approximation to the early exercise boundary (see the end of Section 3) and the decomposition formula of Section 4.1 can be used to develop fast and accurate approximations to the prices of American lookback calls. The case of American lookback puts can be treated similarly by using corresponding results in Section 4.3 instead.

5.1 Computation of the Early Exercise Premium

A first step in dealing with the integrals in (4.5) is to use the change of variables (2.3). Specifically, with \( u = \sigma^2(\tau - T) \) and \( b = \log(y/K) \), the early exercise premium (with \( K = 1 \)) is given by

\[
\Pi(s, z, z^*) = \rho e^{\rho s} \left[ e^{z+(\rho-\gamma\rho-1/2)s} \right]^{1-2\rho(1-\gamma)} \int_\mathcal{A} h(b) \int_s^0 e^{-\rho u} f(z, \bar{z}(u, b), b, u - s) \, du \, db,
\]

\[
+ \rho e^{\rho s}(1 - e^{z^*}) + \int_s^0 e^{-\rho u} g(z, \bar{z}(u, z^*), z^*, u - s) \, du,
\]
where \( h(b) = (1 - e^b)e^{(2\rho(1-\gamma) - 1)b} \) and for fixed \( s \),

\[
f(a, \bar{a}, b, u) = (2\rho(1 - \gamma) - 1)N(-\Delta(a, \bar{a}, b, u)) + \frac{2n(-\Delta(a, \bar{a}, b, u))}{\sqrt{u}},
\]

\[
g(a, \bar{a}, b, u) = N\left(\frac{a - \bar{a}}{\sqrt{u}}\right) - \left[e^{b-a-(\rho-\gamma-1/2)s}\right]^{2\rho(1-\gamma)-1}N(-\Delta(a, \bar{a}, b, u)),
\]

\[
\Delta(a, \bar{a}, b, u) = \frac{a + \bar{a} - 2b + (2\rho(1 - \gamma) - 1)s}{\sqrt{u}}.
\]

In the last paragraph of Section 3, the early exercise boundary \( \bar{z}(u, b) \) has been computed on a grid of points in the \((u, b)\)-plane. We denote this grid by \( \{u_0, u_1, \ldots, u_m\} \times \{b_0, b_1, \ldots, b_n\} \), where \( s = u_m < \cdots < u_0 = 0 \) and \( b^* = b_n < \cdots < b_0 = 0 \). Here, \( b^* \) is a suitably chosen truncation of the integration limit at \( -\infty \) in (5.1) so that \( \int_{b^*}^{0}\frac{0}{\infty} \) can be replaced by \( \int_{b^*}^{0}\frac{0}{\infty} \). To evaluate the integral

\[
I(z, b) := pe^{\rho s} \int_{u}^{u_b} e^{-\rho u} g(z, \bar{z}(u, b), b, u - s) \, du
\]

when \( b = b_j \), we can linearly interpolate \( \bar{z}(u, b_j) \) for \( u_i \leq u \leq u_{i-1} \). Assume that

\[
\bar{z}(u, b_j) = \beta_i u + \alpha_i \quad \text{for} \quad u_i \leq u \leq u_{i-1} \quad (i = 1, \ldots, m),
\]

where \( \beta_i \) (the “time-gradient”) and \( \alpha_i \) (the “time-intercept”) are given by

\[
\beta_i = \frac{\bar{z}(u_{i-1}, b_j) - \bar{z}(u_i, b_j)}{u_{i-1} - u_i}, \quad \alpha_i = \frac{u_{i-1}\bar{z}(u_i, b_j) - u_i\bar{z}(u_{i-1}, b_j)}{u_{i-1} - u_i}.
\]

It follows that

\[
I(z, b_j) = \sum_{i=1}^{m} \left\{ A_i(z) - \left[e^{b_j-z-(\rho-\gamma-1/2)s}\right]^{2\rho(1-\gamma)-1}A_i(-z + 2b_j - (2\rho(1 - \gamma) - 1)s) \right\},
\]

where \( A_i(z) := pe^{\rho s} \int_{u_i}^{u_{i-1}} e^{-\rho u} N((z - \bar{z}(u, b_j))/\sqrt{u - s}) \, du \) can be evaluated by the following closed-form formula (cf. Ju (1998) and Ait-Sahalia and Lai (1999)):

\[
A_i(z) = e^{-\rho \tau_i} N\left(-\beta_i \tau_i^{1/2} + c_i \tau_i^{-1/2}\right) - e^{-\rho \tau_{i-1}} N\left(-\beta_i \tau_{i-1}^{1/2} + c_i \tau_{i-1}^{-1/2}\right)
\]

\[
+ \frac{1}{2} \left( \frac{\beta_i}{a_i} - 1 \right) e^{(\beta_i + a_i)c_i} \left[ N\left(a_i \tau_i^{1/2} + c_i \tau_i^{-1/2}\right) - N\left(a_i \tau_{i-1}^{1/2} + c_i \tau_{i-1}^{-1/2}\right) \right]
\]

\[
+ \frac{1}{2} \left( \frac{\beta_i}{a_i} + 1 \right) e^{(\beta_i - a_i)c_i} \left[ N\left(a_i \tau_i^{1/2} - c_i \tau_i^{-1/2}\right) - N\left(a_i \tau_{i-1}^{1/2} - c_i \tau_{i-1}^{-1/2}\right) \right],
\]

where \( \tau_i = u_i - u_m, a_i = \sqrt{\beta_i^2 + 2\rho} \), and \( c_i = c_i(z) = z - \beta_i u_m - \alpha_i \). When \( b_j < b < b_{j-1} \), we can evaluate \( I(z, b) \) using linear interpolation between \( I(z, b_j) \) and \( I(z, b_{j-1}) \).

The double integral in (5.1) can be evaluated in a similar manner. First for \( b = b_j \), we evaluate

\[
\tilde{I}(z, b) := pe^{\rho s} \int_{u}^{0} e^{-\rho u} f(z, \bar{z}(u, b), b, u - s) \, du
\]

by the same method as before. In fact,

\[
\tilde{I}(z, b_j) = (2\rho(1 - \gamma) - 1) \sum_{i=1}^{m} \tilde{A}_i(-z + 2b_j - (2\rho(1 - \gamma) - 1)s),
\]
where \( \tilde{A}_i(\cdot) \) is essentially the same expression as \( A_4(\cdot) \) given in (5.4), except with the ratio \( \beta_i/a_i \) replaced by \( \{\beta_i - 4\rho/(2\rho(1 - \gamma) - 1)\}/a_i \). This in turn enables us to evaluate \( \tilde{I}(z, b) \) over a number of equally spaced subintervals of \([b^*, 0 \land z^*]\), linearly interpolating between \( \tilde{I}(z, b_j) \) (\( j = 1, \ldots, n \)) where necessary. To compute the integral \( \tilde{I}(z) := \int_{b^*}^{0 \land z^*} h(b)\tilde{I}(z, b) \, db \), let \( \tilde{h}(b) = h(b)\tilde{I}(z, b) \) and partition \([b^*, 0 \land z^*]\) into an even number, \( 2N \), of subintervals of equal width \( w \) so that the integral can be approximated by Simpson’s rule which has an error of order \( O(w^4) \):

\[
\text{Simpson}(\tilde{I}(z)) = \frac{w}{3} \left\{ \tilde{h}(b^*) + \tilde{h}(0 \land z^*) + 4 \sum_{j=0}^{N-1} \tilde{h}(b^* + (2j + 1)w) + 2 \sum_{j=1}^{N-1} \tilde{h}(b^* + 2jw) \right\}.
\]

Collecting terms, we can compute the early exercise premium (5.1) via the approximation

\[
(5.5) \quad \Pi(s, z, z^*) = (1 - e^{s^*})^+ I(z, z^*) + \left[ e^{s^*+(\rho - \gamma \rho - 1/2)s^*} \right]^{1-2\rho(1-\gamma)} \text{Simpson}(\tilde{I}(z)).
\]

In particular, we can use tabulated values of \( \bar{z}(u_i, b_j) \) (e.g., Table 1) to evaluate the early exercise premium (5.1) through the approximations (5.2) and (5.5).

5.2 An Integral Equation for the Boundary

Instead of relying on tabulated values of \( \bar{z}(u_i, b_j) \) that have been computed previously with high precision by the Bernoulli walk method, we can generate them by solving recursively a system of nonlinear equations. To begin with, note that putting \( P(t, \overline{S}(t, S^*), S^*) = K - S^* \) in (4.5) yields the integral equation

\[
K - S^* = P_E(t, \overline{S}(t, S^*), S^*) + r(K - S^*) + \int_t^T e^{-r(t - \tau)} G(\overline{S}(t, S^*), \overline{S}(\tau, S^*), S^*, \tau - t) \, d\tau \\
+ \frac{r}{\overline{S}(t, S^*)^{2\rho(1-\gamma)-1}} \int_0^{K \land S^*} H(y) \int_t^T e^{-r(t - \tau)} F(\overline{S}(t, S^*), \overline{S}(\tau, y), y, \tau - t) \, d\tau \, dy.
\]

Recasting the integral equation in the canonical variables, we can solve for the boundary recursively over a grid of points \((s_k, b_l)\) with \( 0 = s_0 > s_1 > \cdots > s_m = -\sigma^2T \) and \( 0 = b_0 > b_1 > \cdots > b_n = b^* \). Specifically, set \( \bar{z}_{in} = b_n - (\rho - \gamma \rho - 1/2)s_i \) (\( 0 \leq i \leq m \)). The recursive procedure is initialized at \( s_0 = 0 \) by \( \bar{z}_{0j} = b_j \) (\( j = 1, \ldots, n \)). Suppose that \( \bar{z}_{0j}, \ldots, \bar{z}_{k-1,j} \) (\( j = 1, \ldots, n \)) and \( \bar{z}_{k+1,j}, \ldots, \bar{z}_{kn} \) have been determined. Then, applying linear interpolation to \( \bar{z}(u_i, b_j) \) as in (5.2)–(5.3), we obtain from (5.5) the following expression for the early exercise premium when \( b_l < 0 \):

\[
\Pi(s_k, z, b_l) = (1 - e^{b_l})^+ I(z, b_l) + \left[ e^{z^*+(\rho - \gamma \rho - 1/2)s_k} \right]^{1-2\rho(1-\gamma)} \text{Simpson}(\tilde{I}(z)),
\]

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Therefore the integral equation for $\tilde{S}(t, S^*)$ leads to the following nonlinear equation for $\bar{z}_{k\ell} = \bar{z}(s_k, b_\ell)$:

\begin{equation}
1 - e^{b_\ell} = e^{\rho s_k} p_E(s_k, \bar{z}_{k\ell}, b_\ell) + \Pi(s_k, \bar{z}_{k\ell}, b_\ell),
\end{equation}

where $p_E(s, z, z^*)$ is defined by applying the change of variables (2.3) to $P_E(t, S, S^*)$ in Proposition 4.1.

We can use a bracketing technique to search for the solution $\bar{z}_{k\ell}$. Let $D(\cdot)$ denote the difference between the two sides of (5.6), i.e., $D(\cdot) = \text{l.h.s.} - \text{r.h.s.}$ Since $\bar{z}_{k\ell} \geq \bar{z}_{k,\ell+1}$, initializing at $\hat{z}_0 = \max\{\bar{z}_{k,\ell+1}, b_\ell - (\rho - \gamma \rho - 1/2)s_k\}$ yields $D(\bar{z}_0) < 0$. We can then increment $\hat{z}_0$ in steps of $\varepsilon$ to search for the smallest integer $\ell_0$ such that $D(\bar{z}_0 + \ell_0 \varepsilon) > 0$. After bracketing the solution in this way between $\bar{z}_0$ and $\bar{z}_1 := \bar{z}_0 + \ell_0 \varepsilon$ with $D(\bar{z}_0) < 0$ and $D(\bar{z}_1) > 0$, we can use successive linear approximations of $D(z)$ to find the solution within some prescribed tolerance level. Table 2 shows an implementation of this procedure for $\rho = 0.5$, $\gamma = 0$ and $\sigma^2T = 0.3$, in which we use $b^* = -4$ and $w = 0.0025$ for Simpson($\bar{I}(z)$). The tabulated boundary values are in reasonably good agreement to those given in Table 1 using a backward induction procedure. As pointed out in Section 3, this method is preferred to the Bernoulli walk approach for $s$ very close to 0, because the boundary has infinite gradient as $s$ approaches 0 and the Chernoff-Petkau correction cannot be used there and because only a small number $m$ of time steps in (5.2) is needed to produce a highly accurate piecewise linear approximation of $\bar{z}(u, b)$ for $s \leq u \leq 0$ when $s$ is small.

\begin{table}
\centering
\caption{Boundary values for $\bar{z}(u, b)$}
\begin{tabular}{|c|c|c|}
\hline
$s$ & $\bar{z}(u, b)$ & $\bar{z}(u, b)$ \\
\hline
\end{tabular}
\end{table}

\section{6. NUMERICAL COMPARISON WITH OTHER PROCEDURES}

We present a numerical study of the two approximations based on the tabulation-interpolation approach of Section 5.1 and on the integral equation of Section 5.2, respectively. We also compare our results with option prices computed by two procedures, adapted from the literature, that make use of continuity corrections based respectively on a control variate technique and an overshoot approximation. These alternative procedures are essentially enhancements of the Cox-Ross-Rubinstein binomial tree approach to compute option prices which achieve an improved rate of convergence as the number of steps used in the tree is increased. The enhancements involve the number $n$ of time steps and the strike price $K$ in the binomial tree value $P(n; K)$ of the American option at time $t = 0$ for spot price $S_0$. Note that $S_0^* = S_0$. 

\begin{table}
\centering
\caption{American option prices}
\begin{tabular}{|c|c|c|}
\hline
$n$ & $P(n; K)$ & $P(n; K)$ \\
\hline
\end{tabular}
\end{table}
The control variate technique of Hull and White (1988) is based on the same technique for Monte Carlo simulation under certain conditions. It is applicable here because the European value \( P_E \) is known exactly (see Proposition 4.1). The technique consists of building the usual \( n \)-step augmented binomial tree and computing both the European option value \( P_E(n; K) \) and the American value \( P(n; K) \) using the same tree, so that a continuity adjustment for \( P(n; K) \) can be computed as follows:

\[
P_{CV}(n) = P(n; K) + \left[ P_E - P_E(n; K) \right].
\]

The overshoot correction of Broadie, Glasserman, and Kou (1999) is based on an asymptotic theory developed by Asmussen, Glynn, and Pitman (1995). Using the asymptotic distribution of the discretization error associated with discrete sampling of the maximum along a Brownian path, a first-order correction for lookback options can be derived to relate tree-based price estimates to continuous-time prices. In view of Theorem 4 of Broadie, Glasserman, and Kou (1999), we use the following corrected pricing formula for American lookback puts:

\[
P_{BGK}(n) = e^{-0.5\sigma\sqrt{T/n}} P(n; K e^{0.5\sigma\sqrt{T/n}}).
\]

To further enhance the efficiency of the procedure, we use a two-point extrapolation. Specifically, one computes two estimates \( P_{BGK}(n) \) and \( P_{BGK}(2n) \), based respectively on \( n \) and \( 2n \) time steps. An improved price estimate is obtained as follows: \( P_{BGK} = 2P_{BGK}(2n) - P_{BGK}(n) \).

In our numerical comparison, we take \( S_0 = 40, \ q = 0, \) and adopt a representative selection of values for the other parameters (see Table 3). For the control variate technique (CV) we use \( \delta = 10^{-3} \) in the tree. In implementing the overshoot correction (BGK) we take \( n = 2000 \) across all maturities, which is equivalent to taking one of the values \( \delta \in \{0.625, 1.25, 2.5, 5\} \times 10^{-4} \). For our tabulation-interpolation approach, we adopt the canonical formulation (CF), using \( \delta = 10^{-3} \) for boundary determination, and value the options by evaluating (4.5) with (5.5). Finally, we take the integral equation (IE) approach to determine the stopping boundary by solving (5.6) over the coarse grid of \( (s, z^+) \) values given in Table 2 and then use the boundary to value the options with (4.5) and (5.5). For this approach, we use \( b^* = -4 \) and \( w = 0.0025 \) for all calculations. The results are presented in Table 3. For most of the options, the four option price estimates agree to within 0.01 of each other. This is equivalent to a relative error of not more than 0.25%.

As predicted by theory, options with the same values of \( \rho = r/\sigma^2 \) and \( \sigma^2(T-t) \) ought to have the same price. This is always the case with our tabulation-interpolation approach or integral equation approach. There are two such instances in Table 3: \((r, \sigma, T) = (0.04, 0.2, 1.00)\) with \((0.16, 0.4, 0.25)\), and \((r, \sigma, T) = (0.02, 0.2, 1.00)\) with \((0.08, 0.4, 0.25)\).
7. CONCLUSION

This paper describes a Bernoulli walk method to compute both the value and early exercise boundary of an American lookback option. The Bernoulli walk is a natural alternative to the binomial tree after we use a change of variables to reduce the optimal stopping problem to its canonical form indexed by only one (resp. two) parameter(s) in the absence (resp. presence) of dividends. The time horizon in the canonical scale is considerably shorter than in the calendar time scale, thereby requiring less computational effort than corresponding methods based on the binomial tree. Numerical results obtained by the Bernoulli walk method show that the early exercise boundary is well approximated by a piecewise linear function in the canonical scale. This approximate shape and the decomposition formulas (4.5) and (4.10) suggest two fast and accurate approximate methods to compute the values of American lookback options.

One method is based on tabulation of the exercise boundary at a few pre-specified knots over a set of parameter values and the closed-form expressions like (4.5) with (5.5) to compute the option values once the boundary approximation is determined. Such tabulation can be stored in a handheld calculator which can be used to evaluate the closed-form expressions like (4.5) with (5.5). In fact, once the boundary is determined and a tabulation scheme is decided upon, several options with the same $\rho$ and $\gamma$ can be priced in very little time. Recently Joubert and Rogers (1997) also proposed a tabulation-interpolation approach to approximate standard American option values. Instead of tabulating the early exercise boundaries at a few canonical time points as in Ait-Sahalia and Lai (1999), they tabulate option prices and therefore require much larger tables that have to be stored in a computer as a dictionary. For lookback options, there is an additional state variable $S^*$, requiring even a much larger dictionary. In contrast, we only tabulate the boundary $\tilde{z}(s, z^*)$ at a few points $(s_i, z^*_j)$ in the canonical coordinate system.

An alternative to tabulation is to solve, under the piecewise linear approximation to $\tilde{z}(\cdot, \tilde{z}^*)$ with a few pieces, an integral equation characterizing $\tilde{z}$. This alternative method, which was first introduced by Ju (1998) for American vanilla options, is especially effective when $\sigma^2 T$ is small. The numerical study in Section 6 shows that both approximations are sufficiently fast and accurate for pricing American lookback options.
REFERENCES


Table 1
A tabulation of the optimal stopping boundary \( \mathcal{Z}(s, z^*) \) for \( \rho = 0.5 \). The boundary is obtained using backward induction with a Bernoulli walk approximation.

<table>
<thead>
<tr>
<th>( z^* )</th>
<th>(-.3)</th>
<th>(-.2)</th>
<th>(-.15)</th>
<th>(-.1)</th>
<th>(-.05)</th>
<th>(-.025)</th>
<th>(-.01)</th>
<th>(-.005)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-2.0)</td>
<td>-1.8782</td>
<td>-1.8782</td>
<td>-1.8782</td>
<td>-1.8784</td>
<td>-1.8827</td>
<td>-1.9020</td>
<td>-1.9191</td>
<td></td>
</tr>
<tr>
<td>(-1.0)</td>
<td>-0.6055</td>
<td>-0.6112</td>
<td>-0.6216</td>
<td>-0.6448</td>
<td>-0.7000</td>
<td>-0.7593</td>
<td>-0.8354</td>
<td>-0.8795</td>
</tr>
<tr>
<td>(-.5)</td>
<td>0.2165</td>
<td>0.1524</td>
<td>0.1009</td>
<td>0.0286</td>
<td>-0.0890</td>
<td>-0.1889</td>
<td>-0.2910</td>
<td>-0.3555</td>
</tr>
<tr>
<td>(-.25)</td>
<td>0.7121</td>
<td>0.5930</td>
<td>0.5094</td>
<td>0.4029</td>
<td>0.2438</td>
<td>0.1164</td>
<td>-0.0122</td>
<td>-0.0990</td>
</tr>
<tr>
<td>(-.1)</td>
<td>1.1134</td>
<td>0.9368</td>
<td>0.8255</td>
<td>0.6794</td>
<td>0.4738</td>
<td>0.3131</td>
<td>0.1660</td>
<td>0.0773</td>
</tr>
<tr>
<td>(-.05)</td>
<td>1.3294</td>
<td>1.1174</td>
<td>0.9844</td>
<td>0.8153</td>
<td>0.5813</td>
<td>0.4020</td>
<td>0.2282</td>
<td>0.1306</td>
</tr>
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</table>

Table 2
A tabulation of the optimal stopping boundary \( \mathcal{Z}(s, z^*) \) for \( \rho = 0.5 \). The boundary is obtained by solving an integral equation.

<table>
<thead>
<tr>
<th>( z^* )</th>
<th>(-.3)</th>
<th>(-.2)</th>
<th>(-.15)</th>
<th>(-.1)</th>
<th>(-.05)</th>
<th>(-.025)</th>
<th>(-.01)</th>
<th>(-.005)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-2.0)</td>
<td>-1.8955</td>
<td>-1.8920</td>
<td>-1.8898</td>
<td>-1.8840</td>
<td>-1.8758</td>
<td>-1.8598</td>
<td>-1.8950</td>
<td>-1.8675</td>
</tr>
<tr>
<td>(-1.0)</td>
<td>-0.5992</td>
<td>-0.5996</td>
<td>-0.6074</td>
<td>-0.6264</td>
<td>-0.6807</td>
<td>-0.7377</td>
<td>-0.8041</td>
<td>-0.8410</td>
</tr>
<tr>
<td>(-.5)</td>
<td>0.2342</td>
<td>0.1701</td>
<td>0.1189</td>
<td>0.0518</td>
<td>-0.0676</td>
<td>-0.1572</td>
<td>-0.2716</td>
<td>-0.3000</td>
</tr>
<tr>
<td>(-.25)</td>
<td>0.7331</td>
<td>0.6114</td>
<td>0.5294</td>
<td>0.4269</td>
<td>0.2667</td>
<td>0.1441</td>
<td>0.0213</td>
<td>-0.0500</td>
</tr>
<tr>
<td>(-.1)</td>
<td>1.1346</td>
<td>0.9575</td>
<td>0.8437</td>
<td>0.7076</td>
<td>0.5011</td>
<td>0.3569</td>
<td>0.1930</td>
<td>0.1500</td>
</tr>
<tr>
<td>(-.05)</td>
<td>1.3433</td>
<td>1.1320</td>
<td>0.9992</td>
<td>0.8408</td>
<td>0.6082</td>
<td>0.4447</td>
<td>0.2669</td>
<td>0.2151</td>
</tr>
</tbody>
</table>
Table 3

Option prices obtained from the control variate technique of Hull and White (CV), the overshoot correction of Broadie, Glasserman, and Kou (BGK), our tabulation-interpolation approach via a canonical formulation (CF), and our integral equation approach (IE). Prices are calculated at $t = 0$ with $S_0 = 40$ and $q = 0$.

<table>
<thead>
<tr>
<th>$\sigma = 0.2$</th>
<th>$r = 0.08$</th>
<th>$r = 0.04$</th>
<th>$r = 0.02$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T$</td>
<td>$K$</td>
<td>CV</td>
<td>BGK</td>
</tr>
<tr>
<td>0.25</td>
<td>35</td>
<td>0.2134</td>
<td>0.2134</td>
</tr>
<tr>
<td>40</td>
<td>2.6909</td>
<td>2.6909</td>
<td>2.6925</td>
</tr>
<tr>
<td>0.50</td>
<td>35</td>
<td>0.6109</td>
<td>0.6109</td>
</tr>
<tr>
<td>1.00</td>
<td>35</td>
<td>1.245</td>
<td>1.2453</td>
</tr>
<tr>
<td>40</td>
<td>4.4252</td>
<td>4.4265</td>
<td>4.4342</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\sigma = 0.4$</th>
<th>$r = 0.16$</th>
<th>$r = 0.08$</th>
<th>$r = 0.04$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T$</td>
<td>$K$</td>
<td>CV</td>
<td>BGK</td>
</tr>
<tr>
<td>0.25</td>
<td>35</td>
<td>1.6449</td>
<td>1.6449</td>
</tr>
<tr>
<td>40</td>
<td>5.1498</td>
<td>5.1493</td>
<td>5.1535</td>
</tr>
<tr>
<td>0.50</td>
<td>35</td>
<td>2.8656</td>
<td>2.8661</td>
</tr>
</tbody>
</table>
(a) Lookback Put

(b) Lookback Call

**Figure 1**

Schematic illustration of the properties of the optimal stopping boundary $\overline{z}(s, z^*)$ as a function of $z^*$ (left), and of $s$ (right). $S$ denotes the stopping region, $\mathcal{K}$ the continuation region and $\mathcal{A}^c$ the inadmissible region.
FIGURE 3
Cross-sections of optimal stopping boundaries at four values of $z^*$ for five values of $\rho$. 