ON THE MAXIMUM CORRELATION COEFFICIENT

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Abstract
For an arbitrary random vector \((X, Y)\) and an independent random variable \(Z\) it is shown that the maximum correlation coefficient between \(X\) and \(Y + \lambda Z\) as a function of \(\lambda\) is lower semi-continuous everywhere and continuous at zero where it attains its maximum. If, moreover, \(Z\) is in the class of self-decomposable random variables, then the maximal correlation coefficient is right continuous, non-increasing for \(\lambda \geq 0\) and left continuous, non-decreasing for \(\lambda \leq 0\). Independent random variables \(X\) and \(Z\) are Gaussian if and only if the maximum correlation coefficient between \(X\) and \(X + \lambda Z\) equals the linear correlation between them. The maximum correlation coefficient between the sum of \(n\) arbitrary independent identically distributed random variables and the sum of the first \(m < n\) of these, equals \(\sqrt{m/n}\) (previously proved only for random variables with finite second moments, where it amounts also to the linear correlation). Examples provided reveal counterintuitive behavior of the maximum correlation coefficient for more general \(Z\) and in the limit \(\lambda \to \infty\).

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Keywords and phrases: dependence, maximum correlation, self-decomposable random variables.

Running title: Maximum correlation coefficient
1 Introduction and Statement of Results

The maximum correlation coefficient between two random elements $\xi, \eta$ introduced in Hirschfeld (1935) and Gebelein (1941) is

$$\rho(\xi, \eta) = \sup \{ \text{corr}(\phi(\xi), \psi(\eta)) : 0 < E[|\phi(\xi)|^2] < \infty, 0 < E[|\psi(\eta)|^2] < \infty \},$$

(1)

where $\text{corr}(X, Y)$ is the classical (Pearson) correlation between random variables $X$ and $Y$. The definition (1) is equivalent to

$$\rho(\xi, \eta) = \sup E\{\phi(\xi)\psi(\eta)\},$$

(2)

where the supremum in (2) is taken over all $\phi, \psi$ with

$$E\{\phi(\xi)\} = E\{\psi(\eta)\} = 0, E\{|\phi(\xi)|^2\} = E\{|\psi(\eta)|^2\} = 1.$$

(3)

Geometrically, $\rho(\xi, \eta)$ equals the cosine of the angle between the subspaces (of a larger Hilbert space $L^2(\xi, \eta)$) $L^2(\xi) = \{\phi(\xi) : E(\phi) = 0, E(|\phi|^2) < \infty\}$ and $L^2(\eta) = \{\psi(\eta) : E(\psi) = 0, E(|\psi|^2) < \infty\}$. Another well known interpretation of $\rho$ is as the operator norm of the conditional expectation $\phi \mapsto E(\phi(\xi)|\eta)$ acting on the closed subspace of $L_2$ consisting of functions orthogonal to constants. Thus

$$\rho^2(\xi, \eta) = \sup \left\{ E\left\{ |E(\phi(\xi)|\eta)|^2 \right\} : E\{\phi(\xi)\} = 0, E\{|\phi(\xi)|^2\} = 1 \right\}.$$ 

(4)

The main role of $\rho(\xi, \eta)$ is that of a convenient numerical measure of dependence between $\xi$ and $\eta$. In particular, $\rho(\xi, \eta)$ vanishes if and only if $\xi$ and $\eta$ are independent.

Explicit formulas for $\rho(\xi, \eta)$ are available in very few cases. If $(X, Z)$ is a bivariate Gaussian vector then

$$\rho(X, Z) = |\text{corr}(X, Z)|$$

(5)

(for a proof see, for example, Lancaster (1957)).

If $X = X_1 + \ldots + X_m$, $Z = X_1 + \ldots + X_n$, $m \leq n$ where $X_1, \ldots, X_n$ are independent identically distributed, non-degenerate, random variables with finite second moment, then the maximum correlation between $X$ and $Z$

$$\rho(X, Z) = \sqrt{m/n}$$

(6)

(see Dembo et al (2001)) and, thus, does not depend on the distribution of $X_i$. In Corollary 1 below, we show that (6) applies also for any independent
identically distributed, non-degenerate random variables $X_i$. There are a few other isolated cases when $\rho(X, Y)$ is known in an explicit form. Different properties of the maximum correlation were studied in Sarmanov (1958a,b), Rényi (1959), Csáki and Fisher (1963), Breiman and Friedman (1985). If $X, Y, Z$ are Markov-dependent, then it is easy to see that

$$\rho(X, Y) \geq \rho(X, Y + Z). \quad (7)$$

Indeed, since $L^2(Y + Z) \subset L^2(Y, Z)$, one has

$$\rho(X, (Y, Z)) \geq \rho(X, Y + Z).$$

From Markov property $E(\phi(X)|Y, Z) = E(\phi(X)|Y)$ which by (4) implies the well known formula

$$\rho(X, Y) = \rho(X, (Y, Z))$$

(cf. proof of Lemma 1 Rosenblatt (1971), page 207). Thus (7) follows. If bivariate random vectors $(X_1, Y_1)$ and $(X_2, Y_2)$ are independent, then

$$\max\{\rho(X_1, Y_1), \rho(X_2, Y_2)\} \geq \rho(X_1 + X_2, Y_1 + Y_2). \quad (8)$$

This follows from the fact that $\rho(X_1 + X_2, Y_1 + Y_2) \leq \rho((X_1, X_2), (Y_1, Y_2))$ and from the Csáki-Fisher identity

$$\rho((X_1, X_2), (Y_1, Y_2)) = \max\{\rho(X_1, Y_1), \rho(X_2, Y_2)\},$$

see Theorem 1 in Witsenhausen (1975). Inequality (8) yields two implications in the same spirit as (7).

- If $(X', Y')$ is an independent copy of $(X, Y)$ then
  $$\rho(X, Y') \geq \rho(X + X', Y + Y'). \quad (9)$$

- If $(X, Y)$ is an arbitrary bivariate random vector and $Z_1, Z_2$ are independent of each other and of $(X, Y)$, then
  $$\rho(X, Y) \geq \rho(X + Z_1, Y + Z_2).$$

We prove in this paper the following general properties of $\rho(X, Y + \lambda Z)$.

**Theorem 1.** The function $\lambda \mapsto \rho(X, Y + \lambda Z)$ is lower semi-continuous in $\lambda$ for any random variables $X, Y, Z$. In particular, if $Z$ is independent of the pair $(X, Y)$ then $\rho(X, Y + \lambda Z)$ is continuous at $\lambda = 0$. 

4
A random variable $Z$ is in $\mathcal{L}$ if for any $c$, $0 < c < 1$ there exists a random variable $U_c$ independent of $Z$ such that

$$Z \text{ is equidistributed with } cZ + U_c.$$  \hspace{1cm} (10)

Equivalently, a real valued random variable $Z$ belongs to the class $\mathcal{L}$ if its characteristic function

$$f(t) = E(e^{itZ}), \ t \in \mathbb{R}$$

possesses the following property: for any $c$, $0 < c < 1$ there exists a characteristic function $f_c(t)$ such that

$$f(t) = f(ct)f_c(t), \ t \in \mathbb{R}.$$  \hspace{1cm} (11)

The random variables in $\mathcal{L}$ are called self-decomposable. All random variables in $\mathcal{L}$ are infinitely divisible. Necessary and sufficient conditions (in terms of Levy functions) are known for an infinitely divisible random variable to belong to $\mathcal{L}$ (see Lukacs (1970), Ch. 5). In particular, all random variables having stable distributions are in $\mathcal{L}$.

We next detail additional properties of $\rho(X, Y + \lambda Z)$ in case $Z$ is independent of $(X, Y)$ and belongs to the class $\mathcal{L}$.

**Theorem 2** If a random variable $Z \in \mathcal{L}$ is independent of a bivariate random vector $(X, Y)$, then $\lambda \mapsto \rho(X, Y + \lambda Z)$ is a non-increasing right continuous function on $[0, \infty)$ and a nondecreasing left continuous function on $(-\infty, 0]$.

The above results hold for random elements $X, Y, Z$ taking values in an arbitrary separable, Banach space; the proofs remain the same.

The next theorem is a converse of (5) holding when $X$ and $Z$ are Gaussian.

**Theorem 3** If $X$ and $Z$ are independent, non-degenerate, square-integrable real-valued random variables such that for every real $\lambda$

$$\rho(X, X + \lambda Z) = \text{corr}(X, X + \lambda Z),$$  \hspace{1cm} (12)

then $X$ and $Z$ are Gaussian.

Dembo et al (2001) show that equality (12) may hold true for a fixed $\lambda \neq 0$ with independent, non-degenerate square-integrable, non-Gaussian $X$ and $Z$, see (6).

Our next result provides the value of $\rho(X, X + \lambda Z)$ in case both $X$ and $Z$ are symmetric $\alpha$-stable random variables.
Theorem 4 Suppose $X$ and $Z$ are independent copies of an $\alpha$-stable random variable, $\alpha \in (0, 2]$. Then,

$$\rho(X, X + \lambda Z) = \frac{1}{\sqrt{1 + |\lambda|^\alpha}},$$

(13)

for all $\lambda \geq 0$. If $X$ and $Z$ are symmetric, the equality (13) holds also for $\lambda < 0$.

The next lemma which is key to the proof of Theorem 4, is of independent interest.

Lemma 1 Suppose $X$ and $Y$ are non-degenerate independent random variables with characteristic functions $\phi_X(t)$ and $\phi_Y(t)$ such that

$$\liminf_{t \to 0} \frac{1 - |\phi_Y(t)|^2}{1 - |\phi_X(t)|^2} = c.$$

(14)

Then,

$$\rho(X, X + Y) \geq \frac{1}{\sqrt{1 + c}}.$$

(15)

Suppose $X_j$ are independent identically distributed, non-degenerate, random variables with characteristic function $\phi(t)$ (possibly with infinite second moment). The independent random variables $X = \sum_{j=1}^m X_j$ and $Y = \sum_{j=m+1}^n X_j$, have characteristic functions $\phi_X(t) = \phi(t)^m$ and $\phi_Y(t) = \phi(t)^{n-m}$. Applying Lemma 1 for the pair $(X, Y)$, where $c = (n - m)/m$ by the continuity of $|\phi(t)|^2$ at $t = 0$, we get that

$$\rho(X, X + Y) \geq \sqrt{m/n}.$$

(16)

Combining this lower bound with the upper bound of inequality (19) of Dembo et al (2001), we get the following corollary.

Corollary 1 The equality (6) holds for any non-degenerate independent identically distributed $X_1, \ldots, X_n$.

2 Proofs

Proof of Theorem 1. Note that if $\psi_n \to \psi$ in $L^2$ and $\phi_n \to \phi$ in $L^2$, with both $\phi$ and $\psi$ non-zero (in $L^2$), then $\text{corr}(\phi_n, \psi_n) \to \text{corr}(\phi, \psi)$. Consequently, in the definition (1) it suffices to consider $\phi$ and $\psi$ in the dense subsets $BL$ of $L^2(\xi)$ and $L^2(\eta)$, consisting of bounded, Lipschitz functions, using
the notation \( \| \psi \|_{BL} = \| \psi \|_{\infty} + \| \psi \|_{LP} \). Fix \( \phi, \psi \) bounded and Lipschitz having positive variances at \( X \) and \( Y + tZ \) respectively. Define \( \Delta(r) = \min(1, |r| \|Z\|) \), so

\[
|\psi(Y + tZ) - \psi(Y + sZ)| \leq 2\|\psi\|_{BL} \Delta(t - s),
\]

implying that

\[
|\text{Cov}(\phi, \psi(Y + tZ)) - \text{Cov}(\phi, \psi(Y + sZ))| \leq 4\|\phi\|_{\infty} \|\psi\|_{BL} E(\Delta(t - s)),
\]

\[
|\text{Var}(\psi(Y + tZ)) - \text{Var}(\psi(Y + sZ))| \leq 8\|\psi\|_{BL}^2 E(\Delta(t - s)).
\]

By bounded convergence \( E(\Delta(t - s)) \to 0 \) as \( s \to t \), implying that \( \text{Var}\{\psi(Y + sZ)\} \) is bounded away from 0 in neighborhood of \( t \), so also

\[
\text{corr}\{\phi, \psi(Y + sZ)\} \to \text{corr}\{\phi, \psi(Y + tZ)\} \text{ as } s \to t.
\]

Thus

\[
\lambda \mapsto \sup \{\text{corr}(\phi, \psi(Y + \lambda Z)) : \text{Var}(\phi(X)), \text{Var}(\psi(Y + \lambda Z)) \text{ finite}\}
\]

is lower semi-continuous. This ends the proof of the first part of the theorem. Combining the first part of the theorem and the inequality (7) for \( \lambda Z \), we have that

\[
\rho(X, Y) \geq \liminf_{\lambda \to 0} \rho(X, Y + \lambda Z) \geq \rho(X, Y),
\]

proving the continuity of \( \rho(X, Y + \lambda Z) \) at \( \lambda = 0 \) and completing the proof of Theorem 1. \( \square \)

**Proof of Theorem 2.** Let \( \lambda_2 > \lambda_1 > 0 \); write \( \lambda_1 = c \lambda_2 \) where \( 0 < c < 1 \). Let \( U_c \) be a random variable independent of \( X, Y, Z \) such that \( Z \equiv cZ + U_c \). From (7),

\[
\rho(X, Y + \lambda_1 Z) = \rho(X, Y + \lambda_2 cZ) \geq \rho(X, Y + \lambda_2 cZ + \lambda_2 U_c) = \rho(X, Y + \lambda_2 cZ + U_c) = \rho(X, Y + \lambda_2 Z).
\]

If \( \lambda_2 < \lambda_1 < 0 \), then setting \( \lambda'_2 = -\lambda_2 \), \( \lambda'_1 = -\lambda_1 \) one has

\[
\rho(X, Y + \lambda_1 Z) = \rho(-X, -Y - \lambda_1 Z) = \rho(-X, -Y + \lambda'_1 Z) \geq \rho(-X, -Y + \lambda'_2 Z) = \rho(X, Y + \lambda'_2 Z) = \rho(X, Y + \lambda_2 Z).
\]

The inequality (7), applied for \( \lambda Z \), extends the above monotonicity properties of \( \lambda \mapsto \rho(X, Y + \lambda Z) \) to \([0, \infty)\) and \((-\infty, 0]\), respectively. By Theorem
1 this function is lower semi-continuous in \(\lambda\), hence it is right continuous wherever non-increasing, and left continuous wherever non-decreasing. \(\square\)

**Proof of Theorem 3.** We may and shall assume without loss of generality that \(E(X) = E(Z) = 0\), \(E(X^2) = E(Z^2) = 1\). Using (12) with \(\lambda = 1/s > 0\) we get \(\rho(X, Z + sX) = \text{corr}(X, Z + sX)\) and hence

\[
E(X|Z + sX) = \frac{s}{1 + s^2}(Z + sX)
\]  
(17)

(see Dembo et al (2001), page 344). Replacing \(Z\) by \(-Z\) in (12), it is easy to see that (17) holds also for \(s \leq 0\). This implies

\[
E[X \exp\{it(Z + sX)\}] = \frac{s}{1 + s^2}E[(Z + sX) \exp\{it(Z + sX)\}].
\]

Differentiating this relation with respect to \(s\) at \(s = 0\) we get

\[
itE(X^2 \exp(itZ)) = E[Z \exp(itZ)].
\]

Since \(X, Z\) are independent and \(E(X^2) = 1\), this shows that the characteristic function \(\phi(t) = E \exp(itZ)\) satisfies the differential equation \(\phi'(t) = -t\phi(t)\), and hence \(Z\) is Gaussian. With \(u = 1/s\), it follows from (17) that

\[
E(Z|X+uZ) = E(Z|Z+sX) = (Z+sX) - sE(X|Z+sX) = \frac{u}{1+u^2}(X+uZ).
\]

Reversing the roles of \(X\) and \(Z\), by the same argument as before \(X\) is also Gaussian. \(\square\)

**Proof of Lemma 1.** Recall that for characteristic function \(|\phi(t)|^2 \leq 1\). So, fixing \(t \in \mathbb{R}\) such that \(|\phi_X(t)| \neq 1\), and considering separately the real and imaginary parts of \(f(x) = e^{itx}\), it is easy to check that (4) implies

\[
\rho^2(X, X + Y) \geq \frac{E|E(f(X + Y)|X)|^2 - |E f(X + Y)|^2}{E|f(X + Y)|^2 - |E f(X + Y)|^2}.
\]

We have

\[
|E f(X + Y)|^2 = |\phi_X(t)\phi_Y(t)|^2,
|E(f(X + Y)|X)|^2 = |\phi_Y(t)|^2,
|f(X + Y)|^2 = 1.
\]
Thus, if in addition $|\phi_Y(t)| \neq 0$,

$$\rho^2(X, X + Y) \geq \frac{|\phi_Y(t)|^2(1 - |\phi_X(t)|^2)}{1 - |\phi_X(t)|^2|\phi_Y(t)|^2} = \frac{1}{1 + \frac{|\phi_Y(t)|^2 - |\phi_X(t)|^2}{1 - |\phi_X(t)|^2}}.$$  

Taking now the lim sup of the right hand side as $t \to 0$, we get the conclusion (15) out of our assumption (14). \(\square\)

**Proof of Theorem 4.** Applying Lemma 1 to $Y = \lambda Z$ we get that

$$\rho(X, X + \lambda Z) \geq \frac{1}{\sqrt{1 + |\lambda|^\alpha}}. \quad (18)$$

If $X$ and $Z$ are symmetric, the pairs $(X, X + \lambda Z)$ and $(X, X - \lambda Z)$ have the same distribution. Hence, it suffices to prove the converse of (18) for $\lambda > 0$. To this end, fix $0 < \epsilon < \lambda$ and let $m < n$ be positive integers such that $\lambda - \epsilon < (n/m - 1)^{1/\alpha} < \lambda$. Then, by Theorem 2 (and by the invariance of $\rho$ under non-degenerate linear transformations),

$$\rho(X, X + \lambda Z) \leq \rho(X, X + (n/m - 1)^{1/\alpha}Z) = \rho(m^{1/\alpha}X, m^{1/\alpha}X + (n - m)^{1/\alpha}Z) = \rho(S_m, S_n),$$

where $S_n$ denotes the sum of $n$ independent copies of the $\alpha$-stable random variable $X$. Therefore, inequality (19) of Dembo et al (2001) gives

$$\rho(X, X + \lambda Z) \leq \sqrt{\frac{m}{n}} \leq \frac{1}{\sqrt{1 + (\lambda - \epsilon)^\alpha}}.$$

Since $\epsilon > 0$ is arbitrary, this ends the proof. \(\square\)

3 Some counterintuitive examples

Here some examples are constructed that demonstrate counterintuitive features of the maximum correlation.

**Example 1** Let $Z$ be a non-degenerate random variable independent of $(X, Y)$. As $|\lambda| \to \infty$, one may ask when $X$ and $Y + \lambda Z$ become "asymptotically independent", i. e.,

$$\lim_{|\lambda| \to \infty} \rho(X, Y + \lambda Z) = 0. \quad (19)$$
From Bryc and Smolenski (1992) it follows that for bounded $\phi, \psi$

$$\lim_{|\lambda| \to \infty} \text{Cov}\{\phi(X), \psi(Y + \lambda Z)\} = 0$$

if $Z$ has a density, and that

$$\lim_{|\lambda| \to \infty} \text{Cov}\{\phi(X), \psi(Y + \lambda Z)\} = \text{Cov}\{\phi(X), \psi(Y)\}$$

if $Z$ is discrete. The latter shows that (19) does not hold in general. Here is a related explicit example. Let $X$ be an arbitrary (non-degenerate) random variable, the distribution of $Y$ be concentrated on $[-1/2, 1/2]$ and $Z$ be a binary random variable taking values $-1$ and $+1$. For any (known) $\lambda$ with $|\lambda| > 1$, $Y$ can be reconstructed from $Y + \lambda Z$ implying that for $|\lambda| > 1$, $\rho(X, Y + \lambda Z) = \rho(X, Y)$ and does not go to 0 as $|\lambda| \to \infty$.

**Example 2.** These $X, Y, Z$ would give an example of $\rho(X, Y + \lambda Z)$ that does not decrease monotonically in $\lambda \in (0, \infty)$ unless $\rho(X, Y + \lambda Z) \equiv \rho(X, Y)$. We shall now show that if $X = Y$ with $P(X = -1/2) = P(X = 1/2) = 1/2$ and $Z$ is independent of $X$ with $P(Z = -1) = P(Z = 1) = 1/2$, then $\rho(X, X + \lambda Z) = 1/\sqrt{2} < 1$ for $\lambda = 1/2$.

The random variable $X + Z/2$ takes values $-1, 0, +1$ with probabilities $1/4, 1/2, 1/4$, respectively. In view of (3), one may always assume

$$\phi(-1/2) = -1, \quad \phi(1/2) = 1.$$  

Then $E(\phi(X)|X + Z/2 = \pm 1) = \pm 1$ and $E(\phi(X)|X + Z/2 = 0) = 0$. Thus $\rho^2 = \sup_{\phi} E[(E(\phi(X)|X + Z/2)^2] = 1/2$ by (4).

**Example 3.** The random variables $X, Z$ from **Example 2** also are an example of $\rho(X, X + \lambda Z)$ that, as a function of $\lambda$, is discontinuous at $\lambda = 1/2$ and $-1/2$. Indeed, for any $\lambda$ with $|\lambda| \neq 1/2$, $X$ can be reconstructed from $X + \lambda Z$ whence

$$\rho(X, X + \lambda Z) = 1, \quad |\lambda| \neq 1/2$$

while $\rho(X, X \pm Z/2) < 1$ as shown above.

This construction can easily be generalized to $X$ and $Z$ with finite $> 2$ number of values such that the continuity of $\rho(X, X + \lambda Z)$ fails to hold at prescribed $\lambda_i > 0$, $i = 1, 2, \ldots, k$.

Taking $X, Z$ as above and $Y$ concentrated on $[-1/6, 1/6]$, independent of $Z$, such that $1 > \rho(X, Y) > 1/\sqrt{2}$, we now see by **Example 1** that for $\lambda = 1/2$,

$$\rho(X, Y + \lambda Z) = \rho(X, Y) > \rho(X, X + \lambda Z),$$

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that is, $X$ and $Y + \lambda Z$ depend stronger than $X$ and $X + \lambda Z$ at $\lambda = 1/2$, whereas the opposite relationship holds at $\lambda = 0$.

**Example 4.** The asymptotic independence (19) may fail even when $X = Y$ and $Z$ are both in $\mathcal{L}$. Indeed, let $X$ be $\alpha$-stable random variable and $Z$ be $\beta$-stable random variable independent of $X$, for some $0 < \alpha < \beta \leq 2$ ($\beta = 2$ in case $Z$ is Normal). With $(X', Z')$ denoting an independent copy of $(X, Z)$, the distribution of $(X + X', Z + Z')$ equals up to a non-random constant, that of $(2^{1/\alpha}X, 2^{1/\beta}Z)$. Hence, by (9),

$$
\rho(X, X + \lambda Z) \geq \rho(X + X', X + X' + \lambda(Z + Z')) = \rho(X, X + \lambda 2^{1/\beta - 1/\alpha}Z).
$$

Since $2^{1/\beta - 1/\alpha} < 1$, Theorem 2 provides the reverse inequality, implying that $\rho(X, X + \lambda Z)$ is constant on $(0, \infty)$ and constant on $(-\infty, 0)$. By Theorem 1 this function of $\lambda$ is continuous at $\lambda = 0$, hence $\rho(X, X + \lambda Z) = \rho(X, X) = 1$. Obviously, (19) fails to hold in this case.

## 4 Open problems

**I.** As shown in Theorem 2, $Z \in \mathcal{L}$ implies monotonicity of $\rho(X, Y + \lambda Z)$ in $\lambda$ for any $(X, Y)$ independent of $Z$.

It is interesting to find out if the condition $Z \in \mathcal{L}$ is sufficient for continuity of $\rho(X, Y + \lambda Z)$ at $\lambda \neq 0$. More generally, what is the class of $Z$ for which monotonicity or continuity of $\lambda \mapsto \rho(X, Y + \lambda Z)$ apply for all $(X, Y)$?

**II.** More generally, one may consider the properties of

$$
\rho_{p, q}(\xi, \eta) = \sup \left\{ \frac{\text{Cov}(U, V)}{\|U\|_p \|V\|_q} : U \in L^p(\xi), V \in L^q(\eta), U, V \neq 0 \right\}
$$

for $1/p + 1/q \leq 1$, where $\rho = \rho_{2, 2}$. Of particular interest are $\rho_{\infty, \infty}$ corresponding to strong mixing, and $\rho_{\infty, 1}$ and $\rho_{1, \infty}$ corresponding to uniform strong mixing.
References


