ESTIMATING THE TERM STRUCTURE OF CREDIT SPREADS:
CALLABLE CORPORATE DEBT

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ABSTRACT

In this paper I extract credit pricing information from the prices of callable corporate debt, by disentangling the components of callable corporate bond prices associated with discounting at market interest rates, discounting for default risk, and optionality. The value of the call option is sensitive to both interest rate risk and to the risk of changes in credit quality. The results include the first empirical analysis, in the setting of standard arbitrage-free term-structure models, of the time-series behavior of callable corporate bond yield spreads, explicitly incorporating the valuation of the American call options. As an application, I consider medium-quality callable issues of Occidental Petroleum Corporation, using a three-factor model for the term structures of benchmark (LIBOR-dollar) swap rates and for Occidental yield spreads. I study the correlations of these Occidental spreads with various macroeconomic and firm-specific time series, and discuss the implications of the estimated model for the current market practice of pricing callable corporate debt.

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I. Introduction

This paper presents a methodology for extracting credit pricing information from the prices of callable corporate debt. Prior empirical research on corporate bond pricing has avoided a direct treatment of callable corporate bonds. In practice, however, callable debt is popular. As of April 2000, the Fixed Investment Securities Database (FISD (2000)) contained a total of about 18,500 US-dollar fixed-rate corporate bonds, of which roughly 65% (both in number and in offering amount) were callable. In order to extract credit-quality information from yield spreads, one must treat the simultaneous effects of credit risk and optionality.

Figure 1 shows market prices of a Baa3-rated callable bond issued by Occidental over the time period from January 1990 through December 1995. Also plotted are the prices at which this bond would trade if it was noncallable and default-free. The reduction in price of the

![Figure 1. Prices of a callable Occidental bond and its noncallable, default-free equivalent. Source: Datatream.](image)

actual callable, defaultable bond relative to its noncallable, default-free equivalent is due to
two factors: discounting for default risk, and callablity. The main objective of my work is to
disentangle these two components and thus to specify the values of the embedded American
call option and of the noncallable (defaultable) bond. In practice, the problem of valuating the
call option is often approached by pricing it based on a term-structure model of the default-
free yields, possibly adjusted for default risk by adding yield spreads of noncallable bonds
(see, for example, Fan, Haubrich, Ritchken, and Thomson (2002)). This method, however,
reflects a somewhat superficial point of view, because the market value of the call option
depends not only on uncertainty regarding market interest rates, but also on the risk of changes
in the credit quality of the bond. For example, an upgrade of Occidental’s credit quality,
holding default-free yields constant, would increase the value of the call option. The challenge
when estimating the term structure of callable corporate bond yield spreads stems from this
interaction between call-free credit spreads and the prices of the call option, which calls for a
simultaneous solution of both.

This paper includes a model for pricing callable bonds that accounts for both default and
American callability, and allows for compensation for illiquidity. I develop a methodology
that is designed to disentangle the components of callable corporate bond prices associated
with (i) discounting at market interest rates, (ii) discounting for default and illiquidity risk,
and (iii) callability. I build on the framework of Duffie and Singleton (1999), who show that
the cash flows promised by a corporate bond can be priced using a default-adjusted short-term
discount rate that reflects the mean arrival rate of a credit event and the associated loss in
market value upon arrival. Additionally, one can adjust the discount rate by a mean fractional
cost rate to compensate for illiquidity effects. The model incorporates the dependency of the
value of call option on both interest rate risk and the risk of changes in credit quality.

The valuation of callable corporate bonds requires an assumption about the call policy of
the issuer. I will assume that a callable bond is called so as to minimize its market value
despite the fact that firms may exercise bond calls, or fail to exercise them, for many other
reasons. For example, the desire to change the firm’s capital structure or the need to eliminate
restrictive covenants are two reasons why a company might decide to call in the debt even
though its market value is below the strike price. Liquidity constraints, on the other hand, could defer calls. In addition, for the issuer of a portfolio of corporate liabilities, in order to minimize the portfolio's market value, it may not be optimal to call in a particular bond so as to minimize that bond's market value.

In the context of multi-factor affine term-structure models, I use the popular Least-Squares Method (LSM) commonly attributed to Longstaff and Schwartz (2001)\(^1\) to price the American-style options embedded in callable bonds. The LSM is a simulation-based algorithm that solves for the optimal stopping rule. The key to this approach is to use least-squares regression to estimate the conditional expected value of the bond if not called. Ugurlu (2001) suggests an "override" condition that suppresses approximate call exercise strategies whenever it is more valuable at present to commit to exercise at some future date. I find that the LSM algorithm, when accompanied by this override feature, achieves a high degree of both accuracy and robustness for a wide range of parameterizations.

The results of this paper include the first empirical analysis, in the setting of standard arbitrage-free term-structure models, of the time-series behavior of callable corporate bond yield spreads, explicitly incorporating valuations of the American call options. As an application, I consider a Baa3-rated callable issue of Occidental Petroleum Corporation, using a three-factor model for the term structures of benchmark (LIBOR-dollar) swap rates and for Occidental yield spreads. Applying an approximate-maximum-likelihood estimator, I estimate a model of the term structure of noncallable credit spreads, using as data the prices of the callable bond issued by Occidental.

Using the parameters and implied noncallable spreads, I examine some implications of the estimated model for the current market practice of pricing callable corporate debt, and study the correlations of these spreads with various macroeconomic and firm-specific time series, including a U.S. chemicals index and Occidental's leverage ratio. Given a simple model of recovery at default and for the mean fractional liquidity cost rate, one is then in a position to estimate the implied risk-neutral probability of default from corporate bond prices. The actual
probability of default can finally be estimated on the basis of the estimated risk premia (see, for example, Huang and Huang (2002)).

The remainder of this paper is organized as follows. Section II gives a summary of the related literature. Section III presents my valuation framework for callable corporate bonds, discusses how to approximate the optimal strategy for exercising the call option, and describes the estimation strategy applied. Section IV presents the empirical results for Occidental’s bond data.

II. Literature Review

This paper draws on four different lines of literature, namely (i) the theory of valuing callable, defaultable bonds, (ii) simulation-based American option pricing methods in the setting of multi-factor term-structure models, (iii) estimation techniques for latent-factor term-structure models, and (iv) the empirical methods for time-series modeling of credit spreads.

A. Valuation of Callable, Defaultable Coupon Bonds (Theory)

Two major building blocks of any valuation framework for callable, defaultable bonds are the treatments of discounting for default risk, and callability. Corporate default risk has previously been captured by a variety of models. Recent methods are based on either a structural or a reduced-form model. Structural models are based on a model of the firm’s value, as a stochastic process, and on the assumption that default is triggered when the firm’s value falls below some critical value, related to liabilities. The structural approach was pioneered by Black and Scholes (1973), and much of the literature is based on Merton (1974). Reduced-form models generally treat default as the arrival of a counting process with a (stochastic) intensity process. See, for example, Pye (1974), Das and Tufano (1995), Jarrow and Turnbull (1995), Jarrow, Lando, and Turnbull (1997), Madan and Unal (1998), and Duffie and Singleton
(1999). Treatments of loss given default vary, both among reduced-form models and among structural models.

Regarding optimal exercise strategies for the embedded call options, several theories have been developed. In particular, one can distinguish between models using partial differential equation (PDE) techniques and the martingale approach. Regarding the PDE-based literature, models with default were advocated by Merton (1974), who argued that prices of a callable, defaultable bond solve a PDE subject to boundary conditions that describe default and call events. Closed-form solutions do not exist, and finite-difference methods are applied. Kim, Ramaswamy, and Sundareasan (1993) extend this work by allowing for stochastic interest rates. In Sarkar (2000), imperfections in the capital structure (refunding costs, taxes, bankruptcy costs) alter the optimal call policy. Martingale methods simplify the calculation of prices of bonds with embedded options. For example, Duffie and Singleton (1999) rely on the martingale approach to price callable, defaultable bonds assuming that the issuer calls the bond so as to minimize its market value. Acharya and Carpenter (2002) model call and default options as American options written on a noncallable, default-free bond with fixed continuous coupons. In a recent related paper, Guenta (2002) proposes a double-hazard framework to price callable, defaultable coupon bonds. He models call risk and default risk as two correlated hazard processes, while allowing taxes and refunding costs to affect the arrival rate of the call, and firm characteristics to influence the arrival of default.

B. American Option Pricing in Multi-Factor Term-Structure Models

In general, there are no analytic expressions for the prices of a callable bond. Many numerical approaches to price the embedded American-style call options have been suggested. These methods can be classified as either deterministic lattice-based methods or as nondeterministic simulation-based methods. In addition, some analytical approximations have been developed. The literature on lattice-based models took its beginning with the explicit finite difference method pioneered by Brennan and Schwartz (1977) and Couratadon (1982). Cox, Ross, and
Rubinstein (1979) introduced binomial trees, also known as CRR-type models. Boyle (1988), Boyle, Evnine, and Gibbs (1989), He (1990) and others extend the CRR binomial model to two and three state variables. Binomial trees and finite-difference methods have been successfully applied to problems with up to three state variables that have standard dynamics (for example, Gaussian). Pricing methods based on analytical approximations rely on extrapolation methods (Geske and Johnson (1984)), quasi-analytical solutions (Barone-Adesi and Whaley (1987)), or the decomposition of the American option in European option plus early exercise premium (Kim (1990), Carr, Jarrow, and Myneni (1992), Huang, Subrahmanyam, and Yu (1996)). Examples of other hybrid methods include Broadie and Detemple (1997), Carr (1998), Ju (1998), and Su (2000). Most of these methods, however, have been applied only to standard American options depending on one or two state variables. Broadie and Detemple (1996) provide a comparison of various non-simulation-based methods for pricing American options written on a single underlying asset.

C. Estimation Techniques for Latent-Factor Term-Structure Models


Dai and Singleton (2000) classify $N$-factor affine term-structure models (ATMs, see Duffie and Kan (1996) for a general treatment) into $N+1$ non-nested subfamilies $A_M(N)$, $0 \leq M \leq N$, of models and discuss their structural differences. Their canonical parameterization is especially useful when addressing admissibility and identification issues. In this paper, I deal with intermediate-type affine models in the families $A_M(N)$, for $0 < M < N$. Transition densities of the state vectors for many such models are not known in closed-form. Hence, if MLE (in combination with either exact inversion or filtering) is adopted, the likelihood function is normally computed numerically. Umanzov (2001) discusses three applicable methods, (i) Quasi Maximum Likelihood (Fisher and Gilles (1996)), (ii) Approximate Maximum Likelihood (Aït-Sahalia (2002), Duffie, Pedersen, and Singleton (2003)), and (iii) Maximum Likelihood based on the Characteristic Function (Liu, Pan, and Pedersen (2000), Singleton (2001)).

Simulation-based density estimation for affine models is offered by Pedersen (1995) and Santa-Clara (1995). While generally more computationally intensive than analytic methods, this approach is applicable to all affine diffusions.
D. Empirical Estimation of Credit Spreads

Existing empirical literature on the estimation of credit spreads of corporate bonds, such as Duffee (1999), Elton, Gruber, Agrawal, and Mann (2001), or Driessen (2002), investigates expected returns and risk premia of noncallable bonds only. Fan, Haubrich, Ritchken, and Thomson (2002) calibrate the term structure of credit spreads for five large banks to subordinated noncallable bond data, and use the parameter estimates to generate theoretical prices of puttable coupon bonds. Duffie, Pedersen, and Singleton (2003) estimate a time-series model of the term structure of yield spreads for noncallable Russian dollar-denominated bonds. To my knowledge, however, the existing empirical literature on estimating the term structure of credit spreads using observed bond prices is restricted to noncallable bonds.  

III. Methodology

I consider the price \( P_t \), at any time \( t \) before default, of a corporate security that promises to pay a single, possibly random, amount \( Z \) at some time \( T > t \). I introduce a probability space with measure \( P \) (data-generating measure) and an increasing family \( \{ \bar{F}_t : t \geq 0 \} \) of information sets defining the resolution of information over time. I take as given a short-rate process \( r \) and an equivalent martingale measure \( Q \). With respect to this “risk-neutral” measure, prices are discounted expected cash flows, as specified in more detail in what follows. This means that a default-free claim to \( Z \) at time \( T \) has a price at \( t \) of 
\[
E_t^Q \left[ e^{-\int_t^T r_s \, ds} Z \right],
\]
where \( E_t^Q \) denotes \( F_t \)-expectation with respect to \( Q \). Harrison and Kreps (1979) and Delbaen and Schachermayer (1999) show that the existence of such a risk-neutral measure \( Q \) and the absence of arbitrage are essentially equivalent.

Consider a bond issue that defaults at the first arrival of a doubly-stochastic counting process with intensity \( h \) with respect to \( Q \). That is, \( h_t \) is the risk-neutral default intensity. This means, under technical regularity conditions, that the \( F_t \)-conditional risk-neutral probability
of default between $t$ and $t + \Delta$, given that default has not occurred by $t$, is approximately $h_t \Delta$, in the limit as $\Delta$ goes to zero.

For the case of a single credit event, Duffie and Singleton (1999) show that for pricing purposes, under technical regularity conditions, one can treat the issue's promised cash flow $Z$ as default-free, and allow for default risk by replacing the discount rate $r$ with the default-adjusted discount rate $r + hL$, where $L$ is the risk-neutral expected fractional loss in market value in the event of default. Under the assumption that illiquidity of the security can be captured by a fractional cost rate of $l$, where $l$ is a predictable process, the total mean loss rate of the security due to default and illiquidity is

\[ s = hL + l. \]  

I call the process $s$ the short spread. The default- and liquidity-adjusted short-rate process $R$ associated with the bond is then defined as

\[ R = r + s. \]  

Duffie and Singleton (1999) show that, under mild technical conditions,

\[ P_t = E_t^Q \left( e^{-\int_t^T R_u du} Z \right). \]  

I evaluate corporate bonds within the class of Affine Term-Structure Models (ATSMs). The default- and liquidity-adjusted short rate $R$ is modeled as affine with respect to the state vector $X \in \mathbb{R}^3$, which is modeled as a regular affine process under both the data-generating measure $P$ as well as under the equivalent martingale measure $Q$. (See Duffie, Filipovic, and Schachermayer (2001) for a complete definition.) A key property of affine models (generalized by Duffie and Kan (1996)) is that zero-coupon bond prices are exponential-affine
functions of the state. The market value at $t$ of a zero-coupon defaultable bond maturing at time $T$, as a fraction of face value, is given by

$$B(t; T) = E^Q_t e^{-\int_t^T r(s) \, ds} = e^{A(T-t) + B(T-t)X_t},$$

(4)

for coefficients $A(\cdot)$ and $B(\cdot)$ that are available as the solutions of Ricatti equations (Duffie and Kan (1996)). Such ODEs can generally be solved fast and accurately using multistep Runge-Kutta methods. For certain models, including correlated Gaussian (Vasicek (1977), Langetieg (1980)) and independent CIR (Cox, Ingersoll, and Ross (1985)), the solutions are known in closed form. I will take the affine state process $X$ to be a multi-factor diffusion. The canonical parameterization of multi-factor diffusion ATSMs developed in Dai and Singleton (2000) allows me to address the issues of admissibility and identification.

A. Parametric Model of the Adjusted Short Rate

I rely on the term structure of U.S.-dollar LIBOR-quality swap yields as the reference curve based on the short-rate process $r$. Duffie, Pedersen, and Singleton (2003) discuss advantages of swap over Treasury yields, and propose a two-factor affine model for the reference term structure which describes the dynamics of the short-rate process $r$ and the process $\nu$ driving its volatility. Specifically,

$$d\begin{bmatrix} \nu_t \\ r_t \end{bmatrix} = \begin{bmatrix} K^{\nu \nu} & 0 \\ K^{\nu r} & K^{rr} \end{bmatrix} \begin{bmatrix} \theta^\nu \\ \theta^r \end{bmatrix} \begin{bmatrix} \nu_t \\ r_t \end{bmatrix} dt + \sqrt{\nu_t} \begin{bmatrix} 1 & 0 \\ \Sigma^\nu & \Sigma^r \end{bmatrix} \begin{bmatrix} dW^\nu_t \\ dW^r_t \end{bmatrix},$$

(5)

where $(W^\nu, W^r)$ is a two-dimensional standard Brownian motion under the data-generating measure. The distribution of the risk-neutral Brownian motion $(\tilde{W}^\nu, \tilde{W}^r)$ is specified as

$$\begin{bmatrix} d\tilde{W}^\nu_t \\ d\tilde{W}^r_t \end{bmatrix} = -\sqrt{\nu_t} \begin{bmatrix} 0 \\ \kappa^r \end{bmatrix} dt + \begin{bmatrix} d\tilde{W}^\nu_t \\ d\tilde{W}^r_t \end{bmatrix}.$$
Duffie, Pedersen, and Singleton (2003) use weekly data on two- and ten-year swap rates for the period January 1987 through July 1999 to estimate the model parameters \( (K^v = 0.0047, K^r = -0.027, \theta^v = 0.34, \theta^r = 107.40, \sigma^v = 5.68, \sigma^r = 0.044, \lambda^r = 0.11, \lambda^v = -0.076) \) using approximate maximum likelihood estimation. For the purpose of my empirical analysis, I will treat these parameter estimates as the true parameters. The implied short rates \( r_t \) are displayed in Figure 12 of Appendix A.

With regard to the short spread \( s \) of a particular bond, I make the simplifying assumption that the short spread is independent (under both \( P \) and \( Q \)) of the two-factor model \( (\nu, r) \) of the reference curve. Empirical studies of noncallable corporate bonds yield spreads estimate the correlation between the instantaneous reference rate and the short spread to be negative, but only of relatively small absolute value. (For instance, in Duffie (1999) neither of the correlation coefficients between \( s \) and the two factors of the default-free term structure is significant.) Preliminary examinations in my empirical case study have confirmed these results, hence indicating that assuming independence of the reference curve and the short spread will not introduce any significant bias to my results. I assume that \( s \) follows a Vasicek-type (Ornstein-Uhlenbeck) process under both the data-generating and the risk-neutral measure. The dynamics of the short spread are assumed to be given by

\[
    ds_t = K(\theta - s_t) dt + \Sigma dW^s_t,
\]

where \( W_{t}^{vrs} = (W^v, W^r, W^s) \) is a standard three-dimensional Brownian motion under \( P \). The distribution of the risk-neutral Brownian motion \( \tilde{W}_{t}^{vrs} = (\tilde{W}^v, \tilde{W}^r, \tilde{W}^s) \) is specified as

\[
    dW^s_t = -\left(\Lambda + \frac{\tilde{K}}{\Sigma}(s_t - \theta)\right) dt + d\tilde{W}^s_t.
\]

As discussed in Duffie, Pedersen, and Singleton (2003), the fact that the short spread can take on negative values is not necessarily inconsistent with the proposed theoretical model, due to the possibility of a negative liquidity factor. Moreover, as illustrated in Figure 2, even for issues below LIBOR-quality negative yield spreads can be observed in the market.
If the issuer has issued several bonds, each bond’s short spread may incorporate an idiosyncratic component specific to this issue (like special default or liquidity characteristics), and not necessarily to a maturity segment of that firm’s yield curve. I select one bond as a “benchmark”, and model its short spread $s$ as in Equations (7) and (8). For any non-benchmark bond $i$, the short spread process is assumed to be given by $s^i = s + u^i$, where

$$du^i_t = \kappa^i (\theta^i - u^i_t) \, dt + \sigma^i \, dW^u^i_t,$$  \hfill (9)

and $W^u^i$ is a standard Brownian motion independent of $\{W^{vrs}, W^u^j, j \neq i\}$. The distribution of the risk-neutral Brownian motion $\tilde{W}^u^i$ is specified as

$$dW^{	ilde{u}}^i_t = - \left( \lambda^i + \frac{\tilde{\kappa}^i - \tilde{K}^i}{\sigma^i} (s_t - \theta) + \frac{\tilde{\kappa}^i - \kappa^i}{\sigma^i} (u^i_t - \theta^i) \right) dt + d\tilde{W}^u^i_t,$$  \hfill (10)

where $\tilde{W}^u^i$ is a standard Brownian motion under $Q$, independent of $\{\tilde{W}^{vrs}, \tilde{W}^u^j, j \neq i\}$. In Appendix B I show that the specifications of the market price of risk for $s$ and $u^i$ in Equations (8) and (10) are indeed admissible.

The model setup in Equations (7) through (10) represents a trade-off between the aims of capturing important empirical features of corporate bond yield spreads, while maintaining a setting that allows to estimate the model parameters in a feasible fashion. As will become more evident in Sections C and D, for callable bonds the estimation procedure is numerically intensive, and requires time-consuming attention by the user in order to obtain a reasonable fit.

This leaves us with the vector

$$\Theta = (\theta, K, \Sigma, \Lambda, \tilde{K}, \{\theta^i\}, \{\kappa^i\}, \{\sigma^i\}, \{\lambda^i\}, \{\tilde{\kappa}^i\})$$

of unknown parameters, governing the stochastic behavior of corporate bond yield spreads, to be estimated from observed bond prices.
B. Valuation of Callable Corporate Bonds

Suppose that a corporate bond that matures at time $T$ pays a coupon of size $c$, as a fraction of face value, at an increasing sequence $T_1, T_2, \ldots, T_L + 1 = T$ of times. If the bond is noncallable, the market value (ex-coupon) at any time $t$ before maturity, as a fraction of face value, is from Equation (3),

$$
V_t = E_t^Q \left[ \sum_{t < T_i \leq T} ce^{-\int_{T_i}^{T} R_u du} + e^{-\int_{T}^{T} R_u du} \right].
$$

I have implicitly assumed the same model for coupons and principal. One could, in principle, have adopted a different short spread model for each coupon.

Now, suppose the bond is callable at par, possibly only after some initial lock-out period. Throughout, I assume that a callable bond is called so as to minimize its market value. For any time $t$ before maturity, let $T(t, T)$ denote the set of feasible call policies after $t$. Provided default has not occurred by time $t$, the ex-coupon market value at $t$, as a fraction of face value, is

$$
V_t = \inf_{\tau \in T(t, T)} V_t(\tau),
$$

(11)

where

$$
V_t(\tau) = E_t^Q \left[ \sum_{t < T_i \leq \tau} ce^{-\int_{T_i}^{\tau} R_u du} + e^{-\int_{\tau}^{T} R_u du} \right],
$$

(12)

which can be shown by standard arguments for nondefaultable securities. (A slightly different expression applies if the call price is not par.) The pricing relation (11) applies under technical regularity conditions discussed, for example, in Duffie and Singleton (1999). According to Bellman's principle of optimality, at each time $t$, the issuer minimizes the market value of the liability by exercising the option to call in the bond if and only if its market value, if not called, is higher than the call price (here assumed to be par).
C. Simulation-Based American Option Pricing in Multi-Factor ATSMs

In general, there is no closed-form solution available for the price (11) of a callable bond. The underlying dynamic optimization problem is solved numerically, for example by dynamic programming. I now suppose, as is often the case in practice, that the bond is callable at par with first call date at \( T \), and that the times of "callability" include all coupon payment dates thereafter. If the bond is callable at some inter-coupon time, and is not called before that date, the issuer maximizes the value of the call option embedded in the bond (and hence minimizes the market value of the callable bond) by not exercising it until the next coupon payment date. This argument assumes that the short-rate process \( R \) associated with the corporate security is nonnegative and is standard. I will ignore minor adjustments due to market conventions for quoting accrued interest on corporate bonds. The model assumptions for \( R \) in Equations (2) and (5) through (8) allow for negative values of \( R \). Throughout my empirical analysis, however, the (risk-neutral) probability of such occurrences is small. I therefore make the simplifying assumption that the issue is callable only at coupon dates, i.e. \( T(t, T) = \{ T_1, \ldots, T_{L+1} : T_j \geq T, T_j > t \} \) for all \( t \) before maturity.

Initially, I fix some time \( t \) before maturity. Let \( \tau^* \in T(t, T) \) denote an optimal stopping time, characterized by \( V_t = V_t(\tau^*) \) in Equations (11) and (12). Given \( \tau^* \), \( V_t \) can be evaluated accurately by straightforward Monte Carlo (MC) simulation. The optimal exercise strategy \( \tau^* \) is usually not explicitly known, however, and must be approximated. In general, the optimal stopping time will depend in a complicated way on the discount factors \( B(T_i, T_j) \) (as defined in Equation (4)) for all feasible call dates \( T_i \) and \( T_j \) after \( t \). As this optimal stopping time depends on too many variables to be feasibly determined within a standard MC setting, I rely, therefore, on the popular Least-Squares Method (LSM, Longstaff and Schwartz (2001)) to price the American-style options embedded in callable bonds. I implement the LSM algorithm as a two-step procedure. In the first step, I characterize an approximate exercise strategy. This policy is then used in the second step to evaluate the callable bond using standard MC techniques.\(^7\) In the characterization step, Longstaff and Schwartz (2001) find an approximate exercise strategy through a recursive, simulation-based algorithm that proceeds backwards in time and solves
for the stopping rule that minimizes the value of the callable bond at each time point along each path. The key to the approach is that, at each exercise date, least-squares regression is used to estimate the conditional expected value of the bond, if not exercised (value of continuation), as a function in polynomials of the underlying state variables $X$, and possibly of other nonlinear transformations of $X$. Appendix C records the complete algorithm. Appendix D describes how to simulate from the conditional (risk-neutral) distribution of the state vector $X$.

Convergence results for the LSM algorithm are available (see, for example, Clément, Lamberton, and Protter (2002), Tsitsiklis and Roy (2001), and Stentoft (2001)). Under fairly general conditions, Clément, Lamberton, and Protter (2002) prove the almost-sure convergence of the complete LSM algorithm. The argument builds on the fact that the conditional risk-neutral expected value of continuation is an element of the Hilbert space of square-integrable functions relative to the risk-neutral measure. This Hilbert space has a countable orthonormal basis, hence the conditional expectation can be represented as a linear function of the basis elements. Additionally, the authors show that the normalized error of the procedure, after replacing the conditional expected values of continuation by projections on a finite set of basis functions, is asymptotically Gaussian.

To my knowledge, there is no study available as to which and how many basis functions to use in specific situations, or of convergence rates as the number of basis functions goes to infinity. Longstaff and Schwartz (2001) argue that, for many applications, only a few basis functions suffice and that the exact choice of basis functions does not have a significant effect. In my empirical applications, I use the first three powers of all unmatured discount bond prices with final maturity up to and including maturity, as well as products of the unmatured discount bond prices with the shortest and the longest time remaining. In order to obtain an estimator of the decision boundary which is smooth in the state and the parameter vectors, $X$ and $\Theta$, I use all sample paths in the regression step, and not only the in-the-money paths as recommended by Longstaff and Schwartz (2001). Ugłum (2001), however, suggests an override, to exercise, whenever the value of current exercise is dominated by the value of exercising at a later deterministic time. To protect against misspecification of the decision boundary associated with my
implementation of the LSM method, I implement the Uglum override condition. My extensive numerical tests indicate that the LSM algorithm, when accompanied by this override feature, achieves a high degree of accuracy and is robust for a wide range of parameterizations of the term structure of callable bond yield spreads.

D. Estimation Strategy

I outline a procedure for estimating the parameter vector \( \Theta \) associated with the short spread of a callable bond from its observed market prices, using the valuation framework established in the previous sections. The estimation strategy is based on standard maximum likelihood estimation (MLE) techniques for latent-factor affine models of the term structure as discussed, for example, by Duffie, Pedersen, and Singleton (2003) and Umantsvev (2001).

As before, I fix some time \( t \) before maturity. Given the states \( v_t \) and \( r_t \) of the reference curve, the bond value in Equation (11) is a deterministic function, here denoted by \( G_t(\cdot; \cdot) \), of the current short spread \( s_t \) and the parameter vector \( \Theta \) governing its stochastic behavior. That is,

\[
V_t = G_t(s_t; \Theta).
\] (13)

Hence, the estimation problem is typical for latent-factor models in that one directly observes the bond prices \( V_t \) but not the short spreads \( s_t \) which determine the current and near-future term structure of the callable bond yield spreads.

Suppose, I observe the prices of the callable corporate bond at times \( t_0 \) through \( t_N \). Standard change-of-measure arguments lead to the log-likelihood function for the observed bond price vector \( \mathcal{V} = (V_{t_0}, V_{t_1}, \ldots, V_{t_N}) \) given by

\[
I(\mathcal{V} | \Theta) = \frac{1}{N} \sum_{n=1}^{N} \left[ \log P \left( s_n^\Theta | s_{n-1}^\Theta; \Theta \right) - \log \left| \Delta G_{t_n} \left( s_n^\Theta | \Theta \right) \right| \right],
\]
where $s^\Theta_n = G_{t_n}^{-1}(V_n; \Theta)$. Appendix E shows that, for the short-spread model (7) and (8), $G_{t_n}(\cdot; \Theta)$ is indeed invertible for each $\Theta$. Consequently, a MLE for $\Theta$ is given by

$$
\hat{\Theta} \in \arg \max_{\Theta} I(V|\Theta).
$$

(14)

However, faced with the problem of estimating the parameter vector $\Theta$ from the observed prices of corporate bonds with an embedded American-style call option, one encounters several additional challenges, and Equation (14) only holds in an approximate sense. First, the optimal exercise strategy in Equation (11) is, for most cases, not explicitly known. Hence, I approximate it using the LSM algorithm together with Uglum’s override condition outlined in Section C. Second, given an (approximate) stopping rule, often there is still no explicit formula at hand to calculate the callable bond prices and MC simulation has to be employed. And third, computing the likelihood function for the time series of observed callable bond prices involves both inverting the value function $G_{t_n}(\cdot; \Theta)$ in $s_n$ and calculating the sensitivity of the bond prices relative to the short spread, that is $DG_{t_n}(\cdot; \Theta)$. Again, numerical methods have to be applied.9

IV. Empirical Case Study: Occidental Petroleum

In this section, I investigate the term-structure behavior of credit spreads of a callable bond issued by Occidental Petroleum Corporation. Headquartered in Los Angeles, California, Occidental is a large, multinational company with worldwide interests in oil and gas exploration and production, as well as the manufacturing of chemicals. The principal operations of the company are conducted through the company’s subsidiaries, Occidental Oil and Gas Corporation and Occidental Chemical Corporation.

The contractual characteristics of the callable bond issue are summarized in the second column of Table I. The callable ten-year note was issued on 7/1/89 with an initial size of $300 million and a semi-annual coupon at an annual rate of 9.625%. The issue was rated Baa3 by
Table I
Contractual characteristics of two bonds issued by Occidental. Source: FISD.

<table>
<thead>
<tr>
<th>Security</th>
<th>Callable 10yr note</th>
<th>Straight 12yr note</th>
</tr>
</thead>
<tbody>
<tr>
<td>Issue date</td>
<td>7/1/89</td>
<td>11/15/89</td>
</tr>
<tr>
<td>Maturity</td>
<td>7/1/99</td>
<td>11/15/01</td>
</tr>
<tr>
<td>Amount issued (MM)</td>
<td>$300</td>
<td>$330</td>
</tr>
<tr>
<td>Coupon</td>
<td>9.625%</td>
<td>10.125%</td>
</tr>
<tr>
<td>Credit rating (Moodys)</td>
<td>Baa3</td>
<td>Baa3/Baa2</td>
</tr>
<tr>
<td>Seniority</td>
<td>Senior/Unsecured</td>
<td>Senior/Unsecured</td>
</tr>
<tr>
<td>First call date</td>
<td>7/1/96</td>
<td>–</td>
</tr>
<tr>
<td>Redemption</td>
<td>7/1/96 at par</td>
<td>–</td>
</tr>
</tbody>
</table>

Moodys on 3/17/94, and neither downgraded nor upgraded by that rating agency thereafter. The notes were redeemable on or after 7/1/96, at Occidental's option, at a redemption price equal to 100% of the principal amount (see Figure 13 in Appendix F). The entire issue was called on 7/1/96, its first call date. The empirical analysis will also address the relative pricing of callable and noncallable bonds. Hence, I collect data on a straight Occidental bond with the same credit rating and with similar features as the callable issue. For example, on 11/15/89, Occidental issued a straight twelve-year note with an initial size of $330 million and a semi-annual coupon at an annual rate of 10.125%. On 3/17/94, Moodys assigned a Baa3 credit rating to this bond, which was upgraded one notch to Baa2 on 12/16/96, but again downgraded to Baa3 on 2/2/99.10 Neither bond had sinking fund provisions nor any variation in promised coupon payments over time. Both notes were senior-unsecured, nonputtable, nonconvertible, and nonexchangable. They did not default prior to maturity or redemption (see, for example, FISD (2000)).

Datasync provides weekly (each Wednesday) market price information for both issues, for the period 1/10/90 to 12/6/95. Figure 2 shows the corresponding (monthly) yield spreads relative to the U.S.-dollar swap curve.11 Of particular interest is the surge in yield spreads during the Fall of 1990. The callable note lost 9% of its market value between 8/1/90 and 10/10/90, while the value of the straight issue dropped 12% over the same period of time.
Both issues recovered quickly and were back at August 1990 levels by mid-January 1991. The economic background pertinent to these observations is the Gulf War. After Iraq’s invasion of Kuwait in August, 1990, anxiety about Mideast stability caused oil prices to jump from $17 a barrel in July to $36 by October, 1990. On the other hand, confidence in the U.S. economy faltered and consumer spending fell off which forced the U.S. economy into a recession in the Fall of 1990. In general, higher oil prices mean good news to oil producing companies (given they have sufficient reserves or crude-oil production), boosting their stock and bond prices. So why did Occidental’s bond (and stock prices) prices fall instead of rise? Contrary to popular conception, Occidental at that time was more of a chemical company than an oil and gas producer, as illustrated in Figure 14 of Appendix F. Hence, I attribute the sharp rise in Occidental’s yield spreads to a drastic softening in the U.S. chemical markets just prior to the Gulf War.

**Figure 2.** Yield spreads (relative to Dollar swap curve) of the callable and straight Occidental bonds. Source: Datastream.
In Figure 2, one further observes very low yield spreads for the callable bond from December 1994 forward, and even an occasional trading through LIBOR (periods of negative yield spreads). One possible explanation for the observed overpricing when approaching the first call date in July 1996 is that investors had assigned a (significant) positive probability to the event that Occidental would not (or would not be in a position to) redeem this issue at the first possible call date. Interest rates had fallen substantially since the initiation of the bond (see Figure 12 of Appendix A), hence investors would have profited considerably from Occidental’s decision to postpone redemption of the debt beyond the “optimal” date. Another observation is that the callable bond yield spreads appear to be more volatile. While taking into account the shorter effective duration of the callable note, I suspect this to be due, at least partly, to issue-specific illiquidity risks caused by different clientele trading patterns, or asymmetrically informed traders, or data noise. Near the optimal call date, the implied yield spread (and short spread) is of course extremely sensitive to the measured price.

A. Call-Corrected Short Spreads

Preliminary estimates of the parameters in Equations (7) and (8) for the callable Occidental bond are displayed in Table II, together with their MC distribution. I impose three over-

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Estimate$^{13}$</th>
<th>MC parameter distribution</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>1$^{st}$ quartile</td>
</tr>
<tr>
<td>$\theta$</td>
<td>69</td>
<td>-</td>
</tr>
<tr>
<td>$K$</td>
<td>0.18</td>
<td>0.04</td>
</tr>
<tr>
<td>$\Sigma$</td>
<td>20</td>
<td>-</td>
</tr>
<tr>
<td>$\bar{K}$</td>
<td>-0.23</td>
<td>-0.23</td>
</tr>
</tbody>
</table>

identifying restrictions in order to reduce the dimension of the parameter space. First, I set the market price of spread risk factor, $\Lambda$, equal to zero. In preliminary empirical investigations I
found \( \Lambda \) to be of small absolute value, with the data being unable to pin down its value with any reliability. Second, I set \( \Sigma = -\frac{\theta}{\Phi(\varepsilon)\sqrt{(1-e^{-2\tilde{\kappa}})/(2\tilde{\kappa})}} \), where \( \Phi \) denotes the standard normal cumulative distribution function. This ensures that the risk-neutral one-year ahead distribution of the short spread, conditional on the current short spread being equal to \( \theta \), is negative with only a small probability \( \varepsilon \). Lastly, I set the long-run mean under the data-generating measure, \( \theta \), equal to the implied sample mean\(^{14} \) in order to improve the average fit of the spread curve to the bond yield spreads.

I note that the mean-reversion parameter under the risk-neutral measure, \( \tilde{\kappa} \), is negative, which is in line with the results for noncallable debt found by Duffee (1999) for many issuers. Recalling that \( \Lambda \) equals zero, one interpretation of this is that investors have a high degree of risk aversion with respect to an increase of the short spread above its long-run average. The estimated mean term structure of yield spreads\(^{15} \) is slightly downward sloping. Figure 3 shows the time series of the implied short spreads of the callable Occidental note. The basic

![Figure 3. Implied short spreads for the callable Occidental bond.](image)

pattern follows that of the yield spreads in Figure 2. Near the first call date, the effective

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maturity of the callable Occidental bond is basically July 1996. This can be seen in Figure 15 in Appendix F which displays the time series of the estimated median and quartiles of the time of optimal exercise. Near the optimal call date, the implied short spread is extremely sensitive to the measured price.

B. Interpreting the Short Spread

The likelihood of a default by Occidental is influenced by firm-specific balance-sheet and related macroeconomic variables, on which I now focus. As illustrated in Figure 14 in Appendix F, chemicals (rather than oil and gas) was Occidental’s main industry segment throughout the observation period (1/90 to 12/95). Hence, during that time, a boost to the chemicals market would have been expected to lower the probability of default by Occidental, and thereby lower the short spread $s$ (naturally assuming that these are in a monotonic relationship). Further, Occidental’s leverage ratio is defined as the book value of its debt divided by the sum of the market value of Occidental’s equity plus the book value of its debt. A higher leverage ratio should raise the level of the short spread. As a preliminary examination of the potential influence of such covariates, I regress the implied short spread of the callable bond, $s$, onto the Datastream index for U.S chemicals (CHEM) and Occidental’s leverage ratio (LEV) using monthly data from January 1990 to December 1995.\textsuperscript{16}

$$s_t = 69 - 0.21 \times \text{CHEM}_t + 2.78 \times \text{LEV}_t + \varepsilon_t$$

(3.5) (0.04) (1.11)

I obtain an $R^2$ of about 43.6%. Newey-West heteroskedasticity-corrected standard errors are reported in parentheses. Both coefficients show the expected sign, and are significant at conventional levels.\textsuperscript{17}
C. Decomposing Callable Bond Prices

I am now in a position to value the American option embedded in the callable bond, and thereby disentangle the components of the callable bond prices due callability and due to default and illiquidity risk. Figure 4 shows the disentangled components of Occidental’s callable bond prices. It matches Figure 1 in that it shows the market prices of the callable Occidental bond together with the prices at which this bond would trade if it was noncallable and default-free. I am now in a position, however, to show also the prices at which this bond would trade if it was noncallable (but still defaultable). In other words, I can compute the prices of the straight version of the bond, and thereby specify what portion of the total price difference between the callable, defaultable bond and its noncallable, default-free equivalent is the price of the American call option, and what part is due to discounting for default and illiquidity risk.

![Figure 4. Prices of the callable Occidental bond, of its noncallable, defaultable equivalent, and of its noncallable, default-free equivalent.](image)
Figure 5 displays the amounts by which the theoretical price of the noncallable, defaulatable bond and the theoretical price of the noncallable, default-free bond exceed the market price of the callable Occidental bond, respectively. The shaded area underneath the graph associated with the noncallable, defaulatable bond shows the amounts by which Occidental’s bond prices are less than the theoretical values of the underlying straight bond, hence giving the implied values of the American call option embedded in the callable bond. One observes that the call option gains in value throughout time until just prior to the first call date in July 1996. The only exception is the time around the Gulf War, when Occidental’s bond prices were so weak that the early redemption option was far out of the money. The noncallable, default-free bond prices exceed the theoretical values of the noncallable, defaulatable bond by the amounts highlighted in the shaded area between the corresponding two lines, indicating the reduction in price due to discounting for default and illiquidity risk.
D. Relative Pricing of Callable and Straight Debt

In order to investigate the relative pricing of callable and noncallable bonds, I choose the straight Occidental issue as benchmark and treat the callable note as a non-benchmark bond. This selection is based on the observation that the straight note is of higher volume, and that it appears to be more liquid across the observation period. The "recursive" short spread model described in Equations (7) through (10) allows estimation of the parameters of the short spread of the straight bond, $s^s$, in a first stage, followed by the estimation of the parameters of the idiosyncratic factor of the callable Occidental bond, $u^c$, in a second step. Preliminary estimates of the parameters in Equations (7) and (8) for the straight Occidental bond are displayed in Table III, together with their MC distribution. I then treat the estimates of $\theta$, $K$, $\Sigma$, $\Lambda$, and $\check{K}$ as the true parameters, and estimate the parameters of the idiosyncratic spread risk component in Equations (9) and (10), and their MC distribution, for the callable issue. For the benchmark bond, I impose over-identifying restrictions analogous to those described in Section A. In particular, I enforce $\Lambda = 0$. In order to reduce the dimension of the parameter space associated with the short spread of the non-benchmark callable issue, I set the mean-reversion parameter

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Estimate</th>
<th>MC parameter distribution</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1st quartile</td>
<td>median</td>
</tr>
<tr>
<td>$\theta$</td>
<td>85</td>
<td>-</td>
</tr>
<tr>
<td>$K$</td>
<td>2.16</td>
<td>1.73</td>
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<tr>
<td>$\Sigma$</td>
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</tr>
<tr>
<td>$\check{K}$</td>
<td>-0.30</td>
<td>-0.31</td>
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<tr>
<td>$\check{\theta}$</td>
<td>-23</td>
<td>-</td>
</tr>
<tr>
<td>$\kappa$</td>
<td>7.74</td>
<td>6.35</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>72</td>
<td>69</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>10</td>
<td>-</td>
</tr>
</tbody>
</table>

25
of the idiosyncratic factor under the risk-neutral measure, \( \bar{\kappa} \), equal to \( \bar{\kappa} \) (that is, \( \bar{\kappa} = -0.30 \)), and also require that its market price of risk component, \( \lambda \), equals \( \frac{\bar{\kappa} \theta}{\sigma} \). Lastly, I choose \( \theta \) as described in Note 14.

The results suggest that the callable bond short (or yield) spreads were indeed significantly more volatile. Moreover, the estimate of \( \theta \), -23 basis points, suggests that the callable bond traded "rich" relative to the noncallable. This might be due to an assumption by investors of "sub-optimal" calling behavior by Occidental. Figure 6 displays the time series of the implied short spreads of the straight Occidental note. Again, the basic pattern follows that of the associated yield spreads in Figure 2. The model-implied estimates of the idiosyncratic factor \( \epsilon^c \) are shown in Figure 7.

**Figure 6.** Implied short spread for the benchmark straight Occidental bond.

**Figure 7.** Implied idiosyncratic factor, \( \epsilon^c \), for the callable Occidental bond.

In further applications I will compare the yield spread of the straight version of the callable bond that is theoretically implied by the short spread of the callable bond (the "call-corrected" yield spread) with the yield spread that is implied by the short spread of the straight bond. Additionally, given a simple model of expected recovery at default and for the mean fractional liquidity cost rate, I am able to estimate the implied risk-neutral probabilities of default. The actual probabilities of default can be estimated on the basis of estimated risk premia. Finally, using the parameter estimates and implied short spreads from Tables II and III and Figures 3
and 6, I can compute model-implied prices for out-of-sample bonds which were not included in the estimation procedure, but were issued by Occidental and had similar contractual features.

E. Call-Corrected Yield Spreads

For a callable bond, the call-corrected yield spread is defined as the theoretical yield spread, relative to the reference curve, of the underlying straight bond (see Figure 5). Figure 8 shows the call-corrected yield spreads of the callable Occidental bond, using the stand-alone parameter estimates and model-implied short spreads reported in Table II and Figure 3. Unlike the quoted yield spreads of a callable bond (see Note 11), these spreads are computed based on a fixed maturity. Call-corrected yield spreads may be of particular use when communicating credit pricing information among traders and investors, for example when comparing them to the yield spreads of other, callable or straight, bonds of the same issuer, or other issuers. These yield spreads offer a uniform cross-market measure of the credit risk and illiquidity priced into corporate bonds.

I now turn to a comparison of the noncallable yield spreads of Occidental’s callable bond with those based on a common market practice for pricing callable corporate debt. Practitioners often value callable defaultable bonds based on a term-structure model calibrated to (or estimated from) the prices of straight bonds of the same credit rating. As outlined in the introduction to this paper, this does not allow for idiosyncratic risk factors specific to the callable bond under investigation. I implement this approach by using the parameter estimates (Table III) and corresponding short spreads (Figure 6) backed-out from the straight bond prices to value the straight bond underlying the callable Occidental issue. Figure 8 shows the associated not-call-corrected yield spreads, together with the call-corrected yield spreads computed from the parameter estimates and model-implied short spreads for the callable bond as reported in Section A. For example, the not-call-corrected yield spread exceeds its call-corrected counterpart by 76 basis points in June 1994, and on average underprices the callable issue by 14
basis points. Consequently, in the case of Occidental, using not-call-corrected yield spreads as a substitute for call-corrected yield spreads produces misleading results.

![Figure 8](image1.png) ![Figure 9](image2.png)

**Figure 8.** Noncallable yield spreads (relative to Dollar swap curve) of the callable Occidental bond using the short spread estimated from callable bond prices (call-corrected) in Section A, and as quoted (see Figure 2).

**Figure 9.** Noncallable yield spreads (relative to Dollar swap curve) of the callable Occidental bond using the short spread estimated from callable bond prices (call-corrected) in Section A, and estimated from straight bond prices (not call-corrected).

### F. Estimation of Implied Default Probabilities

Given a simple model of recovery at default and for the mean fractional cost rate capturing the illiquidity risk, I am now in a position to estimate the implied risk-neutral probabilities of default. For $t < \bar{t}$, let $\bar{p}(t, \bar{t})$ denote the risk-neutral probability of default before time $\bar{t}$, given that default did not occur by $t$. Under technical conditions discussed, for example, in Duffie (2001), $\bar{p}$ is given by

\[
\bar{p}(t, \bar{t}) = 1 - E_t^Q \left( e^{-\int_t^{\bar{t}} h_u \, du} \right). 
\]  

(15)

Given the stochastic model of the short spread $s$, associated with the estimated parameter vector $\hat{\Theta}$, one can use the relationship $h = \frac{s - l}{l}$ from Equation (1) to calculate the default
probability \( \tilde{p}(t, \tilde{T}) \) from Equation (15) (see comments appendent to Equation (4)). Under the simplifying assumption that the risk-neutral mean fractional loss in market value at default, \( L \), as well as the risk-neutral mean fractional cost rate due to illiquidity, \( l \), are constants, Equations (7) and (8) yield

\[
 dh_t = \tilde{K} \left( \tilde{\Theta} - \frac{l}{L} - h_t \right) dt + \frac{\Sigma}{L} d\tilde{\nu}_t. \tag{16}
\]

In line with Duffee (1999) and Driessen (2002), I assume a constant value of 56% for \( L \) for Occidental’s senior unsecured bonds. Data limitations make it difficult to determine a breakdown of Occidental’s total short spreads into the risk-neutral mean fractional loss rate due to credit risk, \( hL \), and the risk-neutral mean fractional cost rate due to illiquidity, \( l \). Ignoring the component due to illiquidity (that is, assuming that \( l = 0 \)), Figure 10 shows the associated risk-neutral probabilities of default of the callable Occidental bond before its first call date in July 1996. These values are computed according to Equations (15) and (16) using the pa-

![Figure 10](image_url)

Figure 10. Implied risk-neutral probabilities of default by the first call date (7/1/96) for the callable Occidental bond in Section A.
rameter estimates and implied short spreads of the callable Occidental bond from Table II and Figure 6.

The data-generating (actual) probabilities of default can be estimated on the basis of estimated risk premia (see, for example, Driessen (2002) and the literature cited therein). Under the simplifying assumption that the default timing risk has no risk premium, the actual intensity of default is also $h$. (See Jarrow, Lando, and Yu (2001) for a discussion of conditions under which this property holds.) I emphasize that this conclusion does not imply that actual and risk-neutral default probabilities are the same, for while the respective default intensity processes are assumed to have the same outcomes in each state, their probability distributions under $P$ and $Q$ may differ. If the actual intensity of default is also $h$, for $t < \bar{t}$, the actual probability of default before time $\bar{t}$, provided that default did not occur by $t$, is

$$p(t, \bar{t}) = 1 - E_t \left( e^{-\int_t^{\bar{t}} h_u du} \right),$$

which can be estimated using the estimated data-generating mean-reversion parameter $K$. For Occidental, the actual probabilities of default are somewhat lower (higher) than their risk-neutral counterparts given that the current short spread is above (below) the long-run mean.

**G. Pricing Out-of-Sample Bonds**

From Datastream, I obtained market price information for an additional (out-of-sample) U.S.-dollar denominated Occidental debenture, whose contractual characteristics are summarized in Table IV. This callable ten-year note was issued on 5/1/88, with an initial principal of $200 million and a semi-annual coupon at an annual rate of 10.75%. The issue was rated Baa3 by Moody’s on 3/17/94, and neither downgraded nor upgraded by that rating agency thereafter. This debenture was redeemable at any time on or after 5/1/95, at Occidental’s option, at a redemption price of 100% of the principal amount. The entire issue was called in at par effective 5/1/95, its first call date.
Table IV

Contractual characteristics of the out-of-sample bond issued by Occidental. Source: FISD.

<table>
<thead>
<tr>
<th>Security</th>
<th>Callable 10yr note</th>
</tr>
</thead>
<tbody>
<tr>
<td>Issue date</td>
<td>5/1/88</td>
</tr>
<tr>
<td>Maturity</td>
<td>5/1/98</td>
</tr>
<tr>
<td>Amount issued (MM)</td>
<td>$200</td>
</tr>
<tr>
<td>Coupon</td>
<td>10.75%</td>
</tr>
<tr>
<td>Credit rating (Moodys)</td>
<td>Baa3</td>
</tr>
<tr>
<td>Seniority</td>
<td>Senior/Unsecured</td>
</tr>
<tr>
<td>First call date</td>
<td>5/1/95</td>
</tr>
<tr>
<td>Redemption</td>
<td>5/1/95 at par</td>
</tr>
</tbody>
</table>

Figure 11 shows the market prices and the model-implied prices of this out-of-sample callable Occidental bond. The model-implied price that is labeled “model-implied (callable)” is defined as the market value of the out-of-sample bond when priced using the parameter estimates and implied short spreads of the callable Occidental bond, from Table II and Figure 3 respectively. Figure 11 also shows the model-implied prices of the out-of-sample debenture that are computed based on the parameter estimates and the implied short spreads that are associated with the straight Occidental bond. The two models fit the out-of-sample debenture prices reasonably well. In particular, the model-implied prices fluctuate around the actual observed market prices, and do not consistently over- or underestimate them. The average absolute relative pricing error amounts to 1.1% and 1.5% for the model-implied prices relative to the callable and straight bonds, respectively. It is noteworthy, however, that during the two years prior to the first call date of the out-of-sample bond in May 1995, its market prices are comparably high and “jumpy”, while the lower model-implied prices smoothly approach the strike price.
Figure 11. Empirical and model-implied prices of the out-of-sample Occidental bond using the short spread estimated from callable bond prices (callable) in Section A, and estimated from straight bond prices (straight).
References


Huang, J.-Z., and M. Huang, 2002, How Much of the Corporate-Treasury Yield Spread is Due to Credit Risk, Working Paper, Penn State University.


Appendix A. Implied Short Rates

Figure 12 shows the implied levels of the short-rate process $r$, together with the two- and ten-year U.S.-dollar LIBOR-quality swap yields.

![Graph showing implied short rates and swap yields](image)

**Figure 12.** Implied short rates, two-year and ten-year LIBOR-quality swap yields.

Appendix B. Admissibility of Market Price of Spread Risk

In this appendix, I show that the parametrization of the market price of spread risk in Equation (8) is admissible. Let $Z$ solve the stochastic differential equation

$$dZ_t = K(\theta - Z_t) dt + \Sigma dW_t, \quad Z_0 = 0,$$
where \( k, K \) and \( \Sigma \) are constants, and \( W \) is a standard Brownian motion under the data-generating measure. I rewrite \( Z_t \) as

\[
Z_t = \theta (1 - e^{-Kt}) + \Sigma e^{-Kt} \int_0^t e^{Ks} dW_s \\
= \theta (1 - e^{-Kt}) + \Sigma \left( W_t - Ke^{-Kt} \int_0^t e^{Ks} W_s ds \right),
\]

where the last equation uses Itô's formula. Let \( W_t^* = \max_{0 \leq s \leq t} |W_s| \). Then, for each \( T \geq 0 \) and any \( t \in [0, T] \),

\[
|Z_t| \leq |\theta| |1 - e^{-Kt}| + |\Sigma| \left( 1 + |1 - e^{-Kt}| \right) W_t^* \\
\leq \left[ |\Sigma| + (|\theta| + |\Sigma|) e^{K|T|} \right] (1 + W_t^*).
\]

Consequently,

\[
\xi_t = e^{|Z_t| dW_t - \frac{1}{2} |Z_t|^2 ds}, \quad t \geq 0
\]

is a \( F_t \)-martingale (see, for example, Karatzas and Shreve (1998)).

**Appendix C. Implementation of the LSM Algorithm**

With the notation of Section IIIC, I fix some time \( t \) before maturity. The objective is to estimate the market value \( V_t \) of the callable bond, given the state vector \( X_t \) and the parameter vector \( \Theta \). I implement the LSM algorithm as a two-step procedure. The first step is designed to characterize an approximate exercise strategy, which is then used in the second step to value the callable bond using standard Monte Carlo (MC) techniques.

**Characterization of the Approximate Stopping Rule**

1. Discretize problem: \( L - l + 1 \) exercise opportunities at \( T_l < \cdots < T_L \)
2. Simulate, under the risk-neutral measure, $Z$ sample paths of the state vector starting from $X_0, X^1, \ldots, X^Z$, and of the discount factors $D^1, \ldots, D^Z$.

\[
X^z = (X^z_1, \ldots, X^z_T)', D^z = (D^z_1, \ldots, D^z_T)', D_k = e^{-\int_T^k R_u du}
\]

3. At time $T_L$, the risk-neutral expected value from continuation and the value of the callable bond for path $z$, as fractions of face value, are respectively given by:

\[
C^z_L = (1 + c) E^Q \left( e^{-\int_{T_L}^T R_u du} \left| X_L = X^z_L \right. \right)
\]

\[
V^z_L = I_{C^z_L > 1} + C^z_L I_{C^z_L \leq 1}
\]

4. Repeat for $k = L - 1, \ldots, l$:

- At time $T_k$, the value from continuation for path $z$ is given by:

\[
C^z_k = \frac{D^z_{k+1}}{D^z_k} V^z_{k+1}
\]

- Choose $M_k$ and find $a^k = (a^k_1, \ldots, a^k_{M_k})$ so that

\[
a^k = \arg \min_{a \in \mathbb{R}^{M_k}} \sum_{z=1}^Z \left[ C^z_k - a \cdot e^{M_k(X^z_k)} \right]^2,
\]

where $\{e_m(\cdot)\}$ is a sequence of real functions on $\mathbb{R}^3$ and $e^{M(\cdot)} = (e_1(\cdot), \ldots, e_M(\cdot))$.

- Estimate the value of the callable bond for path $z$ as:

\[
V^z_k = I_{a^k \cdot e^{M_k}(X^z_k) > 1} + C^z_k I_{a^k \cdot e^{M_k}(X^z_k) \leq 1}
\]

5. Store $M_k$ and $a^k$ for $k = l, \ldots, L - 1$.

**MC Simulation of Callable Bond Prices**

1. Simulate, under the risk-neutral measure, $\tilde{Z}$ sample paths of the state vector $X^1, \ldots, X^\tilde{Z}$ starting from $X_0$, and of the discount factors $D^1, \ldots, D^\tilde{Z}$.
2. Set the exercise time $\tau^z$ for path $z$ equal to:

$$\min \left\{ T_k : a^k \cdot e^{M_k(X_k^z)} > 1, l \leq k < L, T_k I_{C_k^z > 1} + T I_{C_k^z \leq 1} \right\}$$

3. Price the callable bond at time $t$ using standard MC simulation:

$$\hat{V}_t = \frac{1}{Z} \sum_{Z=1}^{Z} \left[ \sum_{\bar{Y} \leq T_k \leq \tau^z} cD^x_k + D^z_k \right].$$

I particular, I set $Z = 10,000$ and $\bar{Z} = 100,000$. Moreover, in the second step of the algorithm, I apply antithetic sampling to the short spread.

**Appendix D. Simulation of Sample Paths of the State Vector and the Discount Factors**

In this appendix, I describe how to simulate sample paths of an affine diffusion of the form $X = (Y, Z) \in \mathbb{R}_+^n \times \mathbb{R}^m$ which solves the stochastic differential equation

$$dY = (k^y - K^y Y_t) dt + \sqrt{\text{diag}(Y_t)} dW^y_t$$
$$dZ = (k^z - K^z Y_t - K^{zz} Z_t) dt + \sqrt{\text{diag}(\alpha^z + \beta^z Y_t)} dW^z_t,$$

where $k^y$ and $k^z$ are $n$ and $m$-dimensional vectors, $K^y$ is a diagonal $n \times n$-matrix, $K^y$ and $K^{zz}$ are $m \times n$ and $m \times m$ matrices, and $(W^y, W^z)$ is a standard Brownian motion in $\mathbb{R}^n \times \mathbb{R}^m$. $\text{diag}(x)$ denotes a diagonal matrix with the respective elements of the vector $x$ as its diagonal entries.

Given $(Y_t, Z_t)$, I wish to simulate from the risk-neutral distribution of $(Y_{t+h}, Z_{t+h})$. Bayes rule and the Markov property of $Y$ establish that

$$\text{Prob}((Y_{t+h}, Z_{t+h})|(Y_t, Z_t)) = \text{Prob}(Y_{t+h}|Y_t) \text{Prob}(Z_{t+h}|(Y_t, Z_t), Y_{t+h}). \quad \text{(D1)}$$
Equation (D1) indicates that I can generate \((Y_{t+h}, Z_{t+h})\) from \((Y_t, Z_t)\) by first simulating \(Y_{t+h}\) from \(Y_t\), and then, given information for \(Y_t, Y_{t+h}, \text{and } Z_t\), I can generate \(Z_{t+h}\). As I show, one is able to carry out the first simulation explicitly, while, for the second part, in general no explicit expression is known, and one needs to simulate from an approximate distribution instead.

For any component \(Y^i\) of the vector \(Y\) \((i = 1, \ldots, n)\), set \(C_i = \frac{1}{4K_{ii}^y} \left(1 - e^{-K_{ii}^y h}\right)\) and \(M_i = \frac{4K_{ii}^y}{e^{K_{ii}^y h} - 1} Y^i_t\). Under the independence assumption for \(Y\), which is equivalent to assuming that \(K_{ii}^y\) is diagonal, \(\frac{Y^i_{t+h}}{C_i}\) has a non-central \(\chi^2_{v_i}(M_i)\)-distribution whose degrees of freedom \(v_i\) is \(4K_{ii}^y\), and whose non-centrality parameter is \(M_i\). Consequently, the Laplace transform \(\varphi(\cdot)\) of \(\frac{Y^i_{t+h}}{C_i}\) is given by, for \(v > 1\),

\[
\varphi(t) = \frac{1}{(2t + 1)^{v/2}} e^{-\frac{t M_i}{2v+1}} = \left[\frac{1}{(2t + 1)^{(v-1)/2}}\right] \left[\frac{1}{(2t + 1)^{1/2}} e^{-\frac{t M_i}{2v+1}}\right].
\]  

(D2)

For \(v > 1\), the first factor on the right-hand side of Equation (D2) is the Laplace transform of a \(\chi^2_{v_i-1}\)-distribution with \(v_i - 1\) degrees of freedom.\(^{21}\) The second factor is the Laplace transform of the square of a normal distribution with mean \(\sqrt{M}\) and variance 1. Hence, for \(v > 1\),

\[
\chi^2_{v}(M) =_d \chi^2_{v-1} + \left(N(\sqrt{M}, 1)\right)^2,
\]

where \(=_d\) denotes equivalence in distribution. Most (statistical) software packages, such as Matlab or Splus, allow one to generate random numbers from a \(\chi^2_v\) or \(\Gamma(\frac{1}{2}, \frac{v}{2})\) distribution, provided \(v > 0\). I can, therefore, generate \(Y^i_{t+h}\) by generating and then adding two independent random variables, one of which has a \(\chi^2_{v_i-1}\)-distribution and the other of which is the square of a \(N(\sqrt{M_i}, 1)\)-variable.

The distribution of \(Z_{t+h}\) conditional on \((Y_t, Z_t)\) and \(Y_{t+h}\) is generally not explicitly known. However, as Duffie, Pedersen, and Singleton (2003) point out, the distribution of \(Z_{t+h}\) condi-
tional on $Z_t$ and the entire path of $\{Y_s : t \leq s \leq t + h\}$ is normally distributed with mean and variance given by

$$E \left( Z_{t+h} | Z_t, \{Y_s \}_{s=t}^{t+h} \right) = e^{-K\tau h} \left[ Z_t + \int_0^h e^{-K\tau s} (K\xi - K\alpha Y_s) \, du \right],$$

$$Var \left( Z_{t+h} | Z_t, \{Y_s \}_{s=t}^{t+h} \right) = e^{-K\tau h} \int_0^h e^{K\tau u} \text{diag}(\alpha \xi + \beta \alpha Y_t) e^{K\tau u} \, du \, e^{-K\tau h},$$

respectively. I approximate $Y$ on the interval $[t, t + h]$ by its linear interpolation.

**Appendix E. Identification of $s_t$ given $\Theta$**

I consider a callable bond, as described in Section IIIIB, and fix some time $t$ before its maturity. Let $s^1 < s^2$ denote two possible outcomes of $s_t$. The respective optimal exercise strategies $\tau^*_1$ and $\tau^*_2$ trigger exercise of the embedded call option as soon as the immediate exercise value is less than the value of the bond if not called. According to Equations (7) and (8), for any $i > t$,

$$s_i = e^{-\tilde{\theta}(i-t)} \left( s_{i-1} \tilde{\theta}(i-t) e^{\tilde{\theta}(i-t)} - 1 \right) + \sum_{j=t}^{i} e^{\tilde{\theta}(j-t)} d\tilde{W}_j,$$

where $\tilde{\theta} = \theta - \frac{\Delta \Sigma}{\Delta}$. Hence, for some stopping time $\tau$, using the notation for $V_t$ introduced in Section IIIIB,

$$V_t(\tau|v_t, r_t, s_t = s) = E_t^Q \left[ \sum_{i < \tau \leq \tau^*_1} c e^{-\tilde{\theta}(i-t) r_t + s_t d \tilde{W}_t} + e^{-\tilde{\theta}(i-t) r_t + s_i d \tilde{W}_t} \right]$$

$$= E_t^Q \left[ \sum_{i < \tau \leq \tau^*_1} c e^{-\tilde{\theta}(i-t) r_t + e^{-\tilde{\theta}(i-t)} \left( s + \tilde{\theta}(e^{\tilde{\theta}(i-t)} - 1) \right) + \sum_{j=t}^{i} e^{\tilde{\theta}(j-t)} d\tilde{W}_j} \right]$$

$$+ e^{-\tilde{\theta}(\tau^*_1-t) r_t + e^{-\tilde{\theta}(\tau^*_1-t)} \left( s + \tilde{\theta}(e^{\tilde{\theta}(\tau^*_1-t)} - 1) \right) + \sum_{j=t}^{\tau^*_1} e^{\tilde{\theta}(j-t)} d\tilde{W}_j}$$

is strictly decreasing in $s$. Consequently,

$$V_t(v_t, r_t, s_t = s^1) = V_t(\tau^*_1|v_t, r_t, s_t = s^1) > V_t(\tau^*_1|v_t, r_t, s_t = s^2)$$

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\[ V_t(v^*_x | v_t, r_t, s_t = s^2) = V_t(v_t, r_t, s_t = s^2). \]

Appendix F. Supplementary Data on Occidental

Bond Yields

Figure 13 displays the yields-to-maturity and the yields-to-first-call of the callable Occidental issue. The yield-to-first-call exceeds the yield-to-maturity throughout 1990, until January 1991. This relationship inverts, however, after May, 1991, because the option of calling the issue at the first possible date goes consistently deeper into the money.
Industry Segments

Figure 14 shows the annual net sales and operating revenues of Occidental’s two main industry segment, obtained from Global Access. Throughout the observation period of Occidental’s

Figure 14. Net sales and operating revenues from Occidental’s oil and gas operations and its chemicals operations. Source: Global Access.

bond prices, January, 1990, to December, 1995, Occidental was apparently more of a chemical company than an oil and gas producer.

Implied Optimal Exercise Date

Figure 15 shows the time series of the model-implied median and quartiles of the conditional risk-neutral distribution of the optimal exercise date for the callable Occidental bond. These estimates are based on the short spread parameters reported in Table II in Section IVA.
**Figure 15.** Implied conditional risk-neutral distribution of the optimal exercise date for the callable Occidental bond in Section IVA.
Notes

1The idea of using regression methods previously appeared in Carrière (1996), and is also discussed in Tsitsiklis and Roy (1999, 2001).


3In recent contributions to the pricing of European optionality within the affine framework, numerically accurate and computationally efficient approximations to swaption prices have been developed. For example, Singleton and Umantsev (2002) provide theoretical results based on an affine approximation of the decision boundary and Fourier inversion methods, whereas Collin-Dufresne and Goldstein (2002) use an Edgeworth-expansion technique to estimate the probability distribution of the future asset price. These papers, however, do not treat the American case.

4In addition, some of the empirical literature on corporate bonds investigates such features as contractual characteristics, the value of the call option, and the time of call. Features of callable corporate bonds have been studied empirically by Narayanan and Lim (1989), McDonald and de Gucht (1999), Mitchell (1991), Guentay, Prabgala, and Unal (2001), and others. Previous empirical attempts to estimate the value of the embedded call option, such as Jen and Wert (1967), Pye (1967), Bodie and Friedman (1978), Crabbe (1991), and King (2001), do not use an explicit (arbitrage-free) bond pricing model. Some studies of callable bonds have adopted the so-called standard call policy to call the bond whenever the market price exceeds the call price. Several empirical tests, however, find that this policy can be consistently violated (Vu (1986), Longstaff and Tuckman (1994), King and Mauer (2000)).

5This is a filtration satisfying the usual conditions. See Protter (1990).
The short-rate process \( r \) is \( \{F_t : t \geq 0\} \) progressively measurable with \( \int |r_s| \, ds < \infty \text{ a.s.} \), and 
\[
\mathbb{E}^Q \left[ e^{-\int_0^t r_s \, ds} \right] < \infty \text{ for all } t.
\]

Using a new set of sample paths to value the bond in the second step guarantees that, up to MC noise, the estimated price is a "true" upper bound for the value of the callable bond.

In practice, one often estimates the conditional expected value of the bond at \( T_i \), if not exercised, as a function in polynomials of the intrinsic value \( V_{T_i}(T) \) and/or the unmatured discount bond prices \( B(T_i, T_j) \), with maturity dates \( T_j \) up to and including the final maturity date of the bond.

Here, for a given choice of parameters \( \Theta \), I first calculate \( G_{t_n}(\cdot; \Theta) \) by MC simulation for a sufficiently wide range of outcomes of \( s_{t_n} \). Next, I use Chebyshev polynomials to approximate \( G_{t_n}(\cdot; \Theta) \) and \( DG_{t_n}(\cdot; \Theta) \). Finally, I invert \( G_{t_n}(\cdot; \Theta) \) (approximated) in \( V_{t_n} \) to retrieve an approximation of \( s_{t_n} \), at which I evaluate \( DG_{t_n}(\cdot; \Theta) \).

On 12/20/99, Occidental announced the completion of a tender offer of this issue. The amount tendered and repurchased totaled \$240.286 million, leaving an amount of \$89.714 million outstanding at that time. Evaluation was based on the yield to maturity of the 5.875\% U.S. Treasury note due 11/30/01 at the time of the tender plus 3.7 basis points, and plus interest accrued.

The "yield" on the callable bond is recorded, in practice, as the minimum of the yield-to-maturity and the yield-to-first-call.

That is, I treat the parameter estimates in Table II as the true parameters, generate an independent sample of the time series of observed callable Occidental bond prices, and re-estimate the parameters using ML. I repeat this procedure 100 times in order to retrieve an empirical distribution of the estimated parameters.

Numbers are shown in basis points, except for \( K \) and \( \bar{K} \).
That is, I first determine the sample mean of the implied short spread process under the additional assumption that $\bar{K}$ equals zero, and fix $\theta$ at this level (which is 80 basis points). I estimate the model and compute the sample mean of the implied short spread to be 73 basis points, fix $\theta$ at this level and re-estimate. After the fifth iteration, the sample mean of the implied short spread is 69 basis points, and the estimates remain essentially unchanged.

That is, the term structure of zero-coupon yield spreads evaluated at the implied sample mean, $\theta$.

Both covariates were centered around their respective sample means.

I repeat the regression analysis for the time period that excludes the Gulf War, that is April 1991 to December 1995. The resulting estimated regression model is

$$s_t = 63 - 0.20 \times \text{CHEM}_t + 2.96 \times \text{LEV}_t + \varepsilon_t$$

$$(3.6)(0.05) \quad (1.17)$$

with standard error estimates, corrected for heteroskedasticity using the Newey-West method, shown in parentheses. As expected, without the evident macroeconomic effects of the Gulf War, I now find it harder to explain Occidental's short spread. This difficulty is reflected by the lower $R^2$ of 33.6%. Both coefficients, however, show the expected sign, and are significant at conventional levels.

Numbers are shown in basis points, except for $K$, $\bar{K}$ and $\kappa$.

Fan, Haubrich, Ritchken, and Thomson (2002) price American-style puttable fixed-rate bonds relative to credit spreads that are calibrated to the prices of noncallable bank loans.

Duffie, Pedersen, and Singleton (2003) derive an approximation of the transition probability of such affine diffusions $X$. 

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A $\chi^2_{\nu}$-distribution is a special form of a $\Gamma(\lambda, r)$-distribution with parameters $\lambda = \frac{1}{2}$ and $r = \frac{\nu}{2}$. 