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Symmetry Analysis of Reversible Markov Chains

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Abstract

We show how to use subgroups of the symmetry group of a reversible Markov chain to give useful bounds on eigenvalues and their multiplicity. We supplement classical representation theoretic tools involving a group commuting with a self-adjoint operator with criteria for an eigenvector to descend to an orbit graph. As examples, we show that the Metropolis construction can dominate a max-degree construction by an arbitrary amount and that, in turn, the fastest mixing Markov chain can dominate the Metropolis construction by an arbitrary amount.

1 Introduction

In our work on fastest mixing Markov chains on a graph [BDX03, PXBD03], we encountered highly symmetric graphs with weights on the edges. Examples treated below include the graphs shown in figures 1-5. The graphs in figures 2, 3, 4 and 5 have weights chosen so that the stationary distribution of the associated random walk is uniform. We will show that the walk in figure 2 mixes much more rapidly than the walk in figure 3, and that the walk in figure 4 mixes much more rapidly than the walk in figure 5. For general graphs, we seek good bounds for eigenvalues and their multiplicity using available symmetry.

Let a connected graph \((V, E)\) have vertex set \(V\) and undirected edge set \(E\). We allow loops but not multiple edges. Let \(w(e)\) be positive weights on the edges. These ingredients define a random walk on \(V\) which moves from \(v\) to a neighboring \(v'\) with probability proportional to \(w(v, v')\). This walk has transition matrix

\[
K(v, v') = \frac{w(v, v')}{W(v)}, \quad W(v) = \sum_{v''} w(v, v'').
\]  

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Figure 1: \( F_{mn} \) with \( m \) petals, each a cycle of length \( n \). All edges having weight one.

Figure 2: \( F_{mn} \) with all loops having weight \( m - 1 \), and all edges not incident to the center having weights \( m \).

Figure 3: \( F_{mn} \) with all loops having weight \( 2m - 2 \).

Figure 4: \( K_n - K_n \) with center edge and all loops of weight \( n - 1 \).

Figure 5: \( K_n - K_n \) with all edge and loop weights one.
The Markov chain $K$ has unique stationary distribution $\pi(v)$ proportional to the sum of the edge weights that meet at $v$:

$$\pi(v) = \frac{W(v)}{W}, \quad W = \sum_{v'} W(v').$$ (1.2)

By inspection, the pair $K, \pi$ is reversible:

$$\pi(v)K(v, v') = \pi(v')K(v', v).$$ (1.3)

Reversible Markov chains are a mainstay of scientific computing through Markov chain Monte Carlo [Liu01]. Any reversible Markov chain can be represented as random walk on an edge weighted graph. Background on reversible Markov chains can be found in the textbook of Brémaud [Bré99], the lecture notes of Saloff-Coste [SC97] or the treatise of Aldous and Fill [AF03].

Define $L^2(\pi) = \{f : V \to \mathbb{R}\}$ with inner product $\langle f_1, f_2 \rangle = \sum_v f_1(v)f_2(v)\pi(v)$. The matrix $K(v, v')$ operates on $L^2$ by

$$Kf(v) = \sum_{v'} K(v, v')f(v').$$ (1.4)

Reversibility (1.3) is equivalent to self adjointness $\langle Kf_1, f_2 \rangle = \langle f_1, Kf_2 \rangle$. It follows that $K$ is diagonalizable with all real eigenvalues and eigenvectors.

An automorphism of a weighted graph is a permutation $g : V \to V$ such that if $(v, v') \in E$, $(gv, gv') \in E$ and $w(v, v') = w(gv, gv')$. Let $G$ be a group of automorphisms. This group acts on $L^2(\pi)$ by

$$T_g f(v) = f(g^{-1}v).$$ (1.5)

Since $g$ is an automorphism,

$$T_g K = KT_g, \quad \forall g \in G.$$ (1.6)

**Proposition 1.1** For random walk (1.1) on an edge weighted graph, the stationary distribution $\pi$ defined in (1.2) is invariant under all automorphisms.

**Proof**

$$T_g \pi(v) = \frac{1}{W} \sum_u w(g^{-1}v, u) = \frac{1}{W} \sum_u w(g^{-1}v, g^{-1}u)$$

$$= \frac{1}{W} \sum_u w(v, u) = \pi(v)$$

\[\square\]

It follows that $L^2(\pi)$ is a unitary representation of $G$. 

3
Example 1 (Suggested by Robin Forman) Let $F_{mn}$ be the graph of a “flower” with $m$ petals, each containing $n$ vertices, joined at the center vertex $o$. Thus figure 1 shows $m = 3$, $n = 5$. If $w(e) = 1$ for all $e \in E$, the stationary distribution is highly non-uniform. From (1.2),

$$\pi(o) = \frac{1}{n}, \quad \pi(v) = \frac{1}{mn} \text{ for } v \neq o.$$ 

Our work in this area begin by considering two methods of putting weights on the edges of $F_{mn}$ to make the stationary distribution uniform. The Metropolis weights (figure 2) turn out to lead to a more rapidly mixing chain than the max-degree weights (figure 3). In [BDX03, PXBD03], we show how to find optimal weights that give the largest spectral gap. For $F_{mn}$ these improve slightly over the Metropolis weights. Our algorithms give exact numerical answers for fixed $m$ and $n$. In the present paper we give analytical results.

Example 2 (Suggested by Mark Jerrum) Let $K_n - K_n$ be two copies of the complete graph $K_n$ joined by adding an extra edge as in figures 4 and 5 for $n = 4$. Here, the max-degree weights (shown in figure 5) are dominated by the choice of weights shown in figure 4. Our numerical results show that the optimal choice differs only slightly from the choice in figure 4.

In section two, we review the classical connections between the spectrum of a self adjoint operator and the representation theory of a group commuting with the operator. Examples 1 and 2 are treated. We also review the literature on coherent configurations and the centralizer algebra.

Section three gives our first new results. We show how the orbits of various subgroups of the full automorphism group give smaller 'orbit chains' which contain all the eigenvalues of the original chain. A key result is a useful sufficient condition for an eigenvector of $K$ to descend to an orbit chain. One consequence is a simple way of determining which orbit chains are needed.

In section four the random walk on $F_{mn}$ is explicitly diagonalized. Using all the eigenvalues and eigenvectors, we show that order $n^2 \log m$ steps are necessary and sufficient to achieve convergence to stationarity in chi-squared distance while order $n^2$ steps are necessary and sufficient to achieve stationarity in $L^1$. In section five, all the eigenvalues for any symmetric weights on $K_n - K_n$ are determined. In section six, symmetry analysis is combined with geometric techniques to get good bounds on the weighted chains for $F_{mn}$ (figures 2 and 3). These show that the Metropolis chain is better (by a factor of $m$) than the max-degree chain. As shown in [BDX03], this is the best possible.

For background on graph eigenvalues, automorphisms and their interaction, see Babai [Bab95], Chung [Chu97], Cvetković et al. [CDS95], Godsil-Royle [GR01], or Lauri and Scapellato [LS03].

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2 Background in representation theory

2.1 Representation theory

The interaction of spectral analysis of a self adjoint operator with the representation theory of a group of commuting operators is classical. Mackey [Mac78, pages 17-18], Fässler and Stiefel [FS92, pages 40-43] and Sternberg [Ste94] are good references. Graph theoretic treatment appears in Cvetković et al [CDS95, chapter 5]. In present notation, for $K$, $\pi$ defined at (1.1) and (1.2), let $G$ be a group of automorphisms and $T$ the representation of $G$ on $L^2(\pi)$. If $\lambda$ is an eigenvalue of $K$ with eigenspace $M_\lambda = \{ f : Kf = \lambda f \}$, then

$$ L^2(\pi) = \bigoplus_\lambda M_\lambda $$

where the sum is over distinct eigenvalues of $K$. Of course, $L^2(\pi) = \bigoplus_i V_i$ with $V_i$ some choice of irreducible representations of $G$. Since $T_g M_\lambda = M_\lambda$, these combine to give

$$ L^2 = \bigoplus \bigoplus_i M_{\lambda,i} $$

with the sum over distinct eigenvalues $\lambda$ and then over irreducible representations of $G$, $M_{\lambda,i}$ — in the eigenspace $M_\lambda$.

Example 1 ($F_{mn}$)

For the 'flower' $F_{mn}$ defined in section one

(2.2) the automorphism group is $B_m = S_m \ltimes C_m^n$, the hyperoctahedral group.

(2.3) for $n$ odd,

$$ L^2(\pi) = L_0 \bigoplus_{i=1}^{(n-1)/2} \left( L_{i0} \bigoplus L_{i1} \bigoplus L_{i2} \right) $$

with $L_0$, $L_{i0}$ copies of the one dimensional trivial representation, $L_{i1}$ copies of the $m-1$ dimensional permutation representation, $L_{i2}$ copies of the $m$ dimensional reflection representation of $B_m$.

(2.4) for $n$ even,

$$ L^2(\pi) = L_0 \bigoplus L_* \bigoplus_{i=1}^{(n-2)/2} \left( L_{i0} \bigoplus L_{i1} \bigoplus L_{i2} \right) $$

with notation as in (2.3) and $L_* = L_{*0} \bigoplus L_{*1}$

Corollary 2.1 For any choice of invariant weights (with loops allowed), the corresponding Markov chain on $F_{mn}$ has
(a) \((n \text{ odd})\)

\[
1 + (n - 1)/2 \quad \text{one dimensional eigenspaces}
\]

\[
(n - 1)/2 \quad (m - 1) \text{ dimensional eigenspaces}
\]

\[
(n - 1)/2 \quad m \text{ dimensional eigenspaces}
\]

(b) \((n \text{ even})\)

\[
1 + n/2 \quad \text{one dimensional eigenspaces}
\]

\[
n/2 \quad (m - 1) \text{ dimensional eigenspaces}
\]

\[
-1 + n/2 \quad m \text{ dimensional eigenspaces}
\]

**Remark** Of course, in non-generic situations, some of these eigenspaces may coalesce further. In section four, the chain with edge-weights all equal to one is explicitly diagonalized.

**Proof** Label the vertices of \(F_m\) as \(o\) (center) and \((i, j)\), \(1 \leq i \leq m, 1 \leq j \leq n - 1\). The hyperoctahedral group \(B_m = S_m \ltimes C_2^m\) is the group of symmetries of an \(m\)-dimensional hypercube. Elements are written \((\pi, x)\) with \(\pi \in S_m\) permuting the petals \((i\text{-variables})\) and \(x = (x_1, \ldots, x_m)\) with \(x_i = \pm 1\), reflections in the \(i\)-th petal. Thus \((\pi; x)(o) = o, (\pi; x)(i, j) = (\pi(i), x_i j)\) with operations in the second coordinate carried out modulo \(n\). From this, \((\pi; x)(\sigma; y) = (\pi \sigma; x^\sigma y)\) with \(x^\sigma = (x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(m)})\). This is a standard representation of \(B_m\). This proves (2.2). For background on \(B_m\) see James and Kerber [JK81] or Halverson and Ram [HR96].

To prove (2.3) note that the symmetry group splits the vertex set into orbits. These are the central point \(o\), the \(2m\) points at distance one away, the \(2m\) points at distance two away and so on. If \(n\) is even, there are only \(m\) points at distance \(n/2\). We focus on \(n\) odd for the rest of the proof. Thus \(L^2(\pi) = L_0 \bigoplus_{i=1}^{(n-1)/2} L_i\) with \(L_0\) the one dimensional trivial representation and the \(L_i\) \(2m\)-dimensional real vector spaces of functions that vanish off the corresponding orbits. All of these \(L_i, 1 \leq i \leq (n-1)/2\) are isomorphic representations of \(B_m\). To decompose into irreducibles, let \(e_1, e_2, \ldots, e_{2m-1}, e_{2m}\) be the usual basis for \(\mathbb{R}^{2m}\). The group \(B_m\) acts on ordered pairs \((e_1, e_2), (e_3, e_4), \ldots, (e_{2m-1}, e_{2m})\) by permuting pairs using \(\pi\) and using \(\pm 1\) to switch within a pair. Using this, the character \(\chi_{2m}\) on any of these \(L_i\) is

\[
\chi_{2m}(\pi; x) = \sum_{i=1}^{m} \delta_{\text{fix}(i)}(1 + x_i).
\]

Indeed, \(\chi_{2m}\) is simply the trace of a permutation representation. Now \(\sum \delta_{\text{fix}(i)}\) is the number of fixed points of \(\pi\). This is the usual permutation character of the subgroup \(S_m\) extended to \(B_m\). It splits into a one-dimensional trivial representation and an \(n - 1\) dimensional irreducible. Finally, \(\sum \delta_{\text{fix}(i)} x_i\) is the character of the usual \(m\)-dimensional reflection representation of \(B_m\) acting on \(\mathbb{R}^m\) by permuting coordinates and reflecting in each coordinate. It is easy to show this is irreducible, e.g., by computing its inner product with itself. Thus, as claimed, \(\chi_{2m} = \chi_0 + \chi_{m-1} + \chi_m\). This holds for each \(L_i\) proving (2.3). The proof of (2.4) is similar.
Example 2 \((K_n - K_n)\)

For two copies of the complete graph \(K_n\) joined via an extra edge, defined in section one \((2.5)\) the symmetry group is \(G = C_2 \ltimes (S_{n-1} \times S_{n-1})\).

\[(2.6)\]

\[L^2(\pi) = 2L_r \bigoplus L_{2n-4}\]

with \(L_r\) the two-dimensional regular representation of \(C_2\) extended to \(G\) and \(L_{2n-4}\) an irreducible representation of dimension \(2n - 4\).

**Corollary 2.2** For any choice of invariant weights (with loops allowed), the corresponding Markov chain on \(K_n - K_n\) has at most five distinct eigenvalues with one eigenvalue of multiplicity \(2n - 4\).

**Proof** It is clear by inspection that the automorphisms are all possible permutations of the two sets of \(n - 1\) vertices, distinct from the connecting edge, among themselves (this gives an action of \(S_{n-1} \times S_{n-1}\)) and switching the two halves (this gives an action of \(C_2\)). The actions do not commute and the combined action is the semidirect product \(C_2 \ltimes (S_{n-1} \times S_{n-1})\). This proves \((2.5)\).

To prove \((2.6)\), observe first that under \(G\) there are two orbits: the two points connected by the extra edge and the remaining \(2n - 2\) points. The representation of \(G\) on the two-point orbit gives one copy of the regular representation of \(C_2\). Let \(\chi\) be the character of the representation of \(G\) on the remaining \(2n - 2\) points. As a permutation character it is clear that

\[\chi(x; \sigma, \zeta) = \delta_{1x} (\text{FP}(\sigma) + \text{FP}(\zeta))\]

with \(\delta_{1x}\) one or zero as \(x\) is one or \(-1\). and \(\text{FP}(\sigma)\) the number of fixed points in \(\sigma\). Computing the inner product of \(\chi\) with itself gives

\[\langle \chi, \chi \rangle = \frac{1}{2((n-1)!)^2} \sum_{x,\sigma,\zeta} (\delta_{1x} (\text{FP}(\sigma) + \text{FP}(\zeta)))^2\]

\[= \frac{1}{2((n-1)!)^2} \sum_{\sigma,\zeta} (\text{FP}^2(\sigma) + 2\text{FP}(\sigma)\text{FP}(\zeta) + \text{FP}^2(\zeta))\]

\[= \frac{1}{2} (2 + 2 + 2) = 3.\]

The second from last equality follows by interpreting the sum as an inner product of characters on \(S_{n-1} \times S_{n-1}\) and decomposing \(\text{FP}(\sigma)\) as a sum of two irreducibles. Thus \(\chi\) decomposes as a sum of three irreducibles of \(G\). If \(\chi_1\), \(\chi_{-1}\) are the two characters of \(C_2\) extended to \(G\), computing as above gives \(\langle \chi, \chi_1 \rangle = \langle \chi, \chi_{-1} \rangle = 1\). It follows that what is left is a \(2n - 4\) dimensional irreducible of \(G\). This proves \((2.6)\). \(\square\)
Remark The irreducible characters of Wreath products such as $G$ are explicitly described in James and Kerber [JK81, chapter 4]. For our special case the irreducible of $G$ having dimension $2n - 4$ may be seen as induced from the $n - 2$ dimensional representation of $S_{n-1} \times S_{n-1}$. The eigenvalues for all invariant weightings of $K_n - K_n$ are given in section five.

2.2 Centralizer algebras

In our work we often begin with a single weighted graph or Markov chain, calculate its symmetry group and use this to aid in diagonalizing the chain. As examples of figures 1-5 show, there are often several chains of interest with the same symmetry group. It is natural to study all weightings consistent with a given symmetry group. This brings us close to the rich world of coherent configurations and distance regular graphs. To see the connection, let $V$ be a finite set and $G$ a group of permutations of $V$. Let $\Omega_1, \Omega_2, \ldots$ be the orbits of $G$ operating coordinate wise on $V \times V$. If $A_i$ is a $|V| \times |V|$ matrix with $(v, v')$ entry one or zero as $(v, v') \in \Omega_i$ or not, then the matrices $A_i$ satisfy

1. $\sum A_i = J$ (the matrix of all ones)
2. there is a subset $S$ with $\sum_{i \in S} A_i = I$ (the identity)
3. the set $\{A_i\}$ is closed under taking transposes
4. there are numbers $p_{ij}^k$ so that $A_iA_j = \sum_k p_{ij}^k A_k$

A collection of zero-one matrices satisfying (1)-(4) is called a coherent configuration. Cameron [Cam99, Cam03] gives a very clear development with extensive references and connections to association schemes, distance regular graphs and much else. Applications to optimization are developed by Gatermann and Parrilo [GP02]. From (4), the set of real linear combinations of the $A_i$ span an algebra, the centralizer algebra of the action of $G$ on $V$.

The direct connection with our work is as follows: given a graph $(V, E)$ with automorphism group $G$, the set of all labelings of the edges compatible with $G$ gives a sub-algebra of the centralizer algebra. The set of all non-negative weightings gives a convex cone in this sub-algebra. The set of all $G$-invariant Markov chains with a fixed stationary distribution is a convex subset of this cone.

We have not found the elegant developments of this theory particularly helpful in our work — we are usually interested in non-transitive actions and use eigenvalues to bound rates of convergence rather than to show a certain configuration cannot exist. An extremely fruitful application of distance regular graphs to random walk is in Belsley [Bel98]. It is a natural project to extend Belsley’s development to completely general coherent configurations. We may also hope for some synergy between the coding and design developments of Delsarte and the semi-definite tools of [BDX03, GP02].

To conclude this section on a more positive note we give

**Proposition 2.1** Let $V$ be a finite set with $G$ a finite group acting on $V$. The set of all Markov chains on $V$ that commute with the action of $G$ is a convex set with extreme points
indexed by orbits of $G$ on $(V \times V)$. Given such an orbit, the associated extremal chain is constant in positions $(v, v')$ in the orbit and has ones on the diagonal of the other rows.

**Proof** The only thing to prove is that the construction unambiguously specifies a stochastic matrix. For this, consider rows indexed by $v, v'$ which have now diagonal entries. We show that the number of non-zero pairs $(v, w)$ in the orbit is the same as the number of non-zero pairs $(v', w')$ in the orbit. Suppose the orbit is $\{gx, gy\}$ for fixed $x \neq y$. Then $v = gx$, $v' = g'x$ so $g'g^{-1}v = v'$. It follows that if there are $k$ non-zero entries in the row $v$ there are $k$ non-zero entries in row $v'$.

## 3 Orbit theory

Let $K, \pi$ be a reversible Markov chain as in (1.1) and (1.2), with $H$ a group of automorphisms. Often, it is a subgroup of the full automorphism group. The vertex set $V$ partitions into orbits $O_v = \{hv : h \in H\}$. Define an orbit chain by

$$K_H(O_v, O_{v'}) = K(v, O_{v'}) = \sum_{u \in O_{v'}} K(v, u). \quad (3.1)$$

Note that this is well defined (it is independent of which $v \in O_v$ is chosen). Further, the lumped chain (which just reports which orbit the original chain is in) is Markov, with $K_H(O_v, O_{v'})$ as the transition kernel. This follows from what is commonly called Dynkin’s criteria (the lumped chain is Markov if and only if $K(x, O_{v'})$ doesn’t depend on the choice of $x$ in $O_v$). See Kemeny and Snell [KS60, chapter 3] for background. Finally, the chain at (3.1) is reversible with $\pi(O_v)$ as the reversing measure; as a check

$$\pi(O)K_H(O, O') = \sum_{v \in O} \pi(v)K(v, O') = \sum_{v \in O} \sum_{v' \in O'} \pi(v)K(v, v') = \sum_{v \in O} \sum_{v' \in O'} \pi(v')K(v', v)$$

$$= \sum_{v' \in O'} \pi(v')K(v', O) = \pi(O')K_H(O', O).$$

In this section we relate the eigenvalues and eigenvectors of various orbit chains to the eigenvalues and eigenvectors of $K$. This material is related to material surveyed by Chan and Godsil [CG97], but we have not found our results in other literature.

### 3.1 Lifting

**Proposition 3.1** Let $K, \pi$ be a reversible Markov chain with automorphism group $G$. Let $H \subseteq G$ be a subgroup. Let $K_H$ be defined as in (3.1).

(a) If $f$ is an eigenfunction of $K_H$ with eigenvalue $\tilde{\lambda}$, then $\tilde{\lambda}$ is an eigenvalue of $K$ with $H$-invariant eigenfunction $f_v = f(O_v)$.

(b) Conversely, every $H$-invariant eigenfunction appears uniquely from this construction.
Proof For (a), we just check that with \( f \) as given
\[
\sum_{v'} K(v, v') f(v') = \sum_{O'} K(v, O') \hat{f}(O') = \sum_{O'} K(O, O') \hat{f}(O') = \hat{\lambda} \hat{f}(O) = \hat{\lambda} f(v).
\]

For (b), we just check that the only \( H \)-invariant eigenfunctions occur from this construction. This is precisely the content of “the lemma that is not Burnside’s”, see Neumann [Neu79]. The representation of \( H \) on \( L^2(\pi) \) is the permutation representation corresponding to the action of \( H \) on \( V \). A \( H \)-fixed vector \( f \in L^2(\pi) \) corresponds to a copy of the trivial representation. The character \( \chi \) of the representation of \( H \) on \( L^2(\pi) \) is \( \chi(h) = \text{FP}(h) = \#\{ v \in V : hv = v \} \). “Burnside’s lemma” (or Frobenius reciprocity) says that
\[
\frac{1}{|H|} \sum_{h} \text{FP}(h) = \# \text{orbits}.
\]

The left side is the inner product of \( \chi \) with the trivial representation. It thus counts the number of \( H \)-fixed vectors in \( L^2(\pi) \). The right side counts the number of eigenvalues in the orbit chain. Of course, any \( H \)-invariant eigenfunction of \( K \) projects to a non-zero eigenfunction of the orbit chain (see proposition 3.2 below). \( \square \)

Remark We originally hoped to use the orbit chain under the full automorphism group coupled with the multiplicity information of section two to completely diagonalize the chain. To see how wrong this is, consider a graph such as the complete graph \( K_n \) with automorphism group operating transitively on \( V \). Then the orbit chain just has one point and one \( G \)-invariant eigenfunction corresponding to eigenvalue one. For the flower \( F_{mn} \), with edge weights one, the \( C_2^m \) action collapses each petal into a path (with a loop at the end if \( n \) is odd) and then the \( S_m \) action identifies these paths. It follows that the orbit chain corresponds to unweighted random walk on the path shown in figure 6.

![Figure 6: The orbit chain of \( F_{mn} \).]

It is easy to diagonalize this orbit chain and find the \( 1 + (n - 1)/2 \) eigenvalues \( \cos(2\pi j/n), 0 \leq j \leq (n - 1)/2 \) (see section 3.3). These appear with multiplicity one for generic weights. As shown in section four, for weight one, there are non \( G \)-invariant eigenvectors with these same eigenvalues, and many further eigenvalues of the full Markov chain \( K \).

3.2 Projection

As above, \( G \) is the automorphism group of a reversible Markov chain, \( H \subseteq G \) is a subgroup and \( K_H(O, O') \) is the orbit chain of (3.1). We give a useful condition for an eigenfunction of \( K \) to project down to a non-zero eigenfunction of \( K_H \). Several examples and applications follow.
Proposition 3.2 If \( f \) is an eigenfunction of \( K \) with eigenvalue \( \lambda \), let \( \bar{f}(x) = \sum_{h \in H} f(h^{-1}x) \). If \( \bar{f} \neq 0 \), then \( \bar{f} \) is an eigenfunction for \( K_H \) with eigenvalue \( \lambda \).

Proof For any \( H \)-orbit \( O \) write \( \bar{f}(O) \) for the constant value of \( \bar{f} \). For \( x \in O \), \( y_i \in O_i \),

\[
\sum_i K(O, O_i) \bar{f}(O_i) = \sum_i \left( \sum_{y \in O_i} K(x, y) \right) \sum_h f(h^{-1}y) = \sum_i \sum_h \sum_{y \in O_i} K(x, y) f(h^{-1}y) \\
= \sum_h \sum_y K(x, y) f(h^{-1}y) = \lambda \sum_h f(h^{-1}x) = \lambda \bar{f}(O).
\]

\[\square\]

Warning Of course, \( \bar{f} \) can vanish. If, e.g., the original graph is the cycle \( C_9 \) and \( H = C_3 \) acting by \( T_a(j) = j + 3a \), for \( a \in C_3 = \{0, 1, 2\} \). There are 9 original eigenvalues with eigenfunctions \( f_j(k) = e^{2\pi i jk/9} \) (here \( i = \sqrt{-1} \)). Using \( \bar{f}_j \) as in the proposition above

\[
\bar{f}_j(k) = e^{2\pi i jk/9} + e^{2\pi i (j+3)/9} + e^{2\pi i (j+6)/9} \\
= e^{2\pi i jk/9} (1 + e^{2\pi i 3j/9} + e^{2\pi i 6j/9}) \\
= 0 \quad \text{if} \ j \ \text{is relatively prime to} \ 9.
\]

In proposition C in section 3.3 below we give examples where several different eigenfunctions coalesce under projection. The following proposition gives sufficient conditions for an eigenvalue of \( K \) to appear in a projection.

Proposition 3.3 Let \( H \) be a subgroup of the automorphism group \( G \) of a reversible Markov chain \( K \). Let \( f \) be an eigenfunction of \( K \) with eigenvalue \( \lambda \). Then \( \lambda \) appears as an eigenvalue in \( K_H \) if either of the following conditions holds

(a) \( H \) has a fixed point \( v^* \) and \( f(v^*) \neq 0 \).

(b) \( f \) is non-zero at \( v^* \) that is in a \( G \)-orbit containing an \( H \) fixed point.

Proof For (a), \( \bar{f} \) defined in proposition 3.2 above satisfies \( \bar{f}(O_{v^*}) \neq 0 \). For (b), since \( f(v^*) \neq 0 \), let \( g \) map \( v^* \) to \( v^{**} \) an \( H \)-fixed point. Then \( (T_g f)(v^*) \neq 0 \).

\[\square\]

Example 1 \((F_{mn})\) Let \( H = B_{m-1} \), the subgroup of \( B_m \) fixing the first petal. The orbit graph is a lollipop \( L_n \) (see figure 7). We claim all the eigenvalues of the weight one random walk on \( F_{mn} \) occur as eigenvalues of \( L_n \). Indeed, if \( f \) is an eigenfunction of \( F_{mn} \) with \( f(o) \neq 0 \), then we are done by (a) of proposition 3.3. If \( f(o) = 0 \), then \( f(v) \neq 0 \) for some other \( v \). We may map \( v \) to the first petal (fixed by \( H \)). We are done by (b) of proposition 3.3.
Figure 7: The lollipop graph $L_n$ (orbit of $F_{mn}$ under $B_{m-1}$): $n$ odd (left) and $n$ even (right).

Figure 8: The orbit chain of $K_n - K_n$ under the group action $S_{n-2} \times S_{n-1}$.

**Example 2** ($K_n - K_n$) Consider the subgroup $S_{n-2} \times S_{n-1} \subseteq C_2 \ltimes (S_{n-1} \times S_{n-1})$. The orbit graph is shown in figure 8. Arguing as above, we see all eigenvalues of the unweighted walk appear.

It is natural to ask which orbit chains are needed to get all the eigenvalues of the original chain $K$. The following theorem gives a simple answer.

**Theorem 3.1** Let $G$ be the automorphism group of the reversible Markov chain $(K, \pi)$. Suppose $V = O_1 \cup \cdots \cup O_k$ as a disjoint union of $G$-orbits. Represent $O_i = G/H_i$ with $H_i$ the subgroup fixing a point in $O_i$. Then all eigenvalues of $K$ occur among the eigenvalues of $\{K_{H_i}\}_{i=1}^k$. Further, every eigenfunction of $K$ occurs by translating a lift of an eigenfunction of some $K_{H_i}$.

**Proof** Say $f$ is an eigenfunction of $K$ with eigenvalue $\lambda$. Let $f(v) \neq 0$, say $v \in O_i \cong G/H_i$ with $H_i = \{g : gv_i = v_i\}$ for a prechosen $v_i \in O_i$. Choose $g$ with $g^{-1}v = v_i$ and let $f_1 = T_g f$. Then, $f_1$ has $\lambda$ as an eigenvalue and a non-zero $H_i$ invariant point. The result follows from proposition (3.3). \hfill \Box

**Remarks** Observe that if $H \subseteq J \subseteq G$, then the eigenvalues of $K_H$ contain all eigenvalues of $K_J$. This allows disregarding some of the $H_i$. Consider example 1 ($F_{mn}$) with $n$ odd. There are $1 + (n - 1)/2$ orbits $O_0 \cup O_1 \cup \cdots \cup O_{(n-1)/2}$. The corresponding $H_i$ are $G$ for $O_0$ and $B_{m-1}$ for all the other $O_i$. It follows that all the eigenvalues occur in the orbit chain for $B_{m-1}$, this is the lollipop $L_n$ described above. Similarly, for $K_n - K_n$, there are two orbits: the two central points (with $H_1 = S_{n-1} \times S_{n-1}$) and the remaining $2n - 2$ points (with $H_2 = S_{n-2} \times S_{n-1}$). Since $H_2 \subseteq H_1$, we get all eigenvalues from this quotient, as discussed just above theorem 3.1.

There remains the question of relating the orbit theory of this section with the multiplicity theory coming from the representation theory of section two. We have not treated this out neatly. The following classical proposition gives a simple answer in the transitive case.

**Proposition 3.4** Let $G$ be the automorphism group of the reversible Markov chain $(K, \pi)$. Suppose $G$ acts transitively on $V$. Let $L^2(\pi) = V_1 \oplus \cdots \oplus V_k$ be the isotropy decomposition
with $V_i = d_i W_i$ and $W_i$ distinct irreducible representations. Suppose $V \cong G/H$. Then the $H$-orbit chain has $\sum_{i=1}^{k} d_i$ distinct eigenvalues generically, with $d_i$ eigenvalues having multiplicity $\text{Dim}(W_i)$ in the original chain $K$. These eigenvalues may be determined as follows: set $Q(y) = K(H, yH)/|H|$. This is an $H$-bi-invariant probability measure on $G$ ($Q(h_1gh_2) = Q(g)$). Let $\rho_i$ be a matrix representation for $W_i$ with basis chosen so that the first $d_i$ basis vectors are fixed by $H$. Then $\hat{Q}(\rho_i) = \sum_{g} Q(g)\rho_i(g)$ is zero except for the upper left $d_i \times d_i$ block. The eigenvalues of this block are the $d_i$ eigenvalues, each with multiplicity $\text{Dim}(W_i)$.

**Proof** This is standard in the multiplicity free case [Dia88, chapter 3]. Dieudone [Die78, section 22.5] covers the general case.  

**Remark** In the transitive case, $L(V) = \text{Ind}^G_H(1)$. The $H$-orbit chain is indexed by $H$-$H$ double cosets. By the Mackey intertwining theorem [CR62, 44.5], the number of orbits is $\sum d_i^2$. Thus the $H$-orbit chain, which has only $\sum d_i$ distinct eigenvalues, has the $d_i$ eigenvalues each occurring with multiplicity $d_i$.

**Example** Consider the hypercube $C^n$. The automorphism group is $G = B_n$, the hyperoctahedral group. This operates transitively with $C^n = G/H$ for $H = S_n$. Further $L^2(\pi) = \sum_{i=1}^{n} \text{Dim}(W_i) = \binom{n}{i}$. As is well known, random walk on $C^n$ has eigenvalues $1 - \frac{2i}{n}$, $0 \leq i \leq n$ with multiplicity $\binom{n}{i}$. See [Dia88, page 28] for background.

**Remarks**

1. In the non-transitive case, Arun Ram has taken us a step closer to connecting the orbit theory to the representation theory. Suppose that, as a representation of $G$, $L^2 = \sum_{\lambda} d_{\lambda} V_{\lambda}$, with $V_{\lambda}$ irreducible representations of $G$ occurring with multiplicity $d_{\lambda}$. Let $\mathcal{H} = \text{End}_G(L^2)$ be the algebra of all linear transformations that commute with $G$. Then $L^2$ is a $(G, \mathcal{H})$ bi-module and basic facts about double commutators ([Mac78, pages 17-18]) give

\[
L^2 = \bigoplus_{\lambda} V_{\lambda} \otimes W_{\lambda}, \quad \text{as a } (G, \mathcal{H}) \text{ bi-module.} \tag{3.2}
\]

In this decomposition $G$ only acts on $V_{\lambda}$. The $d_{\lambda}$ dimensional space $W_{\lambda}$ is called a multiplicity space. Dually, $\mathcal{H}$ (and so $K$) only acts on $W_{\lambda}$ and each eigenvalue of $K$ on $W_{\lambda}$ occurs with multiplicity $\text{Dim}(V_{\lambda})$. Usually, the action of $K$ on $W_{\lambda}$ (or even an explicit description of $W_{\lambda}$) is not apparent.

If $\mathcal{X} = G/H_1 \cup G/H_2 \cup \ldots \cup G/H_r$ is a union of $G$ orbits, theorem 3.1 says we need only consider the $H_i$ orbit chains. By standard theory, the $H_i$ lumped chain $K_{H_i}$ may be seen as the action of $K$ on

\[
L(H_i \setminus \mathcal{X}) \cong \bigoplus_{\lambda} (V_{\lambda})^{H_i} \otimes W_{\lambda}. \tag{3.3}
\]
Here, $(V^\lambda)^{H_i}$ is the subspace of $H_i$-invariant vectors in the representation of $G$ on $V^\lambda$.

The point is that (as in the examples), we may be able to calculate all the eigenvalues of $K_{H_i}$. Further, we know these occur with multiplicity $\text{Dim}((V^\lambda)^{H_i})$. These numbers are computable from group theory, independently of $K$. If they are distinct, they allow us to identify the eigenvalues of $K$ on $W^\lambda$. For $\lambda$ allowing $H_i$-fixed vectors, the action of $K$ on $W^\lambda$ is the same in (3.2) and (3.3). With several $H_i$, the possibility of unique identification is increased.

2. An example of Ron Graham shows that we should not hope for too much from symmetry analysis. To see this, consider the simplest graph with no symmetries

![Graph](image)

Take $n$ copies of this six vertex graph and join them, head to tail, in a cycle. This $6n$ vertex graph has only $C_n$ symmetry. The orbit graph is

![Graph](image)

By proposition 3.1, each of the six eigenvalues of this orbit graph occur with multiplicity one in the large graph. We have not found any way to get a neat description of the remaining eigenvalues. The quotient of the characteristic polynomial of the big graph by that of the orbit graph is often irreducible for small examples. Of course, we can get good bounds on the eigenvalues with geometric arguments as in section six. However, symmetry does not give complete answers.

### 3.3 Three $C_2$ actions

We now illustrate the orbit theory for three classical $C_2$ actions. The results below are well known, see Kac [Kac47] or Feller [Fel68]. We find them instructive in the present context. Further, we need the very detailed description we provide to diagonalize $F_{mn}$. Pinsky [Pin80, Pin85] gives a much more elaborate example of this type of argument.

![Graphs](image)

Figure 9: Three graphs with $n$ vertices.

Consider the three graphs in figure 9, each on $n$-vertices. It is well known that the nearest neighbor Markov chain on each can be explicitly diagonalized by lifting to an appropriate circle.
Case A  Consider $C_{2(n-1)}$. For example, figure 10 shows the case with $n = 4$. Label the points of $C_{2(n-1)}$ as $0, 1, \ldots, 2(n-1) - 1$. Let $C_2$ act on $C_{2(n-1)}$ by $j \rightarrow -j$. This fixes 0, $n - 1$ and gives $(n-2)$ two-point orbits. The orbit chain is precisely the loopless path of case A in figure 9. The eigenvalues/functions of $C_{2(n-1)}$ are
\begin{align*}
1 & / \text{ constant} \\
-1 & / \cos \left( \frac{2\pi(n-1)k}{2(n-1)} \right) = \cos(\pi k) \\
\cos \left( \frac{2\pi j}{2(n-1)} \right) & / \cos \left( \frac{2\pi jk}{2(n-1)} \right), \sin \left( \frac{2\pi jk}{2(n-1)} \right), \quad 1 \leq j \leq n - 2
\end{align*}
Using proposition 3.2, relabeling vertices of the path as $0, 1, \ldots, n - 1$, we have

Proposition A  The loopless path of length $n$ has eigenvalues $\cos \left( \frac{\pi j}{n-1} \right)$ with eigenfunction $f_j(k) = \cos \left( \frac{\pi jk}{n-1} \right)$, $0 \leq j \leq n - 1$.

Note  Here $-1$ is an eigenvalue, all eigenvalues are distinct and $\cos \left( \frac{\pi j}{n-1} \right) = - \cos \left( \frac{\pi(n-1-j)}{n-1} \right)$.

Case B  Consider $C_{2n-1}$. For example, figure 11 shows the case with $n = 4$. Again $C_2$ acts on $C_{2n-1}$ by $j \rightarrow -j$. This fixes 0 and then there are $n - 1$ orbits of size two. The orbit chain is the single loop chain of case B in figure 9. The eigenvalue/function pairs of $C_{2n-1}$ are:
\begin{align*}
1 & / \text{ constant} \\
\cos \left( \frac{2\pi j}{2n-1} \right) & / \cos \left( \frac{2\pi jk}{2n-1} \right), \sin \left( \frac{2\pi jk}{2n-1} \right), \quad 1 \leq j \leq n - 1
\end{align*}
**Proposition B**  The single loop path of case B has eigenvalues \( \cos\left(\frac{2\pi j}{2n-1}\right) \) with eigenfunction \( f_j(k) = \cos\left(\frac{2\pi j k}{2n-1}\right), 0 \leq j \leq n - 1. \)

**Note**  Here \(-1\) is not an eigenvalue, and all eigenvalues have multiplicity one.

![Figure 12: Case C, n = 4.](image)

**Case C**  Consider \( C_{2n} \). For example, figure 12 show the case with \( n = 4 \). Map \( C_{2n} \rightarrow C_{2n} \) with \( T(k) = 2n - 1 - k = -(k + 1), 0 \leq k \leq 2n - 1 \). Clearly \( T^2(k) = k \) and \( T \) sends edges to edges. There are \( n \) orbits of size two. The orbit chain is the double loop chain of case C in figure 9. The eigenvalue/function pairs of \( C_{2n} \) are

\[
\begin{align*}
1 & \;/ \text{ constant} \\
-1 & \;/ \cos(\pi k) \\
\cos\left(\frac{2\pi j}{2n}\right) & \;/ \cos\left(\frac{2\pi j k}{2n}\right), \sin\left(\frac{2\pi j k}{2n}\right), 1 \leq j \leq n - 1
\end{align*}
\]

Summing over orbits gives

**Proposition C**  The double loop path of length \( n \) has eigenvalues \( \cos\left(\frac{\pi j}{n}\right) \) with eigenfunction \( f_j(k) = \cos\left(\frac{\pi j k}{n}\right) + \cos\left(\frac{\pi j (k+1)}{n}\right), 0 \leq j \leq n - 1, 0 \leq k \leq n - 1. \) Using \( \cos(x) + \cos(y) = 2 \cos\left(\frac{x+y}{2}\right) \cos\left(\frac{x-y}{2}\right) \), we see that \( f_j(k) \) is proportional to \( \cos\left(\frac{\pi j (k + \frac{1}{2})}{n}\right). \)

**Remarks**  In this example, we may also write \( f_j(k) = \sin\left(\frac{\pi j k}{n}\right) - \sin\left(\frac{\pi j (k+1)}{n}\right), 0 \leq j \leq n - 1. \) Note

\[
\sin\left(\frac{\pi j k}{n}\right) - \sin\left(\frac{\pi j (k+1)}{n}\right) = \left(\cos\left(\frac{\pi j k}{n}\right) + \cos\left(\frac{\pi j (k+1)}{n}\right)\right) \left(\tan\left(\frac{\pi j}{2n}\right)\right).
\]

This shows another way that eigenfunctions can collapse. On \( C_{2n} \) we may choose the pairing \( T(k) = a - k \), for any odd \( a \) (even a lend to fixed points).
4 Simple random walk on $F_{mn}$

In this section we give an explicit diagonalization of the random walk on the “flower” $F_{mn}$ with all edge weights one. We have two motivations: first, to give an illustration of the theory developed in a two parameter family of examples; second, the analysis of rates of convergence of the random walk to stationarity needs both eigenvalues and eigenvectors. It clears up a mystery that was troubling us in comparing different weighted walks on $F_{mn}$. Our careful analysis allows us to show the walks have different rates of convergence in $L^2$ and $L^1 = n^2 \log m$ vs $n^2$. We first give the diagonalization, then the $L^2$ analysis, then the $L^1$ analysis. We note that the graph $F_{m3}$ is thoroughly studied as the “friendship graph”. See [ERS66].

4.1 Diagonalizing simple random walk on $F_{mn}$

This walk $K(v, v')$ and stationary distribution $\pi(v)$ were introduced in section one. We suppose throughout this section that $n \geq 3$ is odd and $m \geq 2$ is arbitrary. The state-space has $|V| = 1 + m(n - 1)$.

**Proposition 4.1** For $n \geq 3$ odd and $m \geq 2$, let $K(v, v')$ be simple random walk on the flower $F_{mn}$ with points $o, (i, j), 1 \leq i \leq m, 1 \leq j \leq n$. The walk is reversible with stationary distribution $\pi(o) = \frac{1}{n}, \pi(i, j) = \frac{1}{mn}$. An orthonormal basis of eigenvectors $f(i, j)$ and eigenvalues $\lambda$ is

$$
\begin{align*}
\text{constant} / 1 \\
\sqrt{2} \cos \left( \frac{2\pi a j}{n} \right) / \cos \left( \frac{2\pi a}{n} \right), \quad 1 \leq a \leq \frac{n-1}{2} \\
\sqrt{2m} s_{ab}(i, j) / \cos \left( \frac{2\pi b}{n} \right), \quad 1 \leq a \leq m, \quad 1 \leq b \leq \frac{n-1}{2} \\
c_b^{-1/2} f_{ab}(i, j) / \cos \left( \frac{\pi (2b + 1)}{n} \right), \quad 1 \leq a \leq m-1, \quad 0 \leq b \leq \frac{n-3}{2}
\end{align*}
$$

where

$$
\begin{align*}
s_{ab}(i, j) &= \begin{cases} 
0 & \text{if } i \neq a \\
\sin \left( \frac{2\pi b j}{n} \right) & \text{if } i = a
\end{cases} \\
f_{ab}(i, j) &= \begin{cases} 
\cos \left( \frac{\pi b}{n} \left( \frac{n}{2} - |j| \right) \right) & \text{if } i = a \\
\cos \left( \frac{\pi b}{n} \left( \frac{n}{2} + |j| \right) \right) & \text{if } i \neq a
\end{cases}
\end{align*}
$$

and $4c_b = 2 - \frac{3}{n}, 2 - \frac{5}{n}$, $2$ as $b \equiv 0, 1, 2, 3 \pmod{4}$.

**Remarks** This gives $1 + (n-1)/2 + \frac{m(n-1)/2 + (m-1)(n-1)/2 = 1 + m(n-1)}{2}$ pairs $f/\lambda$. Comparing with the corollary to (2.3), we see the multiplicities check but the eigenvalues for the one-dimensional spaces sometimes equal the eigenvalues for the $m-1$ dimensional eigenspaces. In section six, with weights on the edges to force a uniform distribution, all these 'accidents' disappear. To evaluate the eigenfunctions at zero use the expressions given with $j = 0$.

**Proof** We lift eigenvalues from two distinct orbit chains — a cycle $C_n$ (with $H = S_m$) and a path with loops (with $H = S_{m-1} \ltimes C_2^m$). The argument breaks into the following cases:
• vectors coming from the circle $C_n$ not vanishing at zero;
• vectors coming from the circle $C_n$ vanishing at zero and their shifts;
• vectors coming from a path and their shifts.

The results are developed in this order.

(a) Vectors coming from $C_n$  The symmetric group $S_m$ acts on $F_{mn}$ and the orbit chain is the simple random walk on $C_n$. This has eigenvalues/vectors

$$1 / \text{constant}$$

$$\cos\left(\frac{2\pi a}{n}\right) / f_a(j) = \cos\left(\frac{2\pi a}{n}\right), \sin\left(\frac{2\pi a}{n}\right), \quad 1 \leq a \leq \frac{n-1}{2}.$$

We lift these eigen-vectors up to $F_{mn}$ in two ways:

(a1) The eigenvectors $\cos(2\pi aj/n)$ are lifted to $F_{mn}$ by defining them to be constant on orbits of $S_m$. This gives $(n-1)/2$ eigenvectors. Note since $\cos(2\pi aj/n) = \cos(-2\pi aj/n)$ for all $j$, these are in fact $B_m$ invariant and exactly the eigenvectors accounted for by proposition 3.1. By an elementary computation

$$f_a(i, j) = \sqrt{2} \cos\left(\frac{2\pi a}{n}\right), \quad \lambda_a = \cos\left(\frac{2\pi a}{n}\right)$$

are orthonormal eigen pairs.

(a2) The eigenvectors $\sin(2\pi aj/n)$ vanish at $j = 0$. Because of this, we may define $m$ distinct lifts by installing $\sin(2\pi aj/n)$ on the $i$th petal and define it as zero elsewhere. Thus define

$$s_{ab}(i, j) = \begin{cases} 0 & \text{if } i \neq a \\ \sin\left(\frac{2\pi a}{n}\right) & \text{if } i = a. \end{cases}$$

It is easy to check that this works: for $i \neq a, j \neq 0$, with $K$ the transition kernel of $F_{mn}$,

$$Ks_{ab}(i, j) = 0 = \cos\left(\frac{2\pi a}{n}\right) s_{ab}(i, j).$$

For $i = a, j \neq 0$,

$$Ks_{ab}(i, j) = \cos\left(\frac{2\pi a}{n}\right) s_{ab}(i, j).$$

Finally at 0,

$$Ks_{ab}(0) = \frac{1}{2m} \sum_i (s_{ab}(i, 1) + s_{ab}(i, -1))$$

$$= \frac{1}{m} (s_{ab}(a, 1) + s_{ab}(a, -1)) = 0 = \cos\left(\frac{2\pi a}{n}\right) s_{ab}(0).$$

This gives $m(n-1)/2$ further eigenvectors which have been normalized in the statement.
b) Vectors coming from a path  Fix a petal, say the first \((1,j), 1 \leq j \leq n - 1\). The subgroup \(S_{m-1} \times C_2^m\) acts on \(F_{mn}\) by switching the halves of each petal and permuting petals, keeping the first one fixed. The orbit chain is the path of length \(n\) with two loops described in proposition C in section 3.3. The eigenvalues we will lift with non \(B_m\)-invariant eigenfunctions are

\[ \cos \left( \frac{\pi (2b + 1)}{n} \right), \quad 0 \leq b \leq \frac{n - 3}{2}. \]

These have eigenfunctions

\[ f_b(j) = \cos \left( \frac{\pi b}{2} \left( j + \frac{1}{2} \right) \right). \]

We lift this to \(F_{mn}\) by installing it in the \(i\)th petal, and averaging over the remaining petals. To be precise, let us first lift the eigenvalue from the path with two end loops to \(F_{2n}\). Using the eigenvalue/vectors of proposition C in section 3.3. Define \(\tilde{f}_b\) on \(F_{2n}\) as in figure 13 with \(\tilde{f}_b\)

![Figure 13: Definition of \(\tilde{f}_b\) on \(F_{2n}\).](image)

defined symmetrically between left and right. Note that \(\tilde{f}_b\) on the top cycle is different from \(\tilde{f}_b\) on the bottom cycle. Therefore, it makes sense to define, for \(1 \leq i \leq m - 1\),

\[ f_{ab}(i, j) = \begin{cases} \tilde{f}_b(j) & \text{if } i = a \\ \tilde{f}_b(j) & \text{if } i \neq a. \end{cases} \]

This gives \(m - 1\) eigenvectors for each of the \((n - 1)/2\) eigenvalues \(\cos(\pi b/n), b = 2k + 1, 0 \leq k \leq (n - 3)/2\). The normalized versions are given in the statement. \(\square\)

### 4.2 Rates of convergence

**Theorem 4.1** There exist positive constants \(A, B, C\) such that if \(K(v, v')\) denotes the transition matrix of simple random walk on the unweighted graph \(F_{mn}\), with \(m \geq 2, n \geq 3\) odd,
for all $v$ and $l \geq 1$
\[ Ae^{-B l/(n^2 \log m)} \leq \sum_{v'} \frac{(K^l(v, v') - \pi(v'))^2}{\pi(v)} \leq Ce^{-B l/(n^2 \log m)}. \]

**Remark** The result shows that order $n^2 \log m$ are necessary and sufficient to achieve stationarity in the $L^2$ or Chi-square distance. The constants $A, B, C$ are independent of $m, n$ and explicitly computable. Results in section C below show that order $n^2$ steps are necessary and sufficient for $L^1$ or total variation convergence. We only know a handful of examples where the $L^1$ and $L^2$ rates differ. See, e.g., Stong [Sto91] or Diaconis, Holmes, Neal [DHN00].

**Proof** Let $f_i, \lambda_i$, $1 \leq i \leq m(n - 1)$ denote the non-constant eigenfunction, eigenvalue pairs of proposition 4.1. By a standard identity (see, e.g., Diaconis and Saloff-Coste [DSC93])
\[ \sum_{v'} \frac{(K^l(v, v') - \pi(v'))^2}{\pi(v)} = \sum_{i=1}^{m(n-1)} f_i^2(v) \lambda_i. \]
Using the known values, this sum equals
\begin{align*}
&\sum_{a=1}^{(n-1)/2} 2 \cos^2 \left( \frac{2 \pi a j}{n} \right) \cos^2 \left( \frac{2 \pi a}{n} \right) + \sum_{a=1}^{(n-1)/2} 2 \sin^2 \left( \frac{2 \pi a j}{n} \right) \cos^2 \left( \frac{2 \pi a}{n} \right) \\
&+ \sum_{b=0}^{(n-3)/2} c_b^{-1} \cos^2 \left( \frac{\pi b}{n} \left( \frac{n}{2} - |j'| \right) \right) \cos^2 \left( \frac{\pi (2b + 1)}{n} \right) \\
&+ (m - 1) \sum_{b=0}^{(n-3)/2} c_b^{-1} \cos^2 \left( \frac{\pi b}{n} \left( \frac{n}{2} + |j'| \right) \right) \cos^2 \left( \frac{\pi (2b + 1)}{n} \right)
\end{align*}

For the lower bound, discard all terms except the $b = 0$ term in the final sum. Bounding $c_0^{-1}$ below by 2, the $L^2$ distance is bounded below by
\[ 2(m - 1) \cos^2(\pi/n). \]
This clearly takes $l$ of order $n^2 \log m$ to drive it to zero.

The upper bound proceeds just as for simple random walk on $n$ point circles. See Diaconis [Dia88, page 25] or Saloff-Coste [SC03] for these classical trigonometric inequalities. Further details are omitted.

\[ \square \]

## 4.3 $L^1$ bounds

Let $K(x, y)$ be the simple random walk on the unweighted “flower” $F_{mn}$. This is reversible with unique stationary distribution $\pi(o) = 1/n$, $\pi(i, j) = 1/(mn)$. In this section we show that the $L^1$ or total variation relaxation time has order $n^2$, independent of $m$. Recall that
\[ \|K^l_v - \pi\|_1 = \frac{1}{2} \sum_{v'} |K^l(v, v') - \pi(v')| = \max_{A \subseteq V} |K^l(x, A) - \pi(A)|. \]
Proposition 4.2 There are universal constants $A_1$, $B_i$, $C_1$ such that for every starting state $v$, and all $l$

$$A_1 e^{-B_i l/n^2} \leq \| K_v^l - \pi \|_1 \leq C_1 e^{-B_2 l/n^2}.$$ \hfill (4.2)

Proof The argument uses standard results from random walk on an $n$-point circle. For the lower bound, the walk started at the center has vanishingly small probability of being in the top half of a petal after $en^2$ steps. The walk started inside any petal has a vanishingly small chance of being in the top half of any of the other petals after $en^2$ steps. In either case, there is a set $A$ with $\pi(A) \geq 1/10$ and $K^l(v, A) < 1/10$ for $l \leq en^2$ for suitable $\epsilon$ independent of $n$ or $m$. These statements combine to give the lower bound in (4.2).

For the upper bound, we construct a strong stationary time $T$ as in [DF90]. If $X_0, X_1, X_2, \ldots$ denotes the random walk (with $X_0 = v$), $T$ is a stopping time such that $P\{X_T \in A : T = t\} = \pi(A)$, for all $t$ such that $P(T = t) > 0$. Such stopping times yields

$$\| K_v^l - \pi \|_1 \leq P(T \geq l).$$

Let $T_1$ be the first hitting time of the walk started at $v$ to the state $o$. Let $T_2$ be a strong stationary time for the image of the walk started at $o$. An explicit construction of such a time is in example 1 of Diaconis and Fill [DF90]. Clearly $T = T_1 + T_2$ is a strong stationary time for the original walk on $F_{mn}$. Further, $T_1$ and $T_2$ are independent and

$$P\{T_i \geq l\} \leq A_i e^{-B_i l/n^2}$$

for suitable $A_i, B_i$, $i = 1, 2$, by classical estimates. This proves the upper bound in (4.2). \hfill \Box

5 The graph $K_n - K_n$

Consider the graph $K_n - K_n$ with loops and weights as follows: vertices $(x, y)$ are end points of the extra edge with weight $A$. Edges in the left copy of $K_n$ with $x$ as endpoint have weight $B$. The same for vertices in the right copy of $K_n$ with $y$ as an endpoint. All other edges of $K_n$ have weight $C$. Finally every vertex different from $\{x, y\}$ has a loop with weight $D$. For $n = 4$ the graph is shown in figure 14.

![Figure 14: The graph $K_n - K_n$ with weights $A, B, C, D$.](image_url)
Proposition 5.1 The transition matrix of the Markov chain on the edge weighted graph described above has the following set of eigenvalues

- 1 with multiplicity one
- \(-1 + \frac{A}{A+(n-1)B} + \frac{D+(n-2)C}{B+D+(n-2)C}\) with multiplicity one
- \(\frac{D-C}{B+D+(n-2)C}\) with multiplicity \(2n - 4\)
- \(-AB+EF±\sqrt{(A+E)^2(2B+F)^2-2AE(B^2+BF-F^2)}\) \(2(A+E)(B+F)\) each with multiplicity one

where \(E = (n-1)B\) and \(F = D + (n-2)C\).

Proof All of our graphs have symmetry group \(C_2 \ltimes (S_{n-1} \times S_{n-1})\) as in section two. From the computations of example two in section two (corollary 2.2), the graph has at most five distinct eigenvalues, four with multiplicity one and one with multiplicity \(2n - 4\). We determine the eigenvalues in the list above using a sequence of orbit graphs.

a) The orbit graph under the full automorphism group has two states, one corresponding to the orbit \(\{x, y\}\) and one corresponding to the orbit formed by the remaining \(2n - 2\) points. The transition matrix of this orbit chain is

\[
\begin{bmatrix}
\frac{A}{A+(n-1)B} & \frac{(n-1)B}{A+(n-1)B} \\
\frac{B}{B+D+(n-2)C} & \frac{D+(n-2)C}{B+D+(n-2)C}
\end{bmatrix}
\]

Taking traces gives the second eigenvalue shown. By proposition 3.1 this lifts to an eigenvalue of multiplicity one for the full chain.

b) Consider next the symmetry group \(S_{n-1}\) with two orbits, one of size one and the other of size \(n - 1\). The isotropy subgroup of the large orbit is \(S_{n-2}\). The orbit chain under \(S_{n-2}\) has three states with transition matrix (only diagonals shown)

\[
\begin{bmatrix}
\frac{A}{A+(n-1)B} & \frac{D}{B+D+(n-2)C} \\
\frac{D+(n-3)C}{B+D+(n-2)C}
\end{bmatrix}
\]

Taking traces, and using the eigenvalue found above (which also appears here) we find the third eigenvalue shown with the reported high multiplicity. Indeed the \(C_2\) orbit chain has three eigenvalues with multiplicity \(1, 1, n - 2\) and the second eigenvalue has multiplicity one.
c) To get the last two eigenvalues, consider the orbit chain for the subgroup $S_{n-1} \times S_{n-1}$. This has four orbits: $O_\alpha$ consisting of the $n-1$ points in the left $K_n$, $\{x\}$, $\{y\}$ and $O_\beta$. The transition matrix is

$$
\begin{bmatrix}
\alpha & \beta & x & y \\
\alpha & & & \\
\beta & & & \\
x & & & \\
y & & & \\
\end{bmatrix}
$$

This matrix has the first two displayed eigenvalues known. Solving the resulting quadratic for the last two eigenvalues gives the final result.

The bounds implicit in the following two corollaries were first derived by Mark Jerrum using quite different arguments.

**Corollary 5.1** On $K_n - K_n$, for a uniform stationary distribution, the max-degree construction has $A = B = C = D = 1$. The eigenvalues (in the listed order) are $1, 0, 0, \frac{1}{2} - \frac{1}{n} \pm \sqrt{1 - \frac{4}{n} - \frac{4}{n^2}}$. It follows that the second largest eigenvalue is $1 - \frac{2}{n^2} + O\left(\frac{1}{n^4}\right)$. The Metropolis chain has slightly different weights but the second eigenvalue is $1 - \frac{5}{n^2} + O\left(\frac{1}{n^4}\right)$.

**Corollary 5.2** A good approximation to the fastest mixing Markov chain for a uniform stationary distribution has weights $A = D = n - 1$, $B = C = 1$. The eigenvalues are $1, \frac{1}{2} - \frac{1}{2n-2}, \frac{1}{2} - \frac{1}{2n-2}$ (multiplicity $2n-4$), $\frac{1}{4} - \frac{1}{4n-4} \pm \frac{1}{4} \sqrt{1 - \frac{4}{2n-4} + \frac{4}{(2n-2)^2}}$. We thus see that the second largest eigenvalue is

$$
1 - \frac{1}{3n} + O\left(\frac{1}{n^2}\right).
$$

This shows that the fastest mixing Markov chain has spectral gap a factor of $n$ larger than the Metropolis and max-degree chains. As argued in [BDX03], the fastest mixing Markov chain can only improve over the Metropolis algorithm by a factor of the maximum degree of the underlying graph. Thus this example is best possible.

**Remark** Using our algorithm to find the truly Fastest mixing Markov chain shows it is slightly different than the chain of corollary 5.2. Figure 15 shows the mixing time $1/(1 - \mu)$ (here $\mu$ is the second-largest eigenvalue magnitude) of four different Markov chains, when $n$ varies from 2 to 100.
Figure 15: The mixing time $1/(1 - \mu)$ of $K_n - K_n$. 
6 Uniform stationary distribution on flowers

In this section we give sharp bounds on the spectral gap for two different weightings of the flower graph $F_{mn}$. Both graphs have a uniform stationary distribution $\pi(v) = \frac{1}{m(n-1)+1}$. The max-degree weighting gives the chain (for $(v, v')$ an edge in the graph)

$$K_1(v, v') = \frac{1}{2m}, \quad K_1(v, v) = 1 - \frac{d_v}{2m}. \quad (6.1)$$

A picture of the weighted graph appears in figure 3. The Metropolis weighting gives the chain

$$K_2(o, v) = \frac{1}{2m}, \quad K_2(v, v') = \frac{1}{2} \text{ for } v' \neq o, (i, 1)$$

$$K_2((i, 1), o) = \frac{1}{2m}, \quad K_2((i, 1), (i, 2)) = \frac{1}{2}, \quad K_2((i, 1), (i, 1)) = \frac{1}{2} - \frac{1}{2m}. \quad (6.2)$$

A picture of the weighted graph appears in figure 2. Using the symmetry analysis and geometric eigenvalue bounds we show that the spectral gap for the max-degree chain is a factor of $m$ times smaller than the spectral gap for the Metropolis chain. We have also found (numerically) the spectral gap for the fastest mixing Markov chain. Figure 16 shows a plot when $m = 10$ and $n$ varies from 2 to 20. Figure 17 shows a plot when $n = 10$ and $m$ varies from 2 to 20. In the cases we tried, the Metropolis and optimal chains were virtually identical.

6.1 The max-degree chain

**Proposition 6.1** For $m \geq 2$ and odd $n \geq 3$, on the flower graph $F_{mn}$, the max-degree chain $K_1$ of (6.1) has second absolute eigenvalue satisfying

$$1 - \frac{c_1}{mn^2} \leq \mu(K_1) \leq 1 - \frac{c_2}{mn^2}$$

for universal constants $c_1$ and $c_2$. Further, this eigenvalue appears with multiplicity at least $m$.

**Proof** Using results of section three (see also section four for the unweighted case) all the eigenvalues of $K_1$ appear among the eigenvalues of the two graphs shown in figure 18. The cycle with $n$ vertices has a loop with weight $w = 2(n-1)$ at all points except zero. The path with $n$ vertices has a loop with weight $w$ at all points except the central point. If $v \neq o$ for this graph, $\pi(o) = 2/c, \pi(v) = 2m/c$ with $c = 2(n-1)(m+1)$.

We use path arguments with Poincaré inequalities. See [Bré99] for textbook treatment; we follow the original treatment in [DS91]. The arguments are essentially the same for the path as for the circle. We treat the path and leave the circle for the reader. For each $v \neq v'$ there is a unique path $v = v_0, v_1, \ldots, v_h = v'$ with $(v_i, v_{i+1})$ an edge. Call this path $\gamma_{vv'}$ with $|\gamma_{vv'}| = h$. The basic Poincaré inequality says that the second eigenvalue from the top, $\lambda_2$, is bounded above

$$\lambda_2 \leq \frac{1}{A}, \quad A = \max_e \frac{1}{Q(e)} \sum_{\gamma_{vv'} \ni e} |\gamma_{vv'}| \pi(v)\pi(v'). \quad (6.3)$$
Figure 16: $m = 10$, $n$ varies from 2 to 20

Figure 17: $n = 10$, $m$ varies from 2 to 20.
Here, the maximum is over edges $e = (w, z)$, $Q(e) = \pi(w)K_1(w, z)$, and the sum is over paths $\gamma_{vw}$ containing $e$. Such paths range over choices of $v$ to the left of $w$ and $v'$ to the right of $z$ with $w = v$, $z = v'$ allowed. We treat two cases. First suppose if $(w, z)$ does not contain zero, then $\pi(w)K_1(w, z) = Q(e) = (1/2)(n-1)(m+1) = 1/c$. Suppose, without loss that $(w, z)$ is to the right of zero. Break the sum of paths into those with $v$ contained and not, $I$ and $II$ say, bound $|\gamma_{vw}| \leq n$. Then, since the sum $I$ involves at most $n$ paths,

$$I \leq \frac{2m}{c} \frac{2m}{c} = \frac{4n^2m}{c}.$$

The sum $II$ involves at most $n^2$ paths so that

$$II \leq \frac{2m}{c} \frac{2m}{c} = \frac{4n^2m^2}{c} \leq 2n^2(m + 1).$$

We see $A \leq 4n^2(m+1)$.

For the second case, suppose the edge is $(o, v)$, with $v$ to the right of $o$. Now $\pi(o)K_1(o, v) = Q(e) = 1/c$. Again break the sum into paths that start at $o$ ($I$) and paths that start to the left of $o$. Bound $|\gamma_{ov}| \leq n$. Then

$$I \leq \frac{2m}{c} \frac{2m}{c} = \frac{4n^2m}{c}$$

$$II \leq \frac{2m}{c} \frac{2m}{c} = \frac{4n^2m}{c}.$$

We get the same bound as above. This shows

$$\lambda_2 \leq 1 - \frac{1}{4n^2(m+1)}.$$

For a lower bound on the smallest eigenvalue $\lambda_{\min}$, use proposition 2 of [DS91]. This requires paths of odd length from $v$ to $v'$. For all vertices not equal to zero, choose a loop for zero, go to an adjacent vertex, around the loop and back to 2. The upshot is $\lambda_{\min} \geq -1 + 1/m$. Thus $\mu(K_1) = \lambda_1$. In more detail, we have $\lambda_{\min} \geq -1 + 2/i$ with

$$i = \max_e \frac{1}{Q(e)} \sum_{\sigma_x \in e} |\sigma_x| \pi(v)$$

with paths of odd length $\sigma_x$ from $x$ to $x$. For $K_1$, $\pi(o) = \frac{2}{c}$, $\pi(v) = \frac{2m}{c}$, $c = 2(n-1)(m+1)$, paths of length one except for 0.
• For a loop \( e \) at \( x \), \( Q(e) = \frac{2(m-1)}{c} \cdot \frac{2m}{2m} = \frac{2(m-1)}{cm} \cdot \frac{2m}{2m} = \frac{m^2}{m-1} \).

• For the edge from 0 to 1, \( Q(e) = \frac{2}{c} \cdot \frac{1}{Q(0,1)} = \frac{1}{Q(0,1)} \cdot \frac{1}{Q(0,1)} = c^2, \pi(x) = c^2 = 2 \).

• For the loop at 1,

\[
\frac{1}{Q(1,1)} \{ |\sigma_0| \pi(0) + |\sigma_1| \pi(1) \} = \frac{(n-1)(m+1)}{(n-1)(m+1)} \left\{ \frac{2m}{c} + \frac{2m}{c} \right\} = \frac{3 + m}{m-1}.
\]

The max is thus \( \frac{m^2}{m-1} \leq 2m \), thus \( \lambda_{\text{min}} \geq -1 + \frac{1}{m} \).

To prove the lower bound in proposition 6.1, we use the variational characterization

\[
1 - \lambda_1 = \inf_f \frac{\mathcal{E}(f)}{\text{Var}(f)} \leq \frac{\mathcal{E}(f_0)}{\text{Var}(f_0)}.
\]

Thus, for any test function \( f_0 \), \( \lambda_1 \geq -1 - \mathcal{E}(f_0)/\text{Var}(f_0) \). Choose \( f_0(j) = j \), \( -(n-1)/2 \leq j \leq (n-1)/2 \). Then under \( \pi \), \( f_0 \) has mean zero and variance asymptotic to \( \frac{2m}{c} n^3 A \) for \( A \), a computable constant. Next \( \mathcal{E}(f_0) = \sum_{j=-\frac{(n-1)}{2}}^{\frac{(n-1)}{2}} (f_0(j)-f_0(j+1))^2 \pi(j) K_1(j, j+1) \). The differences are all one, \( \pi(j) = \frac{2m}{c} \), \( K_1(j, j+1) = \frac{1}{2m} \), except when \( j = 0 \). It follows that \( \mathcal{E}(f_0) \) is asymptotic to \( A_2 \frac{2m}{c} \frac{1}{2m} n = A_2 \frac{n^3}{c} \). Putting these results together (\( c = 2(n-1)(m+1) \)) yields \( \mathcal{E}(f_0)/\text{Var}(f_0) \leq A_3/(n^2 m) \).

\[ \square \]

Remarks

1. Each eigenvalue of the orbit chain lifts with multiplicity \( m \) or \( m-1 \). Our numerical calculations show the eigenvalues are all distinct (as opposed to the unweighted case where there is a coalescence). Thus the second eigenvalue has multiplicity \( m \).

2. The max-degree weighted flower is a nice example where conductance arguments (or Cheeger's inequality) give a less satisfactory answer than the Poincaré inequality we have used. To see this without a lot of fuss, consider a slight variant: take an \( n \)-point cycle and put a loop with weight \( m \) at each vertex. Now, the stationary distribution is uniform and standard Fourier arguments show that the eigenvalues are \( \frac{m}{m+2} + \frac{2}{m+2} \cos \left( \frac{2\pi j}{n} \right) \), \( 0 \leq j \leq n-1 \). Thus for large \( n \), the second eigenvalue is \( 1 - \frac{4m^2}{n^2 (m+2)} + O \left( \frac{1}{n^4 (m+2)} \right) \). Cheeger's inequality says \( \lambda_1 \leq 1 - 4h^2 \), with \( h = \min Q(A, A^c)/\pi(A) \) with the minimum taken over connected subsets \( A \) with \( \pi(A) \leq 1/2 \). The min is easily shown to occur for \( A \) any connected half of the cycle. Thus \( \pi(A) = 1/2, \) but \( Q(A, A^c) = c/(nm) \). Because of the square, Cheeger's inequality gives \( \lambda_1 \leq 1 - c/(mn)^2 \). This is off by a factor of \( m \). The Poincaré inequalities used as above give \( \lambda_1 \leq 1 - c/(n^2 m) \).
6.2 The Metropolis chain

Proposition 6.2 For $m \geq 2$ and odd $n \geq 3$, on the flower graph $F_{mn}$, the Metropolis chain $K_2$ of (6.2) has second absolute eigenvalue satisfying

$$1 - c_2' \frac{(n + m)}{n^3} \leq \mu(K_2) \leq 1 - c_2 \frac{(n + m)}{n^3} \quad (6.4)$$

with $c_2, c_2'$ universal constants. This eigenvalue has multiplicity at least $m$.

![Figure 19: Two orbit chain with weight $w = 2(m - 1)$.](image)

**Proof** As for proposition 6.1 above, all the eigenvalues of $K_2$ appear among the eigenvalues of the two graphs shown in figure 19 (all unlabeled edges have weight one). We treat the $n$-point path, the cycle is similar. Label the vertices of the path $(n - 1)/2, \ldots, 0, 1, \ldots, (n - 1)/2$ from left to right. Then $\pi(0) = 1/c, \pi(\pm 1) = 2m/c, \pi(\pm 2) = m/c, \pi(\pm j) = 2/c$ for $j \neq 0, \pm 1, \pm 2$, with $c = n + 2m$. Using the bound (6.3) with the same choice of paths, there are four cases corresponding to edges $e = (0, 1), (1, 2), (2, 3)$ and all other edges. Here $Q(0, 1) = 1/c, Q(1, 2) = (m - 1)/c, Q(2, 3) = 1/c, Q(i, i + 1) = 1/c$ otherwise. In each of the four cases it is necessary to consider separately paths $\gamma_{xy}$ using the edge with start or end a general vertex or one of the three special vertices. In each case the largest term is bounded by the order $n^2$ paths such that $x, y \neq \{0, \pm 1, \pm 2\}$. These sums give:

- $e = (0, 1), \frac{cnn^2n^2}{c} = \frac{4n^3}{c}$
- $e = (1, 2), \frac{c}{m-1} \frac{nn^2n^2}{c} = \frac{4n^3}{cm}$
- $e = (2, 3), \frac{cnn^2n^2}{c} = \frac{4n^3}{c}$
- $e = (i, i + 1), \frac{cnn^2n^2}{c} = \frac{4n^3}{c}$

The arguments for the cycle are similar. In all, we see that the constant $A$ in (6.3) is bounded above by $kn^2/(n + 2m)$ for a universal $k$. Again the second eigenvalue is $\lambda_2$. The same linear test function before gives the lower bound on $\lambda_2$. This completes the proof of proposition (6.2).
Remarks

1. For \( m = o(n) \), \( \mu(K_2) \leq 1 - \frac{c}{n^2} \) and thus the Metropolis construction leads to faster mixing than the max-degree construction.

2. While we will not give details here, a coupling argument shows that in total variation, order \( mn^2 \) steps are necessary and sufficient for mixing of the max-degree chain, while order \( (n^2 + m) \) steps are necessary and sufficient for mixing of the Metropolis chain.

References


