CORRECTED RANDOM WALK APPROXIMATIONS TO FREE BOUNDARY PROBLEMS IN OPTIMAL STOPPING

by

Tze Leung Lai
Yi-Ching Yao
Farid Aitsahlia

Technical Report No. 2005-17
July 2005

Department of Statistics
STANFORD UNIVERSITY
Stanford, California 94305-4065
CORRECTED RANDOM WALK APPROXIMATIONS TO
FREE BOUNDARY PROBLEMS IN OPTIMAL STOPPING

by

Tze Leung Lai
Department of Statistics
Stanford University

Yi-Ching Yao
Institute of Statistical Science
Academia Sinica

Farid Aitsahlia
Department of Industrial and Systems Engineering
University of Florida

Technical Report No. 2005-17
July 2005

This research was supported in part by the
National Science Foundation grant DMS 0305749

Department of Statistics
STANFORD UNIVERSITY
Stanford, California 94305-4065

http://www-stat.stanford.edu
CORRECTED RANDOM WALK APPROXIMATIONS TO
FREE BOUNDARY PROBLEMS IN OPTIMAL STOPPING

TZE LEUNG LAI, *Stanford University
YI-CHING YAO, **Academia Sinica
FARID AITSAHLIA, ***University of Florida

Abstract

Corrected random walk approximations to continuous-time optimal stopping boundaries for
Brownian motion, first introduced by Chernoff and Petkau, have provided powerful computational
tools in option pricing and sequential analysis. This paper develops the theory of these second-order
approximations and describes some new applications.

Keywords: Optimal stopping; free boundary PDE; random walks; renewal theory; singular
stochastic control.

2000 Mathematics Subject Classification: Primary 60G40
Secondary 60H30, 90C39

*Postal address: Department of Statistics, Stanford University, Stanford, CA 94305, USA.
Email address: lait@stat.stanford.edu.
**Postal address: Institute of Statistical Science, Academia Sinica, Taipei 115, Taiwan, ROC.
Email address: yao@stat.sinica.edu.tw.
***Postal address: Department of Industrial and Systems Engineering, University of Florida,
P.O. Box 116595, Gainesville, FL 32611-6595, USA. Email address: farid@ise.ufl.edu.
1 Introduction

Chernoff [9] and Chernoff and Petkau [11] introduced a second-order correction for computing the optimal stopping boundary of Brownian motion via a random walk approximation and the backward induction algorithm of discrete-time dynamic programming. This correction is based on the solution to an associated stopping problem in which the stopping time $0 \geq \tau \geq t$ for Brownian motion $\{B_t, t \leq 0\}$ is chosen to maximize $E\{g(B_{\tau}, \tau) \mid B_t = y_0\}$, where $y_0 < 0$ and

$$g(y, s) = -s1_{\{s < 0\}} + y^21_{\{s = 0, y \leq 0\}}.$$  \hfill (1.1)

Approximating Brownian motion by a zero-mean random walk so that the discrete-time optimal stopping problem maximizes $E\{g(y_0 + \sqrt{\delta}S_N, t + \delta N)\}$ over integer-valued stopping times $N$, Hogan [22] generalized the previous work of Chernoff [9] for normal random walks and Chernoff and Petkau [11] for Bernoulli random walks, and showed that the (discrete-time) optimal stopping boundary $b_0(t)$ for the approximating random walk is related to the (continuous-time) optimal stopping boundary $b(t)$ ($= 0$) in the associated problem for Brownian motion by

$$b_0(t) = b(t) - \rho \sqrt{\delta} + o(\sqrt{\delta}),$$  \hfill (1.2)

where $\rho = E S_{\tau^+}^2/2E S_{\tau^+}$, $\tau^+ = \inf\{n : S_n > 0\}$ and $S_n = \sum_{i=1}^n X_i$ with $EX_i = 0$, $EX_i^2 = 1$. Actually Hogan only proved that $\lim_{\delta \to 0} b_0(s) = -\rho$. However, this result corresponding to the special case $\delta = 1$ can be combined with the scaling property $g(\sqrt{\delta}y, \delta s) = \delta g(y, s)$ of (1.1) to yield (1.2) for all negative $t$ bounded away from 0.

Whereas (1.2) has been derived only for the special payoff function (1.1), Chernoff [10] has given some heuristic argument based on Taylor expansions around the optimal stopping boundary to justify the use of (1.2) for more general payoff functions. In this paper, by making use of results from a canonical optimal stopping problem (see Section 3.2) that generalizes Chernoff’s “associated stopping problem”, we derive (1.2) for general payoff functions that satisfy regularity conditions of the type given in van Moerbeke ([34], [35]).

Besides sequential analysis problems that motivated Chernoff [9] and Chernoff and Petkau [13] to develop corrected random walk approximations to optimal stopping boundaries for Brownian motion, we were also motivated by certain computational problems in singular stochastic control (see Section 5) and option pricing. In the Black-Scholes theory, the price of a standard American option is $V := \sup_{\tau \in \mathcal{T}(0, T)} E\{e^{-\tau T} f(P_{\tau})\}$, where $f(P) = (K - P)^+$ or $(P - K)^+$ for a put or call option, with strike price $K$ and expiration date $T^*$, $\mathcal{T}(a, b)$ is the set of stopping times taking values between $a$ and $b$ with $b > a$, and $P_t := P_0 \exp\{\left(r - d - \sigma^2/2\right)t + \sigma\mathcal{B}_t\}$ is the price (under
the risk-neutral measure) of the risky asset at time \( t \), in which \( r \) is the risk-free interest rate, \( d \) is the dividend rate and \( \sigma \) is the volatility. A standard method to compute American option prices, introduced by Cox et al. [16], is to approximate the geometric Brownian motion \( P_t \) by a binomial tree, for which backward induction can be applied to find the value \( V_n \) in the corresponding discrete optimal stopping problem with \( n \) evenly spaced time steps. Lamberton [27] proved the following error bounds for such approximation when \( d = 0 \) and the binomial tree approximating \( P_t \) corresponds to a symmetric Bernoulli walk approximating \( B_t \):

\[-c_1 n^{-2/3} \leq V_n - V \leq c_2 n^{-3/4},\]

for some positive constants \( c_1 \) and \( c_2 \) (depending on the initial price \( P_0 \)), which was later in [28] extended to \( d \geq 0 \) and improved to

\[|V_n - V| \leq c \left( \sqrt[4]{\log n/n} \right)^{4/5},\]

for some positive constant \( c \) (depending on \( P_0 \)). He also indicated that a major reason why sharper bounds could not be established is that \( \partial^2 V / \partial P^2 \) and the derivative of the optimal stopping boundary become infinite when approaching the expiration date \( T^* \). AitSahalia and Lai [1] circumvented this difficulty by using another method to compute the stopping boundary and the value function near \( T^* \). This method involves numerical solution of the integral equation defining the optimal stopping boundary \( b(t) \) (see (2.9) below), which is reasonably fast when \( t \) is near \( T^* \) although its computation becomes increasingly expensive as \( t \) moves further from \( T^* \). Moreover, instead of binomial trees to approximate \( P_t \), they used symmetric Bernoulli walks to approximate \( B_t \) in computing the stopping boundary and value function on \([0, T^* - \epsilon]\) for some small \( \epsilon > 0 \), on which both functions are smooth and (1.2) holds (see Theorem 2 below).

In Section 3 we prove (1.2) for general payoff functions by using an induction argument that proceeds backwards in time, initializing at time \( T < T^* \) with approximations that are within \( O(\delta) \) of the value function \( v(x, T) \) and \( O(\sqrt{\delta}) \) of the optimal stopping boundary \( b(T) \) at \( T \). The induction argument also establishes that \( v_b(x, t_i) - v(x, t_i) = O(\delta) \) uniformly in \( t_i \), where \( v \) is the value function of the optimally stopped Brownian motion and \( v_b \) is the value function of the discrete-time optimal stopping problem in which stopping can only occur at times \( t_i \) with \( t_{i-1} - t_i = \delta \). Applying these results to American option pricing with \( \delta = 1/n \) yields \( \tilde{V}_n - V = O(n^{-1}) \) as an improvement of (1.3), where \( \tilde{V}_n \) is the value function of the approximating Bernoulli random walk initialized at \( T = T^* - \epsilon \). The proof of the main theorem in Section 3.3 requires renewal theory for random walks related to excess over the boundary (see Section 3.1) and certain basic properties of the value functions of optimal stopping problems for Brownian motion. Section 2 reviews some of these
properties and proves a basic decomposition formula which we use to derive other basic properties and computational methods in Sections 3.3 and 3.4.

For standard normal $X_t$ and constant boundary $b(t) \equiv b$, the boundary correction (1.2) has also been introduced by Broadie et al. [7] to approximate the value of a discretely monitored European barrier option by its continuously monitored counterpart whose value is given by an explicit formula. The value $V(H)$ of a continuously monitored European barrier option, with strike price $K$, barrier $H$ and expiration date $T^*$, is $E \left[ e^{-rT^*} f(P_{T^*} \mathbf{1}_{\{H \leq T^*\}} \right]$ for a knock–out option and $E \left[ e^{-rT^*} f(P_{T^*} \mathbf{1}_{\{H \leq T^*\}} \right]$ for a knock–in option, where $\tau_H^{(D)} = \inf \{ t \geq 0 : P_t < H \}$ is the first time that the asset price falls below the barrier $H$, called a “down barrier” (denoted by $D$), and $\tau_H^{(U)} = \inf \{ t \geq 0 : P_t > H \}$ is the first time that the asset price rises above $H$ in the case of an “up” barrier (denoted by $U$). When the barrier option is monitored only at times $i\delta$ ($i = 0, 1, \ldots, n$), with $\delta = T^*/n$, the stopping times $\tau_H^{(D)}$ and $\tau_H^{(U)}$ are restricted to the discrete set $\{0, \delta, 2\delta, \ldots, T^*\}$ and there are no closed–form expressions for these discretely monitored barrier option prices. Broadie et al. [7] proposed to use a continuity correction for the barrier, similar to (1.2), to approximate the price $V_n(H)$ of the discretely monitored knock–in or knock–out option with barrier $H$ by the explicit formula of the corresponding continuously monitored option price, and showed that

$$V_n(H) = V \left( H e^{+p\sigma\sqrt{T^*/n}} \right) + o(n^{-1/2}), \quad (1.4)$$

where $+$ applies to $\tau_H^{(U)}$ and $-$ to $\tau_H^{(D)}$, and $\rho = ES \sigma^2 / 2ES \sigma^2 = 0.5826$ for normal $X_t$. Note that whereas (1.3) is concerned with approximating the value of an optimal stopping problem for Brownian motion by that of a zero–mean random walk, (1.4) considers the reverse approximation that approximates the stopped value of a random walk by that of Brownian motion. In Section 4 we apply (1.2) in this reverse manner, approximating a discrete–time optimal stopping boundary by its continuous–time counterpart that has an explicit formula in two classical optimal stopping problems.

Section 5 considers another application of (1.2), namely, evaluating the optimal (reflection) boundary in singular stochastic control problems for Brownian motion via corrected random walk approximations.

2 The value function and its Doob-Meyer decomposition

Instead of (1.1), we shall consider more general payoff (reward) functions $g(x, t)$ that satisfy certain regularity conditions. The corresponding value function is

$$v(x, t) = \sup_{\tau \in T(t, T^*)} E \{ g(B_\tau, \tau) \mid B_t = x \}, \quad (2.1)$$
\[ \bar{T} \leq t < T^*, \text{ in which } g(x, t) \text{ has a left-hand limit } g(x, T^*-) \text{ as } t \uparrow T^*. \] Let \( g_*(x) = g(x, T^*) - g(x, T^*-) \). Whereas the optimal stopping rule for (1.1) stops sampling as soon as \( B_t \geq 0 \) (so \( b(t) \equiv 0 \) is the optimal stopping boundary), the optimal stopping rule in the general case has continuation region \( \mathcal{R} := \{(x, t) : v(x, t) > g(x, t)\} \), which need not be a connected set. To extend (1.2) to the present case, we therefore assume that
\[
\mathcal{R} = \{(x, t) : x < b(t), \bar{T} \leq t < T^*\},
\] (2.2)
where the boundary \( b(t) \) converges to a limit \( b(T^*) \) as \( t \uparrow T^* \) and satisfies the smoothness condition (2.4) below. While Theorem 2 in Section 3 shows that (1.2) holds for continuation regions of the form (2.2), a similar correction with \( -\rho \sqrt{\delta} \) in (1.2) replaced by \( \rho \sqrt{\delta} \) is also applicable to continuation regions of the form \( \{(x, t) : x > b(t)\} \), and we can combine the corrections for the upper and lower boundaries when the continuation region is of the form \( \{(x, t) : b_1(t) < x < b_2(t)\} \).

Let \( \mathcal{R}_\xi = \{(x, t) : x < b(t) + \xi, \bar{T} \leq t < T^*\} \) (\( \mathcal{R}_0 = \mathcal{R} \)). Because our derivation of (1.2) relies on Taylor expansions, we shall assume that \( g \) belongs to the class \( C^{2, 1} \) on \( \mathbb{R} \times [\bar{T}, T^*] \) and \( C^{3, 2} \) on \( \mathcal{R}_\xi \) such that
\[
g, g_x, g_{xx}, g_{xxx}, g_t, g_{tt}, g_{xt}, g_*, g'_*, g''_*, g'''_* \text{ are bounded by } e^{\beta(x)} \text{ on } \mathcal{R}_\xi \] (2.3)
for some \( \xi > 0 \), where \( \beta(x) \) satisfies the growth condition \( \beta(x) = O(|x|) \) as \( x \to -\infty \), which is somewhat stronger than van Moerbeke's Tychonov condition [34, p. 115]. We assume furthermore that
\[
b'(t) \text{ is continuous at every } t \in [\bar{T}, T^*], \] (2.4)
so that we can also apply Taylor's theorem to the stopping boundary. Under these assumptions, we have the following result due to van Moerbeke ([34], [35]).

**Lemma 1.** Let \( G(x, t) = (\partial / \partial t + \frac{1}{2} \partial^2 / \partial x^2) g(x, t) \).

(a) If \( G(x, t) > 0 \), then \( (x, t) \) is a continuation point. If \( G(x, t) = 0 \) and \( G_x(x, t) \neq 0 \), then \( (x, t) \) is a continuation point.

(b) \( (\partial / \partial x)(v - g) \) and \( (\partial / \partial t)(v - g) \) are continuous across the stopping boundary \( b(\cdot) \) and vanish at \( x = b(t) \) for \( t \in [\bar{T}, T^*] \).

(c) \( \frac{1}{2}(\partial^2 / \partial x^2)(v - g) \) is continuous in the continuation region \( \mathcal{R} \) up to the boundary \( b(\cdot) \) and equals \( -G \) at \( x = b(t) \) for \( t \in [\bar{T}, T^*] \).

(d) \( (\partial^2 / \partial x \partial t)(v - g) \) is continuous in \( \mathcal{R} \) up to the boundary \( b(\cdot) \) and equals \( 2b'(t)G(b(t), t) \) at \( x = b(t) \) for \( t \in [\bar{T}, T^*] \).
(e) \((\partial^2/\partial t^2)(v - g)(b(t), t) = -2G(b(t), t)(b'(t))^2\) for \(t \in [\tilde{T}, T^*]\).

(f) \(v \in C^\infty(\mathcal{R})\) and \((\partial/\partial t + \frac{1}{2} \partial^2/\partial x^2)v(x, t) = 0\) for \((x, t) \in \mathcal{R}\).

Making use of Lemma 4 of [34] and (2.4) together with Lemma 1(d), we next prove the following important property of \(\partial^5 v/\partial x^5\) that will be used in Section 3.

**Lemma 2.** Assume furthermore that

\[
g_{tt} \in C^{2,1}, \quad g_{xx} \in C^{1,1} \text{ on } \mathcal{R}_\xi \quad \text{for some } \xi > 0.
\]

Then \(\partial^5 v/\partial x^5\) is continuous in the continuation region \(\mathcal{R}\) up to the boundary \(b(\cdot)\).

**Proof.** Define \(u(x, t) = v_{tt}(x, t) - g_{tt}(x, t) + 2(b'(t))^2 G(b(t), t)\) on \(\mathcal{R}\). By Lemma 1(e), \(u(b(t), t) = 0\). We shall show that

\[
(b'(t))^2 G(b(t), t) \text{ is continuously differentiable for } \tilde{T} \leq t < T^*.
\]

Combining this result with (2.5) shows that \((\partial u/\partial t) + \frac{1}{2} (\partial^2 u/\partial x^2)\) is continuous in \(\mathcal{R}\). Hence by Lemma 4 of [34], \(\partial u/\partial x\) is continuous in \(\mathcal{R}\) up to \(b(\cdot)\). In view of (2.5), this implies that \(\partial v_{tt}/\partial x\) is continuous in \(\mathcal{R}\) up to \(b(\cdot)\). Since \(v_t = -\frac{1}{2} v_{xx}\) implies that \(v_{tt} = \frac{1}{4} \partial^4 v/\partial x^4\), it then follows that \(\partial v_{tt}/\partial x = \frac{1}{4} \partial^5 v/\partial x^5\) is continuous in \(\mathcal{R}\) up to \(b(\cdot)\).

To prove (2.6), we apply the integral equation for \(f(t) := -(b'(t))^2 G(b(t), t)\) similar to that in [35, Eq. (37)], which is derived by making use of Lemma 1(e) to arrive at a free boundary problem for \(v_{tt} - g_{tt}\) and then applying Green's theorem. Specifically, for \(T' \in [\tilde{T}, T^*]\) and \(\tilde{T} \leq t \leq T'\),

\[
f(t) = \int_0^{T'-t} \left\{ \int_{-\infty}^{b(T'-\tau)} \psi(b(t); y, \tau), \frac{\partial}{\partial t}G(y, T' - \tau)dy \right\} \frac{d\tau}{\sqrt{T' - t - \tau}}
\]

\[
+ \int_0^{T'-t} \psi(b(t); b(T' - \tau), \tau)f(T' - \tau) \frac{d\tau}{\sqrt{T' - t - \tau}}
\]

\[
- \int_{-\infty}^{b(T')} \psi(b(t); y, 0) \left\{ \frac{1}{2} (v_{tt} - g_{tt})(y, T') \right\} \frac{dy}{\sqrt{T' - t}},
\]

where \(\psi(x; y, \tau) = (\partial/\partial x) \phi((x - y)/\sqrt{T' - t - \tau})\) and \(\phi\) is the standard normal density function.

Since \(b(t)\) is continuously differentiable on \([\tilde{T}, T']\), so is \(f(t)\).

The following theorem provides a representation of \(v\) in terms of the optimal stopping boundary and plays an important role in our subsequent development. Let \(E_{x,t}\) denote the conditional expectation given \(B_t = x\).
Theorem 1. For $\bar{T} \leq t \leq T < T^*$,
\begin{equation}
v(x,t) = E_{x,t}[v(B_T, T) - \int_t^T E_{x,s}[G(B_s, s)1_{\{B_t \geq b(s)\}}]ds].
\end{equation}

The process $\{v(B_t, t), \mathcal{F}_t, \bar{T} \leq t \leq T\}$ is the minimal right-continuous supermartingale that majorizes $\{g(B_t, t), \bar{T} \leq t \leq T\}$, where $\{\mathcal{F}_t, \bar{T} \leq t \leq T\}$ is a filtration to which $\{B_t, \bar{T} \leq t \leq T\}$ is adapted; see Fakeev [18, Theorem 2]. Therefore, the Doob-Meyer decomposition yields
\begin{equation}
v(B_t, t) = M_t - A_t,
\end{equation}
where $M_t$ is a martingale and $A_t$ is a predictable nondecreasing process, and the choice of $M_t - M_{\bar{T}}$ and $A_t - A_{\bar{T}}$ in (2.10) is essentially unique. To find $M$ and $A$, we apply the generalized Ito's formula [26, Theorem 2.10.1] to $v(B_t, t)$. Since $v$ is $C^1$ (in view of Lemma 1(b)) and piecewise $C^{2,1}$ on $\mathbb{R} \times [\bar{T}, T]$, the generalized Ito's formula yields that for $T > t$,
\begin{equation}
v(B_T, T) - v(B_t, t) = \int_t^T (v_t + v_{xx}/2)(B_s, s)ds + \int_t^T v_x(B_s, s)dB_s,
\end{equation}
which gives the martingale component $\int_t^T v_x(B_s, s)dB_s$ in the Doob-Meyer decomposition (2.8).

If $g(x, T^* -) = g(x, T^*)$ for all $x$, then $\lim_{T \uparrow T^*} v(x, T) = v(x, T^*) = g(x, T^*)$, so letting $T \uparrow T^*$ in (2.7) yields
\begin{equation}
v(x,t) = E_{x,t}[g(B_{T^*}, T^*) - \int_t^{T^*} E[G(x + \sqrt{s-t}Z, s)1_{\{x + \sqrt{s-t}Z \geq b(s)\}}]ds],
\end{equation}
where $Z$ is a standard normal random variable. In particular, for American options, (2.10) corresponds to the decomposition formula due to Jacka [23] and Carr et al. [8]:

American option price = European option price + Early exercise premium,
in which the integrand in the integral defining the early exercise premium is an explicit function of the optimal stopping boundary, noting that $-G \geq 0$ at stopping points by Lemma 1(a). Putting $x = b(t)$ in (2.10) leads to the following integral equation defining $b$:
\begin{equation}
g(b(t), t) = E[g(b(t) + \sqrt{T^*-t}Z, T^*)]
- \int_t^{T^*} E[G(b(t) + \sqrt{s-t}Z, s)1_{\{Z \geq (b(s) - b(t))/\sqrt{s-t}\}}]ds.
\end{equation}
Ait-Sahalia and Lai [1] showed how (2.10) and (2.11) can be used to develop a relatively fast and accurate method to compute \( v(x, t) \) and \( b(t) \) for \( t \) near the expiration date \( T^* \) in the case of American options, thereby circumventing difficulties in the Bernoulli walk approximation due to unbounded \( \partial^2 v / \partial x^2 \) and \( b' \) as \( t \uparrow T^* \). A similar idea was used by Brezzi and Lai [6, p. 92] to avoid the singularity at \( T^* \) in initializing the Bernoulli walk method to compute the optimal stopping boundary for bandit problems. In Section 3.3 we give another application of (2.10) to bound the fifth derivative of \( v \) (with respect to \( x \)), which we use to prove (1.2) for general payoff functions \( g \).

3 Theory of corrected random walk approximations

In this section, we first present a theorem showing that under suitable conditions discrete-time random walk approximations to continuous-time optimal stopping problems can approximate the value function with an error of the order \( O(\delta) \) and the stopping boundary with an error of the order \( o(\sqrt{\delta}) \), where \( \delta \) is the interval width in discretizing time for the approximating random walk. The theorem also shows how to correct the discrete-time approximations to continuous-time stopping boundaries. As shown by Hogan [22] in a prototypical special case, the boundary correction involves an "excess over the boundary" term. In this connection, we use renewal theory to derive in Lemma 3 an inequality, which plays an important role in the proof of the theorem, for the excess over the boundary.

Suppose that in the continuous-time optimal stopping problem (2.1) with horizon \( T^* \), one has an approximation \( v_\delta(\cdot, T) \) to the value function \( v(\cdot, T) \) and an approximation \( \beta_\delta \) to \( b(T) \), with \( \bar{T} < t < T < T^* \), such that

\[
\beta_\delta = b(T) + O(\sqrt{\delta}), \quad v_\delta(x, T) - v(x, T) = O(\delta e^{\alpha'(x - b(T))})
\]

(3.1)

uniformly in \( x \) for some positive constant \( \alpha' \) where \( x^- = \max\{-x, 0\} \). (Note that as \( b(T) \) is approximated by \( \beta_\delta \), we have \( v_\delta(x, T) = g(x, T) \) for \( x \geq \beta_\delta \), so \( v_\delta(x, T) = v(x, T) = g(x, T) \) for \( x \geq \max\{b(T), \beta_\delta\} \)). A random walk approximation to the value function and the stopping boundary of (2.1) for \( \bar{T} \leq t \leq T \) can be described as follows. Let \( t_0 = T > t_1 > \cdots > t_{K_\delta} = \bar{T} \) partition the interval into \( K_\delta \) subintervals such that

\[
t_{i-1} - t_i = \delta \quad \text{for} \quad 1 \leq i < K_\delta, \quad t_{K_\delta} - t_{K_\delta} = \delta.
\]

(3.2)

Let \( X, X_1, X_2, \ldots \) be i.i.d. random variables with

\[
EX = 0 = EX^3, \quad EX^2 = 1, \quad E e^{\alpha'' |X|} < \infty \text{ for some } \alpha'' > 0,
\]

(3.3)
and let $S_n = X_1 + \cdots + X_n$ ($S_0 = 0$). Approximating the Brownian motion in the optimal stopping problem (2.1) by the random walk $S_n$, we can use backward induction to compute the value function $v_\delta(x, t_i)$ and stopping boundary $b_\delta(t_i)$ of the discrete-time optimal stopping problem. Specifically, first initialize by defining $b_\delta(t_i) = \beta_\delta$, for $0 \leq i < \lfloor \log \delta \rfloor$. Then define recursively

$$v_\delta(x, t_i) = \begin{cases} g(x, t_i) & \text{if } x \geq b_\delta(t_{i-1}) + \sqrt{\delta} |\log \delta|, \\ \max \left[ g(x, t_i), Ev_\delta(x + \sqrt{\delta} X, t_{i-1}) \right] & \text{if } |x - b_\delta(t_{i-1})| < \sqrt{\delta} |\log \delta|, \\ Ev_\delta(x + \sqrt{\delta} X, t_{i-1}) & \text{if } x \leq b_\delta(t_{i-1}) - \sqrt{\delta} |\log \delta| \end{cases}$$

for $1 \leq i \leq K_\delta$, and define $b_\delta(t_i)$ for $\lfloor \log \delta \rfloor \leq i \leq K_\delta$ recursively by

$$b_\delta(t_i) = \inf \{ x : |x - b_\delta(t_{i-1})| < \sqrt{\delta} |\log \delta|, \quad g(x, t_i) \geq Ev_\delta(x + \sqrt{\delta} X, t_{i-1}) \} \quad (\inf \emptyset = b_\delta(t_{i-1})).$$

**Theorem 2.** With the same notation as in Section 2, assume (2.2)–(2.5) and that

$$G(b(t), t) < 0 \quad \text{for all } \bar{T} \leq t \leq T, \quad (3.5)$$

$$\sup_{x < b(T^*)} e^{-\alpha''|x - b(T^*)|} |g(x, T^*)| < \infty \quad \text{for some } \alpha'' > 0, \quad (3.6)$$

$$\sup_{\bar{T} \leq t \leq T^*, x \geq b(t)} e^{-\beta(x)} \{ |g(x, t)| + |G(x, t)| \} < \infty, \quad (3.7)$$

with $\beta(x) = O(x)$ as $x \to \infty$. Then

$$\sup_{1 \leq i \leq K_\delta, x \in \mathbb{R}} e^{-\alpha|x - b(t_i)|} |v_\delta(x, t_i) - v(x, t_i)| = O(\delta) \quad \text{with } \alpha := \max \{ \alpha', \alpha'' \}, \quad (3.8)$$

$$b_\delta(t_i) = b(t_i) - \rho \sqrt{\delta} + o(\sqrt{\delta}) \quad \text{uniformly in } |\log \delta| \leq i \leq K_\delta, \quad (3.9)$$

where $\tau_+ = \inf \{ n : S_n > 0 \}$ and $\rho = ES_{\tau_+}^2 / 2ES_{\tau_+}$.

**Remark 3.1.** In view of Lemma 1(a), $G(b(t), t) \leq 0$ and condition (3.5) requires the inequality to be strict for all $t \in [\bar{T}, T]$. Conditions (3.6) and (3.7) are assumed in order to establish bounds on higher-order derivatives of $v$ with respect to $x$ (see Lemma 9). By (3.8), we have $v_\delta(x, t_i) - v(x, t_i) = O(\delta)$ uniformly in $i \in \{1, \ldots, K_\delta\}$ and in $x$ over any bounded subset of $\mathbb{R}$. Along the lines of the proof of Theorem 2, it can be shown that

$$\sup_{1 \leq i \leq K_\delta, x \in \mathbb{R}} |v_\delta(x, t_i) - v(x, t_i)| = O(\delta),$$
if (3.1) and (3.6) are strengthened to

$$\sup_{x} |v_6(x, T) - v(x, T)| = O(\delta) \quad \text{and} \quad \sup_{x \in b(T^*)} |g(x, T^*)| < \infty.$$  

The latter condition implies that $v(x, t)$ has a bounded fifth derivative with respect to $x$ in the continuation region (see Lemma 9).

**Remark 3.2.** In many applications, the payoff function may not belong to the class $C^{3,2}$ on the continuation region. For example, the payoff function associated with the standard American call option is given by $g(x, t) = e^{-rt}(P_0e^{(r-d-\sigma^2/2)t+\sigma x} - K)^+$, which is not differentiable in $\{(x, t) : P_0e^{(r-d-\sigma^2/2)t+\sigma x} = K\}$. It can be readily shown that the value function and the stopping boundary remain unchanged if the payoff function is replaced in the continuation region by any function not greater than the value function (cf. [35, p. 143]). Therefore, this payoff function yields the same value function and stopping boundary as

$$\tilde{g}(x, t) = e^{-rt}(P_0e^{(r-d-\sigma^2/2)t+\sigma x} - K) \quad \text{for } t < T^*,$$

$$= g(x, T^*) \quad \text{for } t = T^*,$$

to which Theorem 2 applies. Moreover, such changes in the payoff function (in the continuation region) have little effect on the algorithm (3.3) and (3.4) if $\delta$ is sufficiently small.

**Remark 3.3.** Although $v_6(x, t)$ only gives an approximation to $v(x, t)$ for $t = t_i, i = 0, \ldots, K$, we can extend $v_6(x, t)$ by linear interpolation to approximate $v(x, t)$ for $t_i < t < t_{i-1}$, where we define

$$v_6(x, t) = \frac{t_{i-1} - t}{t_{i-1} - t_i} v_6(x, t_i) + \frac{t - t_i}{t_{i-1} - t_i} v_6(x, t_{i-1}). \quad (3.10a)$$

With this definition of $v_6(x, t)$ at every $t \in [\tilde{T}, T]$, the $O(\delta)$ error in (3.8) still holds if we replace $v_6(x, t_i) - v(x, t_i)$ by $v_6(x, t) - v(x, t)$ and take the supremum over $t \in [\tilde{T}, T]$. In practice, $X$ is usually taken to be lattice-valued to avoid numerical integration in computing $Ev_6(x + \sqrt{\delta}X, t_i-1)$. In this case, with $L$ denoting the lattice for which $P(X \in L) = 1$, $v_6(x, t)$ is only given by (3.3) for $x \in \sqrt{\delta}L$ if $(x, t)$ belongs to the continuation region of the discrete-time optimal stopping problem. For $x_j < x < x_{j+1}$, where $x_j$ and $x_{j+1}$ are two adjacent points of the lattice $\sqrt{\delta}L$ such that at least one of $(x_j, t)$ and $(x_{j+1}, t)$ is a continuation point, we can again use linear interpolation to approximate $v(x, t)$ by

$$\tilde{v}_6(x, t) = \frac{x_{j+1} - x}{x_{j+1} - x_j} v_6(x_j, t) + \frac{x - x_j}{x_{j+1} - x_j} v_6(x_{j+1}, t); \quad (3.10b)$$
see Section 3.4 for further details and discussion.

3.1 Overshoots, ladder heights and a renewal–theoretic lemma

In the non-arithmetic case, as noted by Hogan [22], the constant \( \rho \) in Theorem 2 is the limiting expected overshoot of the boundary \( c \) at the first passage time \( \tau(c) = \inf \{n : S_n > c\} \). Specifically \( \rho = \lim_{c \to \infty} ER_c \) in this case, where \( R_c = S_{\tau(c)} - c \). Moreover, in the case where \( X \) is arithmetic with span \( h \), \( ER_c \to \rho + h/2 \) as \( c \to \infty \) in \( L \); see [19, pages 300 and 371]. Whereas only \( \lim_{c \to \infty} ER_c \) is involved in approximating the difference between \( b(t) \) and \( b(t) \), bounds on \( ER_c \) for all values of \( c \) are needed in the proof of Theorem 2. Lorden [31] proved the upper bound \( \sup_{c \geq 0} ER_c \leq E S^2_{\tau_+}/E S_{\tau_+} \) (\( = 2\rho \)) by making use of Wald’s identity and certain representations of \( \int_0^c R_t dt \). He also obtained similar inequalities for \( \sup_{c \geq 0} ER_c^p \) for any \( p > 0 \); in particular

\[
\sup_{c \geq 0} ER_c^2 \leq 4 E S^3_{\tau_+}/3 E S_{\tau_+},
\]  

(3.11)

which is also used in the proof of Theorem 2. Note that \( E S^p_{\tau_+} < \infty \) if \( EX = 0 \) and \( E(X^+)^{p+1} < \infty \); see [14, Theorem 1]. Besides (3.11), we also need a different inequality which bounds \( ER_c^2 \) by \( \rho \) and \( ER_c \) and which is used in the proof of Lemma 7.

**Lemma 3.** If \( EX > 0 \) and \( E(X^+)^2 < \infty \), or if \( EX = 0 \) and \( 0 < E(X^+)^3 < \infty \), then \( ER_c^2 \leq 2\rho(c + ER_c) \) for all \( c \geq 0 \).

**Proof.** Under the assumptions on \( X \), \( \tau(c) < \infty \) a.s. and \( E S^2_{\tau(c)} < \infty \) (cf. Theorem 2.1 of [20] for the case \( EX > 0 \) and Theorem 1 of [14] for the case \( EX = 0 \)). Note that \( \tau_+ = \tau(0) \). Replacing \( X \) by the ladder variable \( S_{\tau_+} \), we shall assume without loss of generality that \( X > 0 \) and \( EX^2 < \infty \). For the positive random variable \( X \), the desired inequality \( ER_c^2 \leq 2\rho(c + ER_c) \) can be written as

\[
(EX)(ER_c^2) \leq (EX^2)(c + ER_c),
\]

(3.12)

which we shall prove by using the renewal function \( U(x) = \sum_{n=0}^{\infty} P\{S_n \leq x\} \) \( (S_0 = 0) \). Let \( F \) denote the distribution function of \( X \) and let

\[
G(t) = \int_t^\infty x(1 - F(x))dx, \quad H(t) = \int_t^\infty (1 - F(x))dx, \quad t \geq 0.
\]

To prove (3.12), note that for \( y \geq 0 \),

\[
P\{R_c > y\} = \sum_{n=1}^{\infty} P\{S_{n-1} \leq c, S_n > c + y\} = \int_{[0,c]} (1 - F(y + c - x))dU(x),
\]
in which $U$ is interpreted as a measure with $U(\{0\}) = 1$, and therefore

\[
ER_c^2 = 2 \int_0^\infty \int_{[0,c]} \{1 - F(y + c - z)\} dU(z) dy \\
= 2 \int_{[0,c]} \int_{c-z}^\infty (x - c + z)(1 - F(x)) dx dU(z) \\
= 2 \int_{[0,c]} G(c - z) dU(z) - 2 \int_{[0,c]} (c - z) H(c - z) dU(z),
\]

\[
ER_c = \int_0^\infty \int_{[0,c]} \{1 - F(y + c - z)\} dU(z) dy = \int_{[0,c]} H(c - z) dU(z).
\]

Since $\int_{[0,c]} G(c - z) dU(z) = G(c) + \int_{[0,c]} G(c - z) dU(z)$ and since $U(0) = 1, G(0) = \frac{1}{2} EX^2$ and $(d/dz)G(c - z) = (c - z)(1 - F(c - z))$ a.e., integration by parts yields

\[
ER_c^2 = (EX^2)U(c) - 2 \int_{[0,c]} U(z)(c - z)(1 - F(c - z)) dz \\
- 2 \int_{[0,c]} (c - z) H(c - z) dU(z) \leq (EX^2)U(c). \tag{3.13}
\]

Since $H(c - z) = EX - \int_0^{c-z} (1 - F(x)) dx = EX - \int_x^c (1 - F(t - z)) dt$ for $0 \leq z \leq c,$

\[
ER_c = (EX) \int_{[0,c]} dU(z) - \int_{[0,c]} \int_{x}^{c} (1 - F(t - z)) dt dU(z) \\
= (EX)U(c) - \int_{0}^{c} \int_{[0,t]} (1 - F(t - z)) dU(z) dt.
\]

Noting that $\int_{[0,t]} (1 - F(t - z)) dU(z) = U(t) - \int_{[0,t]} F(t - z) dU(z)$ and that $\sum_{n=1}^{\infty} P\{S_n \leq t\} = U(t) - 1,$ we then obtain that

\[
c + ER_c = c + (EX)U(c) - \int_0^c dt = (EX)U(c). \tag{3.14}
\]

From (3.13) and (3.14), the desired conclusion (3.12) follows. $\Box$

### 3.2 A canonical optimal stopping problem

We introduce here a canonical optimal stopping problem, depending on $\delta$ and $0 < \zeta < \gamma < 1/3,$ which is basic to the induction proof of Theorem 2. The payoff function is of the form

\[
g_\delta(y, s) = - s + \tau_\delta(y, s) \quad \text{if } s < 0, \\
= h_\delta(y) \quad \text{if } s = 0. \tag{3.15}
\]
where for some $A > 0$,

$$|r_δ(y, s)| \leq A\{δ s^2 + \sqrt{δ}|s|y| + \sqrt{δ}|y|^3\} \quad \text{for } |y| \leq δ^{-1/6},$$

$$h_δ(y) = y^21_{\{y \leq 0\}} + a_δ(y) \quad \text{with} \quad |a_δ(y)| \leq A \quad \text{for } |y| \leq δ^{-1/6}. \quad (3.16)$$

Assume further that for the random walk $\{S_n, n \geq 1\},$

$$\sup_{|y| \leq δ^{-1/2}, n \leq δ^{-γ}} E|h_δ(y + S_n)|^p + \sup_{|y| \leq δ^{-1/2}, 1 \leq m \leq δ^{-γ}, 1 \leq n \leq δ^{-γ}} E|r_δ(y + S_n, -m)|^p = O(δ^{-q}) \quad (3.17)$$

for some $p > 1$ and $q > 0$. For $-∞ < y < ∞$ and $t \in \{0, -1, -2, \ldots\}$, let

$$V_δ(y, t) = \sup_{τ \in T_δ, t, y} E g_δ(y + S_τ, t + τ) \quad (3.18)$$

where $T_δ, t, y$ is the class of stopping times $τ \leq -t$ such that on $\{-t > τ \geq n\}$,

$$y + S_n ≥ |\log δ| \quad \Rightarrow \text{stopping occurs at } n,$$

$$y + S_n < -|\log δ| \quad \Rightarrow \text{stopping cannot occur at } n. \quad (3.20)$$

This canonical optimal stopping problem is a modification of Chernoff’s [9] associated stopping problem with payoff function $g$ defined in (1.1), which is a special case of (3.15). By Doob’s optimal stopping theorem,

$$\tilde{V}(y, t) := \sup_{0 \leq τ \leq -t} E g(y + S_τ, t + τ) + t = \sup_{0 \leq τ \leq -t} E \left[g^*(y + S_τ)1_{(t+τ=0)} - τ\right], \quad (3.21)$$

where $g^*(y) = y^21_{\{y \leq 0\}}$, so the associated optimal stopping problem with payoff function (1.1) has value function $\tilde{V}(y, t) - t$. As pointed out by Hogan [22], Lemmas 3.1 and 3.2 of [9] hold for general zero-mean, finite-variance random walks. Therefore $\tilde{V}(y, t)$ is monotone decreasing in $-t$ and in $y$. Hogan [22] has shown that there exists a nondecreasing optimal stopping boundary $z(t) < 0$ for $t < 0$ such that $\lim_{t \to -∞} z(t) = -ρ$ and $\tilde{V}(y, t) > 0$ if $y < z(t)$, $\tilde{V}(y, t) = 0$ if $y ≥ z(t)$. The following lemma, whose proof is given in the Appendix, considers instead of $\tilde{V}(y, t)$ the expected reward $u(y, t)$ of the policy that starts at $(y, t)$, takes an observation and proceeds optimally thereafter.

**Lemma 4.** For $t < 0$, let $u(y, t) = -1 + E\tilde{V}(y + X, t + 1)$. Then $u(y, t) < 0$ for $y > z(t)$. Consequently $\lim_{t \to -∞} u(y, t) < 0$ for every $y > -ρ$.

To analyze the canonical optimal stopping problem (3.19), we use Lemmas 3, 4 and the following.
Lemma 5. Let $\tau(y) = \inf\{n : S_n > y\}$, in which $S_n = \sum_{i=1}^{n} X_i$ and $X, X_1, \ldots$ are i.i.d. random variables satisfying (3.2). Then as $m \to \infty$,

(i) $P\{\tau(y) \geq m\} = O((y^2/m)^{\lambda})$ uniformly in $1 \leq y \leq \sqrt{m}$, for every $\lambda < 1/2$;

(ii) $P\{\tau(y) \leq m\} \leq \exp\{-\epsilon \min(y^2/m, \epsilon y/2)\}$ for some $\epsilon > 0$ and all $y \geq \sqrt{m}$.

Part (i) of Lemma 5 follows from (4.6), (4.12) and (4.13) of [29]. Part (ii) follows from

$$P\{\tau(y) \leq m\} = P\{\max_{n \leq m} S_n > y\} \leq e^{-\theta y} \left( Be^{\theta X}\right)^m,$$

by the submartingale inequality and (3.2), with $\theta > 0$ and $\epsilon > 0$ chosen suitably to yield the desired conclusion. In the Appendix we make use of Lemmas 3–5 to prove the following results on the canonical optimal stopping problem (3.19) that are used to prove Theorem 2.

Lemma 6. For every fixed $\epsilon > 0$ and all sufficiently small $\delta > 0$, if $-\delta^{-1} \leq t \leq -|\log \delta|$, then in the optimal stopping problem (3.19), $(y, t)$ is a stopping point for $y > -\epsilon + \delta$.

Lemma 7. For every fixed $\epsilon > 0$ and all sufficiently small $\delta > 0$, if $-\delta^{-1} \leq t \leq -|\log \delta|$ in the optimal stopping problem (3.19), $(y, t)$ is a continuation point for $y < -\epsilon - \delta$.

3.3 Proof of Theorem 2

To prove Theorem 2 we approximate the underlying optimal stopping problem by a recursively defined family of optimal stopping problems. The payoff function in each optimal stopping problem is of the canonical form (3.15), in which $g_s$ is given by the following.

Lemma 8. With the same notation and assumptions as in Theorem 2, let $t^* \in (\bar{T}, T]$ and $x^* = b(t^*)$. Define the quadratic function

$$\pi_g(x, t) = g(x^*, t^*) + \frac{\partial g}{\partial x}(x^*, t^*)(x - x^*) + \frac{1}{2} \frac{\partial^2 g}{\partial x^2}(x^*, t^*)(x - x^*)^2$$

$$- \frac{1}{2} \frac{\partial^2 g}{\partial x \partial t}(x^*, t^*)(t - t^*).$$

(3.22)

Letting $y = \delta^{-1/2}(x - x^*)$, $s = \delta^{-1}(t - t^*)$, define

$$\tilde{g}_s(y, s) = \begin{cases} 
\delta^{-1}\{g(x, t) - \pi_g(x, t)\} & \text{if } t < t^*, \\
\delta^{-1}\{u(x, t) - \pi_g(x, t)\} + C_s(y) & \text{if } t = t^*,
\end{cases}$$

(3.23)
where \( \exp\{-\alpha\sqrt{\delta}y^{-}\}|C_{\delta}(y)|/|G(x^{*},t^{*})| \) is bounded by some positive constant \( A \) for all \( y \) and all sufficiently small \( \delta > 0 \) with \( \alpha \) given in (3.8). Then \( g_{\delta} := \frac{\delta}{|G(x^{*},t^{*})|} \) satisfies (3.15)–(3.18) for \(-\delta^{-\gamma} \leq s \leq 0\), for every choice of \( 0 < \zeta < \gamma < 1/3 \) and \( q \geq p > 1 \). Furthermore, defining \( V_{\delta} \) by (3.19) with this choice of \( g_{\delta} \), we have as \( \delta \to 0 \),

(i) \( V_{\delta}(y,s) = y^{2} - s - E(-\rho + R_{-\rho-y})^{2} + o(1) \) uniformly in \(-\delta^{-\gamma} \leq s \leq -\frac{1}{2}\delta^{-\gamma} \) and \( -\rho \geq y \geq -\delta^{-\zeta/2} \);

(ii) \( V_{\delta}(y,s) = y^{2}1_{\{y \leq 0\}} - s + O(1) \) uniformly in \(-\delta^{-\gamma} \leq s < 0 \) and \( |y| \leq \delta^{-\zeta/2} \).

**Remark 3.4.** For the symmetric Bernoulli case, the term \( E(-\rho + R_{-\rho-y})^{2} \) appearing in Lemma 8(i) plays an important role in Chernoff and Petkau's numerical algorithm [13, p. 48] for computing the boundary \( b(\cdot) \).

**Remark 3.5.** Letting \( y = \delta^{-1/2}(x - x^{*}) \) and \( s = \delta^{-1}(t - t^{*}) \), define

\[
V(y,s) = \delta^{-1}(u(x,t) - \pi_{\delta}(x,t)) / |G(x^{*},t^{*})|.
\]

Recalling that \( y^{2}1_{\{y \leq 0\}} - s \) is the (continuous-time) value function for the optimal stopping problem (1.1), we can make use of Taylor expansions and arguments as in the proof of Lemma 8 (given in the Appendix) to show that

\[
V(y,s) = y^{2}1_{\{y \leq 0\}} - s + o(1)
\]

uniformly in \(-\delta^{-\gamma} \leq s < 0 \) and \( |y| \leq \delta^{-\zeta/2} \). By Lemma 8(ii), there exist \( C > 0 \) and \( \delta_{0} > 0 \) (depending on the constant \( A \) that bounds \( \exp\{-\alpha\sqrt{\delta}y^{-}\}|C_{\delta}(y)|/|G(x^{*},t^{*})| \)) such that for \( 0 < \delta < \delta_{0} \), we have \( |V(y,s) - V_{\delta}(y,s)| \leq C \) for \(-\delta^{-\gamma} \leq s < 0 \) and \( |y| \leq \delta^{-\zeta/2} \). Moreover, by Lemma 8(i), there exists \( \delta_{1} > 0 \) (depending on \( A \)) such that for \( 0 < \delta < \delta_{1} \), we have

\[
|V(y,s) - V_{\delta}(y,s)| \leq 1 + \sup_{y < -\rho} E(-\rho + R_{-\rho-y})^{2}
\]

for \(-\delta^{-\gamma} \leq s \leq -\frac{1}{2}\delta^{-\gamma} \) and \( -\rho \geq y \geq -\delta^{-\zeta/2} \). Since by Lemma 6, \( V_{\delta}(y,s) = -s + o(1) \) for \(-\rho < y \leq \delta^{-\zeta/2} \) and since \( \rho^{2} = E(-\rho + R_{0})^{2} = E(-\rho + S_{r_{s}})^{2} \), the inequality (3.24) actually holds for \(-\delta^{-\gamma} \leq s \leq -\frac{1}{2}\delta^{-\gamma} \) and \( |y| \leq \delta^{-\zeta/2} \).

We next combine Lemma 2 with conditions (3.6) and (3.7) of Theorem 2 to prove the following.
Lemma 9. With the same notation and assumptions as in Theorem 2,

\[ \sup_{\bar{T} \leq t \leq T, x < b(t)} e^{-\alpha''|x-b(t)|} |(\partial^j / \partial x^j) v(x, t)| < \infty \text{ for } j = 0, \ldots, 5, \]

where \((\partial^0 / \partial x^0) v(x, t) := v(x, t)\). Furthermore, if (3.6) is strengthened to \(\sup_{z < b(T^*)} |g(x, T^*)| < \infty\), then we have

\[ \sup_{\bar{T} \leq t \leq T, x < b(t)} |(\partial^j / \partial x^j) v(x, t)| < \infty \text{ for } j = 0, \ldots, 5. \]

Proof. By Lemma 2, \(\sup_{\bar{T} \leq t \leq T, L \leq x \leq b(t)} (\partial^j / \partial x^j) v(x, t) | < \infty \text{ for } j = 0, \ldots, 5\) where \(L := \min\{b(t) : \bar{T} \leq t \leq T\} - 1\). Let \(\Omega = \{(x, t) : x \leq L, \bar{T} \leq t \leq T\}\). Note that \(E_{x,t}g(B_{T^*}, T^*)\) can be expressed as

\[ \int_{-\infty}^{b(T^*)} g(u, T^*) \varphi \left( \frac{u - x}{\sqrt{T^* - t}} \right) \frac{du}{\sqrt{T^* - t}} + \int_{b(T^*)}^{\infty} g(u, T^*) \varphi \left( \frac{u - x}{\sqrt{T^* - t}} \right) \frac{du}{\sqrt{T^* - t}}. \]

Here, in view of (3.7) and \(T < T^*\), the second integral has a bounded \(j\)th derivative with respect to \(x\) over \(\Omega\) (\(j = 0, \ldots, 5\)), whereas the first integral has its \(j\)th derivative bounded by \(O(e^{a''|x|})\) over \(\Omega\) in view of (3.6) (or bounded by \(O(1)\) if \(\sup_{z < b(T^*)} |g(x, T^*)| < \infty\)). Similarly,

\[ \int_{t}^{T^*} \left\{ \int_{b(s)}^{\infty} G(u, s) \varphi \left( \frac{u - x}{\sqrt{s - t}} \right) \frac{du}{\sqrt{s - t}} \right\} ds \]

also has a bounded \(j\)th derivative with respect to \(x\) over \(\Omega, j = 0, \ldots, 5\). Hence the desired conclusion follows from (2.7) and (2.10). \(\square\)

Proof of Theorem 2. Let \(\epsilon > 0\) and take \(0 < \zeta < \gamma < 1/3\). We shall use induction to prove that for sufficiently small \(\delta > 0\),

\[ |b(t_i) - b_s(t_i) - \rho \sqrt{\delta}| \leq \epsilon \sqrt{\delta}, \quad i = [\log \delta], \ldots, K_\delta; \quad (3.25) \]

\[ \max_{b(t_{i-1}) - \delta^{(1/2)^i} \leq x \leq b(t_i) + 2 \delta^{(1/2)^i}} |v(x, t_i) - v_b(x, t_i)| \leq C_0 \delta, \quad i = 1, \ldots, K_\delta; \quad (3.26) \]

\[ \max_{x < b(t_{i-1}) - \delta^{(1/2)^i}} e^{-\alpha x - b(t_i)} |v(x, t_i) - v_b(x, t_i)| \leq e^{\alpha B t_i} (C_0 \delta + C_1 \delta^2), \quad i = 1, \ldots, K_\delta; \quad (3.27) \]

where \(B := \max\{b'(t) : \bar{T} \leq t \leq T\} < \infty, C_0 \text{ and } C_1 \text{ are positive constants that will be specified later. Since } \{\pi_g(x + \sqrt{\delta} S_n, t + \delta n), n \geq 0\} \text{ and } \{\pi_g(x + \sqrt{\delta} B_u, t + \delta u), u \geq 0\} \text{ are martingales, Doob's optional stopping theorem yields} \]

\[ \pi_g(x, t) = E \pi_g(x + \sqrt{\delta} B_\tau, t + \delta \tau) = E \pi_g(x + \sqrt{\delta} S_{\kappa}, t + \delta \kappa) \]

for any bounded stopping times \(\tau\) and (integer-valued) \(\kappa\) (adapted to the filtration generated by \(\{S_n\}\)). In the induction proof of (3.25)–(3.27), we can therefore replace \(g\) and \(v\) by \(g - \pi_g\) and \(v - \pi_g\). 16
for each time interval \([t_i, t_{i-1}]\) with the optimal stopping boundary preserved. Note also that by (2.4), (3.3) and (3.25), we have, for \(i = 1, \ldots, K_\delta,\)

\[
b(t_{i-1}) + 2\delta^{1/2}|\log \delta| > \max\{b(t_i), b_\delta(t_{i-1}) + \delta^{1/2}|\log \delta|\},
\]

for sufficiently small \(\delta > 0,\) from which it follows that \(v(x, t_i) = u_\delta(x, t_i) = g(x, t_i)\) for \(x > b(t_{i-1}) + 2\delta^{1/2}|\log \delta|\).

We first apply Lemma 8 with \(t^* = T(= t_0),\) \(y = \delta^{-1/2}(x - b(T)),\) and \(C_\delta(y) = \delta^{-1}\{u_\delta(x, T) - v(x, T)\}.\) By (3.1) and noting that \(\alpha' < \alpha,\)

\[
A := 1 + \lim_{\delta \to 0} \sup_y \exp\{-\alpha\sqrt{\delta}y\} |C_\delta(y)| / \|G(b(T), T)\| < \infty.
\]

Then Lemmas 6, 7 and 8(ii) show that (3.25) holds for \(i = [\log \delta], \ldots, [\delta^{-\gamma}]\) and that (3.26) holds for \(i = 1, \ldots, [\delta^{-\gamma}]\) for some constant \(C_0\) (see Remark 3.5). We shall choose \(C_0\) to satisfy

\[
C_0 \geq \sup_{\tilde{t} \leq t \leq T} |G(b(t), t)| \max_{y < -\rho} \left\{ A, 1 + \sup_{y < -\rho} E \left( -\rho + R_{-\rho - y} \right)^2 \right\}. \tag{3.28}
\]

We now make use of (3.26) and an induction argument to prove (3.27) for \(i = 1, \ldots, [\delta^{-\gamma}]\).

First consider the case \(i = 1.\) For \(x \leq b(T) - \delta^{1-\gamma}/2,\) \(u_\delta(x, t_1) = Ev_\delta(x + \sqrt{\delta}X, T)\) (by (3.3)), and \(v(x, t_1) = Ev(x + \sqrt{\delta}B_1, T) + O(\delta^p)\) for every \(p > 0.\) It follows that for \(x \leq b(T) - \delta^{1-\gamma}/2\)

\[
v_\delta(x, t_1) - v(x, t_1) = E \left\{ u_\delta(x + \sqrt{\delta}X, T) - v(x + \sqrt{\delta}B_1, T) \right\} + O(\delta^4). \tag{3.29}
\]

By (3.28), \(\sup_x \exp\{-\alpha(x - b(T))\} |u_\delta(x, T) - v(x, T)| \leq C_0\delta\) for sufficiently small \(\delta > 0.\) Since \(EX^r = EB_1^r\) for \(r = 1, 2, 3\) by (3.2), it follows from Lemma 9 together with Taylor's theorem that

\[
E \left\{ v(x + \sqrt{\delta}X, T) - v(x, T) \right\} = E \left\{ v(x + \sqrt{\delta}B_1, T) - v(x, T) \right\}
\]

\[
= \frac{1}{4!} \frac{\partial^4 v}{\partial x^4}(x, T) \left\{ E(\sqrt{\delta}X)^4 - E(\sqrt{\delta}B_1)^4 \right\}
\]

\[
+ \frac{1}{5!} E \left\{ \frac{\partial^5 v}{\partial x^5}(x + \theta'\sqrt{\delta}X, T)(\sqrt{\delta}X)^5 - \frac{\partial^5 v}{\partial x^5}(x + \theta''\sqrt{\delta}B_1, T)(\sqrt{\delta}B_1)^5 \right\}, \tag{3.30}
\]

where \(\theta'\) and \(\theta''\) are between 0 and 1. (Strictly speaking, as \(\partial^2 v(x, T) / \partial x^2\) has a discontinuity at the boundary, we need to replace \(v(x, T)\) by a fifth polynomial of \(x\) in the half line \(\{(x, T) : b(T) \leq x < \infty\}\) so as to make the fifth derivative continuous in order to apply Taylor's theorem. Such replacement yields a negligible error of \(O(\delta^p)\) for every \(p > 0.\)) Setting

\[
C_1 = 1 + \frac{1}{4!} |EX^4 - EB_1^4| \sup_{\tilde{t} \leq t \leq T, x < b(t)} e^{-\alpha(x - b(t))} \left| \frac{\partial^4 v}{\partial x^4}(x, t) \right|,
\]
we have by (3.29) and (3.30) together with Lemma 9 that
\[
\sup_{x < b(T) - \delta^{(1-\gamma)/2}} e^{-\alpha|x - b(T)|} |v_\delta(x, t_1) - v(x, t_1)| \leq C_0 \delta + C_1 \delta^2
\]
for sufficiently small \(\delta > 0\), implying that
\[
\sup_{x < b(T) - \delta^{(1-\gamma)/2}} e^{-\alpha|x - b(t_1)|} |v_\delta(x, t_1) - v(x, t_1)| \leq e^{\alpha B\delta} (C_0 \delta + C_1 \delta^2).
\]
This shows that (3.27) holds for \(i = 1\). Replacing \(T(= t_0)\) in the preceding argument by \(t_1\) then shows (3.27) for \(i = 2\), recalling that (3.26) has already been established for \(i = 1\). Proceeding inductively in this way proves (3.27) for \(i = 1, \ldots, [\delta^{-\gamma}].\)

To proceed inductively beyond \([\delta^{-\gamma}]\), suppose (3.25)–(3.27) have been shown to hold for all \(i \leq j\) with \(j \geq [\delta^{-\gamma}]\). To show that (3.25) and (3.26) also hold for \(i = j + 1\), we apply Lemma 8 with \(t^* = t_{j-1}\) and \(C_\delta(y) = \delta^{-1} \{v_\delta(x, t_{j-1}) - v(x, t_{j-1})\}\), where \(J = [\delta^{-\gamma}/2]\). By the induction hypothesis,
\[
\sup_y e^{-\alpha \sqrt{\delta y}} |C_\delta(y)|/|G(b(t_{j-1}), t_{j-1})| < e^{\alpha B(T - \bar{T})} \left( C_0 + C_1 (T - \bar{T}) \right) / \inf_{T \leq t \leq T} |G(b(t), t)| < \infty
\]
for sufficiently small \(\delta > 0\). Thus we can apply Lemmas 6, 7, and 8(i) to establish (3.25) and (3.26) for \(i = j + 1\). (In view of Remark 3.5, recall that \(C_0\) has been chosen to satisfy (3.28) and note that the bound \(\exp[\alpha B(T - \bar{T})]\) \(\{C_0 + C_1 (T - \bar{T})\}\) does not depend on \(j\) so that the same \(\delta_1\) (referred to in the remark) can be chosen for all \(j\).) Finally, making use of Lemma 9 and noting by the induction hypothesis that \(\sup_x e^{-\alpha |x - b(t_j)|} |v_\delta(x, t_j) - v(x, t_j)| \leq e^{\alpha B\delta} (C_0 \delta + C_1 \delta^2)\), we can proceed as in (3.29) and (3.30) with \(T = t_0\) and \(t_1\) replaced by \(t_j\) and \(t_{j+1}\) to establish (3.27) for \(i = j + 1\).

\begin{proof}
3.4 The Chernoff–Petkau method for symmetric Bernoulli walks
\end{proof}

In the backward induction scheme (3.3)–(3.4) considered in Theorem 2, the search for the optimal stopping boundary \(b_\delta(t_i)\) is carried out over the interval \(b_\delta(t_i-1) \pm \sqrt{\delta} |\log \delta|\), for \(i = 1, \ldots, K_\delta\). In practice, however, the backward induction procedure (3.3) to compute \(v_\delta(x, t_i)\) is usually carried out over a fixed grid of points instead of within the entire interval. Particularly popular is the symmetric Bernoulli walk approximation with the set of grid points \(L = \{\beta_\delta + \sqrt{\delta} k : k = 0, \pm 1, \ldots\}\), for which at any time \(t_i\) every \(x \in L\) is characterized as either a stopping point, if \(g(x, t_i) \geq Ev_\delta(x + \sqrt{\delta} X, t_{i-1})\), or a continuation point if \(Ev_\delta(x + \sqrt{\delta} X, t_{i-1}) > g(x, t_i)\), where \(X\) is a symmetric Bernoulli random variable. A simple way to approximate the optimal stopping boundary \(b_\delta(t_i)\) is to let
\[
x_1 = \inf \{x \in L : g(x, t_i) \geq Ev_\delta(x + \sqrt{\delta} X, t_{i-1})\},
\]
\[ x_0 = x_1 - \sqrt{\delta}, \] so that \( b_\delta(t_i) \) is approximated by the value of \( x \) satisfying

\[ \Delta(x_1, t_i)(x - x_0) + \Delta(x_0, t_i)(x_1 - x) = 0, \]

where \( \Delta(x, t_i) = \frac{1}{2} \left[ u_\delta(x + \sqrt{\delta}, t_i-1) + u_\delta(x - \sqrt{\delta}, t_i-1) \right] - g(x, t_i). \) Note that \( \Delta(x_1, t_i) \leq 0 \leq \Delta(x_0, t_i). \)

Besides this linear interpolation method, Chernoff and Petkau ([12], [13]) also introduced an extrapolation method which does not explicitly estimate \( b_\delta(t_i). \) This extrapolation method uses (3.9) to arrive at a system of equations from which the continuous-time optimal stopping boundary \( b(t_i) \) is computed. Specifically, with the \((y, s)\) scale as defined in Lemma 8, Chernoff and Petkau [13] argued that for large negative \( s \) and bounded \( y < -\rho, \) the \( V_\delta(y, s) \) in Lemma 8(i) can be expressed as

\[ V_\delta(y, s) = y^2 - s - E(-\rho + R_{-\rho-y})^2 + o(1) \]

with \( \rho = 0.5 \) and \( R_x = 1+[x] - x \) for the symmetric Bernoulli case. Since \( y^2 - E(-0.5 + R_{-0.5-y})^2 = -h_1(y) \) for \( y < 0.5, \) where

\[ h_1(y) = \begin{cases} 0 & \text{for } y \geq -0.5 \\ -y^2 + \inf_i(y + i)^2 & \text{for } y < -0.5, \end{cases} \]

(3.32)

it follows that for a continuation point \((x*, t^*)\) in the vicinity of a boundary point \((x^*, t^*)\), with \( 0 > x - x^* = O(\sqrt{\delta}) \) and such that \((t - t^*)/\delta \) is negatively large but \( b(t_i) - x^* = b(t_i) - b(t^*) = o(\sqrt{\delta}), \)

\[ D(x, t_i) := u_\delta(x, t_i) - g(x, t_i) \approx a\delta h_1 \left( (x - x^*)/\sqrt{\delta} \right) \approx a\delta h_1 \left( (x - b(t_i))/\sqrt{\delta} \right), \]

(3.33)

where \( a = G(x^*, t^*) < 0. \) Recalling that \( x_0 = x_1 - \sqrt{\delta} \) and that \( x_1 \) is the smallest stopping point on the grid at time \( t = t_i, \) let \( D_0 \) and \( D_1 \) be the values \( D(x_0, t_i) \) and \( D(x_0 - \sqrt{\delta}, t_i), \) respectively. Then \( b(t_i) \) is between \( x_0 \) and \( x_1. \) Since \( b_\delta(t) = b(t) - 0.5\sqrt{\delta} + o(\sqrt{\delta}), \) which implies that (approximately) \( b(t) - 0.5\sqrt{\delta} < x_1 < b(t) + 0.5\sqrt{\delta}, \) we use the representation

\[ x_0 = x_1 - \sqrt{\delta} = b(t) + uv\sqrt{\delta}, \quad x_0 - \sqrt{\delta} = b(t) + (v-1)\sqrt{\delta}, \]

(3.34)

with \(-1.5 < v \leq -0.5, \) to yield via (3.33) that

\[ D_0 \approx a\delta h_1(v) = a\delta(2v + 1), \quad D_1 \approx a\delta h_1(v - 1) = a\delta(4v). \]

(3.35)

The solution to the above linear system is \( v = D_1/(4D_0 - 2D_1), \) which then leads to the following continuity correction for the approximation of \( b(t_i) \) by the stopping point \( x_1 \) defined in (3.31):

\[ b(t_i) = x_0 + \left\{ \frac{D_1}{2(D_1 - 2D_0)} \right\} \sqrt{\delta}. \]

(3.36)
Chernoff and Petkau ([12], [13]) have found the corrected Bernoulli-walk approximation (3.36) to work well when applied to a number of well studied optimal stopping problems. In Section 5, we apply the approximation to a stochastic control problem with an explicit solution and thereby illustrate its accuracy. The function \( h_1(y) \) in (3.32) first arose in [11, p. 882] as the limit of \(-V_1(y, n) + n\), as \( n \to \infty \), where \( V_1(y, n) \) is the value function for the symmetric Bernoulli case \((\delta = 1)\) with payoff (1.1). Since the payoffs (1.1) and (3.15) differ by \( r_\delta(y, s) \) defined in (3.16) that is vanishingly small (see (A.2)-(A.6) in the Appendix), we can use the same arguments as those in the proof of Theorem 2 to show that the approximation (3.36) still has the \( o\left(\sqrt{\delta}\right) \) error.

4 Applications to discrete-time optimal stopping

In this section we show that for two well-known optimal stopping problems in discrete time, Chernoff’s theory of continuity correction can be used in conjunction with the explicit formula of the corresponding continuous-time optimal stopping boundary to yield a second-order approximation to the discrete-time stopping boundary.

4.1 The \( S_n/n \) problem

Consider the payoff function \( g(x, t) = x/t \). For a given initial point \((x_0, t_0)\), the classical \( S_n/n \) problem is to find a stopping time \( \tau \) taking values in \( \{0, 1, \ldots \} \) so as to maximize \( E\{(x_0 + S_\tau)/(t_0 + \tau)\} \), where \( S_n = X_1 + \cdots + X_n, n = 1, 2, \ldots \), and \( X, X_1, X_2, \ldots \) are iid with mean 0 and variance 1; see [15] and [17] which prove that it is optimal to stop at the first \( n \) at which \( x_0 + S_n \geq \beta(t_0 + n) \) for some function \( \beta(t) \) (stopping boundary). Shepp [32] and Walker [36] have independently shown that \( \lim_{n \to \infty} \beta(n)/n^{1/2} = c \) and that the corresponding optimal stopping problem for Brownian motion has stopping boundary \( b(t) = ct^{1/2} \), where \( c = 0.83992\ldots \) is the unique root of the equation

\[
\phi(c) = (1 - c^2) \Phi(c),
\]

(4.1)

\( \phi \) and \( \Phi \) being the standard normal density function and distribution function, respectively. To obtain a second-order approximation to \( \beta(n) \) as \( n \to \infty \), we introduce a family of stopping problems indexed by \( \delta > 0 \). For each \( \delta > 0 \), the problem is to maximize \( E g(x_0 + \sqrt{\delta} S_N, t_0 + \delta N) \) over all stopping times \( N \) taking values in \( \{0, 1, 2, \ldots \} \), so the case \( \delta = 1 \) corresponds to the original \( S_n/n \) problem. Denote by \( b_\delta(t) \) the optimal stopping boundary (i.e. to stop at the first \( n \) at which \( x_0 + \sqrt{\delta} S_n \geq b_\delta(t_0 + n \delta) \)).

To begin with, consider the case where \( X \) is standard normal. Then \( \sqrt{\delta} S_n \) has the same distribution as \( B_{\delta n} \). The optimal stopping problem associated with \( b_\delta \) is therefore the same as
that for Brownian motion with the stopping time restricted to \( \{0, \delta, 2\delta, \ldots \} \). The argument used in Section 5 of [9] can be used to show that, as in Theorem 2, \( u_\delta(x,t) - v(x,t) = O(\delta) \) (see Remark 4.1 below) and that

\[
b_\delta(t) = b(t) - \rho \sqrt{\delta} + o(\sqrt{\delta}) = ct^{1/2} - \rho \sqrt{\delta} + o(\sqrt{\delta}),
\]

where \( \rho = 0.583 \) for the assumed normal \( X \); see in particular Chernoff’s [9] Theorem 5.1. Moreover, since \( g(x,t) = \sqrt{\delta} g(\sqrt{\delta} x, \delta t) \), it follows that

\[
\beta(n) = b_\delta(n) = b_\delta(n\delta)/\sqrt{\delta}.
\]

Setting \( \delta = 1/n \) in (4.2) and (4.3) yields

\[
\beta(n) = b_\delta(1)n^{1/2} = cn^{1/2} - \rho + o(1) \quad \text{as } n \to \infty.
\]

**Remark 4.1.** For the case of standard normal \( X \), we claim that

\[
0 \leq v(x,t) - u_\delta(x,t) \leq ct^{-3/2}\delta.
\]

To see this, let \( \tau := \inf\{s \geq 0 : x + B_s \geq c(t + s)^{1/2}\} \), the optimal stopping time for Brownian motion starting at \( (x,t) \), and \( \tau_\delta := \delta(\lceil \tau/\delta \rceil + 1) \), a stopping time taking values in \( \{\delta, 2\delta, \ldots \} \) with \( 0 < \Delta := \tau_\delta - \tau \leq \delta \). Then we have \( v(x,t) = E[(x + B_t)/(t + \tau)] = E[c/(t + \tau)^{1/2}] \), and

\[
v_\delta(x,t) \geq E\left( \frac{x + B_{t_\delta}}{t + \tau_\delta} \right) = E\left( \frac{x + B_\tau + (B_{t + \Delta} - B_\tau)}{t + \tau + \Delta} \right) = E\left( \frac{c(t + \tau)^{1/2}}{t + \tau + \Delta} \right),
\]

so that

\[
v(x,t) - u_\delta(x,t) \leq E\left( \frac{c(t + \tau)^{1/2}\Delta}{(t + \tau)(t + \tau + \Delta)} \right) \leq \frac{c\delta}{t^{3/2}}.
\]

**Remark 4.2.** The preceding argument for \( S_n/n \) can also be applied to develop second-order approximations to the optimal stopping boundaries for problems with payoff functions of the form \( g(x,t) = t^\alpha f(x/\sqrt{t}) \); see [30] and [32] for some specific examples. However, it depends on the assumption of normal \( X \) in a crucial way: it makes use of normal \( X \) to reduce the optimal stopping problem to that of Brownian motion with the stopping time taking values in \( \{0, \delta, 2\delta, \ldots \} \). For non-normal \( X \), new ideas are needed to establish the second-order expansion (4.4), which we believe should hold in view of the proof of Theorem 2 that can handle the non-normal \( X \) for the finite-horizon case, and in view of the theoretical and extensive numerical results in the following closely related problem.
4.2 Shepp's urn problem

Given $n \geq m \geq 0$, consider an urn with $n$ balls, $m$ of which have payoff $-1$ and $n - m$ have payoff $+1$. A player draws at random from the urn without replacement until he stops; stopping can occur at time 0 (with no draw at all). As noted by Shepp [32, p. 1000], the urn problem is computationally simpler than the infinite-horizon $S_n/n$ problem as one can use backward induction to compute the optimal stopping rule. Instead of $m$, it is more convenient to consider the parameter $k = 2m - n = m - (n - m)$, which is the negative total worth of the urn; see Simons [33]. It is also more convenient to extend $k$ to a continuous variable $w \in [-n, n]$. For any $w \in [-n, n]$, define a $(w, n)$ urn as follows. Let $k$ be the unique integer such that $k - 2 < w \leq k$ and $k$ has the same parity as $n$, so $|k - w - 1| \leq 1$. The $(w, n)$ urn consists of $(n + k)/2 - 1$ balls with payoff $-1$, $(n - k)/2$ balls with payoff $1$, and the remaining ball with payoff $k - w - 1$. The expected return under optimal stopping from a $(w, n)$ urn, which will be called its “maximal expected return”, is continuous and nonincreasing in $w$.

Let $\beta(n)$ be the smallest $w$ for which the $(w, n)$ urn has maximal expected return 0. For a $(\beta(n), n)$ urn, there are two optimal policies; one does not draw at all and the other draws one ball and then proceeds optimally. Both policies yield maximal expected return 0. Given a $(w, n)$ urn, let $X_1^{(n)}, X_2^{(n)}, \ldots$ denote the payoffs of the successive draws (without replacement) from the urn and let $S_j^{(n)} = X_1^{(n)} + \cdots + X_j^{(n)}$. An optimal policy is to stop at the first $j$ at which $w + S_j^{(n)} \geq \beta(n - j)$. Let $u = n^{-1/2}w$ (fixed) and $W_n(j/n) = n^{-1/2}S_j^{(n)}$, $j = 0, 1, \ldots$. The optimal policy maximizes $EW_n(\tau_n)$ over stopping times $\tau_n$ that take values in $\{0, 1/n, \ldots, 1\}$. As $n \to \infty$, $W_n$ converges weakly to Brownian bridge $\{W(t), 0 \leq t \leq 1\}$ with $W(1) = -u$. Shepp [32] has shown that $\sup E(W(\tau)$ is attained by the stopping time $\tau = \inf\{0 \leq t \leq 1 : u + W(t) \geq b(1 - t)\}$, where $b(t) = ct^{1/2}$ is the continuous-time optimal stopping boundary for the $(x_0 + B_t)/(t_0 + t)$ problem considered in Section 4.1, with $c$ given by (4.1). In fact, by using a space-time transformation, Shepp [32] reduced this optimal stopping problem for Brownian bridge to that in Section 4.1 for Brownian motion. With this approximation, we have $\beta(n(1 - t))/n^{1/2} \approx b(1 - t)$ for $t \in \{0, 1/n, \ldots, 1\}$, and in particular $\beta(n) = (c + o(1))n^{1/2}$. This was already noted by Shepp [32] for a slightly different sequence $\tilde{\beta}(n)$, where $\tilde{\beta}(n)$ is the smallest $k$ for which an urn with $n$ balls and total worth $-k$ has maximal expected return 0. Note that $\tilde{\beta}(n) - 2 < \beta(n) \leq \tilde{\beta}(n)$.

For a $(w, n)$ urn with $n$ large and $w$ close to $cn^{1/2}$ (i.e. $(w, n)$ is near the stopping boundary), any fixed number of successive payoffs $X_j^{(n)}$ sampled from the $(w, n)$ urn are approximately independent symmetric Bernoulli random variables. The proof of Theorem 2 can be modified to extend the results to $X_j^{(n)}$ that are asymptotically independent and uniformly bounded, with $EX_j^{(n)} = 0$ and
\[ \text{Var}(X_j^{(n)}) = 1. \] Formal application of Theorem 2 then yields \( \beta(n)/n^{1/2} = c - 0.5n^{-1/2} + o(n^{-1/2}) \), or equivalently,

\[ \beta(n) = cn^{1/2} - 0.5 + o(1). \] (4.5)

A technical difficulty in applying Theorem 2 is that the derivative of \( b(t) = ct^{1/2} \) becomes infinite as \( t \to 0 \). Although we believe that the special structure of the present problem can be used to circumvent this difficulty, we have not been able to provide a rigorous proof at this point. However, we have carried out an extensive numerical study of the accuracy of \( cn^{1/2} - 0.5 \) as an approximation to \( \beta(n) \). The results show that this second-order approximation is accurate even for small \( n \). A small subset of these results is given in Table 1.

**INSERT TABLE 1 ABOUT HERE**

Simons [33] proposed to use \( cn^{1/2} - 0.5 \) as an approximation to \( \bar{\beta}(n) \). Based on an extensive numerical study, he conjectured that

\[ \#\{n \leq N : \beta(n) - 2 < cn^{1/2} \leq \bar{\beta}(n)\}/N \to 0.75, \] (4.6)

\[ \#\{n \leq N : \beta(n) - 2 < cn^{1/2} - 0.5 \leq \bar{\beta}(n)\}/N \to 1, \] (4.7)

as \( N \to \infty \). Note that (4.6) and (4.7) are consequences of (4.5), since the sequence \( \{cn^{1/2}, n = 1, 2, \ldots\} \) is uniformly distributed modulo 1 by Fejér’s Theorem [21, p. 23]).

5 Boundary correction in singular stochastic control

In this section we show how the second-order boundary correction of Section 3 can be extended to compute the optimal boundary in singular stochastic control problems, for which the process does not stop when it hits the boundary but is brought back to the no-action region by the control. Specifically, the control problem is to minimize

\[ E \left[ \int_0^T h(X_t, t)dt + \int_{[0,T]} f(t)|d\xi_t| + k(X_T) \right], \] (5.1)

over bounded variation processes \( \{\xi_t\} \) that are adapted to the filtration generated by a standard Brownian motion \( \{B_t\} \) such that \( X_t = z + B_t + \xi_t \); see [2] and [3]. In (5.1) the functions \( h, f \) and \( k \) represent, respectively, the deviation of \( X_t \) from 0, the unit cost of effort \( \xi_t \) and the terminal penalty when the state is away from the target state 0. In addition, \( h(\cdot, t) \) and \( k(\cdot) \) are assumed to be even, convex functions and \( f \) to be nonnegative. Expressing \( \xi_t \) as the difference \( \xi_t^+ - \xi_t^- \), where
\( \{ \xi^+_t \} \) and \( \{ \xi^-_t \} \) are nonnegative, nondecreasing in \( t \), the so-called monotone follower problem is to minimize (5.1) subject to the constraint \( \xi^+_t \equiv 0 \), which can be solved via a related optimal stopping problem of the form

\[
\min_{\tau \in T[0,T]} E \left[ \int_0^\tau h_x(x + B_t, t) \, dt + f(\tau)1_{\{\tau < T\}} + k'(x + B_T)1_{\{\tau = T\}} \right],
\]

(5.2)

where \( T[0,T] \) is the class of stopping times taking values in \([0,T]\); see [24].

The special case of the monotone follower problem with \( h(x,t) = x^2 \), \( f \equiv 0 \) and \( k \equiv 0 \) has been solved explicitly by Benes et al. [5], who showed that the optimal (nondecreasing) process \( \{ \xi^*_t \} \) is singular with respect to the Lebesgue measure. It is characterized as the local time of the controlled process \( X^*_t = x + B_t - \xi^*_t \) at the boundary \( \partial D = \{(x,t): 0 \leq t \leq T, x = d\sqrt{T-t}\} \), where \( d = 0.63883... \) is the solution of a transcendental equation. The optimal control \( \{ \xi^*_t \} \) is applied at the boundary to keep \( X^*_t \) within \( D \), which corresponds to the continuation region of the optimal stopping problem (5.2). If \( x \in D^c \) at time 0, then \( \xi^*_0 \) is such that \( X^*_0 \) is the closest point on \( \partial D \) to which \( x \) is moved instantaneously.

Let \( U(x,T) \) denote the value function. In this special case, (5.2) reduces to the optimal stopping problem

\[
V(x,T) = \inf_{\tau \in T[0,T]} E \left\{ \int_0^\tau 2(x + B_t) \, dt \right\}.
\]

(5.3)

The optimal boundary for the monotone follower problem with value function \( U \) is the same as the optimal stopping boundary associated with the value function \( V \); moreover, \( U_x = V \). This connection dates back to Bather and Chernoff [4] and has led to many generalizations and an extensive literature; see [2] and [3] and the references therein.

The optimal stopping problem (5.3) is not of the form (2.1) because the payoff has to be integrated over time up to \( \tau \), instead of at the stopping time \( \tau \). On the other hand, an application of Ito’s formula can reduce (5.3) to the form (2.1). More generally, consider the optimal stopping problem (5.2). Let \( \kappa(x,t) \) be a solution to the nonhomogeneous heat equation

\[
\frac{\partial \kappa}{\partial t} + \frac{1}{2} \frac{\partial^2 \kappa}{\partial x^2} - h_x = 0.
\]

Then by Ito’s formula we have

\[
E \{ \kappa(x + B_t, t) - \kappa(x,0) \} = E \int_0^t h_x(x + B_u, u) \, du.
\]

Thus (5.2) can be expressed in the form (2.1) with

\[
-g(z,t) = f(t)1_{\{t < T\}} + k'(z)1_{\{t = T\}} + \kappa(z,t).
\]
In particular, we can write the optimal stopping problem (5.3) as maximizing

\[-E \left\{ \int_0^T (x + B_t) \, dt \right\} = E \{ (T - \tau)(x + B_\tau) - Tx \}. \quad (5.4)\]

Thus the optimal stopping problem (5.3) associated with the monotone follower case of the control problem (5.1) has a payoff of the form (2.1). We can therefore apply the corrected random walk approximation of Section 3 to compute the optimal reflection boundary. As an illustration, Table

2 (with \( \delta = 10^{-4} \) and correction started at \( t = T - 0.0020 \)) shows the dramatic effect of the correction when comparing the corrected Bernoulli walk boundary \( b(t) \) in (3.36) of Section 3.4 and the uncorrected Bernoulli walk boundary \( b_0(t) \) described in Section 3.4 against the actual boundary in the monotone follower problem solved explicitly by Benes et al. [5].

INSERT TABLE 2 ABOUT HERE

**Appendix**

**Proof of Lemma 4.** Note that \( u(y, t) = E\{g^*(y + S_{\sigma(y,t)})1_{\{\sigma(y,t)=-t\}} - \sigma(y, t)\} \), where \( g^*(y) = y^21_{\{y \leq 0\}} \) and

\[ \sigma(y, t) = \inf\{n : 1 \leq n < -t, y + S_n \geq z(t + n)\} \quad (\inf \emptyset = -t). \]

For \( t < 0 \) and \( y > z(t) \), suppose that \( u(y, t) \geq 0 \). Then \( P\{\sigma(y, t) = -t, y + S_{-t} < 0\} > 0 \). Moreover, since \( z(t) + S_{-t} < y + S_{-t} \),

\[ E\{g^*(z(t) + S_{\sigma(y,t)})1_{\{\sigma(y,t)=-t\}} - \sigma(y, t)\} - u(y, t) \geq \int_{\{\sigma(y,t)=-t, y+S_{-t}<0\}} [(z(t) + S_{-t})^2 - (y + S_{-t})^2]dP > 0. \quad (A.1) \]

On the other hand, since \( u(z(t), t) \leq \tilde{V}(z(t), t) = 0 \) and since \( u(z(t), t) \) is the optimal expected reward of policies that start at \( (z(t), t) \) and take at least one observation,

\[ E\{g^*(z(t) + S_{\sigma(y,t)})1_{\{\sigma(y,t)=-t\}} - \sigma(y, t)\} \leq u(z(t), t) \leq 0. \]

Combining this with (A.1) yields \( u(y, t) < 0 \), contradicting the assumption \( u(y, t) \geq 0 \).

Since \( \tilde{V}(y + X, t + 1) \) is monotone decreasing in \(-t\), it follows that \( u(y, t) = -1 + E\tilde{V}(y + X, t + 1) \) is monotone decreasing in \(-t\). For \( y > -\rho = \lim_{s \to -\infty} z(s), u(y, t) < 0 \) for all sufficiently large negative \( t \), and therefore \( \lim_{t \to -\infty} u(y, t) < 0 \).

**Proof of Lemma 6.** By the definition of \( T_{\delta,y}, \) \( y \) is a stopping point if \( y \geq |\log \delta| \). While we need only consider \( y < |\log \delta| \), most of the inequalities below hold for \( |y| \leq \delta^{-\xi/2} \) that will be used in
the proofs of Lemmas 7 and 8. In addition to Lemma 4, we shall make use of the following uniform bounds for all stopping times $\tau \leq -t \leq \delta^{-\gamma}$ and $|y| \leq \delta^{-\zeta/2}$:

$$\delta E(y + S_\tau)^2 \leq \delta(y^2 + \delta^{-\gamma}) \leq \delta(\delta^{-\zeta} + \delta^{-\gamma}) \rightarrow 0, \quad (A.2)$$

$$\sqrt{\delta} E(|y + S_\tau|^3) \leq 4\sqrt{\delta}(|y|^3 + \delta^{-3/2}) \leq 4\sqrt{\delta}(\delta^{-1/2} + \delta^{-3/2}) \rightarrow 0, \quad (A.3)$$

$$\delta E(|t + \tau|^2) \leq \delta^{-1-2\gamma} \rightarrow 0, \quad (A.4)$$

$$P\{\max_{n \leq \delta^{-\gamma}} |S_n| \geq \delta^{-\frac{1}{2}} - \delta^{-\frac{5}{2}}\} = o(\delta^{q}) \quad (A.5)$$

for all $q > 0$ by Lemma 4(ii), recalling (3.2) and that $0 < \zeta < \gamma < \frac{1}{3}$; moreover, for $|y| \leq \delta^{-\frac{1}{2}}$,

$$\sqrt{\delta} E(|t + \tau| |y + S_\tau|) \leq \delta^{\frac{1}{2}-\gamma} E|y + S_\tau| \leq \delta^{\frac{1}{2}-\gamma}\{\delta^{-\frac{1}{2}} + \delta^{-\gamma/2}\} \rightarrow 0. \quad (A.6)$$

Let $u_\delta(y, t)$ be the value of the policy that starts at $(y, t)$, takes one observation and proceeds optimally thereafter in the optimal stopping problem (3.19). Let $\tilde{g}(y) = h_\delta(y)$ for $|y| \leq \delta^{-1/6}$, and $\tilde{g}(y) = y^21_{\{y \leq 0\}}$ for $|y| > \delta^{-1/6}$. In view of (3.17), $\sup_y |\tilde{g}(y) - y^21_{\{y \leq 0\}}| = O(1)$ as $\delta \rightarrow 0$. Therefore Theorem 4.1 of [9] on a modified version of the associated problem holds for general random walks satisfying (3.2) and can be applied to show that as $t \rightarrow -\infty$,

$$u_\delta(y, t) + t \leq u(y, t) + o(1) + \Delta(y, t) \text{ for } \delta^{-\zeta/2} \geq y > -\rho + \varepsilon, \quad (A.7)$$

where $u$ is defined in Lemma 4 and

$$\Delta(y, t) := \sup_{1 \leq \tau \leq -t} E[r_\delta(y + S_\tau, t + \tau)|1_{\{\tau < -t\}}]$$

$$+ E\left(|h_\delta(y + S_\tau)|1_{\{|y + S_\tau| > \delta^{-1/6}\}} + (y + S_\tau)^21_{\{|y + S_\tau| < \delta^{-1/6}\}}\right).$$

By (3.16),

$$\sup_{1 \leq \tau \leq -t} E[r_\delta(y + S_\tau, t + \tau)|1_{\{\tau < -t\}} \leq A \sup_{1 \leq \tau \leq -t} E\left\{\delta(y + S_\tau)^2 + \sqrt{\delta}|y + S_\tau|^3 + \sqrt{\delta}|t + \tau| |y + S_\tau|\right\}$$

$$+ \sum_{n=1}^{-t-1} E[r_\delta(y + S_n, t + n)|1_{\{|y + S_n| > \delta^{-1/6}\}}].$$

From (3.16)–(3.18), (A.2)–(A.6) and Hölder’s inequality, it follows that

$$\sup_{-\rho + \varepsilon < y \leq \delta^{-\zeta/2}, \delta^{-\gamma} \leq t \leq -1} \Delta(y, t) \rightarrow 0 \text{ as } \delta \rightarrow 0.$$

It then follows from Lemma 4 and (A.7) that if $-\delta^{-\gamma} \leq t \leq -|\log \delta|$, then $(y, t)$ is a stopping point (with payoff $-t + o(1)$ upon stopping in view of (3.15)) for every $y > -\rho + \varepsilon$ and for all sufficiently
small $\delta > 0$. \(\square\)

**Proof of Lemma 7.** If $y < -|\log \delta|$, then $(y, t)$ is a continuation point, in view of the definition of $T_{\delta, t, y}$. For $-|\log \delta| \leq y < -\rho - \epsilon$ and $-\delta^{-\gamma} \leq t \leq -|\log \delta|$, to show that $(y, t)$ is a continuation point if $\delta$ is sufficiently small, we make use of (A.2)–(A.6) and that

$$E|h_{\delta}(y + S_{-t})1_{\{y + S_{-t} > \delta^{-1/\alpha}\}} + \sum_{n=1}^{-t-1} E|r_{\delta}(y + S_n, t + n)1_{\{y + S_n > \delta^{-1/\alpha}\}}| \to 0,$$  \hspace{1cm} (A.8)

which follows from (3.18), (A.5) and Hölder’s inequality. It then suffices to show that $\bar{V}(y) := \lim_{t \to -\infty} \bar{V}(y, t) > 0$ if $y < -\rho - \epsilon$, recalling that $\bar{V}(y, t)$ is monotone decreasing in $y$ and in $-t$; see (3.21) and the paragraph preceding Lemma 4. Letting $\tau(y, t) = (-t) \wedge \inf\{n : y + S_n > -\rho\}$, we can apply Doob’s optional stopping theorem as in (3.4) and (3.5) of [11] to obtain that

$$Eg(y + S_{\tau(y, t)}, t + \tau(y, t)) = y^2 - t - E(y + S_{\tau(y, t)})^2 1_{\{\tau(y, t) < -t\}} - E(y + S_{\tau(y, t)})^2 1_{\{\tau(y, t) = -t, y + S_{\tau(y, t)} \geq 0\}}.$$  \hspace{1cm} (A.9)

Fix $y < -\rho - \epsilon$ and let $c = -\rho - y$. Note that as $-t \to \infty$,

$$E(y + S_{\tau(y, t)})^2 1_{\{\tau(y, t) < -t\}} = E(-\rho + R_c)^2 + o(1) \hspace{1cm} (A.10)$$

$$= \rho^2 - 2\rho ER_c + ER_c^2 + o(1),$$

$$E(y + S_{\tau(y, t)})^2 1_{\{\tau(y, t) = -t, y + S_{\tau(y, t)} \geq 0\}} = E(-\rho + R_c)^2 1_{\{-\rho + R_c \geq 0, \tau(y, t) = -t\}} \to 0 \hspace{1cm} (A.11)$$

by the uniform integrability of $R_c^2$ (cf. [22]). Moreover, $y^2 = (\rho + c)^2 = \rho^2 + 2\rho c + c^2$. Putting this in (A.9) and applying (A.10) and (A.11) then yields

$$Eg(y + S_{\tau(y, t)}, t + \tau(y, t)) = -t + 2\rho(c + ER_c) - ER_c^2 + o(1) \hspace{1cm} (A.12)$$

as $t \to \infty$. From this and Lemma 3, $\bar{V}(y, t) \geq Eg(y + S_{\tau(y, t)}, t + \tau(y, t)) + t \geq c^2 + o(1)$. \(\square\)

**Proof of Lemma 8.** Taylor expansion of $g - \pi_g$ around $(x^*, t^*)$ yields that for $t < t^*$

$$(g - \pi_g)(x, t) = G(x^*, t^*)(t - t^*) + O(|t - t^*|^2 + |x - x^*|^3 + |t - t^*||x - x^*|).$$  \hspace{1cm} (A.13)

We can apply Lemma 1 to conclude that $\partial^2 v/\partial x^2$ and $\partial^2 v/\partial x \partial t = -\frac{1}{2} \partial^3 v/\partial x^3$ are continuous in the continuation region up to the boundary $b(\cdot)$; moreover, $\frac{1}{2} (\partial^2 / \partial x^2)(v - g) = -G$ at $x = b(t)$. Note that $v(x, t^*) = g(x, t^*)$ for $x \geq x^*$ and that $(\partial^2 / \partial x^2)v(x, t^*)$ has a discontinuity at $x = x^*$. Taylor expansion around $x^*$ yields

$$(v - \pi_g)(x, t^*) = -G(x^*, t^*)(x - x^*)^2 1_{\{x \leq x^*\}} + O(|x - x^*|^3).$$  \hspace{1cm} (A.14)
In view of (3.5), \(-G(x^*, t^*) = |G(x^*, t^*)| > 0\). From (A.13) and (A.14), it follows that \(g_\delta\) satisfies (3.15)–(3.17) for \(-\delta^{-\gamma} \leq s \leq 0\), for every choice of \(0 < \gamma < 1/3\). To prove that (3.18) holds (with \(h_\delta\) thus defined) for this range of \(\gamma\) and any \(q \geq p > 1\) and \(0 < \zeta < \gamma\), it suffices to show that

\[
\sup_{|x-x^*| \leq \delta^{(1-\zeta)/2}, t^*_x-\delta^{-\gamma} \leq t \leq t^*_n, n \leq \delta^{-\gamma}} E|g(x + \sqrt{\delta} S_n, t)|^p = O(1),
\]

(A.15)

\[
\sup_{|x-x^*| \leq \delta^{(1-\zeta)/2}, n \leq \delta^{-\gamma}} E|v(x + \sqrt{\delta} S_n, t^*)|^p = O(1).
\]

(A.16)

From (2.3), (3.7) and (3.2), (A.15) follows, noting that for some large constant \(M > 0\),

\[
E|g(x + \sqrt{\delta} S_n, t)|^p \leq Me^{M|x|} \left\{ Ee^{M\delta^{\zeta}} + Ee^{-M\delta^{\zeta}} \right\} = O(1).
\]

Similarly as \(\sup_{x \leq b(t^*)} e^{-\alpha''|x-b(t^*)|} |v(x, t^*)| < \infty\) by Lemma 9 and since \(v(x, t^*) = g(x, t^*)\) for \(x \geq b(t^*)\), (A.16) follows from (3.7) and (3.2).

To prove (i), recall that \(\delta^{-\gamma} = o(\delta^{-\gamma})\) and that \(\sup_{T \leq t \leq T^*} \sup_{x \geq b(t)} |g(x, t)| e^{-\beta(x)} < \infty\) by (3.7) with \(\beta(x) = O(x)\) as \(x \to \infty\), and apply Lemmas 5(i), 6 and 7 together with (A.2)–(A.6), (A.9)–(A.11), (3.23) and (3.2). To prove (ii), let \(V^{(c)}(y, s)\) denote the discrete-time optimal value function with respect to the payoff function \(-s1_{(s<0)} + (c + y^21_{(y\leq0)}) 1_{(s=0)}\), which reduces to (1.1) when \(c = 0\).

By Theorem 4.1 of [9] (applicable to random walks satisfying (3.2)) together with (A.2)–(A.6) and (3.16)–(3.18), it follows as \(\delta \to 0\),

\[
V^{(-A)}(y, s) + o(1) \leq V(y, s) \leq V^{(A)}(y, s) + o(1)
\]

uniformly in \(-\delta^{-\gamma} \leq s < 0\) and \(|y| \leq \delta^{-\gamma/2}\). Observing that \(|V^{(c)}(y, s) - V^{(0)}(y, s)| \leq |c|\), part (ii) then follows from Lemma 10 below. □

**Lemma 10.** Let \(V^{(0)}(y, s)\) denote the discrete-time optimal value function for Chernoff’s associated stopping problem (1.1). Then for \(s = -1, -2, \ldots\) and \(-\infty < y < \infty\),

\[
0 \leq y^21_{(y\leq0)} - s - V^{(0)}(y, s) \leq \rho^2 + 4E_S^3/3E_S^+.
\]

**Proof.** Hogan [22] has shown that there exists a nondecreasing optimal stopping boundary \(0 > z(-1) \geq z(-2) \geq \cdots \geq -\rho\). The lemma holds trivially for \((y, s)\) with \(y \geq z(s)\). For \((y, s)\) with \(y < z(s)\), we have, as in (A.9),

\[
V^{(0)}(y, s) = y^2 - s - E(y + S_r)^2 1_{(r < -s)} - E(y + S_r)^2 1_{(r = -s, y + S_r \geq 0)}.
\]

28
where $\tau = (-s) \wedge \inf\{n : y + S_n > z(s + n)\}$ is the optimal stopping time. Letting $\tau(c) = \inf\{n : S_n > c\}$ and noting that $0 > z(-1) \geq z(-2) \geq \cdots \geq -\rho$, we have

$$(y + S_{\tau})^2 1_{\{\tau< -s\}} + (y + S_{\tau})^2 1_{\{\tau=-s,y+S_{\tau} \geq 0\}} \leq \rho^2 + (y + S_{\tau}(-y))^2,$$

which together with (3.11) proves the lemma. $\square$

Acknowledgement

The first author gratefully acknowledges support from the National Science Foundation under grant DMS-0305749. The second author gratefully acknowledges support by the National Science Council of Taiwan, ROC.

References


Table 1: Optimal boundary and its second-order approximation for the urn problem

<table>
<thead>
<tr>
<th>n</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
<th>12</th>
<th>14</th>
<th>16</th>
</tr>
</thead>
<tbody>
<tr>
<td>β((n))</td>
<td>0.000</td>
<td>0.500</td>
<td>1.000</td>
<td>1.000</td>
<td>1.500</td>
<td>1.894</td>
<td>2.153</td>
<td>2.370</td>
<td>2.586</td>
<td>2.797</td>
</tr>
<tr>
<td>(cn^{1/2} - 0.5) difference</td>
<td>-0.340</td>
<td>-0.188</td>
<td>0.045</td>
<td>-0.180</td>
<td>-0.057</td>
<td>0.018</td>
<td>-0.003</td>
<td>-0.040</td>
<td>-0.057</td>
<td>-0.063</td>
</tr>
</tbody>
</table>

| n   | 18    | 20    | 22    | 24    | 25    | 26    | 27    | 28    | 29    | 30    |
|-----|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| \(cn^{1/2} - 0.5\) difference | -0.065 | -0.050 | -0.032 | -0.012 | -0.045 | 0.008 | -0.044 | 0.029 | -0.046 | 0.014 |
Table 2: Uncorrected and corrected random walk approximations vs. actual boundary for the monotone follower problem

<table>
<thead>
<tr>
<th>$T - t$</th>
<th>Uncorrected Boundary</th>
<th>Corrected Boundary</th>
<th>Actual Boundary</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0020</td>
<td>0.0235</td>
<td>0.0285</td>
<td>0.0286</td>
</tr>
<tr>
<td>0.0025</td>
<td>0.0281</td>
<td>0.0331</td>
<td>0.0319</td>
</tr>
<tr>
<td>0.0030</td>
<td>0.0301</td>
<td>0.0351</td>
<td>0.0350</td>
</tr>
<tr>
<td>0.0035</td>
<td>0.0326</td>
<td>0.0376</td>
<td>0.0378</td>
</tr>
<tr>
<td>0.0040</td>
<td>0.0357</td>
<td>0.0407</td>
<td>0.0404</td>
</tr>
<tr>
<td>0.0045</td>
<td>0.0391</td>
<td>0.0441</td>
<td>0.0429</td>
</tr>
<tr>
<td>0.0050</td>
<td>0.0403</td>
<td>0.0453</td>
<td>0.0452</td>
</tr>
<tr>
<td>0.0060</td>
<td>0.0445</td>
<td>0.0495</td>
<td>0.0495</td>
</tr>
<tr>
<td>0.0070</td>
<td>0.0497</td>
<td>0.0547</td>
<td>0.0534</td>
</tr>
<tr>
<td>0.0080</td>
<td>0.0520</td>
<td>0.0570</td>
<td>0.0571</td>
</tr>
<tr>
<td>0.0090</td>
<td>0.0558</td>
<td>0.0608</td>
<td>0.0606</td>
</tr>
<tr>
<td>0.0100</td>
<td>0.0601</td>
<td>0.0651</td>
<td>0.0639</td>
</tr>
<tr>
<td>0.0150</td>
<td>0.0731</td>
<td>0.0781</td>
<td>0.0782</td>
</tr>
<tr>
<td>0.0200</td>
<td>0.0854</td>
<td>0.0904</td>
<td>0.0903</td>
</tr>
<tr>
<td>0.0250</td>
<td>0.0962</td>
<td>0.1012</td>
<td>0.1010</td>
</tr>
<tr>
<td>0.0300</td>
<td>0.1058</td>
<td>0.1108</td>
<td>0.1106</td>
</tr>
</tbody>
</table>