MOMENTS OF MINORS OF WISHART MATRICES

by

Mathias Drton
Helene Massam
Ingram Olkin

Technical Report No. 2006-3
May 2006

Department of Statistics
STANFORD UNIVERSITY
Stanford, California 94305-4065
MOMENTS OF MINORS OF WISHART MATRICES

by

Mathias Drton
Department of Statistics
The University of Chicago

Helene Massam
Department of Mathematics & Statistics
York University

Ingram Olkin
Department of Statistics
Stanford University

Technical Reports No. 2006-3
May 2006

This research was supported in part by National Science Foundation grant DMS-0505612 and NSERC Discovery Grant A8946.

Department of Statistics
STANFORD UNIVERSITY
Stanford, California 94305-4065

http://www-stat.stanford.edu
Moments of minors of Wishart matrices

Mathias Drton*  
Hélène Massam  
Ingram Olkin  

The University of Chicago  
York University  
Stanford University  

April 25, 2006

Abstract

For a random matrix following a Wishart distribution, we derive formulas for the expectation and the covariance matrix of compound matrices. The compound matrix of order $m$ is populated by all $m \times m$-minors of the Wishart matrix. Our results permit to obtain first and second moments of the minors of the sample covariance matrix for multivariate normal observations. This work is motivated by the fact that such minors arise as constraints on the covariance matrix of distributions in classic models from multivariate statistics, including factor analysis.

1 Introduction

Consider the covariance matrix $\Sigma \in \mathbb{R}^{r \times r}$ of a multivariate normal distribution in a factor analysis model with one factor. Such a covariance matrix can be written as

$$\Sigma = \Delta + \lambda \lambda^t \in \mathbb{R}^{r \times r},$$

where $\Delta \in \mathbb{R}^{r \times r}$ is diagonal, and $\lambda$ is a (column) vector in $\mathbb{R}^r$. Assume that $r \geq 4$ such that we can choose two disjoint index sets $\{i, j\}$ and $\{k, \ell\}$. Then it is well-known (Spearman, 1927) that the minor $\det(\Sigma_{\{ij\} \times \{k\ell\}})$ must vanish,

$$\det(\Sigma_{\{ij\} \times \{k\ell\}}) = \sigma_{ik}\sigma_{j\ell} - \sigma_{i\ell}\sigma_{jk} = (\lambda_i\lambda_k)(\lambda_j\lambda_\ell) - (\lambda_i\lambda_\ell)(\lambda_j\lambda_k) = 0. \quad (1)$$

The $2 \times 2$-minor $\sigma_{ik}\sigma_{j\ell} - \sigma_{i\ell}\sigma_{jk}$ is known as a tetrad. Since $\{i, j\} \cap \{k, \ell\} = \emptyset$, the tetrad does not involve any diagonal elements of $\Sigma$. We call a minor with this property an off-diagonal minor of $\Sigma$.

Presented with data, the vanishing of the tetrad in (1) can be tested. Rejection of this hypothesis would imply that the factor analysis model with one factor is inappropriate for the data. Thus, testing for vanishing of the tetrad yields a simple test for evaluating one aspect of the fit of a one-factor analysis model. Considering several different tetrads provides a more holistic picture of the model's goodness-of-fit. Spearman's original work on factor analysis relied heavily on the idea of testing vanishing of tetrads, but the approach constitutes a useful tool also in modern days; see for example Bollen and Ting

*Department of Statistics, The University of Chicago, 5734 S. University Avenue, Chicago, IL 60637.
(1993); Hipp et al. (2005); Spirtes et al. (2000). The route commonly taken when testing the vanishing of a tetrad is to evaluate the tetrad over the sample covariance matrix, to standardize the result, and to compare the standardized sample tetrad to a quantile of a standard normal distribution. The difficulty involved in this procedure is clearly the problem of how to standardize the sample tetrad. However, Wishart (1928) computed the sampling variance of the tetrad rendering the standardization step straightforward.

If a factor analysis model with two or more factors is considered, then higher-order polynomials in the covariance matrix \( \Sigma \) vanish instead of the quadratic tetrads. If \( m \) is the number of factors plus one and we observe \( r \geq 2m \) variables, then the higher-order vanishing polynomials include the \( m \times m \) off-diagonal minors of \( \Sigma \) (Drton et al., 2005). Tetrads present the special case of minor size \( m = 2 \) in the model with \( m - 1 = 1 \) factor. Hence, aspects of the fit of a factor analysis model with two or more factors can be evaluated by testing for vanishing of appropriate minors. As in the case of tetrads and one factor, testing the vanishing of a general \( m \times m \) off-diagonal minor is greatly facilitated by knowledge of the minor's sampling variance. Motivated by this problem, we study in this paper the first and second moments of minors of Wishart matrices. The formulas we derive yield in particular that minors of population covariance matrices can be estimated unbiasedly by scaling the corresponding minor of the sample covariance matrix. We remark that while the above motivation is given merely in terms of factor analysis, vanishing tetrads and higher-order minors also appear in more generally structured multivariate normal models (Spirtes et al., 2000).

The minors of Wishart matrices we consider constitute a special case of random determinants, which have received considerable attention. The article by Fortet (1951) is an example of early work on this topic. Book-length treatments with further references can be found in Girko (1990) and Mehta (2004).

This paper starts with a presentation of simple but powerful invariance arguments that can be applied to determine much of the structure of moments of minors of a standard Wishart matrix (Section 2). Here “standard” refers to a Wishart distribution with the identity matrix as scale parameter. However, the invariance arguments determine the moments only up to constants. We use the Choleski-decomposition of a standard Wishart matrix to work out formulas for these constants (Section 3). By considering compound matrices the results on the standard Wishart distribution can be lifted to Wishart distributions with arbitrary scale parameter matrix (Section 4). Finally, we use conditioning arguments to give a simplified formula for the variance of an off-diagonal minor of a general Wishart matrix (Section 5).

2 Invariance under orthogonal transformations

Let \( X \in \mathbb{R}^{r \times n} \) be a matrix whose columns are independent random vectors distributed according to the multivariate normal distribution \( N_r(0, \Sigma) \) with positive definite covariance matrix \( \Sigma \in \mathbb{R}^{r \times r} \). Then \( S = XX^T \) is distributed according to the Wishart distribution with scale parameter matrix \( \Sigma \) and \( n \) degrees of freedom, in symbols, \( S \sim W_r(n, \Sigma) \). We refer to the Wishart distribution \( W_r(n, I_r) \) with the identity matrix \( I_r \in \mathbb{R}^{r \times r} \) as scale parameter, as standard Wishart distribution.
In this paper we study the expectation and variance-covariance structure of minors of a Wishart matrix \( S \sim W_r(n, \Sigma) \). We begin our investigation by turning to a standard Wishart matrix \( W_r(n, I_r) \). Simple invariance arguments based on ideas from Olkin and Rubin (1962) permit to learn much about the standard case; in this context compare also Casalis and Letac (1996).

Let \( O(r) \) be the group of orthogonal matrices in \( \mathbb{R}^{r \times r} \), that is, \( G \in O(r) \) if \( GGT^T = I_r \).

**Definition 2.1.** The distribution of a symmetric random matrix \( V \in \mathbb{R}^{r \times r} \) is orthogonally invariant, if for all \( G \in O(r) \), the distribution of \( GVGT^T \) is identical to the distribution of \( V \).

When transforming a Wishart matrix \( S \sim W_r(m, \Sigma) \) using a matrix \( G \), the distribution changes to \( GSGT^T \sim W_r(n, G\Sigma G^T) \). Hence, the standard Wishart distribution \( W_r(n, I_r) \) is orthogonally invariant. However, other distributions exhibit such invariance. For example, if \( Y \) is a random matrix whose columns are random vectors drawn from a spherical distribution (Muirhead, 1982, §1.5), then \( YY^T \) has a distribution that is orthogonally invariant. Specific examples of distributions arising in this way include the multivariate Beta- and the Inverse-Wishart distribution with the identity as scale parameter matrix; see e.g. Muirhead (1982) for the definition of these distributions.

The objects of our study are minors \( \det(W_{I \times J}) \) or \( \det(S_{I \times J}) \) that are specified by two subsets \( I, J \subset [r] = \{1, \ldots, r\} \) of equal cardinality \( |I| = |J| = m \). We introduce the notation

\[
\binom{r}{m} = \{ I \subset [r] : |I| = m \}, \quad m \in [r].
\]

**Proposition 2.2.** Let \( I, J \in \binom{r}{m} \). If the distribution of the symmetric random matrix \( V \in \mathbb{R}^{r \times r} \) is orthogonally invariant, then

\[
E[\det(V_{I \times J})] = \begin{cases} 
E[\det(V_{[m] \times [m]})] & \text{if } I = J, \\
0 & \text{otherwise}. 
\end{cases}
\]

**Proof.** We extend the proof of Olkin and Rubin (1962, Lemma 1), which treats the case \( m = 1 \), in which the minors reduce to individual entries of \( V \).

Let \( I, J \in \binom{r}{m} \) be two distinct subsets. Choose \( j \in J \setminus I \), let \( D_j \in O(r) \) be the diagonal matrix which is entry-wise equal to the identity matrix except for entry \((j, j)\) which is equal to minus one. The matrix \( D_jVD_j^T \) differs from \( V \) in that all off-diagonal entries of the \( j \)-th row and column have been negated. Since \( j \in J \) but \( j \notin I \) it follows that \( \det[(D_jVD_j^T)_{I \times J}] = -\det(V_{I \times J}) \). Thus, \( E[\det(V_{I \times J})] = 0 \) is implied by

\[
E[\det(V_{I \times J})] = E[\det(D_jVD_j^T)_{I \times J}] = -E[\det(V_{I \times J})].
\]

Since \( |I| = |J| \), we can find a permutation that maps the indices in \( I \) to those in \( J \). Let \( P = P_{I \leftrightarrow J} \in O(r) \) be the matrix representing this permutation. Then, \((PVPT^T)_{I \times I} = V_{J \times J} \), and

\[
E[\det(V_{I \times I})] = E[\det((PVPT^T)_{I \times I})] = E[\det(V_{J \times J})].
\]

It follows that \( E[\det(V_{I \times I})] = E[\det(V_{[m] \times [m]})] \) for all \( I \in \binom{r}{m} \). \( \Box \)
The invariance approach can also be used to determine the structure of the second moments. Let \( I \Delta J = (J \setminus I) \cup (I \setminus J) \) denote the symmetric difference of two sets.

**Proposition 2.3.** Let \( I, J, K, L \in \binom{[r]}{m} \), and let the distribution of the symmetric random matrix \( V \in \mathbb{R}^{r \times r} \) be orthogonally invariant. If \( I \Delta J \neq K \Delta L \), then

\[
E[\det(V_{I \times J}) \det(V_{K \times L})] = 0.
\]

Moreover, under any permutation \( \sigma \in S_r \) of the indices in \([r]\),

\[
E[\det(V_{I \times J}) \det(V_{K \times L})] = E[\det(V_{\sigma(I) \times \sigma(J)}) \det(V_{\sigma(K) \times \sigma(L)})].
\]

**Proof.** Again we extend the ideas in Olkin and Rubin (1962, Lemma 1).

Let \( I \Delta J \neq K \Delta L \). We can assume without loss of generality that there exists an index \( j \in (I \Delta J) \setminus (K \Delta L) \). Let \( D_j \) be the diagonal matrix defined in the proof of Proposition 2.2. Recall that the action of \( D_j \) negates the \( j \)-th row and column in \( V \). By choice of \( j \in I \Delta J \), it holds that

\[
\det[(D_j V D_j^T)_{I \times J}] = -\det(V_{I \times J}).
\]

Since either \( j \in K \cap L \) or \( j \notin K \cup L \), it holds further that

\[
\det[(D_j V D_j^T)_{K \times L}] = \det(V_{K \times L}).
\]

Using orthogonal invariance of the distribution of \( V \), we can deduce that

\[
E[\det(V_{I \times J}) \det(V_{K \times L})] = 0.
\]

The second claim is immediate from the assumed orthogonal invariance for \( V \). \( \square \)

### 3 Choleski-decomposition of a standard Wishart matrix

The invariance arguments presented in Section 2 determine the first and second moments of minors of random matrices with orthogonally invariant distribution only up to constants. In this section we use the Choleski-decomposition to determine these constants for the standard Wishart distribution. In the Wishart-context, the Choleski-decomposition is also known as Bartlett decomposition (Bartlett, 1933), and the non-trivial elements of the Choleski-factor are often called rectangular coordinates (Mahalanobis et al., 1937).

**Lemma 3.1** (Muirhead, 1982, Thm. 3.2.14). Let \( W \sim W_r(n, I_r) \) have a standard Wishart distribution. Let \( T = (t_{ij}) \in \mathbb{R}^{r \times r} \) be lower-triangular with positive entries such that \( W = TT^T \). Then the non-trivial entries \( t_{ij}, \ i \geq j \), are independent random variables distributed as \( t_{ii}^2 \sim \chi_{n-i+1}^2 \) for \( i = 1, \ldots, r \), and \( t_{ij} \sim N(0, 1) \) for \( 1 \leq j < i \leq r \).

Since \( \det(TT^t) = \det(T) \det(T^t) = \prod_{i=1}^r t_{ii}^2 \) and \( E[t_{ii}^2] = E[\chi_{n-i+1}^2] = n - i + 1 \), we obtain the following corollary to Proposition 2.2 and Lemma 3.1.
Corollary 3.2. Let $I, J \in \{m\}$. If $W \sim W_r(n, I_r)$, then

$$
\operatorname{E}[\det(W_{I, J})] = \begin{cases} 
n! & \text{if } I = J, \\
(n-m)! & \text{otherwise.} 
\end{cases}
$$

The Choleski-decomposition $W = T T^T$ of a standard Wishart matrix $W \sim W_r(n, I_r)$ reveals additional information. In the sequel assume that $T$ is of full rank, which occurs with probability one.

Lemma 3.3. Let $c \in \{0, \ldots, m\}$ be an integer such that there exists a subset $\bar{J} \subset \{m+1, \ldots, r\}$ of cardinality $|\bar{J}| = m - c$. Then

$$
\det(W_{[m] \times ([c] \cup \bar{J})}) = \left( \prod_{i=1}^c t_{ii}^2 \right) \cdot \left( \prod_{j=c+1}^m t_{jj} \right) \cdot \det(T_{\{c+1, \ldots, m\} \times \{c+1, \ldots, m\}}).
$$

Proof. Let $\bar{I} = \{c+1, \ldots, m\} = [m] \setminus [c]$. From the partitioning

$$
W_{[m] \times ([c] \cup \bar{J})} = \begin{pmatrix} W_{[c] \times [c]} & W_{[c] \times \bar{J}} \\
W_{\bar{J} \times [c]} & W_{\bar{J} \times \bar{J}} \end{pmatrix},
$$

we obtain that

$$
\det(W_{[m] \times ([c] \cup \bar{J})}) = \det(W_{[c] \times [c]}) \cdot \det(W_{\bar{I} \times \bar{J}} - W_{\bar{I} \times [c]} W_{[c] \times [c]}^{-1} W_{[c] \times \bar{J}}).
$$

(2)

Clearly, $\det(W_{[c] \times [c]}) = \prod_{i=1}^c t_{ii}^2$ such that we are left with studying the second factor on the right hand side of equation (2).

For a subset $D \subseteq [r]$, let $T_D = T_{D \times [r]}$ be the submatrix comprising all rows of $T$ with index in $D$. Then we can write

$$
\det(W_{\bar{I} \times \bar{J}} - W_{\bar{I} \times [c]} W_{[c] \times [c]}^{-1} W_{[c] \times \bar{J}}) = \det\left\{ T_{\bar{I} \times [r]} - T_{[c]} T_{[c]}^T T_{[c]}^{-1} T_{[c]} \right\}.
$$

The matrix $I_r - T_{[c]}^T (T_{[c]} T_{[c]}^T)^{-1} T_{[c]}$ represents the orthogonal projection on the kernel of the matrix $T_{[c]}$. Since $T$ is lower-diagonal and assumed to have non-zero diagonal elements, and $[c]$ collects the first $c$ indices, we have that $\ker(T_{[c]}) = \{0\}^c \times \mathbb{R}^{r-c}$, which means that the considered projection replaces the first $c$ entries of a vector in $\mathbb{R}^r$ by zeros. Therefore,

$$
\det\left\{ T_{\bar{I} \times [r]} - T_{[c]} (T_{[c]} T_{[c]}^T)^{-1} T_{[c]} \right\} = \det\left( T_{\bar{I} \times (c+1, \ldots, r)} T_{\bar{J} \times (c+1, \ldots, r)} \right).
$$

The latter determinant can be evaluated using the Cauchy-Binet theorem (see Aitken, 1956, Chap. V, or Marshall and Olkin, 1979, p. 503), which yields that

$$
\det\left( T_{\bar{I} \times (c+1, \ldots, r)} T_{\bar{J} \times (c+1, \ldots, r)} \right) = \sum_{D \subseteq \{c+1, \ldots, r\}, \#D = c} \det(T_{\bar{I} \times D}) \det(T_{\bar{J} \times D})
$$

$$
= \det(T_{\bar{I} \times \bar{I}}) \det(T_{\bar{J} \times \bar{J}}).
$$

The second equality in (3) holds because if $D \neq \bar{I} = \{c+1, \ldots, m\}$, then the matrix $T_{\bar{I} \times D}$ contains a column entirely zero and thus $\det(T_{\bar{I} \times D}) = 0$. Since $T_{\bar{I} \times \bar{I}}$ is a lower-triangular matrix, our claim follows from $\det(T_{\bar{I} \times \bar{I}}) = \prod_{j=c+1}^m t_{jj}$. \hfill \square
From Lemma 3.3 we can deduce the second moment of a minor \( \det(W_{I \times J}) \).

**Theorem 3.4.** Let \( I, J \in \binom{r}{m} \) have a (possibly empty) intersection of cardinality \( |I \cap J| = c \geq 0 \). If \( W \sim W_r(n, I_r) \), then

\[
\det(W_{I \times J}) \sim \left( \prod_{i=1}^{c} W_i \right) \cdot \left( \prod_{i=c+1}^{m} \sqrt{W}_i \right) \cdot \det(Z),
\]

where \( W_1, \ldots, W_m \) are independent random variables distributed as \( W_i \sim \chi^2_{n-i+1} \), and \( Z = (Z_{ij}) \in \mathbb{R}^{(m-c) \times (m-c)} \) is a random matrix of independent \( N(0,1) \) random variables that are also independent of \( (W_1, \ldots, W_m) \). In particular,

\[
\mathbb{E}[\det(W_{I \times J})^2] = \frac{n!}{(n-m)!} \cdot \frac{(n+2)!}{(n+2-|I \cap J|)!} \cdot (m-|I \cap J|)!
\]

and

\[
\text{Var}[\det(W_{I \times J})] = \begin{cases} 
\frac{n!}{(n-m)!} \cdot \frac{(n+2)!}{(n+2-m)!} - \frac{n!}{(n-m)!} & \text{if } I = J, \\
\frac{n!}{(n-m)!} \cdot \frac{(n+2)!}{(n+2-|I \cap J|)!} \cdot (m-|I \cap J|)! & \text{if } |I \cap J| < m.
\end{cases}
\]

**Proof.** By orthogonal invariance of the standard Wishart distribution, we can permute rows and columns of \( W \) such that \( I = [m] \) and \( J = [c] \cup \{m+1, \ldots, 2m-c\} \). Thus the claim about the distribution of \( \det(W_{I \times J}) \) follows from Lemma 3.3 and 3.1.

For the derivation of the second moment, recall that \( \mathbb{E}[\chi^2_{n}] = n \) and \( \mathbb{E}[(\chi^2_{n})^2] = n(n+2) \). Let \( S_n \) be the group of permutations of \([h]\). Then

\[
\mathbb{E}[\det(Z)^2] = \sum_{\sigma \in S_{m-c}} \sum_{\tau \in S_{m-c}} \mathbb{E} \left[ \prod_{i=1}^{m-c} Z_{i\sigma(i)} Z_{i\tau(i)} \right] = \sum_{\sigma \in S_{m-c}} \prod_{i=1}^{m-c} \mathbb{E} \left[ Z_{i\sigma(i)}^2 \right] = (m-c)!,
\]

which yields the claimed formula. The variance is obtained using Corollary 3.2. \( \square \)

**Example 3.5.** If \( m = 2 \) and \( r \geq 4 \), then up to permutation there are three cases for the second moment of a minor \( \det(W_{I \times J}) \), which differ according to the cardinality of \( I \cap J \). Picking one representative from each one of the three classes, we have

\[
\mathbb{E}[\det(W_{12 \times 12})^2] = n(n-1)(n+2)(n+1), \\
\mathbb{E}[\det(W_{12 \times 13})^2] = n(n+2)(n-1), \\
\mathbb{E}[\det(W_{12 \times 34})^2] = 2n(n-1).
\]

The associated variances are

\[
\text{Var}[\det(W_{12 \times 12})] = 2n(2n+1)(n-1), \\
\text{Var}[\det(W_{12 \times 13})] = n(n+2)(n-1), \\
\text{Var}[\det(W_{12 \times 34})] = 2n(n-1).
\]
We next turn to second moments of the form \( \mathbb{E}[\det(W_{I \times J}) \det(W_{K \times L})] \) with \((I, J) \neq (K, L)\). Recall that, by Proposition 2.3, this expectation is non-zero only if \(I \triangle J = K \triangle L\). In contrast to previous results, it now matters in which order the elements of \(I, J, K, \) and \(L\) are listed when forming the submatrices \(W_{I \times J}\) and \(W_{K \times L}\). Different orderings may lead to different signs. For example,

\[
\mathbb{E}[\det(W_{12 \times 14}) \det(W_{23 \times 34})] = -\mathbb{E}[\det(W_{12 \times 14}) \det(W_{23 \times 43})].
\]

(4)

In the following theorem, we adopt the convention that submatrices are formed by listing rows and columns according to a total order of the index sets \([r]\) that achieves certain order relationships across the sets \(I, J, K, L\). We write \(A < B\) if all elements of the index set \(A \subset [r]\) are smaller than the elements of \(B \subset [r]\), or if \(A\) or \(B\) is the empty set.

**Theorem 3.6.** Let \(I, J, K, L \in \{1, \ldots, r\}\) such that \(I \triangle J = K \triangle L\). Let

\[
\bar{I} = I \setminus (I \cap J), \quad \bar{K} = K \setminus (K \cap L),
\]

\[
\bar{J} = J \setminus (I \cap J), \quad \bar{L} = L \setminus (K \cap L).
\]

Moreover, assume that the indices in \(I, J, K, \) and \(L\) are listed according to a total order in \([r]\) under which

\[
(I \cap J) \setminus (K \cap L) < \bar{I} < \bar{J} < (K \cap L) \setminus (I \cap J), \quad \bar{I} \cap \bar{K} < \bar{I} \cap \bar{L}, \quad \bar{J} \cap \bar{K} < \bar{J} \cap \bar{L}.
\]

Under these conventions it holds that if \(W \sim W_r(n, I_r)\), then

\[
\mathbb{E}[\det(W_{I \times J}) \det(W_{K \times L})] = \frac{n!}{(n-m)!} \cdot \frac{(n+2)!}{(n+2 - |I \cap J \cap K \cap L|)!} \cdot \frac{\binom{n-m + |(I \cap J) \setminus (K \cap L)|}{n-m}}{(n-m)!} \cdot |\bar{I} \cap \bar{K}|! \cdot |\bar{I} \cap \bar{L}|!.
\]

Theorem 3.6 yields for example that

\[
\mathbb{E}[\det(W_{12 \times 13}) \det(W_{24 \times 34})] = n(n-1)^2.
\]

However, it does not yield directly the value of \(\mathbb{E}[\det(W_{12 \times 14}) \det(W_{23 \times 34})]\) in (4). Instead, we can obtain that

\[
\mathbb{E}[\det(W_{12 \times 14}) \det(W_{23 \times 43})] = n(n-1)^2.
\]

Hence, by (4), we find

\[
\mathbb{E}[\det(W_{12 \times 14}) \det(W_{23 \times 34})] = -n(n-1)^2.
\]

**Example 3.7.** If \(m = 2\) and \(r \geq 4\), then up to permutation and sign change there are four cases for the expected value of \(\det(W_{I \times J}) \det(W_{K \times L})\) that are not already covered in Example 3.5. Selecting one representative from each one of the four classes, we have

\[
\begin{align*}
\mathbb{E}[\det(W_{12 \times 12}) \det(W_{13 \times 13})] &= n(n+2)(n-1)^2, \\
\mathbb{E}[\det(W_{12 \times 12}) \det(W_{34 \times 34})] &= n^2(n-1)^2, \\
\mathbb{E}[\det(W_{12 \times 13}) \det(W_{24 \times 34})] &= n(n-1)^2, \\
\mathbb{E}[\det(W_{12 \times 24}) \det(W_{13 \times 24})] &= n(n-1).
\end{align*}
\]
The associated covariances are

\[
\begin{align*}
\text{Cov}[\det(W_{12 \times 12}), \det(W_{13 \times 13})] &= 2n(n - 1)^2, \\
\text{Cov}[\det(W_{12 \times 12}), \det(W_{34 \times 34})] &= 0, \\
\text{Cov}[\det(W_{12 \times 13}), \det(W_{24 \times 34})] &= n(n - 1)^2, \\
\text{Cov}[\det(W_{12 \times 34}), \det(W_{13 \times 24})] &= n(n - 1).
\end{align*}
\]

The remainder of this section is devoted to the proof of Theorem 3.6, in which we can assume that \( r = \max(I \cup J \cup K \cup L) \). We note that \( |(I \cap J) \setminus (K \cap L)| = |(K \cap L) \setminus (I \cap J)| \), which means that, as should be the case, the formula in Theorem 3.6 is not changed if the order of \((I, J)\) and \((K, L)\) is reversed.

**Lemma 3.8.** If \( I \cap J \cap K \cap L = C \neq \emptyset \), \( |C| = c \geq 1 \), then

\[
E[\det(W_{I \times J}) \det(W_{K \times L})] = E[\det(\tilde{W}_{I \times J, K \times L}) \det(W_{C \times C})^2],
\]

where \( A^c = A \setminus C \) for any subset \( A \subseteq [r] \), and

\[
\tilde{W} = W_{[r] \setminus [C]} - W_{[r] \setminus [C]} W_{C \times C} W_{C \times [r]}^{-1} W_{[r] \setminus [C]} \sim W_{r-c}(n-c, I_{r-c}).
\]

**Proof.** The claim follows from the fact that

\[
\det(W_{I \times J}) \det(W_{K \times L}) = \det(W_{C \times C})^2 \det(W_{I \times J, K \times L})
\]

in conjunction with the independence of \( W_{C \times C} \) and \( \tilde{W} \) (compare Lemma 5.2 below).

Since Theorem 3.4 permits to evaluate the term \( E[\det(W_{C \times C})^2] \) appearing in Lemma 3.8, the proof of Theorem 3.6 is complete if the following Lemma is established.

**Lemma 3.9.** Let \( I, J, K, L \in \binom{[n]}{m} \) such that \( I \triangle J = K \triangle L \) and \( I \cap J \cap K \cap L = \emptyset \). Define \( \bar{I}, \bar{J}, \bar{K}, \bar{L} \) as in Theorem 3.6, and assume furthermore that \( I \cap J < \bar{I} < \bar{J} < K \cap L \), \( \bar{I} \cap \bar{K} < \bar{I} \cap \bar{L} \), and \( \bar{J} \cap \bar{K} < \bar{J} \cap \bar{L} \). If \( W \sim W_r(n, I_r) \), then

\[
E[\det(W_{I \times J}) \det(W_{K \times L})] = \frac{n! |(n-m+c)|!}{|(n-m)|!^2} \cdot p!(m-c-p)!,
\]

where \( c = |I \cap J| = |K \cap L| \) and \( p = |\bar{I} \cap \bar{K}| = |\bar{J} \cap \bar{L}| \).

**Proof.** First, we emphasize that \( \bar{I} \cap \bar{J} = \emptyset \), \( \bar{K} \cap \bar{L} = \emptyset \), \( \bar{I} \cup \bar{J} = I \triangle J = K \triangle L = \bar{K} \cup \bar{L} \), and \( |\bar{I}| = |\bar{J}| = |\bar{K}| = |\bar{L}| = m - c \). Defining \( q = |\bar{I} \cap \bar{L}| = |\bar{J} \cap \bar{K}| \), it also holds that

\[
p + q = |\bar{I} \cap \bar{K}| + |\bar{I} \cap \bar{L}| = |\bar{I}| = m - c.
\]

Moreover, since \( |\bar{I} \cap \bar{K}| + |\bar{J} \cap \bar{K}| = |\bar{K}| = m - c \), we have that

\[
|\bar{I} \cap \bar{K}| = p = |\bar{J} \cap \bar{L}|, \quad |\bar{I} \cap \bar{L}| = q = m - c - p = |\bar{J} \cap \bar{K}|.
\]

8
By permuting the indices in $[r]$ if necessary (Proposition 2.3), we can assume that

$$I \cap J = \{1, \ldots, c\},$$
$$\bar{I} \cap \bar{K} = \{c + 1, \ldots, c + p\},$$
$$\bar{I} \cap \bar{L} = \{c + p + 1, \ldots, m = c + p + q\},$$
$$\bar{J} \cap \bar{K} = \{m + 1, \ldots, m + q + 1\},$$
$$\bar{J} \cap \bar{L} = \{m + q + 1, \ldots, 2m - c = m + q + p\},$$
$$K \cap L = \{2m - c + 1, \ldots, 2m\}.$$

These index choices are depicted in Figure 1, which helps visualize some of the arguments presented in this proof. As another convention, we enumerate the elements of the sets $K$ and $L$ as $K = \{k_1, \ldots, k_m\}$ and $L = \{\ell_1, \ldots, \ell_m\}$, respectively, while choosing $k_i = \ell_i$ for all $i \in [c]$.

Let $W = TT^T$ be the Choleski-decomposition of $W$ whose Choleski-factor $T = (t_{ij})$ is lower-triangular with positive diagonal elements. By Lemma 3.3,

$$\det(W_{I \times J}) = \left(\prod_{i=1}^{c} t_{ii}^2\right) \left(\prod_{i=c+1}^{m} t_{ii}\right) \det(T_{J \times J}).$$

While $\det(W_{I \times J})$ has the simple representation in (5), this is not the case for $\det(W_{K \times L})$. However, since we are interested in the expectation of $\det(W_{I \times J}) \det(W_{K \times L})$ some simplification is possible based on the following fact. Since $t_{ij}$, $i > j$, are independent $N(0,1)$ random variables, if $(\alpha_{ij} \mid 1 \leq j \leq i \leq r)$ contains an entry $\alpha_{ij}$ that is odd and
such that \( i > j \), then
\[
E \left[ \prod_{i \geq j} t_{ij}^{\alpha_j} \right] = \prod_{i \geq j} E \left[ t_{ij}^{\alpha_j} \right] = 0.
\]
(6)

Let \( f, g \in \mathbb{R}[t_{ij} \mid i \geq j] \) be two polynomial expressions in the random variables \( t_{ij} \). Then we write \( f \equiv g \) if \( E[f] = E[g] \).

By the Cauchy-Binet Theorem,
\[
\det(W_{K \times L}) = \sum_{H \in \{m\}} \det(T_{K \times H}) \det(T_{L \times H})
\]
\[
= \sum_{H \in \{m\}} \sum_{\sigma \in S_m} \sum_{\tau \in S_m} (-1)^{\sigma + \tau} \prod_{a=1}^{m} t_{ka, h_{\sigma(a)} \ell_{a, h_{\tau(a)}}},
\]
where \( H = \{h_1, \ldots, h_m\} \). Since \( k_a = \ell_a \) for \( a \in [c] \),
\[
\prod_{a=1}^{c} t_{ka, h_{\sigma(a)} \ell_{a, h_{\tau(a)}}} = \prod_{a=1}^{c} t_{ka, h_{\sigma(a)}} t_{ka, h_{\tau(a)}}.
\]
(7)

We claim that
\[
f_1 = \sum_{H \in \{m\}} \sum_{\sigma \in S_m} \sum_{\tau \in S_m} (-1)^{\sigma + \tau} \left( \prod_{a=1}^{c} t_{ka, h_{\sigma(a)}}^{2} \right) \left( \prod_{b=c+1}^{m} t_{ka, h_{\sigma(b)}} t_{ka, h_{\tau(b)}} \right)
\]
(8)
satisfies
\[
\det(W_{I \times J}) \det(W_{K \times L}) \equiv E \det(W_{I \times J}) \cdot f_1.
\]

For a contradiction, fix \( H \) and \( \sigma \), and assume that \( \tau \) is such that there exists \( a \in [c] \) for which \( \sigma(a) \neq \tau(a) \). Then \( h_{\sigma(a)} \neq k_a \) or \( h_{\tau(a)} \neq k_a \). Without loss of generality, assume that \( h_{\sigma(a)} \neq k_a \). If \( h_{\sigma(a)} > k_a \) then \( t_{ka, h_{\sigma(a)}} = 0 \) because \( T \) is lower-triangular. If \( h_{\sigma(a)} < k_a \) then \( t_{ka, h_{\sigma(a)}} \) appears linearly, i.e. power one, in the monomial \( \prod_{a=1}^{m} t_{ka, h_{\sigma(a)}} t_{ka, h_{\tau(a)}} \cdot h_{\sigma(a)} \) appears linearly in
\[
\det(W_{I \times J}) \cdot \prod_{a=1}^{m} t_{ka, h_{\sigma(a)}} t_{ka, h_{\tau(a)}}.
\]

Therefore, only monomials \( \prod_{a=1}^{m} t_{ka, h_{\sigma(a)}} t_{ka, h_{\tau(a)}} \) appearing in \( f_1 \) may contribute to the expected value of \( \det(W_{I \times J}) \det(W_{K \times L}) \).

We can rewrite (8) as
\[
f_1 = \sum_{H \in \{m\}} \sum_{\sigma \in S_m} (-1)^{\sigma} \left( \prod_{a=1}^{c} t_{ka, h_{\sigma(a)}}^{2} \right) \left( \prod_{b=c+1}^{m} t_{ka, h_{\sigma(b)}} \right) \left[ \sum_{\tau \in S_m} (-1)^{\tau} \prod_{b=c+1}^{m} t_{ka, h_{\tau(b)}} \right].
\]
Now,

$$\sum_{\tau \in S_m, \tau(a) = \sigma(a) \forall a \in [c]} (-1)^{\tau} \left( \prod_{b=c+1}^{m} t_{k_b h_{\tau(b)}} \right) = (-1)^{\sigma} \sum_{\tau \in S_m, \tau(a) = \sigma(a) \forall a \in [c]} (-1)^{\tau \circ \sigma^{-1}} \left( \prod_{b=c+1}^{m} t_{k_b h_{\tau \circ \sigma^{-1}(b)}} \right)$$

$$= (-1)^{\sigma} \sum_{\tau \in S_{\sigma(c+1, \ldots, m)}} (-1)^{\tau} \left( \prod_{b=c+1}^{m} t_{k_b h_{\tau(b)}} \right)$$

$$= (-1)^{\sigma} \det(T_{L \times \{h_{\sigma(c+1)}, \ldots, h_{\sigma(m)}\}}).$$

Therefore,

$$f_1 = \sum_{H \in \binom{\sigma}{m}} \sum_{\sigma \in S_m} \left( \prod_{a=1}^{c} t_{k_a h_{\sigma(a)}}^2 \right) \left( \prod_{b=c+1}^{m} t_{k_B h_{\sigma(b)}} \right) \det(T_{L \times \{h_{\sigma(c+1)}, \ldots, h_{\sigma(m)}\}}).$$

We have thus shown that

$$\det(W_{J \times J}) \det(W_{K \times L}) \stackrel{E}{=} \sum_{H \in \binom{\sigma}{m}} \sum_{\sigma \in S_m} \left( \prod_{a=1}^{c} t_{k_a h_{\sigma(a)}}^2 \right) \left( \prod_{b=c+1}^{m} t_{k_B h_{\sigma(b)}} \right)$$

$$\times \det(T_{L \times \{h_{\sigma(c+1)}, \ldots, h_{\sigma(m)}\}}) \left( \prod_{i=1}^{c} t_{i_{a_i b_i}}^2 \right) \left( \prod_{b=c+1}^{m} t_{i_{b b_i}} \right) \det(T_{J \times I}). \quad (9)$$

We next claim that the expectation of the right hand side of (9) does not change when dropping all terms associated with pairs $(H, \sigma)$ for which $\{h_{\sigma(c+1)}, \ldots, h_{\sigma(m)}\} \neq \bar{I}$. To see this, choose $b \in \{c+1, \ldots, m\}$ for which $h_{\sigma(b)} \in \{h_{\sigma(c+1)}, \ldots, h_{\sigma(m)}\} \setminus \bar{I}$. Now consider three cases. First, if $h_{\sigma(b)} \in (K \cap L) \cup (\bar{J} \cap \bar{I})$, then $h_{\sigma(b)} > k_B \in K$, and it follows that $t_{k_B h_{\sigma(b)}} = 0$, which leads to the vanishing of the term associated with $H$ and $\sigma$. Second, if $h_{\sigma(b)} \in \bar{J} \cap \bar{K}$, then every non-zero term in the expansion of $\det(T_{L \times \{h_{\sigma(c+1)}, \ldots, h_{\sigma(m)}\}})$ involves an off-diagonal element of $T$ that does not appear in $\det(T_{J \times I})$. Hence, every monomial of the term associated with $(H, \sigma)$ features an off-diagonal element of $T$ raised to the power one. Therefore, by (6), the term associated with $(H, \sigma)$ has expectation zero. The third case in which $h_{\sigma(b)} \in I \cap J$ is similar to the second case just discussed.

The claim just verified allows us to rewrite (9) as

$$\det(W_{I \times J}) \det(W_{K \times L}) \stackrel{E}{=} \sum_{H \in \binom{\sigma}{m} : I \subseteq H \subseteq S_m, h_{\sigma(c+1, \ldots, m)} = \bar{I}} (-1)^{\nu_{\sigma}} \left( \prod_{a=1}^{c} t_{k_a h_{\sigma(a)}}^2 \right)$$

$$\times \left( \prod_{a=1}^{c} t_{i_{a_i b_i}}^2 \right) \left( \prod_{b=c+1}^{m} t_{k_B h_{\sigma(b)}} \right) \left( \prod_{b=c+1}^{m} t_{i_{b b_i}} \right) \det(T_{L \times I}) \det(T_{J \times I}), \quad (10)$$

where $\nu_{\sigma}$ is the permutation of $\bar{I} = \{c+1, \ldots, m\}$ that arranges $h_{\sigma(c+1)}, \ldots, h_{\sigma(m)}$ in increasing order, i.e.

$$c+1 = h_{\sigma(\nu(c+1))} < \cdots < h_{\sigma(\nu(m))} = m.$$
We now argue that the expectation of the right hand side of (10) does not change when replacing \( \det(T_{L \times I}) \) by
\[
\left( \prod_{t \in I \cap I} t_{\ell \ell} \right) \det(T_{(J \cap L) \times (I \cap K)}).
\] (11)

In fact, we find (11) from the Laplace expansion along the diagonal \( t_{\ell \ell}, \ell \in \bar{L} \cap \bar{I} \), for which \( \det(T_{(J \cap L) \times (I \cap K)}) \) serves as a cofactor. Every term in \( \det(T_{L \times I}) \) that does not appear in (11) involves an off-diagonal entry in \( T \) of the form \( t_{ab} \) with \( a \in \bar{L} \cap I \) and \( b \in \bar{I}, a > b \). Such \( t_{ab} \), however, does not appear in \( \det(T_{J \times I}) \) since clearly \( a < \min(J) \).

Now, an appeal to (6) closes the argument.

Next, recall that \( k_b \in \bar{I} \cap K \) if \( b \in \{c+1, \ldots, c+p\} \). Hence, if \( b \in \{c+1, \ldots, c+p\} \) but \( h_{\sigma(b)} \neq k_b \), then the term \( t_{k_b, h_{\sigma(b)}} \) does not appear in \( \det(T_{J \times I}) \). In other words, a term in (10) based on \( (H, \sigma) \) with \( h_{\sigma(b)} \neq k_b \) has zero expectation by (6). Combining this observation with the replacement in (11), we define
\[
f_2 = \sum_{H \in \{^r_m\}_{I \subseteq H}} \sum_{\sigma \in \mathfrak{S}_m: h_{\sigma(\{c+1, \ldots, m\})} = \bar{I}, h_{\sigma(c+j)} = \bar{k}_j \cap I \cap K \forall j \in [p]} (-1)^{\nu_{\sigma}} \left( \prod_{a=1}^{c} t_{k_a h_{\sigma(a)}} \right) \left( \prod_{b=c+p+1}^{m} t_{k_b h_{\sigma(b)}} \right) \left( \prod_{k \in \bar{I} \cap K} t_{kk} \right) \left( \prod_{\alpha=1}^{c} t_{i_{ \alpha \alpha a}} \right) \left( \prod_{b=c+1}^{m} t_{i_{b ib}} \right) \left( \prod_{t \in L \cap I} t_{\ell \ell} \right) \det(T_{(J \cap L) \times (I \cap K)}) \det(T_{J \times I}), \tag{12}
\]
for which it holds that that
\[
\det(W_{I \times J}) \det(W_{K \times L}) \overset{E}{=} f_2.
\]

In our next simplification, we claim that if we replace \( \det(T_{J \times I}) \) in \( f_2 \) by
\[
\det(T_{(J \cap L) \times (I \cap K)}) \det(T_{(J \cap K) \times (I \cap L)}),
\] (13)
then the expectation of \( f_2 \) does not change. This follows from (6) because every term in the expansion of \( \det(T_{J \times I}) \) that does not appear in (13) involves some \( t_{ab} \) with \( a \in \bar{J} \cap \bar{L} \) and \( b \in \bar{I} \cap \bar{L} \), and such \( t_{ab} \) appears neither in \( \det(T_{(J \cap L) \times (I \cap K)}) \) nor in \( \prod_{b=c+p+1}^{m} t_{k_b h_{\sigma(b)}} \) because \( k_b \in K < \min(J \cap L) \).

Recall that in \( f_2 \), \( \{^r_m\} \) is such that \( \bar{I} \subseteq H \) and \( h_{\sigma(\{c+1, \ldots, m\})} = \bar{I} \) and therefore
\[
H = \{h_1, h_2, \ldots, h_c\} \cup \bar{I}.
\] (14)

Using (13) and the fact that for \( b \in \{c+p+1, \ldots, m\} = \bar{I} \cap \bar{L}, h_{\sigma(b)} \in \bar{I} \cap \bar{K} = \bar{I} \cap \bar{L} \), we obtain that
\[
\det(W_{I \times J}) \det(W_{K \times L}) \overset{E}{=} \left[ \sum_{h_1 \in \bar{I} \cap L} \sum_{h_2 \in \bar{I} \cap K \setminus \{h_1\}} \cdots \sum_{h_c \in \bar{I} \cap K \setminus \{h_1, \ldots, h_{c-1}\}} \left( \prod_{a=1}^{c} t_{k_a h_a}^2 \right) \right] \times \left( \prod_{\mu \in \mathfrak{S}_m} (-1)^{\mu} \prod_{b=c+p+1}^{m} t_{k_b \mu(b)} \right) \left( \prod_{i \in \bar{I}} t_{i i}^2 \right) \times \det(T_{(J \cap L) \times (I \cap K)})^2 \det(T_{(J \cap K) \times (I \cap L)}),
\] (15)
In the simplification from (12) to (15) we replaced the two sums over $H$ and $\sigma$ by the sums over $h_1, \ldots, h_c$. This is possible because of (14) and because by choosing appropriate $H$ and $\sigma$, $h_{\sigma(a)}$ can take on any value in $[k_a \setminus I]$ while respecting that all $h_{\sigma(a)}$, $a \in [c]$, must be different. In the simplification from (12) to (15) we also replaced the permutation $\nu_\sigma$ by a new permutation $\mu$. For this step, recall that $\nu_\sigma$ in (12) is the permutation that brings $h_{\sigma(c+1)}, \ldots, h_{\sigma(m)}$ in increasing order with $h_{\sigma(c+1)} = k_{c+1} < h_{\sigma(c+2)} = k_{c+2} < \cdots < h_{\sigma(c+p)} = k_{c+p}$ which implies that $\nu_\sigma(j) = j$ for all $j \in \{c+1, \ldots, c+p\}$, which in turn implies that the sign of $\nu_\sigma$ is equal to the sign of $\nu_\sigma|_{\{c+p+1, \ldots, m\}}$. The latter restriction is simply denoted by $\mu$ in (15).

Noting that $k_b \in J \cap K$ if $b \in \{c+p+1, \ldots, m\}$, we see that

$$\sum_{\mu \in S_{I \setminus I}} (-1)^\mu \prod_{b=c+p+1} t_{k_{b\mu}(b)} = \det(T_{(J \cap K) \times (I \setminus I)})$$

Thus, we have shown that

$$\det(W_{(I \times J)} \det(W_{K \times L}) \overset{E}{=} \left( \prod_{i \in I} t_{ii}^2 \right) \det(T_{(I \cap I) \times (J \cap K)})^2 \det(T_{(I \setminus I) \times (I \setminus K)})^2 \left[ \prod_{a=1}^c \left( \sum_{h=1}^{[k_a \setminus I]} t_{ka}^2 \right) \right]$$

Since $t_{ii}^2 \sim \chi_n^2$, and moreover,

$$\sum_{h=1}^{[k_a \setminus I]} t_{ka}^2 \sim \chi_n^2(n-k_a+1) + (k_a - a - |I|) = \chi_n^2(n-a+1) - (m-c) = \chi_n^2(m+c-a+1),$$

this proof can be completed using the results on expected values from the proof of Theorem 3.4. \qed

4 Compounds of Wishart matrices

Consider a general Wishart matrix $S \sim W_r(n, \Sigma)$, and let $\Sigma^{1/2} \in \mathbb{R}^{r \times r}$ be a square root of $\Sigma \in \mathbb{R}^{r \times r}$. In other words, $\Sigma^{1/2}(\Sigma^{1/2})^T = \Sigma$. Then

$$W = \Sigma^{-1/2}S(\Sigma^{-1/2})^T \sim W_r(n, \Sigma^{-1/2}\Sigma^{-1/2}(\Sigma^{-1/2})^T) = W_r(n, I_r).$$

In order to use this result to transfer our previous results about standard Wishart matrices to results about general Wishart matrices, we consider compound matrices.

For a matrix $A \in \mathbb{R}^{r \times r}$ and an integer $m \in \{r\}$, the $m$-th compound of $A$ is the matrix

$$A^{(m)} = \left( \det(A_{I \times J}) \right)_{I,J \in \binom{r}{m}} \in \mathbb{R}^{(\binom{r}{m}) \times (\binom{r}{m})}$$

that is populated with all $m \times m$-minors of $A$ (Aitken, 1956, Chap. V). If $m = 0$, we set $A^{(0)} = 1 \in \mathbb{R}$. The reason why compounds allow us to use (17) to make the transfer from standard to general Wishart matrices is the Cauchy-Binet theorem, which yields that a product of compounds is the compound of the product,

$$(AB)^{(m)} = A^{(m)}B^{(m)}.$$
Theorem 4.1. If \( S \sim W_r(n, \Sigma) \), then

\[
E[S^{(m)}] = \frac{n!}{(n-m)!} \cdot \Sigma^{(m)}, \quad m \in [r].
\]

Proof. Let \( W \) be defined as in (17). By (17), (18) and Corollary 3.2, we obtain that

\[
E[S^{(m)}] = (\Sigma^{1/2})^{(m)} E[W^{(m)}] ((\Sigma^{1/2}T)^{(m)} = \frac{n!}{(n-m)!} \cdot \Sigma^{(m)}.
\]

Note that the above result implies that \( \frac{(n-m)!}{n!} S^{(m)} \) is an unbiased estimator of \( \Sigma^{(m)} \).

The Kronecker product \( S^{(m)} \otimes S^{(m)} \) has the entries

\[
\det(S_{I\times J}) \det(S_{K\times L}), \quad I, J, K, L \in \{1, \ldots, r\},
\]

which are exactly the quantities of interest for studying the variance-covariance structure of minors of \( S \). Since

\[
\left( (\Sigma^{-1/2})^{(m)} \otimes (\Sigma^{-1/2})^{(m)} \right) \cdot (S^{(m)} \otimes S^{(m)}) \cdot \left( ((\Sigma^{-1/2}T)^{(m)} \otimes ((\Sigma^{-1/2}T)^{(m)} \right)
\]

\[
= (\Sigma^{-1/2}S(\Sigma^{-1/2}T)^{(m)} \otimes (\Sigma^{-1/2}S(\Sigma^{-1/2}T)^{(m)},
\]

we obtain the following result from (17). Recall that the entries of \( E[W^{(m)} \otimes W^{(m)}] \) are the expectations \( E[\det(W_{I\times J}) \det(W_{K\times L})] \) discussed in Theorem 3.6.

Proposition 4.2. Let \( S \sim W_r(n, \Sigma) \) and \( W \sim W_r(n, I_r) \). The second moment structure of the compound matrix \( S^{(m)} \) can be obtained from that of \( W^{(m)} \) in that

\[
E[S^{(m)} \otimes S^{(m)}] =
\]

\[
[(\Sigma^{1/2})^{(m)} \otimes (\Sigma^{1/2})^{(m)}] \cdot E[W^{(m)} \otimes W^{(m)}] \cdot \left[ ((\Sigma^{1/2}T)^{(m)} \otimes ((\Sigma^{1/2}T)^{(m)} \right].
\]

Similarly, the covariance matrix of \( S^{(m)} \) can be derived as

\[
\text{Cov}[S^{(m)}] := E[S^{(m)} \otimes S^{(m)}] - \left( E[S^{(m)}] \otimes E[S^{(m)}] \right)
\]

\[
= [(\Sigma^{1/2})^{(m)} \otimes (\Sigma^{1/2})^{(m)}] \cdot \text{Cov}[W^{(m)}] \cdot \left[ ((\Sigma^{1/2}T)^{(m)} \otimes ((\Sigma^{1/2}T)^{(m)} \right].
\]

Example 4.3. If \( m = 2 \) and \( r = 4 \), then the covariance matrix \( \text{Cov}[W^{(m)}] \), which determines \( \text{Cov}[S^{(m)}] \), is a symmetric matrix of size \( 36 \times 36 \). Since \( \text{Cov}[W^{(m)}] \) is derived from the symmetric matrix \( W \), we can restrict ourselves to unordered pairs of sets \( (I, J) \in \{1, \ldots, r\} \times \{1, \ldots, r\} \) with possible equality \( I = J \). There are \( 21 \) such unordered pairs. The resulting representation of \( \text{Cov}[W^{(m)}] \) as (symmetric) \( 21 \times 21 \) matrix is block-diagonal with blocks formed according to \( I \triangle J \). Proposition 2.3 implies such block-diagonal structure also for the general case of arbitrary \( m \) and \( r \).
The first block is indexed by the six pairs \((I, I), I \in \{^{r}_m\}\), involves the principal minors and takes on the form
\[
\begin{pmatrix}
12, 12 & 13, 13 & 14, 14 & 23, 23 & 24, 24 & 34, 34 \\
12 & 13 & 14 & 23 & 24 & 34 \\
1 & 1 & 1 & 1 & 1 & 1
\end{pmatrix}
\]
where \(f_1 = 2n(2n+1)(n-1)\), and \(f_4 = 2n(n-1)^2\); compare Examples 3.5 and 3.7. Next, we have a series of six blocks of size \(2 \times 2\), each involving two pairs \((I, J)\) and \((K, L)\) for which \(I \triangle J = K \triangle L\) and \(|I \cap J| = 1\), or equivalently, \(|I \triangle J| = 2\). Two representatives of these six blocks are
\[
\begin{pmatrix}
12, 13 & 24, 34 \\
12 & 13
\end{pmatrix}
\text{and}
\begin{pmatrix}
12, 14 & 23, 34 \\
12 & 14
\end{pmatrix}
\]
with \(f_2 = n(n + 2)(n - 1)\) and \(f_5 = n(n - 1)^2\). The last block is obtained for the pairs \((I, J)\) with \(I, J\) disjoint, or equivalently, \(I \triangle J = [r] = \{1, 2, 3, 4\}\). It takes the form
\[
\begin{pmatrix}
12, 34 & 13, 24 & 14, 23 \\
12 & 13 & 14
\end{pmatrix}
\]
with \(f_3 = 2n(n - 1)\) and \(f_6 = n(n - 1)\).

5 Variances of minors

The results from Sections 2-4 give the entire covariance matrix of the compound \(S^{(m)}\), but due to the involved square roots \(\Sigma^{1/2}\) the structure of the individual entries of \(\text{Cov}[S^{(m)}]\) is not transparent. In this section, we show that explicit formulas can be given for the variances of the minors of a general Wishart matrix \(S \sim W_r(n, \Sigma)\).

We begin by reviewing the well-known formula for the variance of a principal minor.

**Proposition 5.1.** If \(S \sim W_r(n, \Sigma)\) and \(I \in \{^{r}_m\}\), then
\[
\text{Var}[\det(S_{I \times I})] = \frac{n!}{(n-m)!} \left\{ \frac{(n+2)!}{(n+2-m)!} - \frac{n!}{(n-m)!} \right\} \det(\Sigma_{I \times I})^2.
\]

**Proof.** Apply (17) with the submatrix \(S_{I \times I}\) replacing the full Wishart matrix \(S\) to obtain that
\[
\text{Var}[\det(S_{I \times I})] = \det(\Sigma)^2 \cdot \text{Var}[\det(W_{I \times I})],
\]
which in conjunction with Theorem 3.4 yields the claim. □
Next, we derive an explicit formula for the variance of off-diagonal minors of a general Wishart matrix $S \sim W_r(n, \Sigma)$. From this formula and Proposition 5.1, a formula for the variance of arbitrary minors of $S$ is obtained in Theorem 5.7.

Let $I, J \in \binom{r}{m}$ be two disjoint subsets. Then the minor $\det(S_{I \times J})$ is off-diagonal in that it does not involve any diagonal elements of $S$. Let $S_{I \times J}$ and $\Sigma_{I \times J}$ be the $(I \cup J) \times (I \cup J)$-submatrix of $S$ and $\Sigma$, respectively. We partition these $2m \times 2m$-submatrices into four $m \times m$-submatrices as

$$S_{IJ} = \begin{pmatrix} S_{II} & S_{IJ} \\ S_{JI} & S_{JJ} \end{pmatrix} \quad \text{and} \quad \Sigma_{IJ} = \begin{pmatrix} \Sigma_{II} & \Sigma_{IJ} \\ \Sigma_{JI} & \Sigma_{JJ} \end{pmatrix},$$

where we adopt the shorthand notation $S_{I \times I} = S_{II}$, $S_{I \times J} = S_{IJ}$, etc. Let

$$S_{II, J} = \Sigma_{II} - \Sigma_{IJ} \Sigma_{JJ}^{-1} \Sigma_{JI} \quad \text{and} \quad \Sigma_{II, I} = \Sigma_{II} - \Sigma_{IJ} \Sigma_{JJ}^{-1} \Sigma_{JI}.$$

Our line of attack in computing the variance of the off-diagonal minor $\det(S_{I \times J}) = \det(S_{IJ})$ is to employ the decomposition

$$\text{Var}[\det(S_{IJ})] = \text{Var} \left[ \mathbb{E}[\det(S_{IJ}) \mid S_{JJ}] \right] + \mathbb{E} \left[ \text{Var}[\det(S_{IJ}) \mid S_{JJ}] \right].$$

(19)

The following well-known Lemma is central to this conditioning approach.

**Lemma 5.2** (Muirhead, 1982, Thm. 3.2.10). If $S \sim W_p(n, \Sigma)$, then $S_{IJ} \sim W_m(n, \Sigma_{IJ})$, $S_{II, J} \sim W_m(n-m, \Sigma_{II, J})$, and the random matrix $S_{II, J}$ is independent of $(S_{IJ}, S_{JJ})$. Finally, the conditional distribution of $S_{IJ}$ given $S_{JJ}$ is normal (column-wise vectorization),

$$S_{IJ} \mid S_{JJ} \sim N_{m^2} \left( \Sigma_{IJ} \Sigma_{JJ}^{-1} \Sigma_{JI}, \Sigma_{II, J} \otimes S_{JJ} \right),$$

(20)

$$\iff S_{IJ} S_{IJ}^{-1/2} \mid S_{JJ} \sim N_{m^2} \left( \Sigma_{IJ} \Sigma_{JJ}^{-1/2} S_{JJ}^{-1/2}, \Sigma_{II, J} \otimes I_m \right),$$

(21)

$$\iff (\Sigma_{II, J}^{-1/2} S_{IJ} S_{IJ}^{-1/2} \mid S_{JJ}) \sim N_{m^2} \left( \Sigma_{II, J}^{-1/2} \Sigma_{IJ} \Sigma_{JJ}^{-1} \Sigma_{JI}, I_m \otimes I_m \right).$$

(22)

**Lemma 5.3.** For the first term in the sum on the right hand side of (19), it holds that

$$\text{Var} \left[ \mathbb{E}[\det(S_{IJ}) \mid S_{JJ}] \right] = \frac{n!}{(n-m)!} \left( \frac{(n+2)!}{(n+2-m)!} - \frac{n!}{(n-m)!} \right) \cdot \det(\Sigma_{IJ})^2.$$

**Proof.** By (21) in Lemma 5.2,

$$\text{Var} \left[ \mathbb{E}[\det(S_{IJ}) \mid S_{JJ}] \right] = \text{Var} \left[ \mathbb{E}[\det(S_{IJ} S_{IJ}^{-1/2}) \mid S_{JJ}] \cdot \det(S_{IJ}^{-1/2}) \right]$$

$$= \text{Var} \left[ \det(\Sigma_{IJ} \Sigma_{JJ}^{-1}) \cdot \det(S_{JJ}) \right]$$

$$= \det(\Sigma_{IJ})^2 \text{det}(\Sigma_{JJ})^{-2} \text{Var} \left[ \det(S_{JJ}) \right].$$

Now the claim follows from Proposition 5.1. \hfill \Box

**Lemma 5.4.** For the second term in the sum on the right hand side of (19), it holds that

$$\mathbb{E} \left[ \text{Var}[\det(S_{IJ}) \mid S_{JJ}] \right] =$$

$$\det(\Sigma_{IJ} \otimes I_m) \cdot \sum_{k=0}^{m-1} \frac{(m-k)!}{(n-m)!} \cdot \frac{n!}{(n+2-m)!} \cdot \frac{(n+2)!}{(n+2-k)!} \cdot \text{tr} \left( \Sigma_{JJ,J}^{-1} \Sigma_{JJ} - I_m \right)(k).$$

16
Proof. First note that

\[ E[\text{Var}[\det(S_{I,J}) \mid S_{J,J}]] = \det(\Sigma_{I,I,J}) \cdot E[\text{Var}[\det(\Sigma_{I,I,J}^{-1/2}S_{I,J}S_{J,J}^{-1/2}) \mid S_{J,J}] \cdot \det(S_{J,J})]. \quad (23) \]

It follows from (22) in Lemma 5.2 that conditional on \(S_{J,J}\), the entries of the matrix \(\Sigma_{I,I,J}^{-1/2}S_{I,J}S_{J,J}^{-1/2}\) are independent normal random variables with variance one, albeit these entries are not identically distributed as their means may differ in arbitrary fashion.

We are led to the problem of computing \(\text{Var}[\det(X)]\), where the matrix \(X \in \mathbb{R}^{m \times m}\) is distributed according to the multivariate normal distribution

\[ X \sim \mathcal{N}_{m^2}(A, I_m \otimes I_m), \quad A = (a_{ij}) = \Sigma_{I,I,J}^{-1/2}\Sigma_{I,J}^{1/2}\Sigma_{I,J}^{-1/2} \in \mathbb{R}^{m \times m}. \]

Lemma A.1 solves this problem, and from (23) we find

\[ E[\text{Var}[\det(S_{I,J}) \mid S_{J,J}]] \]

\[ = \det(\Sigma_{I,I,J}) \sum_{k=0}^{m-1} (m-k)! \cdot E \left[ \text{tr} \left\{ \left( \Sigma_{I,I,J}^{-1/2}\Sigma_{I,J}^{1/2}\Sigma_{I,J}^{-1/2} \right)^{(k)} \right\} \cdot \det(S_{J,J}) \right] \]

\[ = \det(\Sigma_{I,I,J}) \sum_{k=0}^{m-1} (m-k)! \cdot E \left[ \text{tr} \left\{ \left( \Sigma_{I,I,J}^{-1}\Sigma_{I,I,J}^{-1}\Sigma_{I,J}^{-1}\Sigma_{I,J}^{-1} \right)^{(k)} \right\} \cdot \det(S_{J,J}) \right]. \quad (24) \]

Let

\[ C = \Sigma_{I,J}^{-1}\Sigma_{I,I,J}^{-1}\Sigma_{I,J}^{-1}. \]

Using the facts that

\[ \Sigma_{I,I,J}^{-1} = \Sigma^{II}, \quad \Sigma^{II}\Sigma_{I,J} = -\Sigma^{IJ}\Sigma_{J,J}, \quad \Sigma_{J,J}\Sigma^{IJ} = I_m - \Sigma_{J,J}\Sigma^{J,J}, \]

if

\[ \Sigma^{-1} = \begin{pmatrix} \Sigma^{II} & \Sigma^{IJ} \\ \Sigma^{JI} & \Sigma^{JJ} \end{pmatrix}, \]

we can rewrite

\[ C = -\Sigma_{J,J}^{-1}\Sigma_{J,J}^{-1}\Sigma_{J,J}^{-1}\Sigma_{J,J}^{-1} = \Sigma_{J,J}^{-1}(\Sigma_{J,J}\Sigma^{J,J} - I_m) = \Sigma_{J,J}^{-1} - \Sigma_{J,J}^{-1}. \quad (25) \]

Taking up (24) we get

\[ E[\text{Var}[\det(S_{I,J}) \mid S_{J,J}]] = \det(\Sigma_{I,I,J}) \sum_{k=0}^{m-1} (m-k)! \cdot E \left[ \text{tr} \left\{ C^{(k)}S_{J,J}^{(k)} \cdot \det(S_{J,J}) \right\} \right] \]

\[ = \det(\Sigma_{I,I,J}) \sum_{k=0}^{m-1} (m-k)! \cdot \text{tr} \left\{ C^{(k)} E \left[ S_{J,J}^{(k)} \cdot \det(S_{J,J}) \right] \right\}. \]
Now, let $W_{JJ} = (\Sigma_{JJ})^{-1/2} S_{JJ} (\Sigma_{JJ})^{-1/2}$. As in (17), $W_{JJ} \sim W_m(n, I_m)$. Thus

$$E[\text{Var}(S_{JJ}) | S_{JJ}] = \det(\Sigma_{II,J}) \det(\Sigma_{JJ}) \times \left( \sum_{k=0}^{m-1} (m-k)! \cdot \text{tr} \left\{ C^{(k)}(\Sigma_{JJ}^{1/2})^{(k)} E \left[ W_{JJ}^{(k)} \cdot \det(W_{JJ}) \right] (\Sigma_{JJ}^{1/2})^{(k)} \right\} \right).$$

The distribution of $W_{JJ}^{(k)} \cdot \det(W_{JJ})$ has the invariance property that for $G \in O(m)$,

$$G^{(k)} \left( W_{JJ}^{(k)} \cdot \det(W_{JJ}) \right) (G^T)^{(k)} = (G W_{JJ} G^T)^{(k)} \cdot \det(G W_{JJ} G^T) \sim W_{JJ}^{(k)} \cdot \det(W_{JJ}).$$

It follows, analogously to Proposition 2.2, that the expectation $E[W_{JJ}^{(k)} \cdot \det(W_{JJ})]$ is a diagonal matrix. In analogy to the derivation of Theorem 3.4, it holds in fact that

$$E[W_{JJ}^{(k)} \cdot \det(W_{JJ})] = \frac{n!}{(n-m)!} \cdot \frac{(n+2)!}{(n+2-k)!} \cdot I_{\binom{n}{k}}.$$

Since $\det(\Sigma_{II,J}) \det(\Sigma_{JJ}) = \det(\Sigma_{IJxIJ})$, we therefore have that

$$E[\text{Var}(S_{JJ}) | S_{JJ}] =$$

$$\det(\Sigma_{IJxIJ}) \cdot \left( \sum_{k=0}^{m-1} (m-k)! \cdot \frac{n!}{(n-m)!} \cdot \frac{(n+2)!}{(n+2-k)!} \cdot \text{tr} \left\{ C^{(k)} \right\} \right).$$

The claim now follows from (25). \hfill \Box

Combining the two parts of (19) determined in Lemmas 5.3 and 5.4 yields the following formula.

**Proposition 5.5.** If $I, J \subseteq \binom{m}{r} \setminus \{m\}$ be two disjoint subsets, then the off-diagonal minor $\det(S_{IJxJ}) = \det(S_{IJ})$ of the Wishart matrix $S \sim W_r(n, \Sigma)$ has variance

$$\text{Var}(\det(S_{IJ})) = \frac{n!}{(n-m)!} \cdot \det(\Sigma_{IJ})^2 \left\{ \frac{(n+2)!}{(n+2-m)!} - \frac{n!}{(n-m)!} \right\} +$$

$$\frac{n!}{(n-m)!} \cdot \det(\Sigma_{IJxIJ}) \left( \sum_{k=0}^{m-1} (m-k)! \cdot \frac{(n+2)!}{(n+2-k)!} \cdot \text{tr} \left\{ (\Sigma_{JJ}^{-1} \Sigma_{IJ} - I_m)^{(k)} \right\} \right).$$

**Corollary 5.6** (Wishart, 1928). In the special case $m = 2$, in which the off-diagonal minor $\det(S_{IJxJ}) = \det(S_{IJ})$ is known as tetrad, it holds that

$$\text{Var}(\det(S_{IJ})) = n(n-1) \left[ (n+2) \det(\Sigma_{II}) \det(\Sigma_{JJ}) - n \det(\Sigma_{IJxIJ}) + 3n \det(\Sigma_{IJ})^2 \right].$$

**Proof.** The claim follows from Proposition 5.5, and the fact that if $m = 2$ then

$$\text{tr}(\Sigma_{JJ}^{-1} \Sigma_{IJ}) \det(\Sigma_{IJxIJ}) = \det(\Sigma_{II}) \det(\Sigma_{JJ}) + \det(\Sigma_{IJxIJ}) - \det(\Sigma_{IJ})^2. \hfill \Box$$
Theorem 5.7. Let \( I, J \in \binom{\{\tau\}}{m} \) be two subsets with intersection \( C = I \cap J \) of cardinality \( c = |C| = |I \cap J| \). Define \( \bar{I} = I \setminus (I \cap J) \), \( \bar{J} = J \setminus (I \cap J) \), and \( \bar{I} \bar{J} = \bar{I} \cup \bar{J} \). Then the minor \( \det (S_{I \times J}) = \det (S_{I \bar{J}}) \) of the Wishart matrix \( S \sim W_r(n, \Sigma) \) has variance

\[
\text{Var} [ \det (S_{I \times J}) ] = \frac{n!}{(n - m)!} \left\{ \frac{(n + 2)!}{(n + 2 - c)!} - \frac{n!}{(n - c)!} \right\} \det (\Sigma_{C \times C})^2 \times \\
\left[ \det (\bar{\Sigma}_{I \times \bar{J}})^2 \left\{ \frac{(n + 2 - c)!}{(n + 2 - m)!} - \frac{(n - c)!}{(n - m)!} \right\} \right] + \\
\det (\bar{\Sigma}_{I \times \bar{J}}) \left( \sum_{k=0}^{m-c-1} (m - c - k)! \cdot \frac{(n + 2 - c)!}{(n + 2 - c - k)!} \cdot \text{tr} \left\{ (\bar{\Sigma}_{I \times \bar{J}}^{-1} \bar{\Sigma}_{I \bar{J}} - I_{m-c})^{(k)} \right\} \right),
\]

where

\[
\bar{\Sigma} = \Sigma_{(\{\tau\} \setminus C) \times (\{\tau\} \setminus C)} - \Sigma_{(\{\tau\} \setminus C) \times C} \Sigma_{C \times C}^{-1} \Sigma_{C \times (\{\tau\} \setminus C)}.
\]

Proof. Define \( \bar{S} \) in analogy to \( \bar{\Sigma} \). Then we can decompose the minor as \( \det (S_{I \times J}) = \det (S_{C \times C}) \det (\bar{S}_{I \times \bar{J}}) \). Thus the claim follows from Propositions 5.1 and 5.5, and the independence of \( S_{C \times C} \) and \( \bar{S}_{I \times \bar{J}} \) (Lemma 5.2). \( \square \)

6 Conclusion

In this paper we studied first and second moments of minors of a Wishart matrix, relying fundamentally on the properties of compound matrices. Theorem 4.1 gives the expected value of a compound of a Wishart matrix, while the covariance matrix can be determined by combining Theorem 3.6 and Proposition 4.2. We obtained these results by building on the Cauchy-Binet theorem, which enabled us to reduce the problem from an arbitrary Wishart distribution to a standard Wishart distribution with the identity as the scale parameter matrix. For the standard Wishart distribution, we extended classic invariance arguments due to Olkin and Rubin (1962) to the case of compounds, which yielded the moments up to certain constants. Using the Choleski-decomposition of a standard Wishart matrix, we found simple formulas for the desired constants. The reduction step from general to standard Wishart distribution involved a square root matrix \( \Sigma^{1/2} \) for the scale parameter matrix \( \Sigma = \Sigma^{1/2} (\Sigma^{1/2})^T \). As shown in Theorem 5.7, however, the variances of minors of general Wishart matrices can be expressed without reference to such square roots.

Our results greatly generalize a classic result of Wishart (1928) about the variance of off-diagonal \( 2 \times 2 \)-minors known as tetrads; see Corollary 5.6. As detailed in the introduction, tetrads have been applied to test goodness-of-fit of one-factor analysis and other Gaussian models with hidden variables. We believe that our results may help improve tetrad-based procedures as they provide for the first time the full (finite sample) covariance matrix for a vector of several tetrads. Moreover, Gaussian hidden variable models may constrain covariance matrices by requiring higher-order minors to vanish; see Drton et al. (2005). We thus hope that the results in this paper will stimulate further development of goodness-of-fit tests based on testing for vanishing of minors of covariance matrices.
Acknowledgments

Mathias Drton was supported by the US National Science Foundation (DMS-0505612). Hélène Massam was supported by NSERC Discovery Grant A8946.

References


A Expectation of a non-central Wishart determinant

Consider the matrix $X \in \mathbb{R}^{m \times m}$ that is distributed according to the multivariate normal distribution

\[ X \sim \mathcal{N}_m(A, I_m \otimes I_m), \quad A = (a_{ij}) \in \mathbb{R}^{m \times m}. \]

Since the entries of $X$ are independent, we immediately obtain that

\[ \mathbb{E}[\det(X)] = \det(A). \quad (26) \]

If $A$ is non-zero, then the matrix $XXX^T$ is said to follow a non-central Wishart distribution (Muirhead, 1982, §10.3). Theorem 10.3.7 in Muirhead (1982) provides a general formula for moments of the determinant of a non-central Wishart matrix in terms of hypergeometric functions with matrix argument. In the present setup, $X$ is a square matrix so that we could use the results in Muirhead (1982) to obtain $\mathbb{E}[\det(X)^2] = \mathbb{E}[\det(XX^T)]$. However, in our case we can give a simple formula involving only traces and compounds.

The linearity of the trace and the fact that compounds give access to the Cauchy-Binet theorem are important for the application of the formula in Section 5.

Lemma A.1. The expectation of $\det(XX^T) = \det(X)^2$ can be expressed as

\[ \mathbb{E}[\det(X)^2] = \sum_{k=0}^{m} (m-k)! \cdot \text{tr} \left[ (AA^T)^{(k)} \right]. \]

Here, $(AA^T)^{(0)} := 1 \in \mathbb{R}$. Note also that $(AA^T)^{(m)} = \det(AA^T) = \det(A)^2$. From (26), we obtain

\[ \text{Var}[\det(X)] = \sum_{k=0}^{m-1} (m-k)! \cdot \text{tr} \left[ (AA^T)^{(k)} \right]. \]

Proof. Let $S_m$ be the group of permutations of $[m]$. Then,

\[ \mathbb{E}[\det(X)^2] = \sum_{\sigma \in S_m} \prod_{j=1}^{m} \mathbb{E} \left[ X_{\sigma(j)j} X_{\tau(j)j} \right] \]

\[ = \sum_{\sigma \in S_m} \sum_{\tau \in S_m} (-1)^{\sigma+\tau} \prod_{j=1}^{m} \left( \delta_{\sigma(j)\tau(j)} + a_{\sigma(j)j}a_{\tau(j)j} \right), \]

where $\delta_{ij}$ is the Kronecker delta. The product

\[ \prod_{j=1}^{m} \left( \delta_{\sigma(j)\tau(j)} + a_{\sigma(j)j}a_{\tau(j)j} \right) = \sum_{J \subseteq [m]} \left( \prod_{j \in J} a_{\sigma(j)j}a_{\tau(j)j} \right) \cdot \left( \prod_{j \notin J} \delta_{\sigma(j)\tau(j)} \right). \]
Therefore, if we define

\[ g_J(\sigma) = \sum_{\tau \in S_m} (-1)^{\sigma + \tau} \prod_{j \in J} a_{\sigma(j)j} a_{\tau(j)j}, \]

then

\[ \mathbb{E}[\det(X)^2] = \sum_{k=0}^{m} \sum_{J \in \binom{[m]}{k}} \sum_{\sigma \in S_m} g_J(\sigma). \]

Note that the permutations \( \tau \) appearing in the definition of \( g_J(\sigma) \) satisfy \( \tau(J) = \sigma(J) \).

Let \( \sigma_1, \sigma_2 \in S_m \) be two permutations such that \( \sigma_1(j) = \sigma_2(j) \) for all \( j \in J \). Moreover, let \( \tau_1, \tau_2 \in S_m \) satisfy \( \tau_1(j) = \tau_2(j) \) for all \( j \in J \), \( \tau_1(j) = \sigma_1(j) \) for all \( j \not\in J \), and \( \tau_2(j) = \sigma_2(j) \) for all \( j \not\in J \). Then it holds for the permutation signs that

\[ (-1)^{\sigma_1} (-1)^{\tau_1} = (-1)^{\sigma_2} (-1)^{\tau_2}. \]

This implies that \( g_J(\sigma_1) = g_J(\sigma_2) \). We thus obtain that

\[ \mathbb{E}[\det(X)^2] = \sum_{k=0}^{m} (m - k)! \cdot \sum_{J \in \binom{[m]}{k}} \sum_{I \in \binom{[m]}{k}} \sum_{\sigma \in S_m} \sum_{\tau \in S_m} (-1)^{\sigma + \tau} \prod_{h=1}^{k} a_{\sigma(h)h} a_{\tau(h)h}. \quad (27) \]

In this formula, \( J = \{j_1, \ldots, j_k\} \) and the set \( I = \{i_1, \ldots, i_k\} \) is introduced to represent the possible images \( I = \sigma(J) = \tau(J) \). By the definition of the determinant and by the Cauchy-Binet theorem, we can rewrite

\[ \mathbb{E}[\det(X)^2] = \sum_{k=0}^{m} (m - k)! \cdot \sum_{J \in \binom{[m]}{k}} \sum_{I \in \binom{[m]}{k}} \det(A_{IJ})^2 \]

\[ = \sum_{k=0}^{m} (m - k)! \cdot \sum_{I \in \binom{[m]}{k}} \det(A_{I \times [m]} A_{I \times [m]}^T) \]

\[ = \sum_{k=0}^{m} (m - k)! \cdot \text{tr} \left[ (AA^T)^{(k)} \right]. \]