CRITICAL SURFACE OF AN EPIDEMIC SPREADING ON A GROWTH NETWORK

By

Dong Han
Tze Leung Lai

Technical Report No. 2011-02
March 2011

Department of Statistics
STANFORD UNIVERSITY
Stanford, California 94305-4065
CRITICAL SURFACE OF AN EPIDEMIC SPREADING ON A GROWTH NETWORK

By

Dong Han
Shanghai Jiao Tong University

Tze Leung Lai
Stanford University

Technical Report No. 2011-02
March 2011

This research was supported in part by
National Science Foundation grant DMS 0805879.

Department of Statistics
STANFORD UNIVERSITY
Stanford, California 94305-4065

http://statistics.stanford.edu
Critical Surface of an Epidemic Spreading on a Growth Network

Dong Han∗
Department of Mathematics
Shanghai Jiao Tong University†

Tze Leung Lai‡
Department of Statistics
Stanford University§

Abstract

By making use of a continuous-time Markov chain to describe both an epidemic spreading on a random growing network and the network growth, we analyze how the epidemic spreading on the growing network can affect the topological structure (degree distribution) of the random environment represented by the growth network. This analysis provides an expression of critical surface of the spreading epidemic in terms of the intensity $\alpha$ of connecting an edge to a node unaffected by the epidemic, the intensity $\beta$ of connecting an edge to a node with the epidemic disease, the initial attractiveness $\theta$ when the degree of the node is 0, the rate $\lambda$ of the epidemic spreading, and the rate $\gamma$ an infected node becoming susceptible. It is shown that the infections will become endemic, or eventually die out, when $(\alpha, \beta, \theta, \gamma, \lambda)$ is above, or below, the critical surface.

Keywords: Markov chain, degree distribution, epidemic spreading, critical surface.

1 Introduction

Many networks in the real world, such as the Internet, World Wide Web, the scientific collaboration network, social networks, etc., have scale-free structures in the sense that the degree distribution of the network follows a power law (see Newman, Barabási, and Watts, 2006), i.e., the probability of a randomly chosen node having $k$ neighbors is asymptotically equal to $Ck^{−\tau}$ for two positive constants $C$ and $\tau$. One of the most fundamental of epidemic models is the SIS (susceptible-infected-susceptible) model. Harris (1974) introduced this model on the $d$-dimensional integer lattice and named it the contact process. Most of the known results can be found in Liggett (1985, 1999). The contact process, or the susceptible-infected-susceptible (SIS) model in the physics literature, is usually used in the study of the spread of some virus on network structures. For example, computer viruses spread over the Internet. Pastor-Satorras and Vespignani (2001, 2002) initiated the study of the contact processes on random networks with power law degree distribution. They argued that the the critical value (the threshold) $\lambda_c$ of the infection rate $\lambda$ is zero if the power $2 < \tau \leq 3$ by applying simulation and mean-field methods. Using the random graph model of Chung and Lu (2002), Chen and Lui (2010) prove the same result rigorously by using the neighborhood expansion method. Chatterjee and Durrett (2009) have shown further that $\lambda_c = 0$ for all $\tau > 3$. These works are based on the assumption that the degree distribution of the network is known and that the contact process

∗Supported by National Natural Science Foundation of China grant 10531070.
†Shanghai 200030, China. Email: donghan@sjtu.edu.cn
‡Supported by National Science Foundation grant DMS 0805879.
§Stanford, CA 94305-4065. Email: lait@stanford.edu
has no influence on the degree distribution of the network. They show that the topological structure (degree distribution) of the network can greatly affect the behavior of the stochastic processes on the network. However, relatively little attention has been paid to the important question concerning how the dynamic process (epidemic spreading) taking place on a growing network shapes the topological structure of the network. In fact, this is one of the ten leading questions proposed by Amaral, et al. (2004).

The main purpose of this paper is to investigate the interaction between the degree distribution of a growing network and the epidemic spreading on the network. There are three main results of this investigation: (1) Present a continuous-time Markov chain which can describe not only the epidemic spreading on a random growing network but also the network growth in the environment of the epidemic spreading. (2) Show that the degree distribution of the network will be subject to the exponential distribution (the power law) if one connects an edge only to a node without (with) epidemic disease. (3) Give an expression of critical surface of the epidemic spreading. It is shown that the critical surface becomes a hyperplane when the maximum degree is unbounded.

The remainder of this paper is organized as follows. The continuous-time Markov chain of a growing network and epidemic spreading is presented in Section 2. Section 3 derives the degree distributions of the growing network and the condition probability that a node with degree \( k \) has the epidemic virus. The critical surface of the spreading epidemic is analyzed in Section 4.

## 2 A Markov chain of network growth and epidemic spreading

Denote the set of natural numbers by \( \mathbb{N} \). Let

\[
X_0 = \left\{ x = (x_{ij}) \in \{0, 1\}^{\mathbb{N}^2} : x_{ji} = x_{ij}, x_{ii} = 0 \text{ and } \sum_j x_{ij} < \infty \right\}.
\]

Then \( X_0 = \bigcup_{n=2}^{\infty} E_n \), where

\[
E_n = \left\{ x \in \{0, 1\}^{\mathbb{N}^2} : \sum_{j=1}^{n} x_{ij} \geq 1 \text{ for } 1 \leq i \leq n \text{ and } x_{ij} = 0 \text{ for all } \max(i, j) > n \right\}.
\]

for \( n \geq 2 \). An element \( x = (x_{ij}) \in E_n \) can be regard as an \( n \)-by-\( n \) matrix, which is the adjacency matrix of a simple graph \( G \) with all \( n \)-vertex labeled. In this sense, the space \( X_0 \) can be considered as the set of finite vertex labeled networks. We use \( |x| \) for the total number of edges of the network \( x \in E_n \), and \( x_i = \sum_j x_{ij} \) the degree of the node \( i \).

Let \( y_i = 1 \) denotes the node \( i \) has an epidemic disease, otherwise \( y_i = 0 \). Denote by \( y(x) = \{y_i(x) \in Y_0, 1 \leq i \leq n(x)\} \) the distribution of the epidemic disease on \( x \in X_0 \), where

\[
Y_0 = \left\{ y = (y_i, i \geq 1) \in \{0, 1\}^\mathbb{N} : \sum_i y_i < \infty \right\}
\]

and \( n(x) = \max\{ i : x_i \geq 1 \} \) denotes number of nodes with at least one edge.

Let \( Z_0 = X_0 \times Y_0 \). Now we define a continuous-time Markov chain \( \{Z(t), t \geq 0\} \) which describes the epidemic spreading on a growing network, where \( Z(t) = (X(t), Y(t)) \in Z_0 \), and both \( X(t) = (X_{ij}(t), i, j \leq n(X(t))) \in X_0 \) and \( Y(t) = (Y_i(t), i \leq n(X(t))) \) denote the state of network growing and epidemic spreading at time \( t \), respectively.
Network growth with preferences: Starting with an initial state \( x \in E_0 \), at every one-step jump (the sojourn time in any state is exponentially distributed), we add a new node and one edge that links the new node to one of the nodes already present in the network. We assume that the new node will be connected to node \( i \) with probability proportional to a nonnegative function \([\alpha(1 - y_i) + \beta y_i] (x_i + \theta) \wedge m\), where the two nonnegative numbers \( \alpha \) and \( \beta \) denote the intensity of connecting an edge to node without and with epidemic disease respectively, the nonnegative number \( \theta \) represents initial attractiveness when the degree, \( x_i \), of node \( i \) is zero, \( m \) is a natural number and \((x_i + \theta) \wedge m = \min\{x_i + \theta, m\}\) denotes that the degree of node \( i \) is at most \( m \). Here the preferences means that the bigger degree, the more possibility of connecting an edge. All healthy nodes (or infected nodes) with degree \( \geq m \) have an equal connecting probability. The inequality \( \alpha > \beta \) (\( \alpha < \beta \)) means that the probability of connecting a healthy node is great (less) than that of connecting infected node. Especially, the topology structure (type of degree distribution) of the growth network does not depend on the epidemic spreading on it when \( \alpha = \beta \). Here, we assume that the new added node has no epidemic disease. The \( m \) most healthy nodes (or infected nodes) with degree \( m \) most have an equal connecting probability. The inequality \( \alpha > \beta \) (\( \alpha < \beta \)) means that the probability of connecting a healthy node is great (less) than that of connecting infected node. Especially, the topology structure (type of degree distribution) of the growth network does not depend on the epidemic spreading on it when \( \alpha = \beta \). Here, we assume that the new added node has no epidemic disease. The evolving network corresponds to Barabási–Albert (1999) model when \( \alpha = \beta = 1, \theta = 0 \) and \( m = \infty \), and to Dorogovtsev et al. (2000) model when \( \alpha = \beta = 1, \theta > 0 \) and \( m = \infty \).

Epidemic dynamics: The epidemic spreading on the evolving network considered in the paper is the susceptible-infected-susceptible (SIS) model, in which each susceptible node \( i \) becomes infected and therefore has an epidemic disease with the rate of the epidemic spreading \( \lambda > 0 \) if at least one of neighbors \( \mathcal{N}_i = \{ j : x_{ij} = 1 \} \) has the epidemic disease. Infected nodes, on the other hand, recover and become susceptible again with the rate \( \gamma \).

According to the description above, the continuous-time Markov chain \( Z(t), t \geq 0 \), considered in this paper is a jump process defined by the following one-step jump probability matrix \((p(z, z'))\),

\[
p(z, z') = \begin{cases} 
\frac{[\alpha(1 - y_l) + \beta y_l] (x_l + \theta) \wedge m}{S(z)} & \text{if } z' = z + (e_{i,n(x)} + 1, 0) \\
\frac{\lambda(\sum_{j \in \mathcal{N}_i} x_{ij} y_j) \wedge m}{S(z)} & \text{if } z' = z + (0, e_i), y_i = 0 \\
\frac{\gamma}{S(z)} & \text{if } z' = z + (0, -e_i), y_i = 1 \\
0 & \text{otherwise}
\end{cases}
\]

for \( z, z' \in \mathbb{Z}_0 \), where \( \alpha + \beta > 0 \), \( \theta > -1 \), both \( e_{i,j} \) and \( e_i \) denote respectively the matrix \( e_{i,j}(k, l) \) and vector \( e_i(k) \) satisfying \( e_{i,j}(k, l) = 1 \) if \( k = i, l = j \), \( e_i(k) = 1 \) if \( k = i \), \( e_{i,j}(k, l) = 0 \), \( e_i(k) = 0 \) otherwise, and

\[
S(z) = \sum_{i=1}^{n(x)} \left[ (\alpha(1 - y_i) + \beta y_i)x_i \wedge m + \lambda(1 - y_i) \left( \sum_{j \in \mathcal{N}_i} x_{ij} y_j \right) \wedge m + \gamma y_i \right]
\]

is a normalization factor. Obviously, \( p(z, z) = 0, \sum_{z'} p(z, z') = 1 \), and the sojourn time in any state is exponentially distributed with rate 1. Here, we assume that \( \theta > -1 \) since \( x_i \geq 1 \) for all \( i \), and therefore, \( x_i + \theta > 0 \).

As can be seen that the Markov chain defined above describes the epidemic spreading on the random growth network, and simultaneously, the network grows in the environment of epidemic virus disturbance. Especially, the network growth is independent with the epidemic spreading when \( \alpha = \beta \). Moreover, if \( Y(0) = 0 \) at initial time \( t = 0 \), then \( Y(t) = 0 \) for all \( t > 0 \), and therefore, \( Z(t) = (X(t), 0) \) for all \( t \geq 0 \). In this case we write \( Z(t) = X(t) \) simply.
The continuous-time transition probabilities
\[ P(t; z, z') = P\{Z(t) = z'|Z(0) = z\} = P_z\{Z(t) = z'\} \]
is generated by a linear bounded operator \( q(z, z') = p(z, z') - \delta_{z, z'} \), which is called the infinitesimal transition rates satisfying \( p(z, z') \geq 0, p(z, z) = 0 \) and \( \sum_{z'} p(z, z') = 1 \), since
\[ P(t; z, z') = q(z, z')t + o(t). \]
for \( z' \neq z \).

The transition probabilities \( P(t; z, z') \) are differentiable and they satisfy the Kolmogorov differential equations:
\[ P'(t; z, z') = \sum_{y \in E_0} q(z, y)P(t; y, z') = \sum_{y \in E_0} P'(t; z, y)q(y, z'). \]

Let \( f : Z_0 \to \mathbb{R} \) be a Lipschitz function: i.e. there is a constant \( M \), such that for all \( x, y \in Z_0 \), we have \( |f(x) - f(y)| \leq M \|x - y\| \). Then \( f(Z(t)) \) is a stochastic process, and
\[ E_z[f(Z(t))] = \sum_{z'} P'(t; z, z')f(z') = \sum_{z', w} P(t; z, w)q(w, z')f(z') = E_z[Qf(Z(t))] \] (2)
where
\[ Qf(z) = \sum_{z' \in Z_0} q(z, z')f(z') = \sum_{z' \in Z_0} q(z, z')[f(z') - f(z)]. \]

### 3 Degree distribution of the evolving network

In this section we will discuss the degree distribution of the evolving networks in the environment of epidemic virus disturbance. Let
\[ S_1(z) = \sum_{i=1}^{n(z)} \left[ \alpha(1 - y_i) + \beta y_i \right] x_i + \theta \wedge m, \]
\[ S_2(z) = \sum_{i=1}^{n(z)} \left[ \sum_{j \in N_i} x_{ij}y_j \right] \wedge m + \gamma y_i. \]

Now we present three conditions in the following.

(I) \( \lim_{t \to \infty} \frac{E_z(S_1(Z(t)))}{t} = s_1 > 0 \), \( \lim_{t \to \infty} \frac{E_z(S_2(Z(t)))}{t} = s_2 \)

(II) \( \lim_{t \to \infty} \text{Cov} \left( (1 - Y_i(t))I_k(X_i(t)), m \wedge \sum_{j \in N_i} X_{ij}(t) + \theta Y_j(t) \right) = 0 \), \( k, i \geq 1 \)

(III) \( \lim_{t \to \infty} \frac{E_z \left( m \wedge \sum_{j \in N_i} X_{ij}(t)Y_j(t) \right)}{m \wedge X_i(t)} = \lim_{t \to \infty} \frac{E_z(m \wedge \sum_{j \in N_i} X_{ij}(t)Y_j(t))}{E_z(m \wedge X_i(t))} = \rho_m, \quad i \geq 1. \)

Note that
\[ \min\{\alpha, \beta\}n(X(t)) \leq S_1(Z(t)) \leq \sum_{i=1}^{n(X(t))} \alpha(1 - Y_i(t))X_i(t) + \beta Y_i(t)X_i(t) \leq 2 \max\{\alpha, \beta\}n(X(t)), \]
\[ 0 \leq S_2(Z(t)) \leq \sum_{i=1}^{n(X(t))} \left[ \lambda(1 - Y_i(t)) \sum_{j=1}^{n(X(t))} X_{ij}(t)Y_j(t) + \gamma Y_i(t) \right] \leq 2 \max\{\lambda, \gamma\}n(X(t)) \]
and $\mathbb{E}_z(n(X(t)) \leq t$. Hence, it is acceptable to assume the condition (I). The condition (II) means that both random variables $(1 - Y_i(t))I_k(X_i(t))$ and $\frac{m\wedge\sum_{i\in N_i} X_{ij}(t)Y_{ij}(t)}{m\wedge X_i(t)}$ are asymptotic uncorrelated. The number $\rho_m$ in condition (III) denotes the probability that an edge linked from a node points to an infected node.

Let $\tau_k$ be the $k$-th time of state transition in the process, that is,

$$\tau_0 = 0, \tau_k = \inf\{t > \tau_{k-1} | Z(t) \neq Z(\tau_{k-1})\}, \quad k \geq 1,$$

and $N(t)$ be the number of state transition in the time interval $[0, t]$, i.e., $N(t) = \max\{n|\tau_n \leq t\}$ is a Poisson process. Since the one-step jump probability matrix $(\rho - \mathbb{I})$ from a node points to an infected node.

Thus $\rho - \mathbb{I}$ means that both random variables (1

$$\lim_{t \to \infty} \frac{N(t)}{t} = 1, \text{ a.s. } \mathbb{P}_z. \quad (3)$$

**Theorem 1.** Suppose the conditions (I)–(III) hold. Let $s = s_1 + s_2$ and $p = s_1/s$. Then

$$\lim_{t \to \infty} \frac{S_1(Z(t))}{t} = s_1, \quad \lim_{t \to \infty} \frac{S_2(Z(t))}{t} = s_2, \quad \lim_{t \to \infty} \frac{n(X(t))}{t} = p, \text{ a.s. } \mathbb{P}_z. \quad (4)$$

**Proof.** We first prove that $\lim_{t \to \infty} \frac{S_1(Z(t))}{t} = s_1$, a.s. Let $\sigma_t = \sigma\{Z(s)|0 \leq s \leq t\}$ be the smallest $\sigma$-algebra which includes all events occurred before $t$. Let $[t]$ denotes the biggest positive integer less than or equal to $t$. For any fixed $t > 0$, the stochastic process $M(s) = \mathbb{E}_z[S_1(Z(t))|\sigma_{[t]}]$ is a martingale for $0 \leq s \leq t$, and therefore, $\mathbb{E}_z(M(s_4) - M(s_3))(M(s_2) - M(s_1)) = 0$ for $0 \leq s_1 < s_2 < s_3 < s_4 \leq t$. Then, we have

$$\mathbb{E}_z[S_1(Z(t)) - \mathbb{E}_z(S_1(Z(t)))^2 = \mathbb{E}_z[M(t) - M([t])] - \sum_{k=1}^{[t]} (M(k) - M(k-1))^2$$

$$= \mathbb{E}_z(M(t) - M([t]))^2 + \sum_{k=1}^{[t]} \mathbb{E}_z(M(k) - M(k-1))^2.$$ 

Since $|M(t) - M([t])| \leq 2\max\{\alpha, \beta, m\}N(t - [t])$ and $|M(k) - M(k-1)| \leq 2\max\{\alpha, \beta, m\}N(1)$ for $1 \leq k \leq [t]$, it follows that

$$\mathbb{E}_z[S_1(Z(t)) - \mathbb{E}_z(S_1(Z(t)))^2 \leq 4(\max\{\alpha, \beta, m\})^2\mathbb{E}_zN^2(t - [t]) + [t]\mathbb{E}_zN^2(1) = 8(\max\{\alpha, \beta, m\})^2t.$$ 

Thus

$$\mathbb{E}_z\left(\frac{S_1(Z(n^2))}{n^2} - \frac{\mathbb{E}_zS_1(Z(n^2))}{n^2}\right)^2 \leq \frac{8(\max\{\alpha, \beta, m\})^2}{n^2}$$

and therefore, $\lim_{n \to \infty} \frac{S_1(Z(n^2))}{n^2} = s_1$, a.s. For $t > 0$, taking natural number $n$ such that

$$n^2 < t \leq (n + 1)^2$$

we have

$$\left|\frac{S_1(Z(t))}{t} - s_1\right| \leq \left|\frac{S_1(Z(t))}{t} - \frac{S_1(Z(n^2))}{n^2}\right| + \left|\frac{S_1(Z(n^2))}{n^2} - s_1\right|$$

$$\leq \frac{t - n^2}{n^2} \frac{S_1(Z(t))}{t} + \frac{\left|S_1(Z(t)) - S_1(Z(n^2))\right|}{n^2} + \frac{S_1(Z(n^2))}{n^2} - s_1$$

$$\leq \max\{\alpha, \beta\}m\frac{2n + 1}{n^2} \frac{N(t)}{t} + 2(\alpha + \beta)m\frac{2n + 1}{n^2} \frac{N(2n + 1)}{2n + 1} + \left|\frac{S_1(Z(n^2))}{n^2} - s_1\right|.$$
From (3) it follows that
\[ \lim_{t \to \infty} \frac{S_1(\mathbf{Z}(t))}{t} = s_1, \text{ a.s. -} \mathbb{P}_z. \] (5)

By the same method we can prove \( \lim_{t \to \infty} \frac{S_2(\mathbf{Z}(t))}{t} = s_2, \text{ a.s.} \). Since \( 2n(x) = \sum_{i=1}^{n(x)} x_i \), it follows from (2) that
\[
\mathbb{E}_z n(X(t)) = \frac{1}{2} \mathbb{E}_z \left( \sum_{i=1}^{n(X(t))} X_i(t) \right) = \mathbb{E}_z \left( \frac{S_1(\mathbf{Z}(t))}{S_1(\mathbf{Z}(t)) + S_2(\mathbf{Z}(t))} \right).
\]

Hence, \( \frac{1}{t} \mathbb{E}_z n(X(t)) \to p = s_1/s \). As (5), we can further prove that \( \lim_{t \to \infty} \frac{n(X(t))}{t} = p \) a.s. This completes the proof of Theorem 1.

Let \( D_k(x) = \sum_{i=1}^{n(x)} I_k(x_i) \) and \( E_k(z) = \sum_{i=1}^{n(z)} y_i I_k(x_i) \) denote respectively the number of nodes with degree \( k \) and the number of nodes with degree \( k \) having the epidemic virus, where \( I_k(\cdot) \) is the indicator function. Let \( \cdot^T \) and \( \cdot^{-1} \) denote respectively the transform and reverse of a matrix \( \cdot \). Denote by \( I \) the unit matrix.

Let \( P_k \) and \( Q_k \) denote respectively the limit probabilities that a node has the degree \( k \), and that a node has degree \( k \) and the epidemic virus. That is, the degree distributions \( P_k \) and \( Q_k \) can be written as
\[
P_k = \lim_{t \to \infty} \frac{D_k(X(t))}{n(X(t))}, \quad Q_k = \lim_{t \to \infty} \frac{E_k(Z(t))}{n(X(t))}, \text{ a.s. -} \mathbb{P}_z.
\]

Theorem 2. Suppose the conditions (I)–(III) hold. Let \( W_m(x) = (x + \theta) \wedge m \). Then the two limits \( P_k = \lim_{t \to \infty} \frac{D_k(X(t))}{n(X(t))} \) and \( Q_k = \lim_{t \to \infty} \frac{E_k(Z(t))}{n(X(t))} \) exist, and the two probabilities \( P_k \) and \( Q_k \) can be expressed in the following vector form:
\[
(P_k, Q_k)^T = (A(k) + I)^{-1} \left[ \prod_{i=1}^{k-1} B(i)(A(i) + I)^{-1} \right] (1, 0)^T
\] (6)
for \( k \geq 1 \), where \( \prod_{j=i}^{k} 1 = 1 \) for \( i > k \) and
\[
A(k) = s^{-1} \begin{pmatrix}
\alpha W_m(k) & (\beta - \alpha) W_m(k) \\
-\lambda(m \wedge k) \rho_m & \beta W_m(k) + \lambda(m \wedge k) \rho_m + \gamma
\end{pmatrix},
\] (7)
\[
B(k) = s^{-1} \begin{pmatrix}
\alpha W_m(k) & (\beta - \alpha) W_m(k) \\
0 & \beta W_m(k)
\end{pmatrix}.
\] (8)

Proof. Let \( D_k(t) = \mathbb{E}_z[D_k(X(t))] \) and \( E_k(t) = \mathbb{E}_z[E_k(Z(t))] \). It follows from (1) and (2) that
\[
D_k(t) = \alpha W_m(k - 1) \mathbb{E}_z \left( \frac{D_{k-1}(X(t))}{S(Z(t))} \right) - \alpha W_m(k) \mathbb{E}_z \left( \frac{D_k(X(t))}{S(Z(t))} \right)
+ (\beta - \alpha) W_m(k - 1) \mathbb{E}_z \left( \frac{E_{k-1}(Z(t))}{S(Z(t))} \right) - (\beta - \alpha) W_m(k) \mathbb{E}_z \left( \frac{E_k(Z(t))}{S(Z(t))} \right)
+ S_1(Z(t)) \mathbb{E}_z \delta_{k1}.
\]
\[ E_k'(t) = \beta W_m(k - 1)E_z \left( \frac{E_{k-1}(Z(t))}{S(Z(t))} \right) - \beta W_m(k)E_z \left( \frac{E_k(Z(t))}{S(Z(t))} \right) - \gamma E_z \left( \frac{E_k(Z(t))}{S(Z(t))} \right) + \lambda E_z \left( \sum_{i=1}^{m(\chi(t))}(1 - Y_i(t))I_k(X_i(t))(\sum_{j \in N_i} X_{ij}(t)Y_j(t)) \right) \frac{m}{S(Z(t))}. \]

By Theorem 1 and conditions (II) and (III) we can rewrite \( D'_k(t) \) and \( E'_k(t) \) as

\[ D'_k(t) = \alpha W_m(k - 1)\frac{D_{k-1}(t)}{st} - \alpha W_m(k)\frac{D_k(t)}{st} + (\beta - \alpha)W_m(k - 1)\frac{E_{k-1}(t)}{st} - (\beta - \alpha)W_m(k)\frac{E_k(t)}{st} + \delta_k p + \epsilon_k(t) \tag{9} \]

and

\[ E'_k(t) = \beta W_m(k - 1)\frac{E_{k-1}(t)}{st} - \beta W_m(k)\frac{E_k(t)}{st} - \gamma E_k(t) + \lambda (m \wedge k)\frac{D_k(t) - E_k(t)}{st} + \epsilon_k(t) \tag{10} \]

for large \( t \), where \( \epsilon_k(t) \to 0 \) and \( \epsilon_k(t) \to 0 \) as \( t \to \infty \) for all \( k \geq 1 \).

Let \( U_k(t) = (D_k(t), E_k(t))^T, \Xi_k(t) = (\epsilon_k(t), e_k(t))^T \) and \( P_k = (\delta_k p, 0)^T \). We can rewrite the above two equations in the matrix form

\[ U'_k(t) = B(k - 1)\frac{U_{k-1}(t)}{t} - A(k)\frac{U_k(t)}{t} + P_k + \Xi_k(t). \tag{11} \]

To solve the matrix equation we need the following series of matrix functions:

\[ e^{A\log t} = \sum_{i=0}^{\infty} \frac{(A \log t)^i}{i!}, \quad e^{-I \log t} = \sum_{i=0}^{\infty} \frac{(-I \log t)^i}{i!} = \frac{1}{t} I \]

where \( A \) is a matrix and \( I \) is the unit matrix.

Let (11) multiplied by \( e^{A(k)\log t} \) we have

\[ (e^{A(k)\log t}U_k(t))^T = e^{A(k)\log t} \left[ B(k - 1)\frac{U_{k-1}(t)}{t} + P_k + \Xi_k(t) \right] \]

and therefore

\[ e^{A(k)\log t}U_k(t) - e^{A(k)\log t}U_k(t_0) = \int_{t_0}^{t} e^{A(k)\log s} \left[ B(k - 1)\frac{U_{k-1}(s)}{s} + P_k + \Xi_k(s) \right] ds. \tag{12} \]

Since \( B(0) = 0, P_{11} = (p, 0)^T \) and \( \Xi_1(s) \to 0 \) as \( s \to \infty \), we have

\[ e^{A(1)\log t}U_1(t) - e^{A(1)\log t}U_1(t_0) = \left[ e^{(A(1)+I)\log t} - e^{(A(1)+I)\log t_0} \right] (A(1) + I)^{-1}(P_{11} + o(1)) \]

for large \( t_0 \). Thus

\[ \lim_{t \to \infty} \frac{U_1(t)}{t} = (A(1) + I)^{-1}(p, 0)^T. \]
By the similar way we can solve the equation (12) and obtain

\[
\lim_{t \to \infty} \frac{U_k(t)}{t} = (A(k) + I)^{-1}[B(k-1)(A(k-1) + I)^{-1}] \ldots [B(1)(A(1) + I)^{-1}] (p, 0)^T
\]

for \( k \geq 2 \). Note that \( \lim_{t \to \infty} \frac{n(X(t))}{t} = p \). By the same way of proving Theorem 1 we can prove that

\[
(P_k, Q_k)^T = \lim_{t \to \infty} \left( \frac{D_k(X(t))}{n(X(t))}, \frac{E_k(Z(t))}{n(X(t))} \right)^T
\]

\[
= (A(k) + I)^{-1} \left[ \prod_{i=1}^{k-1} B(i)(A(i) + I)^{-1} \right] (1, 0)^T, \quad \text{a.s. -}\mathbb{P}_z.
\]

That is, the degree distributions \((P_k, Q_k)^T, k \geq 1\), are given by (6).

**Corollary 3.** If \( \rho_m = 0 \) and \( \alpha > 0 \), then \( Q_k = 0 \) and

\[
P_k = \frac{\mu_m}{W_m(k)} \prod_{i=1}^{k-1} \frac{W_m(i)}{W_m(i) + \lambda}
\]

for \( k \geq 1 \), where \( \mu_m = \sum_{k=1}^{\infty} W_m(k)P_k \).

**Proof.** It follows from (6) that

\[
(P_k, Q_k)^T = (A(k) + I)^{-1} B(k-1)(P_{k-1}, Q_{k-1})^T
\]

\[
= \begin{pmatrix}
\frac{\alpha W_m(k-1)}{s} & \frac{W_m(k-1)(\beta - \alpha)(\gamma + s)}{\alpha W_m(k) + s}
\end{pmatrix}
\begin{pmatrix}
\frac{\beta W_m(k-1)}{V_m(k)}
\end{pmatrix}

\]

for \( k \geq 2 \) when \( \rho_m = 0 \), where \( V_m(k) = \beta W_m(k) + \lambda(m \land k)\rho_m + \gamma + s \). Since \( Q_1 = \lambda \rho_m / s = 0 \), it follows that \( Q_k = 0 \) for \( k \geq 2 \) and

\[
P_k = \frac{\alpha W_m(k-1)}{s} P_{k-1} = \frac{\alpha W_m(k) + s}{\alpha W_m(k)} \prod_{i=1}^{k-1} \frac{\alpha W_m(i)}{\alpha W_m(i) + s}
\]

for \( k \geq 1 \). Note that

\[
\lim_{t \to \infty} \frac{1}{t} \mathbb{E}_z \left( \sum_{i=1}^{n(X(t))} W_m(X_i(t)) \right) = \lim_{t \to \infty} \frac{1}{t} \mathbb{E}_z \left( \sum_{k=1}^{n(X(t))} W_m(k) \sum_{i=1}^{n(X(t))} I_k(X_i(t)) \right) = p \mu_m
\]

\[
\lim_{t \to \infty} \frac{1}{t} \mathbb{E}_z \left( \sum_{i=1}^{n(X(t))} Y_i(t)W_m(X_i(t)) \right) = \lim_{t \to \infty} \frac{1}{t} \mathbb{E}_z \left( \sum_{k=1}^{n(X(t))} W_m(k) \sum_{i=1}^{n(X(t))} Y_i(t)I_k(X_i(t)) \right)
\]

\[
= p \sum_{k=1}^{\infty} W_m(k)Q_k = 0
\]

for \( \rho_m = 0 \), and therefore,

\[
s = \alpha \lim_{t \to \infty} \frac{1}{t} \mathbb{E}_z \left( \sum_{i=1}^{n(X(t))} W_m(X_i) \right) = \alpha p \mu_m, \quad p = s_1 / s = 1
\]

for \( \rho_m = 0 \). Thus, we have (13).
Remark 1. By (13) we have

\[
P_k = \begin{cases} 
\frac{\mu_m \Gamma(m \rho_m)}{(m + \mu_m)^{k-m}} \Gamma(k+1) \sim C k^{-(1+\mu_m)} & \text{if } k + \theta \leq m \\
\frac{\mu_m}{m} \left( \frac{m}{m + \mu_m} \right)^{k-m} \Gamma(m) \Gamma(m+1+\mu_m) & \text{if } k + \theta > m.
\end{cases}
\]

(15)

This means that the degree distribution \( \{P_k, k \geq 1\} \) is subject to the power law when \( m \to \infty \), which is the same as Barabási–Albert model (1999) for \( \theta = 0 \) since \( \mu_m \to 2+\theta = \sum_{k=1}^{\infty} (k+\theta) P_k \) as \( m \to \infty \).

Corollary 4. Let \( R_k = \frac{Q_k}{P_k} \) which denotes the condition probability that a node with degree \( k \) has the epidemic virus. If \( \alpha = \beta \) and \( \rho_m > 0 \), then

\[
P_k = \frac{s}{\alpha W_m(k) + s} \prod_{i=1}^{k-1} \frac{\alpha W_m(i)}{\alpha W_m(i) + s}
\]

\[
R_k = \lambda \rho_m \sum_{j=1}^{k} \frac{m \wedge j}{V_m(j)} \prod_{i=j+1}^{k} \frac{\alpha W_m(i)}{V_m(i)}
\]

(16) (17)

for \( k \geq 1 \).

Proof. It follows from (6) that

\[
(P_k, Q_k)^T = (A(k) + I)^{-1} B(k-1)(P_{k-1}, Q_{k-1})^T
\]

\[
= \begin{pmatrix}
\frac{\alpha W_m(k-1)}{\alpha W_m(k)+s} & 0 \\
\frac{\lambda m \wedge k \rho_m}{\alpha W_m(k) + V_m(k)} & \frac{\alpha W_m(k-1)}{V_m(k)}
\end{pmatrix}
(P_{k-1}, Q_{k-1})^T
\]

for \( k \geq 1 \) when \( \alpha = \beta \). Hence

\[
P_k = \frac{\alpha W_m(k-1)}{\alpha W_m(k)+s} P_{k-1}
\]

\[
Q_k = \frac{\alpha W_m(k-1)}{\alpha W_m(k)+s} \frac{\lambda m \wedge k \rho_m}{V_m(k)} P_{k-1} + \frac{\alpha W_m(k-1)}{V_m(k)} Q_{k-1}
\]

\[
R_k = \frac{\alpha W_m(k)+s}{V_m(k)} R_{k-1} + \frac{\lambda m \wedge k \rho_m}{V_m(k)}
\]

for \( k \geq 1 \). This implies (16) and (17).

As can be seen that the type of degree distribution for \( \rho_m = 0 \) is the same as that one for \( \alpha = \beta \). It follows from (17) that

\[
\lim_{k \to \infty} R_k = \frac{\lambda m \rho_m}{V_m(m)} \sum_{j=0}^{\infty} \left( \frac{\alpha m + s}{V_m(m)} \right)^j = 1 - \frac{\gamma}{\lambda m \rho_m + \gamma}.
\]

(18)

Hence, \( \{Q_k, k \geq 1\} \) has the same tail distribution as \( \{P_k, k \geq 1\} \) when \( \alpha = \beta \) and \( \rho_m > 0 \).

The following theorem gives some results on the distribution of \( \{P_k, k \geq 1\} \) for \( \alpha \neq \beta \) and \( \rho_m > 0 \).
Theorem 5. Suppose the conditions (I)–(III) hold. Let \( \alpha \neq \beta, \rho_m > 0 \) and \( P_k/P_{k-1} \to q \). Then (i)

\[
\lim_{k \to \infty} R_k = \begin{cases} 
1 - \frac{\gamma}{\lambda m_{\rho_m+\gamma+s}} & \text{if } \alpha = 0 \\
1 - \frac{\gamma}{\lambda m_{\rho_m+\gamma+s}} & \text{if } \beta = 0
\end{cases}
\]

(19)

\[
q = \begin{cases} 
1 - \frac{s\lambda_m}{\beta \lambda m_{\rho_m+s(\lambda m_{\rho_m}+\beta)+s(\gamma+s)/m}} & \text{if } \alpha = 0 \\
1 - \frac{s\lambda_m}{\beta \lambda m_{\rho_m+\alpha(\gamma+s)+s(\gamma+s)/m}} & \text{if } \beta = 0.
\end{cases}
\]

(20)

(ii) If there is a natural number subseries \( \{m_j\} \) such that \( m_j \to \infty \) and \( m_j \rho_{m_j} \to \infty \) as \( j \to \infty \), then

\[
\lim_{k \to \infty} R_k = 1 - \frac{\gamma}{\lambda m_{\rho_m}} (1 + o(1))
\]

(21)

\[
q = 1 - \frac{s}{\beta m_j} (1 + o(1))
\]

(22)

for large \( j \).

(iii) If there is a natural number subseries \( \{m_i\} \) such that \( m_i \to \infty \) and \( m_i \rho_{m_i} \to 0 \) as \( i \to \infty \), then

\[
\lim_{k \to \infty} R_k = \begin{cases} 
(\beta - \alpha)\gamma - \alpha \gamma (1 + o(1)) & \text{if } (\beta - \alpha)s > \alpha \gamma \\
o(1) & \text{if } (\beta - \alpha)s \leq \alpha \gamma
\end{cases}
\]

(23)

\[
q = \begin{cases} 
1 - \frac{s}{\beta m_i} (1 + o(1)) & \text{if } (\beta - \alpha)s > \alpha \gamma \\
1 - \frac{s}{\beta m_i} (1 + o(1)) & \text{if } (\beta - \alpha)s \leq \alpha \gamma
\end{cases}
\]

(24)

for large \( i \).

(iv) If there are two positive constants \( C_1 \) and \( C_2 \) such that \( C_1 \leq m \rho_m \leq C_2 \) for all \( m \), then

\[
\lim_{k \to \infty} R_k = \frac{d - a + \sqrt{(d - a)^2 + 4bc}}{2b}.
\]

(25)

\[
q = 1 - \frac{\lambda \beta m_{\rho_m} + \alpha(\gamma + s) + \beta s - \sqrt{(d - a)^2/m^2 + 4bc/m^2}}{2(\alpha \beta m + \lambda \beta m_{\rho_m} + s \beta + \alpha(\gamma + s))} (1 + o(1))
\]

(26)

for large \( m \), where

\[
a = \alpha m((\beta + \lambda m)m + \gamma + s), \quad b = m(\beta - \alpha)(\lambda m_{\rho_m} m + \gamma + s),
\]

(27)

\[
c = \alpha \lambda m_{\rho_m} m^2, \quad d = \lambda m_{\rho_m}(\beta - \alpha)m^2 + \alpha \beta m^2 + \beta sm,
\]

(28)

\[f = \beta(\alpha + \lambda m)m^2 + s(\beta + \lambda m_{\rho_m})m + (s + \alpha m)(\gamma + s).
\]

(29)

Proof. By using (7) and (8) we have

\[
(A(k) + I)^{-1} B(k - 1) = \frac{1}{f_k} \begin{pmatrix} a_k & b_k \\ c_k & d_k \end{pmatrix}
\]

(30)
Hence, the solutions can be written as

\[ P_k = \frac{1}{f}(aP_{k-1} + bQ_{k-1}) \]

\[ Q_k = \frac{1}{f}(cP_{k-1} + dQ_{k-1}) \]

and therefore

\[ 1 = \frac{1}{f}(a + bR_{k-1}) \frac{P_{k-1}}{P_k} \] (31)

\[ R_k = \frac{1}{f}(c + dR_{k-1}) \frac{P_{k-1}}{P_k} \] (32)

for \( k > m \). Since \( \lim_{k \to \infty} P_k/P_{k-1} = q \), it follows from (31) that \( \lim_{k \to \infty} R_k = R = (qf - a)/b \).

By (31) and (32) we have

\[ qf = (a + bR), \quad qfR = (c + dR). \] (33)

If \( q = 0 \), then \( R = -a/b = -c/d \), and therefore, \( ad - bc = 0 \). But, from (27) and (28) it follows that

\[ ad - bc = \alpha\beta^2(\alpha + \lambda\rho_m)m^4 + \alpha\beta(\alpha\gamma + s(\alpha + \beta + \lambda\rho_m))m^3 + \alpha\beta s(\gamma + s)m^2 > 0. \]

This means that \( q > 0 \). Thus, by (33) we have \( bR^2 - (d - a)R - c = 0 \). Solving this equation the solutions can be written as

\[ R_{1,2} = \frac{d - a \pm \sqrt{(d-a)^2 + 4bc}}{2b}. \]

Note that \( b > 0 \) and \( bc > 0 \) for \( \beta > \alpha \). It follows that \( d - a - \sqrt{(d-a)^2 + 4bc} < 0 \), and therefore \( R = R_1 \) for \( \beta > \alpha \), otherwise, \( R = R_2 < 0 \) which is contradictory with \( 0 \leq R \leq 1 \) since \( R \) is the limit probability.

Let \( \alpha > \beta \). Note that

\[ d - a = \lambda\rho_m(\beta - 2\alpha)m^2 + (\beta - \alpha)(\gamma + s)m - \beta\gamma m \]

\[ = 2b - [\lambda\beta\rho_m m + \beta\gamma - (\alpha - \beta)(\gamma + s)]m \]

\[ (d-a)^2 + 4bc = [\lambda\beta\rho_m m + \beta\gamma - (\alpha - \beta)(\gamma + s)]^2 m^2 \]

\[ + 4\lambda\beta\gamma\rho_m(\alpha - \beta)m^3 + 4\beta\gamma(\alpha - \beta)(\gamma + s)m^2. \]

Hence

\[ 2b - (d - a) + \sqrt{(d-a)^2 + 4bc} > 0. \]
for $\alpha > \beta$. It follows that
\[
R_2 = \frac{d - a - \sqrt{(d - a)^2 + 4bc}}{2b} = \frac{- (d - a) + \sqrt{(d - a)^2 + 4bc}}{-2b} > 1
\]
since $-b > 0$ for $\alpha > \beta$. This means that $R = R_1$ for $\alpha > \beta$. Thus
\[
R = R_1 = \frac{d - a + \sqrt{(d - a)^2 + 4bc}}{2b}
\]
for $\alpha \neq \beta$. This is (25).

Since $\sqrt{(d - a)^2 + 4bc} = \beta (\lambda m \rho_m + s) m$ for $\alpha = 0$ and $\sqrt{(d - a)^2 + 4bc} = \alpha (\gamma + s) m$ for $\beta = 0$, (19) follows by taking $\alpha = 0$ and $\beta = 0$ respectively in (34). Similarly, taking $m_j \rho_{m_j} \to \infty$ and $m_i \rho_{m_i} \to 0$ respectively in (34), we can obtain (21) and (23) respectively. Plugging (34) into (33) we have
\[
q = \frac{a + d + \sqrt{(d - a)^2 + 4bc}}{2f} = 1 - \frac{\lambda \beta \rho_m + 2s \lambda \rho_m + \alpha (\gamma + s) + \beta s - \sqrt{(d - a)^2/m^2 + 4bc/m^2}}{2f/m}.
\]
This is (26) for large $m$. Taking $\alpha = 0$, $\beta = 0$, $m_j \rho_{m_j} \to \infty$ and $m_i \rho_{m_i} \to 0$ respectively in (35), we can obtain (20), (22) and (24) respectively. This completes the proof of Theorem 3.

The condition $P_k/P_{k-1} \to q$ in Theorem 3 usually holds since $\{P_k, k \geq 1\}$ is a probability distribution.

Remark 2. From (20) in Theorem 3 it follows that the tail distribution of $\{P_k, k \geq 1\}$ is subject to the exponential distribution for $\beta = 0, \rho_m \geq c > 0$, i.e., $P_k \sim q^k = e^{-\varepsilon k}$ for large $k$, where $0 < q < 1$ and $\varepsilon = -\ln q$. On the other hand, the tail distribution of $\{P_k, k \geq 1\}$ is asymptotically subject to the power law for $\alpha = 0$ and $m \rho_m \to \infty$ as $m \to \infty$. In other words, if one connects an edge only to a node without (with) epidemic disease, then the topology structure (degree distribution) of the network will be asymptotically subject to the exponential distribution (the power law) for $\rho_m \geq c > 0$ as $m \to \infty$.

4 The critical surface of epidemic spreading

Note that the number $\rho_m = \rho_m(\alpha, \beta, \gamma, \theta, \lambda)$ is dependent on the five parameters, $\alpha, \beta, \gamma, \theta$ and $\lambda$. Now we define a critical value $\lambda_c(m)$ for every $m \geq 1$ in the following.

Definition 1. For fixed $\alpha, \beta, \gamma$ and $\theta$, the epidemic critical value $\lambda_c(m) = \lambda_c(\alpha, \beta, \gamma, \theta, m)$ of $\lambda(m) = \lambda(\alpha, \beta, \gamma, \theta, m)$ for $m \geq 1$ is defined by
\[
\lambda_c(m) = \inf \{\lambda(m) > 0 : \rho_m(\alpha, \beta, \gamma, \theta, \lambda) > 0\}.
\]
The critical value means that if $\lambda(m) > \lambda_c(m)$, the infection spreads and becomes endemic. Below it, i.e., $\lambda(m) < \lambda_c(m)$, the infection dies out finally ($\rho_m = 0$). The function $\lambda_c(\alpha, \beta, \gamma, \theta, m)$ on $\alpha, \beta, \gamma$ and $\theta$ can be seen as the critical surface for any fixed $m$. 
It follows from (6) and (30) that

\[ Q_k = \frac{c_k}{f_k} P_{k-1} + \frac{d_k}{f_k} Q_{k-1} = Q_1 \prod_{j=2}^{k} \frac{d_j}{f_j} + \sum_{j=2}^{k} \frac{c_j}{f_j} P_{j-1} \prod_{i=j+1}^{k} \frac{d_i}{f_i} \]

where

\[ q_m(k) = \frac{1}{s f_1} \prod_{j=2}^{k} \frac{d_j}{f_j} + \alpha \sum_{j=2}^{k} (m \land j) W_m(j-1) \prod_{i=j+1}^{k} \frac{d_i}{f_i} \]

\[ W_m(k) = (k + \theta) \land m. \]

Let

\[ \sigma_m = \frac{\sum_{k=1}^{\infty} (k - m) q_m(k)}{\sum_{k=1}^{\infty} (m \land k) q_m(k)}, \quad \delta_m = \frac{\sum_{k=1}^{\infty} W_m(k) q_m(k)}{\sum_{k=1}^{\infty} (m \land k) q_m(k)}. \]  

**Theorem 6.** Suppose the conditions (I)–(III) hold. Let \( \Lambda_m = (\alpha \mu_m + \gamma)(1 + \sigma_m) - \beta \delta_m. \) If \( \lim_{\lambda \to \lambda_c(m)} \rho_m(\alpha, \beta, \gamma, \theta, \lambda) = 0, \) then the critical value \( \lambda_c(m) \) can be expressed as

\[ \lambda_c(m) = \begin{cases} \frac{\Lambda_m \sum_{k=1}^{\infty} (m \land k) P_k}{\sum_{k=1}^{\infty} (m \land k) P_k} & \text{if } \Lambda_m > 0 \\ 0 & \text{if } \Lambda_m \leq 0. \end{cases} \]  

**Proof.** Denote \( \lambda(m) \) by \( \lambda \) briefly. Let \( \lambda > \lambda_c(m) \), therefore, \( \rho_m > 0 \). Let \( f(Z(t)) = \sum_{i=1}^{n(X(t))} Y_i(t) X_i(t) \) and \( f_m(Z(t)) = \sum_{i=1}^{n(X(t))} Y_i(t) W_m(X_i(t)) \). It follows from (2) that

\[ \mathbb{E}_z'(f(Z(t))) = \beta \mathbb{E}_z \left( \frac{f_m(Z(t))}{S(Z(t))} \right) - \gamma \mathbb{E}_z \left( \frac{f(Z(t))}{S(Z(t))} \right) \\
+ \lambda \mathbb{E}_z \left( \sum_{i=1}^{n(X(t))} (1 - Y_i(t)) X_i(t) P_i(t) \right) \]

By Theorem 1 and conditions (II) and (III) we can rewrite \( \mathbb{E}_z'(f(Z(t))) \) as follows:

\[ \mathbb{E}_z'(f(Z(t))) = \beta \mathbb{E}_z \left( \frac{f_m(Z(t))}{S(t)} - \gamma \mathbb{E}_z \left( \frac{f(Z(t))}{S(t)} \right) \right) \\
+ \lambda \rho_m \mathbb{E}_z \left( \sum_{i=1}^{n(X(t))} (1 - Y_i(t)) X_i(t) (X_i(t) \land m) \right) + \varepsilon(t) \]

or

\[ \mathbb{E}_z(f(Z(t))) - f(z) = \int_0^t \left( \beta \mathbb{E}_z \left( \frac{f_m(Z(u))}{S(u)} - \gamma \mathbb{E}_z \left( \frac{f(Z(u))}{S(u)} \right) \right) \\
+ \lambda \rho_m \mathbb{E}_z \left( \sum_{i=1}^{n(X(u))} (1 - Y_i(u)) X_i(u) (X_i(u) \land m) \right) \right) du, \]  

\[ \int_0^t \left( \beta \mathbb{E}_z \left( \frac{f_m(Z(u))}{S(u)} - \gamma \mathbb{E}_z \left( \frac{f(Z(u))}{S(u)} \right) \right) \\
+ \lambda \rho_m \mathbb{E}_z \left( \sum_{i=1}^{n(X(u))} (1 - Y_i(u)) X_i(u) (X_i(u) \land m) \right) \right) du, \]  

13
where \( \epsilon(t) \to 0 \) as \( t \to \infty \). Note that

\[
2n(X(t)) = \sum_{i=1}^{n(X(t))} X_i(t) = \sum_{k=1}^{n(X(t))} k \sum_{i=1}^{k} I_k(X_i(t)) = \sum_{k=1}^{n(X(t))} kD_k(X(t))
\]

\[
f(Z(t)) = \sum_{i=1}^{n(X(t))} kY_i(t) \sum_{k=1}^{n(X(t))} I_k(X_i(t)) = \sum_{k=1}^{n(X(t))} kE_k(Z(t))
\]

\[
f_m(Z(t)) = \sum_{k=1}^{n(X(t))} W_m(k)E_k(Z(t))
\]

and

\[
\sum_{i=1}^{n(X(t))} (1 - Y_i(t))X_i(t)(X_i(t) \land m) = \sum_{k=1}^{n(X(t))} k(k \land m)(D_k(X(t)) - E_k(Z(t))).
\]

It follows from Theorems 1 and 2 that \( 2 = \sum_{k=1}^{\infty} kP_k \).

\[
\lim_{t \to \infty} \frac{\mathbb{E}_z(f(Z(t)))}{t} = p \sum_{k=1}^{\infty} kQ_k, \quad \lim_{t \to \infty} \frac{\mathbb{E}_z(f_m(Z(t)))}{t} = p \sum_{k=1}^{\infty} W_m(k)Q_k
\]

and

\[
\lim_{t \to \infty} \frac{1}{t} \mathbb{E}_z \left[ \sum_{i=1}^{n(X(t))} (1 - Y_i(t))X_i(t)(X_i(t) \land m) \right] = p \sum_{k=1}^{\infty} k(k \land m)(P_k - Q_k).
\]

By using the results above we can obtain from (39) that

\[
p \sum_{k=1}^{\infty} kQ_k = \frac{p\beta}{s} \sum_{k=1}^{\infty} W_m(k)Q_k - \frac{P\gamma}{s} \sum_{k=1}^{\infty} kQ_k + \frac{p\lambda \rho_m}{s} \sum_{k=1}^{\infty} k(k \land m)(P_k - Q_k)
\]

and therefore

\[
(s + \gamma) \sum_{k=1}^{\infty} (k \land m)Q_k + (s + \gamma) \sum_{k \geq m}^{\infty} (k - m)Q_k - \beta \sum_{k=1}^{\infty} W_m(k)Q_k
\]

\[
= \lambda \rho_m \sum_{k=1}^{\infty} k(k \land m)(P_k - Q_k). \quad (40)
\]

By condition (III) we know that

\[
\rho_m = \frac{\sum_{k=1}^{\infty} R_k(k \land m)P_k}{\sum_{k=1}^{\infty} (k \land m)P_k} = \frac{\sum_{k=1}^{\infty} (k \land m)Q_k}{\sum_{k=1}^{\infty} (k \land m)P_k}.
\]

Plugging (41) into (40) and using (36) and (37) we have

\[
[(s + \gamma)(1 + \sigma_m) - \beta \delta_m] \sum_{k=1}^{\infty} (k \land m)P_k = \lambda \sum_{k=1}^{\infty} k(k \land m)(P_k - Q_k).
\]

\[
\text{14}
\]
Note that
\[ s = \alpha \mu_m, \quad \sum_{k=1}^{\infty} k(k \wedge m)Q_k = 0 \]
\[ d_k = \frac{\alpha \beta W_m(k)W_m(k-1) + s \beta W_m(k-1)}{\alpha \beta W_m(k)^2 + s \beta W_m(k) + (s + \alpha W_m(k)) (\gamma + s)} \]
when \( \rho_m = 0 \). This means that both \( \sigma_m \) and \( \delta_m \) are independent of \( \lambda \) when \( \rho_m = 0 \). Thus, taking \( \lambda \searrow \lambda_c(m) \) in (42) we can obtain (38). \( \square \)

It can be checked that \( \lim_{\alpha \to 0, \beta \to 0} \Lambda_m = \gamma \). Hence,
\[ \lambda_c = \lim_{m \to \infty} \lambda_c(m) = \frac{\sum_{k=1}^{\infty} kP_k}{\sum_{k=1}^{\infty} k^2 P_k} \]
for \( \alpha \to 0, \beta \to 0 \) and \( \gamma = 1 \). The result is the same as in Pastor-Satorras and Vespignani (2002).

**Remark 3.** Since the critical value \( \lambda_c(m) \) depends on \( \alpha, \beta, \gamma \) and \( \theta \), we can define the critical surface \( \Gamma_c \) as follows:
\[ \Gamma_c = \left\{ (\alpha, \beta, \gamma, \theta, \lambda) : \lambda \sum_{k=1}^{\infty} k(m \wedge k)P_k - \Lambda_m \sum_{k=1}^{\infty} (m \wedge k)P_k = 0, \Lambda_m > 0 \right\} \]
Let
\[ \Gamma^+ = \left\{ (\alpha, \beta, \gamma, \theta, \lambda) : \lambda \sum_{k=1}^{\infty} k(m \wedge k)P_k - \Lambda_m \sum_{k=1}^{\infty} (m \wedge k)P_k > 0, \Lambda_m > 0 \right\} \]
and
\[ \Gamma^- = \left\{ (\alpha, \beta, \gamma, \theta, \lambda) : \lambda \sum_{k=1}^{\infty} k(m \wedge k)P_k - \Lambda_m \sum_{k=1}^{\infty} (m \wedge k)P_k < 0, \Lambda_m > 0 \right\} \]
Thus, the infection spreads and becomes endemic for \( (\alpha, \beta, \gamma, \theta, \lambda) \in \Gamma^+ \), and the infection dies out finally when \( (\alpha, \beta, \gamma, \theta, \lambda) \in \Gamma^- \).

It follows from (15) that
\[ \lim_{m \to \infty} \sum_{k=1}^{\infty} k^2 P_k \sim C \sum_{k=1}^{\infty} k^2 k^{-3-\theta} = \infty \]
for \(-1 < \theta \leq 0\). Moreover, it can be checked that \( \max_{m \geq 1} \sigma_m < \infty \) and \( \delta_m \leq 1 + \theta \) for all \( m \geq 1 \). Thus, we have the following corollary.

**Corollary 7.** If \(-1 < \theta \leq 0\) and \( \lim_{m \to \infty} \rho_m(\alpha, \beta, \gamma, \theta, \lambda) = 0 \) for all \( m \geq 1 \), then
\[ \lim_{m \to \infty} \lambda_c(m) = \lambda_c = 0 \]
for any fixed \( \alpha, \beta \) and \( \gamma \).
That is to say, the infection can spread and become endemic on the scale-free network with the power \( \tau = 3 + \theta, 2 < \tau \leq 3 \), as long as there is a small rate of the epidemic spreading when the maximum degree \( m \) is large. This result was found first by Pastor-Satorras and Vespignani (2001).

Now we assume that \( \theta > 0 \) and \( \alpha \beta > 0 \). From (15) it follows that

\[
\lim_{m \to \infty} \sum_{k=1}^{\infty} k W_m(k) P_k \sim C \sum_{k=1}^{\infty} k^2 k^{-3-\nu} < \infty.
\]

for \( \theta > 0 \). Let \( \nu_m = (\alpha \mu_m + \gamma)/\beta \) and \( \nu = (\alpha (2 + \theta) + \gamma)/\beta \). Since

\[
\frac{d_k}{f_k} \sim \left\{ \begin{array}{ll}
1 - \frac{1+\nu_m}{k+\theta} & \text{if } k + \theta < m \\
1 - \frac{\nu_m}{m} & \text{if } k + \theta \leq m
\end{array} \right.
\]

for large \( m \) and \( \nu_m \to \nu \) as \( m \to \infty \), we have

\[
q_m(k) \sim \left\{ \begin{array}{ll}
\frac{A_k(\nu)}{(k+\theta)^{1+\nu}} & \text{if } k \leq m \\
A_k(\nu)(1-\frac{\nu}{\nu})^{k-m} + \frac{P_{k-1}}{\beta} \sum_{j=m+1}^{k} P_{j} (1-\frac{\nu}{m})^{k-j} & \text{if } k > m
\end{array} \right.
\]

for large \( k \), where

\[
A_k(\nu) = \left\{ \begin{array}{ll}
\frac{A(\nu)}{\alpha(2+\theta)f_1} + \frac{1}{\beta(1+\theta-\nu)} & \text{if } \nu < 1 + \theta \\
\frac{1}{\alpha(2+\theta)f_1} + \frac{\ln k}{\beta} & \text{if } \nu = 1 + \theta \\
\frac{1}{\alpha(2+\theta)f_1} + \frac{k^{\nu-1-\theta}}{\beta(\nu-1-\theta)} & \text{if } \nu > 1 + \theta.
\end{array} \right.
\]

Moreover, by (15) and (43) we have

\[
\sum_{k=1}^{m} k q_m(k) \sim \left\{ \begin{array}{ll}
\frac{A(\nu) m^{1-\nu}}{1-\nu} & \text{if } \nu < 1 \\
A(\nu) \ln m & \text{if } \nu = 1 \\
\frac{1}{\alpha(2+\theta)f_1} + \frac{1}{\beta \theta^{2}} & \text{if } \nu = 1 + \theta \\
\frac{1}{\alpha f_1 (2+\theta)(\nu-1)} + \frac{1}{\beta(\nu-1-\theta) \theta} & \text{if } \nu > 1 + \theta
\end{array} \right.
\]

and

\[
\sum_{k=m+1}^{\infty} m q_m(k) \sim \left\{ \begin{array}{ll}
\frac{A(\nu) m^{1-\nu}}{\nu} & \text{if } \nu < 1 + \theta \\
\frac{A_m(\nu)}{\nu m^{\nu-1}} & \text{if } \nu \geq 1 + \theta
\end{array} \right.
\]

and

\[
\sum_{k=m+1}^{\infty} (k-m) q_m(k) \sim \left\{ \begin{array}{ll}
\frac{A(\nu) m^{1-\nu}}{\nu^2} & \text{if } \nu < 1 + \theta \\
\frac{A_m(\nu)}{\nu^2 m^{\nu-1}} + \frac{2}{\beta (2+\theta)^2 m^{1+\theta}} & \text{if } \nu \geq 1 + \theta
\end{array} \right.
\]
for large $m$. Thus, $\lim_{m \to \infty} \delta_m = 1$, $\lim_{m \to \infty} \sigma_m = \nu^{-1} - 1$ for $\nu < 1$ and $\lim_{m \to \infty} \sigma_m = 0$ for $\nu \geq 1$, and therefore,

$$\lim_{m \to \infty} \Lambda_m = \begin{cases} 0 & \text{if } \nu \leq 1 \\ \alpha(2 + \theta) + \gamma - \beta & \text{if } \nu > 1. \end{cases}$$

Now we can obtain the following corollary.

**Corollary 8.** If $\theta > 0$, $\alpha \beta > 0$ and $\lim_{\lambda \to \lambda_c(m)} \rho_m(\alpha, \beta, \gamma, \theta, \lambda) = 0$ for all $m \geq 1$, then

$$\lim_{m \to \infty} \lambda_c(m) = \begin{cases} 0 & \text{if } \beta \geq \alpha(2 + \theta) + \gamma \\ \alpha(2 + \theta) + \gamma - \beta & \text{if } \beta < \alpha(2 + \theta) + \gamma. \end{cases}$$

Note that $\sum_{k=1}^{\infty} kP_k = 2$. Let $\lambda_c = \lim_{m \to \infty} \lambda_c(m)$,

$A = -2(2 + \theta), \quad B = 2, \quad C = -2, \quad D = \sum_{k=1}^{\infty} k^2 P_k.$

Then, (38) can be written as $A\alpha + B\beta + C\gamma + D\lambda_c = 0$ as $m \to \infty$ for $\theta > 0$.

**Remark 4.** For a fixed $\theta > 0$, we can define the critical hyperplane $\Gamma_c$ as follows

$$\Gamma_c = \{ (\alpha, \beta, \gamma, \lambda) : A\alpha + B\beta + C\gamma + D\lambda = 0, \alpha(2 + \theta) + \gamma > \beta, \alpha \beta > 0, \gamma, \lambda \geq 0 \}.$$ 

Let

$$\Gamma^+ = \{ (\alpha, \beta, \gamma, \lambda) : A\alpha + B\beta + C\gamma + D\lambda > 0, \alpha(2 + \theta) + \gamma > \beta, \alpha \beta > 0, \gamma, \lambda \geq 0 \},$$

$$\Gamma^- = \{ (\alpha, \beta, \gamma, \lambda) : A\alpha + B\beta + C\gamma + D\lambda < 0, \alpha(2 + \theta) + \gamma > \beta, \alpha \beta > 0, \gamma, \lambda \geq 0 \}.$$ 

Thus, the infection spreads and becomes endemic for $(\alpha, \beta, \gamma, \lambda) \in \Gamma^+$, and the infection dies out finally for $(\alpha, \beta, \gamma, \lambda) \in \Gamma^-$ when $m \to \infty$.

**References**


