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Abstract. Functions preserving Loewner positivity when applied entrywise to positive semidefinite matrices have been widely studied in the literature. Following the work of Schoenberg [Duke Math. J. 9], Rudin [Duke Math. J. 26], and others, it is well-known that functions preserving positivity for matrices of all dimensions are absolutely monotonic (i.e., analytic with nonnegative Taylor coefficients). In this paper, we study functions preserving positivity when applied entrywise to sparse matrices, with zeros encoded by a graph $G$ or a family of graphs $G_n$. Our results generalize Schoenberg and Rudin’s results to a modern setting, where functions are often applied entrywise to sparse matrices in order to improve their properties (e.g. better conditioning, graphical models). The only such result known in the literature is for the complete graph $K_2$. We provide the first such characterization result for a large family of non-complete graphs. Specifically, we characterize functions preserving Loewner positivity on matrices with zeros according to a tree. These functions are multiplicatively midpoint-convex and super-additive. Leveraging the underlying sparsity in matrices thus admits the use of functions which are not necessarily analytic nor absolutely monotonic. We further show that analytic functions preserving positivity on matrices with zeros according to trees, can contain arbitrarily long sequences of negative coefficients, thus obviating the need for absolute monotonicity in a very strong sense. This result leads to the question of exactly when absolute monotonicity is necessary when preserving positivity for an arbitrary class of graphs. We then provide a stronger condition in terms of the numerical range of all symmetric matrices, such that functions satisfying this condition on matrices with zeros according to any family of graphs with unbounded degrees, are necessarily absolutely monotonic.

1. Introduction and main results

Functions preserving Loewner positivity when applied entrywise to positive semidefinite matrices have been well-studied in the literature (see e.g. Schoenberg [24], Rudin [23], Herz [15], Horn [18], Christensen and Ressel [5], Vasudeva [25], FitzGerald et al [7]). An important characterization of functions $f : (-1,1) \to \mathbb{R}$ such that $f[A] := (f(a_{ij}))$ is positive semidefinite for all positive semidefinite matrix $A = (a_{ij})$ of all dimensions $n$ with entries in $(-1,1)$ has been obtained by Schoenberg and Rudin ([24], [23]). Their results show that such functions are absolutely monotonic (i.e., analytic with nonnegative Taylor coefficients).

In modern applications, functions are often applied entrywise to positive semidefinite matrices (e.g. covariance/correlation matrices) in order to improve their properties such as better conditioning or to induce a Markov random field structure (see [13], [12]). Understanding if and how positivity is preserved is critical for these procedures to be widely applicable. In such settings, various distinguished submanifolds of the cone of positive semidefinite matrices are of particular interest. Two important cases naturally arising in modern applications involve (1) constraining the...
rank, and (2) constraining the sparsity of correlation matrices. The rank of a sample correlation matrix corresponds to the sample size of the population used to estimate it. It is thus natural to ask which functions preserve Loewner positivity when applied entrywise to positive semidefinite matrices of a given rank. This analysis was carried out in [10]. There it was shown that functions preserving positivity when applied entrywise to matrices of rank 1 or 2 are automatically absolutely monotonic. Thus, preserving positivity for small subsets of the cone of positive semidefinite matrices immediately forces the function to be absolutely monotonic. The converse of this result (i.e., that every absolutely monotonic function preserves Loewner positivity) follows immediately from the Schur product theorem.

In this paper, we study the second important problem: preserving positivity when sparsity constraints are imposed. The sparsity pattern of a matrix $A = (a_{ij})$ is naturally encoded by a graph $G = (V, E)$ where $V = \{1, \ldots, n\}$ and $(i, j) \notin E$ if $a_{ij} = 0$. Thus, our goal is to study functions preserving positivity when applied entrywise to positive semidefinite matrices with zeros according to a fixed graph $G$, or a family of graphs $(G_n)_{n \geq 1}$. In particular, when $G_n = K_n$ (the complete graph on $n$ vertices) for all $n$, the problem reduces to the classical problem studied by Schoenberg, Rudin, and others.

Positive semidefinite matrices with zeros according to graphs arise naturally in many applications. For example, in the theory of Markov random fields in probability theory ([19], [26]), the nodes of a graph $G$ represent components of a random vector, and edges represent the dependency structure between nodes. Thus, absence of an edge implies marginal or conditional independence between the corresponding random variables, and leads to zeros in the associated covariance or correlation matrix (or its inverse). Such models therefore yield parsimonious representations of dependency structures. Characterizing entrywise functions preserving Loewner positivity for matrices with zeros according to a graph is thus of tremendous interest for modern applications. Obtaining such characterizations is, however, much more involved than the original problem considered by Schoenberg and Rudin, as one has to enforce and maintain the sparsity constraint. The problem of characterizing functions preserving positivity for sparse matrices is also intimately linked to problems in spectral graph theory and many other problems (see e.g. [17], [18], [22], [4]).

We now state the main results in this paper. To do so, we first introduce some notation. Let $G = (V, E)$ be a graph with vertex set $V = \{1, \ldots, n\}$. Denote by $|G| := |V|$ and by $\Delta(G)$ the maximum degree of the vertices of $G$. Given a subset $I \subseteq \mathbb{R}$, let $S_n(I)$ denote the space of $n \times n$ symmetric matrices with entries in $I$, and $P_n(I)$ be the cone of real positive semidefinite matrices with entries in $I$. Define $S_G(I)$ and $P_G(I)$ to be the respective subsets of matrices with zeros according to $G$:

$$S_G(I) := \{ A \in S_{|G|}(I) : a_{ij} = 0 \text{ for every } (i, j) \notin E, i \neq j \}, \quad P_G(I) := P_{|G|}(I) \cap S_G(I).$$

We denote $S_n(\mathbb{R})$ and $P_n(\mathbb{R})$ respectively by $S_n$ and $P_n$ for convenience. Given a function $f : \mathbb{R} \to \mathbb{R}$ and $A \in S_{|G|}(\mathbb{R})$, denote by $f_G[A]$ the matrix

$$f_G[A]_{ij} := \begin{cases} f(a_{ij}) & \text{if } (i, j) \in E \text{ or } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

In the case where $G = K_n$, the complete graph on $n$ vertices, we denote $f_{K_n}[A]$ by $f[A]$. Schoenberg and Rudin’s result can now be rephrased by saying that $f_{K_n}[A] \in P_{K_n}(\mathbb{R})$ for all $n \geq 1$ and all $A \in P_{K_n}(-1, 1)$ if and only if $f$ has a power series representation with nonnegative coefficients.

In this paper, we generalize Schoenberg and Rudin’s result by considering functions $f$ mapping $P_G$ into itself for other important families of graphs. As we show, this problem is much more involved for non-complete graphs than the special case considered by Schoenberg, Rudin, and others. In fact, such characterization results are only known for (a) the family of all complete graphs $K_n$ — by the work of Schoenberg and Rudin; see Theorem 2.3 and (b) the single graph $K_2$ — by the work of Vasudeva [25, Theorem 2] — see Theorem 2.6. However, to our knowledge, no other characterization
Theorem A. Suppose \( I = [0, R) \) for some \( 0 < R \leq \infty \), and \( f : I \to [0, \infty) \). Let \( G \) be a tree with at least 3 vertices, and let \( A_3 \) denote the path graph on 3 vertices. Then the following are equivalent:

1. \( f_G[A] \in \mathbb{P}_G \) for every \( A \in \mathbb{P}_G(I) \);
2. \( f_T[A] \in \mathbb{P}_T \) for all trees \( T \) and all matrices \( A \in \mathbb{P}_T(I) \);
3. \( f_{A_3}[A] \in \mathbb{P}_{A_3} \) for every \( A \in \mathbb{P}_{A_3}(I) \);
4. The function \( f \) satisfies:

\[
(1.3) \quad f(\sqrt{xy})^2 \leq f(x)f(y), \quad \forall x, y \in I
\]
and is superadditive on \( I \), i.e.,

\[
(1.4) \quad f(x + y) \geq f(x) + f(y), \quad \forall x, y, x + y \in I.
\]

Note that some sources refer to (1.3) as mid(point)-convexity for the function \( x \mapsto \log f(e^x) \), albeit on an interval different from \((0, R)\). Thus functions preserving positivity for trees coincide with the class of midpoint convex superadditive functions.

Recall that previous results by Schoenberg and Rudin show that entrywise functions preserving positivity for all matrices (i.e., according to the family of complete graphs \( K_n \) for \( n \geq 1 \)) are necessarily absolutely monotonic on the positive axis. It is not clear if functions satisfying (1.3) and (1.4) in Theorem A are necessarily absolutely monotonic, or even analytic. We show below in Proposition 4.2 that such functions need not be analytic. Our second main result demonstrates that even if the function is analytic, it can in fact have arbitrarily long strings of negative Taylor coefficients.

Theorem B. There exists a function \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) analytic on \( \mathbb{C} \) such that

1. \( a_n \in [-1, 1] \) for every \( n \geq 0 \);
2. The sequence \( (a_n)_{n \geq 0} \) contains arbitrarily long strings of negative numbers;
3. For every tree \( G \), \( f_G[A] \in \mathbb{P}_G \) for every \( A \in \mathbb{P}_G([0, \infty)) \).

In particular, if \( \Delta(G) \) denotes the maximum degree of the vertices of \( G \), then there exists a family \( G_n \) of graphs and an analytic function \( f \) that is not absolutely monotonic, such that:

1. \( \sup_{n \geq 1} \Delta(G_n) = \infty \);
2. \( f_{G_n}[A] \in \mathbb{P}_{G_n} \) for every \( A \in \mathbb{P}_{G_n}([0, \infty)) \).

As we will show, it is even possible to choose \( f \) to be a real polynomial of degree \( n \geq 6 \) preserving \( \mathbb{P}_G \) for all trees \( G \), and with up to \( n - 5 \) negative coefficients.

Theorem B demonstrates that functions preserving positivity for a general family of graphs \( G_n \) with unbounded degree are not necessarily absolutely monotonic. It is natural to seek minimal additional restrictions on a family of graphs \( \{G_n\}_{n \geq 1} \) and a function \( f \) mapping \( \mathbb{P}_{G_n} \) into itself for all \( n \geq 1 \), in order to conclude that \( f \) is analytic and absolutely monotonic on \([0, \infty)\). Our last main result provides such a sufficient condition.

Theorem C. Let \( \{G_n\}_{n \geq 1} \) be a family of graphs such that

\[
\sup_{n \geq 1} \Delta(G_n) = \infty.
\]

Let \( I := [0, R) \) for some \( 0 < R \leq \infty \) and let \( f : I \to \mathbb{R} \) be a function such that for every \( n \geq 1 \), \( \beta^T f_{G_n}[M] \beta \geq 0 \) for every symmetric matrix \( M \in \mathbb{S}_{G_n}(I) \), and every \( \beta \in \mathbb{R}^{|G_n|} \) such that \( \beta^T M \beta \geq 0 \). Then \( f \) is analytic and absolutely monotonic on \( I \).

In other words, if one wants to preserve a weaker form of positivity as given in Theorem C and simultaneously to be able to use functions that are not absolutely monotonic, then the sequence of graphs \( \{G_n\}_{n \geq 1} \) has to be of bounded degree. Thus this notion of preserving positivity necessitates a specific form of sparsity in terms of the degrees of the associated nodes.
Remark 1.1. Recall that the numerical range of a $n \times n$ matrix $A$ is given by
\[ W(A) := \{ \beta^* A \beta : \beta \in \mathbb{C}^n, \beta^* \beta = 1 \}, \]
where $\beta^*$ denotes the conjugate transpose of $\beta$. When $A$ is Hermitian, it is clear that $W(A) \subset \mathbb{R}$. Moreover, $A$ is positive semidefinite if and only if $W(A) \subset [0, \infty)$, i.e., $W(A) = W(A)_+$ where $W(A)_+ := W(A) \cap [0, \infty)$. Thus $f$ preserves positivity on $\mathbb{P}_n(\mathbb{R})$ if and only if $W(f(A)) \subset [0, \infty)$ for all matrices $A \in \mathbb{S}_n(\mathbb{R})$ such that $W(A) = W(A)_+$. In Theorem C, this condition is strengthened in the hypothesis by considering the effect of $f$ on the positive part of the numerical range of all matrices $A \in \mathbb{S}_n(\mathbb{R})$.

The remainder of the paper is structured as follows. Section 2 reviews many important characterizations of functions preserving positivity in various settings. In Section 3 we study the properties of positive semidefinite matrices with zeros according to a tree, and prove Theorem A. As an application of Theorem A in Section 4, we show that $x \mapsto x^\alpha$ preserves $\mathbb{P}_G$ for any tree $G$ if and only if $\alpha \geq 1$. Thus the phase transition, or critical exponent for preserving positivity on $\mathbb{P}_G$ occurs at $\alpha = 1$ (see e.g. [6, 3, 16, 8, 11, 9] for more details about critical exponents). We then prove Theorem B by showing that there exist polynomials and more general analytic functions with large numbers of negative coefficients, which preserve $\mathbb{P}_G$ for every tree $G$. This provides a negative answer to a natural generalization of Schoenberg and Rudin’s results when the problem of preserving positivity is restricted to sparse positive semidefinite matrices. Finally in Section 5, we present natural stronger conditions for preserving positivity, such that the functions satisfying them are necessarily absolutely monotonic.

Notation: In this paper, all graphs $G = (V, E)$ are finite, undirected, with no self-loops. We denote by $|G|$ the cardinality of $V$. We let $K_n$ and $A_n$ denote the complete graph and the path graph on $n$ vertices respectively. The $n \times n$ identity matrix is denoted by $\text{Id}_n$. We denote by $0_{m \times n}$ and $1_{m \times n}$ the $m \times n$ matrices with all entries equal to 0 and 1 respectively.

2. Literature review

Characterizing functions which preserve some form of positivity of matrices has been studied by many authors in the literature including Schoenberg, Rudin, Herz, Horn, Vasudeva, Christensen and Ressel, FitzGerald, Micchelli, and Pinkus, and more recently, Hansen, Hiai, Bharali and Holtz, as well as the authors. The notion of absolute monotonicity is crucial in many of these results. We begin by reviewing important properties of these functions.

Definition 2.1. Let $I \subset \mathbb{R}$ be an interval with interior $I^\circ$. A function $f \in C(I)$ is said to be absolutely monotonic on $I$ if it is in $C^\infty(I^\circ)$ and $f^{(k)}(x) \geq 0$ for every $x \in I^\circ$ and every $k \geq 0$.

It is not immediate that if $f$ is absolutely monotonic on $[0, \infty)$, then $f$ is entire - however, the following result shows that this is indeed true. Recall that the $n$-th forward difference of a function $f$, with step $h > 0$ at the point $x$, is given by
\[ \Delta_h^n[f](x) := \sum_{i=0}^n (-1)^i \binom{n}{i} f(x + (n-i)h). \]

Theorem 2.2 (see [27] Chapter IV, Theorem 7]). Let $0 < R \leq \infty$ and let $f : [0, R) \to \mathbb{R}$. Then the following are equivalent:

1. $f$ is absolutely monotonic on $[0, R)$.
2. $f$ can be extended analytically to the complex disc $D(0, R) := \{ z \in \mathbb{C} : |z| < R \}$, and $f(z) = \sum_{n=0}^\infty a_n z^n$ on $D(0, R)$, for some $a_n \geq 0$.
3. For every $n \geq 1$, $\Delta_h^n[f](x) \geq 0$ for all non-negative integers $n$ and for all $x$ and $h$ such that $0 \leq x < x + h < \cdots < x + nh < R$. 
One of the main results in the literature on preserving positive semidefiniteness was proved under various restrictions by multiple authors. We only write down the most general version here.

**Theorem 2.3** (see Schoenberg [24], Rudin [23], Vasudeva [25], Herz [15], Horn [18], Christensen and Ressel [5], FitzGerald et al. [7], Hiai [16]). Suppose \( 0 < R \leq \infty \), and \( f : (-R, R) \to \mathbb{R} \). Set \( I := (-R, R) \). Then the following are equivalent:

1. For all \( n \geq 1 \) and \( A \in \mathbb{P}_n(I) \), \( f[A] \in \mathbb{P}_n \).
2. \( f \) is analytic on the complex disc \( D(0, R) \) and absolutely monotonic on \( (0, R) \). Equivalently, \( f \) admits a power series representation \( f(x) = \sum_{n=0}^{\infty} a_n x^n \) on \( (-R, R) \) for some coefficients \( a_n \geq 0 \).

The statement of Theorem 2.3 for \( R = \infty \) is very similar to earlier results by Vasudeva [25], which were extended in previous work [10]. Once again we write down the most general version here.

**Theorem 2.4** (Vasudeva [25]; Guillot, Khare, and Rajaratnam [10]). Let \( 0 \leq a < b \leq \infty \). Assume \( I = (a, b) \) or \( I = [a, b) \) and let \( f : I \to \mathbb{R} \). Then each of the following assertions implies the next:

1. The function \( f \) can be extended analytically to \( D(0, b) \) and \( f(z) = \sum_{n=0}^{\infty} c_n z^n \) on \( D(0, b) \), for some \( c_n \geq 0 \);
2. For all \( n \geq 1 \) and \( A \in \mathbb{P}_n(I) \), \( f[A] \in \mathbb{P}_n \);
3. \( f \) is absolutely monotonic on \( I \).

If furthermore, \( 0 \in I \), then (3) \( \Rightarrow \) (1) and so all the assertions are equivalent.

Note that in all the previous results, the dimension \( n \) is allowed to grow to infinity. When the dimension is fixed, the problem is much more involved and very few results are known. The following necessary condition was shown by Horn [18] (and attributed to Loewner).

**Theorem 2.5** (Horn [18]). Suppose \( f : (0, \infty) \to \mathbb{R} \) is continuous. Fix \( 2 \leq n \in \mathbb{N} \) and suppose that \( f[A] \in \mathbb{P}_n \) for all \( A \in \mathbb{P}_n((0, \infty)) \). Then \( f \in C^{n-3}((0, \infty)) \),

\[
f^{(k)}(x) \geq 0, \quad \forall x > 0, \ 0 \leq k \leq n - 3,
\]

and \( f^{(n-3)} \) is a convex non-decreasing function on \( (0, \infty) \). In particular, if \( f \in C^{n-1}((0, \infty)) \), then \( f^{(k)}(x) \geq 0 \) for all \( x > 0 \) and \( 0 \leq k \leq n - 1 \).

Note that preserving positivity on only a small subset of the matrices in \( \mathbb{P}_n \) (for fixed \( n \)) guarantees that \( f \) is highly differentiable on \( I \) with nonnegative derivatives. Moreover, applying Theorem 2.5 for all \( n \in \mathbb{N} \) easily yields Theorem 2.4 for \( I = (0, \infty) \) as a special case. When \( n = 2 \), the following characterization of entrywise functions preserving positivity on \( \mathbb{P}_2((0, \infty)) \) was shown by Vasudeva [23, Theorem 2]. To the authors’ knowledge, no characterization is known when \( n > 2 \).

**Theorem 2.6** (Vasudeva [23]; Guillot, Khare, and Rajaratnam [10]). Let \( I \subset \mathbb{R} \) be an interval such that \( \inf I \leq \sup I > 0 \), \( I \cap (0, \infty) \) is open, and let \( f : I \to \mathbb{R} \). Then the following are equivalent:

1. \( f[A] \in \mathbb{P}_2 \) for every \( 2 \times 2 \) matrix \( A \in \mathbb{P}_2(I) \).
2. \( f \) satisfies: \( f(\sqrt{xy})^2 \leq f(x)f(y) \) for all \( x, y \in I \cap [0, \infty) \), and \( |f(x)| \leq f(y) \) whenever \( |x| \leq y \in I \).

In particular, if (1) holds, then either \( f \equiv 0 \) on \( I \setminus \{ \pm \sup I \} \), or \( f(x) > 0 \) for all \( x \in I \cap (0, \infty) \). Moreover \( f \) is continuous on \( (0, \infty) \cap I \).

**Remark 2.7.** If \( G \) is a graph with at least one edge and \( f_G[-] \) preserves \( \mathbb{P}_G([0, R]) \), then \( f_{K_2}[-] \) preserves \( \mathbb{P}_2([0, R]) \) by considering matrices of the form \( A \oplus \text{Id}_{n-2} \). Hence all of the assertions in Theorem 2.6 hold when \( I = [0, R) \) and \( G \) is nonempty.
Recall that in applications, functions are often applied entrywise to covariance/correlation matrices to improve properties such as their condition number (see e.g. [13, 12]). In that setting, the rank of a sample correlation matrix corresponds to the sample size of the population used to estimate the matrix. With this application in mind, the following characterization in fixed dimension was obtained in [10] under additional rank constraints. Define \( S_n^k(I) := \{ A \in S_n(I) : \text{rank } A \leq k \} \) and \( P_n^k(I) := \{ A \in P_n(I) : \text{rank } A \leq k \} \).

**Theorem 2.8** (Guillot, Khare, and Rajaratnam, [10] Theorem B). Let \( 0 < R \leq \infty \) and \( I = [0, R) \) or \((-R, R)\). Fix integers \( n \geq 2, 1 \leq k < n - 1, \) and \( 2 \leq l \leq n \). Suppose \( f \in C^k(I) \). Then the following are equivalent:

1. \( f[A] \in S_n^k \) for all \( A \in P_n^l(I) \);
2. \( f(x) = \sum_{i=1}^r a_t x_i^t \) for some \( a_t \in \mathbb{R} \) and some \( i_t \in \mathbb{N} \) such that

\[
\sum_{t=1}^r \left( i_t + l - 1 \right) \leq k.
\]

Similarly, \( f[-] : P_n^l(I) \to P_n^k \) if and only if \( f \) satisfies (2) and \( a_t \geq 0 \) for all \( i_t \). Moreover, if \( I = [0, R) \) and \( k \leq n - 3 \), then the assumption that \( f \in C^k(I) \) is not required.

Many other interesting characterizations have also been obtained in other settings. In [2], Bharali and Holtz characterize entire functions \( f \) that \( f(A) \) is entrywise nonnegative for every entrywise nonnegative and triangular matrix \( A \) (here \( f(A) \) is computed using the functional calculus). In [13], Guillot and Rajaratnam generalize the classical results of Schoenberg and Rudin to the case where the function is only applied to the off-diagonal elements of matrices (as is often the case in applications when regularizing positive semidefinite matrices). Hansen [14] and Micchelli and Willoughby [20] also characterize functions preserving entrywise nonnegativity when applied to symmetric matrices using the functional calculus.

### 3. Characterizing functions preserving positivity for trees

In this section we examine the effect the degree of a graph \( G \) plays in characterizing functions preserving positivity on \( P_G \) when applied entrywise. The simplest graph with a vertex of a given degree is a star graph. Thus we begin by studying functions preserving positivity on \( P_G \) for star graphs \( G \), and more generally, for \( G \) a tree.

#### 3.1. Positive semidefinite matrices on star graphs

Recall that a star graph has \( d + 1 \) vertices for some \( d \geq 0 \), \( d \) edges, and a unique vertex of degree \( d \). The following result characterizes positive semidefinite matrices with zeros according to a star. Note that every nonempty graph contains a star subgraph, so the result yields useful information about \( P_G \) for all nonempty \( G \), and will be crucial in proving Theorem A.

**Proposition 3.1.** Suppose \( d \geq 0 \) and

\[
A = \begin{pmatrix}
p_1 & \alpha_2 & \cdots & \alpha_{d+1} \\
\alpha_2 & p_2 & & 0 \\
\vdots & & \ddots & \\
\alpha_{d+1} & 0 & & p_{d+1}
\end{pmatrix}
\]

is a real-valued symmetric matrix with zeros according to a star graph. Then \( A \) is positive semidefinite if and only if the following three conditions hold:

1. \( p_i \geq 0 \) for all \( 1 \leq i \leq d + 1 \);
2. for all \( 2 \leq i \leq d + 1, p_i = 0 \implies \alpha_i = 0 \);
Corollary 3.2. If $A$ is as in (3.1), then

$$\det A = \prod_{i=1}^{d+1} p_i - \sum_{i>1} \alpha_i^2 \prod_{j=2}^{d+1} p_j.$$
In particular, if \( p_2 = p_3 = \cdots = p_{d+1} \), then the eigenvalues of \( A \) are \( p_2 \) with multiplicity \( d-1 \), and the following two eigenvalues with multiplicity one each (or multiplicity two if they are equal):

\[
\frac{p_1 + p_2 \pm \sqrt{(p_1 - p_2)^2 + 4\sum_{i=2}^{d+1} \alpha_i^2}}{2}
\]

**Proof.** It is clear that if \( p_2, \ldots, p_{d+1} > 0 \) and \( \alpha_1 > 0 \), then \( \det A = \det L_1 L_1^T = (\det L_1)^2 \) by Proposition 3.1, where \( L_1 \) was defined in Equation (3.2). Note that \( (\det L_1)^2 \) is precisely the claimed expression (3.3). Now the determinant is a polynomial in the \( 2d+1 \) entries \( p_1, p_i, \alpha_i \) (for \( 2 \leq i \leq d+1 \)), which equals the polynomial expression (3.3) for a Zariski dense subset of \( \mathbb{R}^{2d+1} \). Hence it equals the polynomial (3.3) at all points in \( \mathbb{R}^{2d+1} \). Finally, to determine the eigenvalues when \( p_2 = \cdots = p_{d+1} \), compute the characteristic polynomial \( \det(A - \lambda \text{Id}_n) \) using Equation (3.3), and solve for \( \lambda \).

Theorems 2.3 and 2.4 characterize functions mapping \( \mathbb{P}_n(I) \) into \( \mathbb{P}_{Kn} \) for every \( n \geq 1 \). Before proceeding to study the case of trees, it is natural to ask which functions \( f \) map \( \mathbb{P}_n \) into \( \mathbb{P}_G \) when \( G \) is a non-complete graph on \( n \) vertices. Proposition 3.3 below shows that such functions have to satisfy many restrictions. In particular, when \( I = (-R, R) \) for some \( R > 0 \), the only such function is \( f \equiv 0 \).

**Proposition 3.3.** Suppose \( 0 \in I \subset \mathbb{R} \) is an interval with \( \sup I \not\in I \) and \( |\inf I| \leq \sup I \). Let \( G \) be a graph and \( f : I \rightarrow \mathbb{R} \) such that \( f \not\equiv 0 \). Suppose \( f_G[-] \) sends all of \( \mathbb{P}_{|G|}(I) \) to \( \mathbb{P}_G \). Then every connected component of \( G \) is complete.

Note that the condition \( |\inf I| \leq \sup I \) is assumed in Theorem 2.6 because no \( 2 \times 2 \) matrix in \( \mathbb{P}_2(I) \) can have any entry in \( (-\infty, -\sup I) \).

**Proof.** Suppose \( f_G[-] \) sends all of \( \mathbb{P}_{|G|}(I) \) to \( \mathbb{P}_G \). Assume to the contrary that not every component of \( G \) is complete. Then, without loss of generality, \( (1,2), (1,3) \in E \) but \( (2,3) \not\in E \). Suppose \( a \in I \cap [0, \infty) \); since \( B := a1_{|G| \times |G|} \in \mathbb{P}_{|G|}(I) \), hence the principal \( 3 \times 3 \) submatrix of \( f_G[B] \) is in \( \mathbb{P}_3 \). But this is precisely the matrix \( f(a)B(1,1,1) \), where

\[
(3.4) \quad B(\mu, \alpha, \beta) := \begin{pmatrix}
\mu & \alpha & \beta \\
\alpha & \alpha & 0 \\
\beta & 0 & \beta
\end{pmatrix}, \quad \mu, \alpha, \beta \in \mathbb{R}.
\]

Thus, the diagonal entries and determinant of \( f(a)B(1,1,1) \) must be nonnegative; this yields \( f(a) \geq 0 \) and \( -f(a)^3 \geq 0 \). Therefore \( f(a) = 0 \) for every \( a \in I \cap [0, \infty) \). Now if \( a \in I \) is negative, apply \( f_G[-] \) to the matrix \( \begin{pmatrix} |a| & a & a \\ a & |a| & a \end{pmatrix} \oplus 0_{(|G|-2) \times (|G|-2)} \in \mathbb{P}_G(I) \), and consider the leading principal \( 2 \times 2 \) submatrix. Since \( f(|a|) = 0 \) from above, hence \( f(a) = 0 \) as well, which contradicts the assumption that \( f \not\equiv 0 \).

**Remark 3.4.** Applying Proposition 3.3 with \( f(x) \equiv x \) and any interval \( 0 \in I \subset \mathbb{R} \) shows that \( f_G[-] \) does not send all of \( \mathbb{P}_{|G|}(I) \) to \( \mathbb{P}_G \) if \( G \) is not a union of disconnected complete components. In other words, thresholding according to a non-complete connected graph, an important procedure in applications in high-dimensional probability and statistics, does not preserve positive definiteness (see [12, Theorem 3.1]).

### 3.2. Characterization for trees

We now use the analysis in Section 3.1 to show Theorem A. We first need the following preliminary result.

**Proposition 3.5.** Suppose \( 0 \in I \subset \mathbb{R} \) is an interval with \( \sup I \not\in I \) and \( |\inf I| \leq \sup I \). Let \( G \) be a non-complete connected graph and \( f : I \rightarrow \mathbb{R} \). If \( f_G[-] \) sends \( \mathbb{P}_G(I) \) to \( \mathbb{P}_G \), then \( f(0) = 0 \) and \( f \) is superadditive on \( I \cap (0, \infty) \).
Proof. Suppose without loss of generality that \( V(G) = \{1, 2, 3, \ldots, n\} \), \((1, 2), (1, 3) \in E(G)\), but \((2, 3) \notin E(G)\). Applying \( f_G[-] \) to the matrix \( 0_{[G] \times [G]} \) shows that \( f(0)B(1, 1, 1) \) (defined in Equation (3.4)) is indefinite semidefinite. This is only possible if \( f(0) = 0 \).

We now show that \( f(\alpha + \beta) \geq f(\alpha) + f(\beta) \) whenever \( \alpha, \beta, \alpha + \beta \in I \). This is clear if either \( \alpha \) or \( \beta \) is zero, since \( f(0) = 0 \); so we now assume that \( \alpha, \beta > 0 \). By Theorem 2.6, we may also assume that \( f(x) > 0 \) on \( I \cap (0, \infty) \). Then \( f_G[B(\alpha + \beta, \alpha, \beta) \oplus 0_{([G] - 3) \times ([G] - 3)}] \in \mathbb{P}_G \). Recall that \( f(\alpha), f(\beta), f(\alpha + \beta) > 0 \) by Theorem 2.6. Now applying Proposition 3.1 to the leading principal \( 3 \times 3 \) submatrix of \( f_G[B(\alpha + \beta, \alpha, \beta) \oplus 0_{([G] - 3) \times ([G] - 3)}] \), we obtain that \( f(\alpha + \beta) \geq f(\alpha) + f(\beta) \), which concludes the proof.

\[ \square \]

**Remark 3.6.** Proposition 3.5 shows that if \( f_G[-] \) maps \( \mathbb{P}_G \) into itself, then \( f(0) = 0 \); as a consequence, \( f_G[-] \) reduces to the standard entrywise function \( f[-] \).

We can now prove Theorem A.

**Proof of Theorem A.** Clearly (2) \( \Rightarrow \) (1) \( \Rightarrow \) (3). We now prove that (3) \( \Rightarrow \) (4) and (4) \( \Rightarrow \) (2).

(3) \( \Rightarrow \) (4). If \( f \equiv 0 \) on \( I \) then the result is obvious. Now assume \( f[A_3] \in \mathbb{P}_{A_3} \) for every \( A \in \mathbb{P}_{A_3}(I) \). In particular, \( f[A] \in \mathbb{P}_{K_2} \) for every \( A \in \mathbb{P}_{K_2}(I) \). Therefore, by Theorem 2.6 \( f \) satisfies (1.3) on \( I \). Now consider the matrix \( A = \) Equation (3.1) for \( d = 2 \). By Proposition 3.1 for \( 0 < p_1, \alpha_1 \in I \), we have \( A \in \mathbb{P}_{A_3}(I) \) if and only if \( p_1 \geq \alpha_2^2/p_2 + \alpha_3^2/p_3 \). Now suppose \( 0 < \alpha_2, \alpha_3, \alpha_2 + \alpha_3 \in I \); then \( f(\alpha_2), f(\alpha_3) > 0 \) by Theorem 2.6. Now \( B(\alpha_2 + \alpha_3, \alpha_2, \alpha_3) \) (defined in Equation (3.4)) lies in \( \mathbb{P}_{A_3}(I) \), so \( f[A_3][B(\alpha_2 + \alpha_3, \alpha_2, \alpha_3)] \in \mathbb{P}_{A_3} \). Thus, by Proposition 3.1

\[ f(p_1) = f(\alpha_2 + \alpha_3) \geq \frac{f(\alpha_2)^2}{f(p_2)} + \frac{f(\alpha_3)^2}{f(p_3)} = f(\alpha_2) + f(\alpha_3). \]

This proves \( f \) is superadditive. The case when \( \alpha_2 \) or \( \alpha_3 \) is zero follows from Proposition 3.5.

(4) \( \Rightarrow \) (2).

Once again, if \( f \equiv 0 \) on \( I \) then the result is immediate. Now suppose \( f \) is superadditive, not identically zero on \( I \), and satisfies (1.3) on \( I \). Let \( 0 \leq y < x \in I \). Then \( x - y \in (0, x) \subset I \), so by the superadditivity of \( f \),

\[ f(x) = f(y + x - y) \geq f(y) + f(x - y) \geq f(y). \]

Moreover, if \( 0 \in I \), then \( 0 \leq f(0) \geq f(0) + f(0) \) by superadditivity, so \( f(0) = 0 \). This shows that \( f \) is nonnegative and nondecreasing on \( I \). Hence by Theorem 2.6 \( f[A] \in \mathbb{P}_{K_2} \) for every \( A \in \mathbb{P}_{K_2}(0, \infty) \).

Now since \( f \not\equiv 0 \) on \( I \), hence \( f(p) > 0 \) for all \( 0 < p \in I \) by Theorem 2.6. Moreover, Equation (1.3) trivially holds if \( x \) or \( y \) is zero (and \( 0 \in I \)). Now assume that \( x, y > 0 \); then (1.3) can be restated as:

\[ p, \frac{\alpha_2^2}{p_2} \in I, \ p > 0 \quad \implies \quad f\left(\frac{\alpha_2^2}{p_2}\right) \geq f\left(\frac{\alpha_2^2}{p}\right). \]

We now prove that (2) holds for any tree \( T \) by induction on \(|T| \geq 3\). Suppose first that \( T \) is a tree with 3 vertices, i.e., \( T = A_3 \). Then, by Proposition 3.1 \( f[A_3] \in \mathbb{P}_{A_3} \) for every \( A \in \mathbb{P}_{A_3} \) if and only if

\[ f\left(\frac{\alpha_2^2}{p_2} + \alpha_3^2/p_3\right) \geq f(\alpha_2)^2/f(p_2) + f(\alpha_3)^2/f(p_3), \]

(or if one of \( p_2, p_3 \) is zero, in which case the assertion is easy to verify). Now suppose \( 0 < p_2, p_3 \in I \).

If \( A \in \mathbb{P}_{A_3}(I) \), then \( p_1 \in I \), so \( \frac{\alpha_2^2}{p_2} + \frac{\alpha_3^2}{p_3} \in [0, p_1] \) is also in \( I \). Hence (3.6) follows immediately by the superadditivity of \( f \) and by (3.5).

Therefore (4) \( \Rightarrow \) (2) holds for a tree with \( n = 3 \) vertices. Now assume that \( A \in \mathbb{P}_{T'}(I) \) implies \( f_{T'}[A] \in \mathbb{P}_{T'} \) for any tree \( T' \) with \( n \) vertices, and consider a tree \( T \) with \( n + 1 \) vertices. Let \( T \)
be a sub-tree obtained by removing a vertex connected to only one other node. Without loss of
generality, assume the vertex that is removed is labeled \( n + 1 \) and its neighbor is labeled \( n \). Let
\( A \in \mathbb{P}_T(I) \); then \( A \) has the form
\[
A = \begin{pmatrix}
\tilde{A}_{n \times n} & 0_{(n-1)\times 1} \\
0_{1 \times (n-1)} & a \\
\end{pmatrix}.
\]
If \( \alpha = 0 \) then \( a = 0 \) since \( A \) is positive semidefinite, and thus \( f_T[A] \in \mathbb{P}_G \) since \( f(0) = 0 \). When
\( \alpha \neq 0 \), the Schur complement \( S_A \) of \( \alpha \) in \( A \) is \( S_A = \tilde{A} - (a^2/\alpha)E_{n,n} \). Here, \( E_{i,j} \) denotes the
\( n \times n \) elementary matrix with the \((i,j)\) entry equal to 1, and every other entry equal to 0. Since
\( A \in \mathbb{P}_T(I) \), hence \( \tilde{A} \in \mathbb{P}_T(I) \), and \( S_A \in \mathbb{P}_T(I) \) from the above analysis (since \( (S_A)_{nn} = \tilde{a}_{nn} - a^2/\alpha \in [0, \tilde{a}_{nn}] \subset I \)). Therefore, by the induction hypothesis, \( f_T[A], f_T[S_A] \in \mathbb{P}_T \). Consider now the matrix
\( f_T[A] \). Using Schur complements, \( f_T[A] \in \mathbb{P}_T \) if and only if \( f_T[A] \in \mathbb{P}_T \) and the Schur complement
\( S_{f_T[A]} \) of \( f(\alpha) > 0 \) in \( f_T[A] \), given by
\[
S_{f_T[A]} = f_T[A] - \frac{f(a)^2}{f(\alpha)}E_{n,n},
\]
belongs to \( \mathbb{P}_T \). Now, notice that \( f_T[S_A] = f_T[A] + [f(b) - f(\tilde{a}_{nn})]E_{n,n} \), where \( b := (S_A)_{nn} = \tilde{a}_{nn} - a^2/\alpha \in I \) from the above analysis. Since \( f_T[S_A] \in \mathbb{P}_T \) from above, to conclude the proof, it
suffices to show that
\[
(3.7) \quad - \frac{f(a)^2}{f(\alpha)} \geq f(b) - f(\tilde{a}_{nn}).
\]
Indeed, by using the superadditivity of \( f \) and (3.5), we compute:
\[
f(\tilde{a}_{nn}) = f \left( \frac{a^2}{\alpha} + b \right) \geq f \left( \frac{a^2}{\alpha} \right) + f(b) \geq \frac{f(a)^2}{f(\alpha)} + f(b),
\]
which proves (3.7). Therefore (4) \( \Rightarrow \) (2) holds for a tree with \( n + 1 \) vertices. This completes the
induction and the proof of the theorem. \( \square \)

**Remark 3.7.** Hiai suggests in [16] Remark 3.4 that optimal conditions for \( f \) to preserve \( \mathbb{P}_3(-R, R) \)
for \( 0 < R \leq \infty \) could be that \( f \) is continuous on \(( -R, R) \). However, note from Theorem \( \overline{A} \) that
any such \( f \) for which \( f(0) = 0 \), also preserves \( \mathbb{P}_{A_3}(0, R) \), and hence is necessarily continuous,
nondecreasing, positive, super-additive, and satisfies (1.3) on \( (0, R) \). These conditions place severe
restrictions on the set of admissible \( f \) preserving \( \mathbb{P}_3(-R, R) \).

**Corollary 3.8.** Let \( I = [0, R) \) for some \( 0 < R \leq \infty \). Let \( f : I \to \mathbb{R} \) and assume \( f_G[A] \in \mathbb{P}_G \)
for every \( A \in \mathbb{P}_G(I) \) for some non-complete connected graph with at least 3 vertices. Then \( f \) is
superadditive and multiplicatively mid-point convex (see (1.3)).

**Proof.** The proof follows by noticing that \( G \) contains a copy of \( A_3 \) as an induced subgraph. \( \square \)

## 4. Fractional Hadamard powers and absolute monotonicity

Recall from Theorem \( \overline{A} \) that general functions preserving positivity on \( \mathbb{P}_G \) for a tree \( G \) are nec-
essarily multiplicatively mid-point convex and superadditive. We now explore a special sub-family
of these functions in greater detail: the power functions \( x^\alpha \). We do so for various reasons: first,
recall that by the Schur product theorem, every integer entrywise power of a positive semidefinite
matrix is positive semidefinite. Studying which powers \( \alpha > 0 \) preserve Loewner positivity on \( \mathbb{P}_G \)
for non-complete graphs \( G \) is a natural extension of this problem. Additionally, power functions are
natural to study since they are tractable as compared to more general families of functions. Finally,
there are also precedents in the literature for studying power functions preserving positivity; see
generally, Theorem B provides an example of a function not of the form (4.2) that maps into $P$.

**Theorem 4.1** (FitzGerald and Horn, [6, Theorem 2.2]). Suppose $A \in \mathbb{P}_n([0, \infty))$ for some $n \geq 2$, and $\alpha \geq n - 2$. Then $A^{\alpha} := ((a_{ij}^{\alpha}))_{i,j} \in \mathbb{P}_n$. If $\alpha \in (0, n - 2)$ is not an integer, then there exists $A \in \mathbb{P}_n([0, \infty))$ such that $A^{\alpha} \notin \mathbb{P}_n$.

A natural generalization of the aforementioned problem would be to characterize the powers preserving positivity for matrices with zeros according to a graph. Using Theorem [A] we now prove an analogue of Theorem [4.1] for $P_G$ when $G$ is a tree.

**Proposition 4.2.** Let $G$ be a tree with $n \geq 3$ vertices. Suppose $A \in \mathbb{P}_G([0, \infty))$. Then $A^{\alpha} := ((a_{ij}^{\alpha}))_{i,j} \in \mathbb{P}_G$ for every $\alpha \geq 1$. If $0 < \alpha < 1$ and $0 < R \leq \infty$, then there exists $A_R \in \mathbb{P}_G([0, R))$ such that $A_R^{\alpha} \notin \mathbb{P}_G$.

**Proof.** Say $f(x) := x^{\alpha}$. By Theorem [A], $f[-]$ preserves positivity on $\mathbb{P}_G([0, \infty))$ if and only if it preserves positivity on $\mathbb{P}_{A_1}([0, \infty))$, which by Proposition [3.1] holds if and only if for every $p_1, p_2, p_3 \geq 0$ and every $\alpha_2, \alpha_3 > 0$,

$$p_1 \geq \frac{\alpha_2^2}{p_2} + \frac{\alpha_3^2}{p_3} \implies f(p_1) \geq \frac{f(\alpha_2)^2}{f(p_2)} + \frac{f(\alpha_3)^2}{f(p_3)}.$$

Since $f$ is increasing on $(0, \infty)$, the previous condition is equivalent to

$$f \left( \frac{\alpha_2^2}{p_2} + \frac{\alpha_3^2}{p_3} \right) \geq \frac{f(\alpha_2)^2}{f(p_2)} + \frac{f(\alpha_3)^2}{f(p_3)},$$

which holds for the multiplicative function $f(x) = x^{\alpha}$, if and only if $\alpha \geq 1$. This proves the result when $\alpha \geq 1$, while for $\alpha < 1$, it implies that there exists $A \in \mathbb{P}_G([0, \infty))$ such that $A^{\alpha} \notin \mathbb{P}_G$. Rescaling $A$ by a small enough constant $c_R > 0$ such that $c_RA \in \mathbb{P}_G([0, R))$, we obtain the desired counterexample $A_R := c_RA \in \mathbb{P}_G([0, R))$. \qed

Recall from Section [2] that characterizing entrywise functions preserving positivity in a fixed dimension is a difficult problem. Theorem [4.1] provides a large family of functions mapping $\mathbb{P}_n([0, \infty))$ into itself, for any $n \geq 1$. Namely, given a nonnegative measure $\mu_n$ on $[n - 2, \infty)$, the function

$$(4.1) \quad f^{\mu_n}(x) := \sum_{i=1}^{n-3} a_i x^i + \int_{n-2}^{\infty} x^\alpha \, d\mu_n(\alpha), \quad x > 0,$$

preserves $\mathbb{P}_n([0, \infty))$ for all choices of nonnegative scalars $a_1, \ldots, a_{n-3}$ (see [6, Corollary 2.3]). In particular, if one imposes the condition that $f[-]$ preserves $\mathbb{P}_n([0, \infty))$ for all $n$ (or equivalently $\mathbb{P}_{K_n}([0, \infty))$ for all $n$), then the intersection of the above families over all $n > 2$ is precisely the set of absolutely monotonic functions; see Theorem [2.3]. Given the above observations, it is natural to ask if every function $f[-] : \mathbb{P}_n([0, \infty)) \to \mathbb{P}_n$ is necessarily of the form (4.1). Note that this is indeed the case if one imposes rank constraints on $f$; see Theorem [2.8].

Similarly, if $G$ is a tree with $n \geq 3$ vertices, Proposition [4.2] implies that for any nonnegative measure $\mu$ on $[1, \infty)$, functions of the form

$$(4.2) \quad f^{\mu}(x) := \int_{1}^{\infty} x^\alpha \, d\mu(\alpha), \quad x > 0,$$

map $\mathbb{P}_G([0, \infty))$ into itself. We ask if every function preserving $\mathbb{P}_G([0, \infty))$ has to be of this form. Theorem [13] provides a negative answer to these questions. First, note that entrywise functions mapping $\mathbb{P}_n$ into itself are not necessarily of the form (4.1) when $n = 2$ since by Theorem [13] there exists an analytic function $f$ with some negative coefficients, which maps $\mathbb{P}_2([0, \infty))$ into $\mathbb{P}_2$. More generally, Theorem [13] provides an example of a function not of the form (4.2) that map $\mathbb{P}_T([0, \infty))$ into $\mathbb{P}_T$ for all trees $T$. 

\[\text{\[3 [6 [11 [16].\] The following important result characterizes the powers preserving positivity for symmetric matrices with nonnegative entries.} \]
4.1. Proof of Theorem 4.3. We now proceed to prove the second main result of this paper. The proof requires constructing and working with multiplicatively convex polynomials with negative coefficients. We first collect together some basic properties of these functions.

**Definition 4.3.** Given an interval $I \subset [0, \infty)$, a function $f : I \to [0, \infty)$ is said to be multiplicatively convex if $f(x^{1-\lambda}y^{\lambda}) \leq f(x)^{1-\lambda}f(y)^{\lambda}$ for all $x, y \in I$ and $0 \leq \lambda \leq 1$. (Here we set $0^0 = 1$.)

Clearly, a function $f$ is multiplicatively convex if and only if log $f$ is a convex function of log $x$, i.e., the function $g(x) = \log f(e^x)$ is convex.

**Theorem 4.4** (Properties of multiplicatively convex functions, [21]). Let $I \subset [0, \infty)$ be an interval, and $f, g : I \to [0, \infty)$.

1. If $f, g$ are multiplicatively convex, then so are $f + g$, $fg$, $\alpha f$ for all $0 \leq \alpha \in \mathbb{R}$. In particular, every polynomial with nonnegative coefficients is multiplicatively convex.

2. $f[A]$ is positive semidefinite for every $A \in \mathbb{P}_2(I)$ of rank 1, if and only if
   \begin{equation}
   f(\sqrt{xy})^2 \leq f(x)f(y) \quad \forall x, y \in I. \tag{4.3}
   \end{equation}

3. $f[A]$ is positive semidefinite for every $A \in \mathbb{P}_2(I)$ if and only if $f$ satisfies (4.3) and is nondecreasing on $I$.

4. If $0 \not\in I$ and $f$ is continuous, then $f$ satisfies (4.3) if and only if $f$ is multiplicatively convex.

5. If $I$ is open and $f$ is twice differentiable on $I$, then $f$ is multiplicatively convex on $I$ if and only if
   \begin{equation}
   \Psi_f(x) := x \left[ f''(x)f(x) - (f'(x))^2 \right] + f(x)f'(x) \geq 0 \quad \forall x \in I. \tag{4.4}
   \end{equation}

These properties are all proved in [21]. The first part follows from Exercises 2.1.3, 2.1.4, and Proposition 2.3.3 in loc. cit. (the last is attributed to Hardy, Littlewood, and Pólya). The second part is obvious, while the third part follows from Theorem 2.6. The fourth and fifth parts follow from Theorem 2.3.2 and Exercise 2.4.4 in [21] respectively.

Note that by continuity, a polynomial $p$ is multiplicatively convex if and only if it satisfies (4.3). If in addition, $p$ takes only positive values on $(0, \infty)$, then its first and last coefficients are necessarily positive.

**Proposition 4.5.** Let $p(x) = \sum_{k=0}^{n} a_k x^k$ be a polynomial of degree $n \geq 3$. Assume $p(x) > 0$ for every $x > 0$ and $p$ satisfies (4.3) on $(0, \infty)$. Then $a_0, a_1, a_{n-1}, a_n \geq 0$.

**Proof.** Since $p(x) > 0$ for every $x > 0$, then $a_0, a_n > 0$. Now consider (4.3) with $y = x/2$. Then,
   \[ q(x) := p(x^2)p(x^2/4) - p(x^2/2)^2 = \frac{a_{n-1}a_n}{4^n}x^{4n-2} + \cdots + \frac{a_0a_1}{4}x^2, \]
where only the lowest and highest order terms are displayed. Since $q(x) \geq 0$ for every $x > 0$, then $a_n a_{n-1} \geq 0$ and $a_0 a_1 \geq 0$. Since $a_0, a_n > 0$, then it follows that $a_{n-1}, a_n \geq 0$. \qed

We now show that Proposition 4.5 is the best possible result along these lines, in the sense that apart from the first two and last two coefficients, every other coefficient of a positive multiplicatively convex polynomial can be negative.

**Theorem 4.6.** Fix $0 < r < s < \infty$, $B \subset (r, s)$, and $a_r, a_s > 0$. Now let
   \begin{equation}
   f(x) = a_r x^r + a_s x^s + \int_B h(\beta) x^\beta \, d\mu(\beta), \tag{4.5}
   \end{equation}
where $\mu$ is a nonnegative measure on $B$ such that $\mu(B) > 0$, and $h : B \to \mathbb{R}$ is such that $\beta \mapsto h(\beta)x^\beta$ is $\mu$-measurable on $B$.

1. Suppose $r > 1$. Then there exists $\nu > 0$ such that if $h(\beta) > -\nu \forall \beta \in B$, then $f(x)$ is nonnegative and super-additive on $[0, R)$.
Suppose $0 \leq r' < r < s < s'$, and let $a_r, a_s > 0$. Then there exists $\lambda > 0$ such that $h(\beta) > -\lambda \forall \beta \in B$, then $g(x) := f(x) + a_r x^{r'} + a_s x^s$ is multiplicatively convex on $[0,R)$.

**Proof.** Define for each $\beta \in B$:

$$f_\beta(x) := \frac{a_r x^{r'} + a_s x^s}{\mu(B)} + h(\beta)x^\beta, \quad g_\beta(x) := f_\beta(x) + \frac{a_r x^{r'} + a_s x^s}{\mu(B)}.$$  

It is clear that sums and integrals of super-additive functions are super-additive. Thus, if $f_\beta$ is super-additive on $[0,R)$ whenever $h(\beta) > -\nu$, then so is

$$\int_B f_\beta(x) \, d\mu(\beta) \equiv f(x).$$

Similarly, we claim that multiplicatively convex functions are closed under taking sums and integrals. Indeed, simply note that $g : [0,R) \to \mathbb{R}$ is multiplicatively convex if and only if $g(A) \in \mathbb{P}_2$ for all $A \in \mathbb{P}_2([0,R))$. Therefore, it suffices to prove the second part of the theorem for functions of the form $g_\beta$.

**Proof of (1).** Suppose as in Equation (4.6) that $f(x) = c_r x^r + c_s x^s + c_\beta x^\beta$ for some $1 < r < \beta < s < \infty$, and where $c_r, c_s > 0$. We show the result in this special case, when $R = \infty$. Define

$$\nu' := \frac{r(r-1)}{s(s-1)} \min(c_r, c_s).$$

Note that if $c_\beta \geq 0$ then the function $f$ is clearly nonnegative and super-additive on $[0,\infty)$. Suppose now that $-\nu' < c_\beta < 0$. Observing that $x^{\beta-1} < x^{r-1} + x^{s-1}$ for all $x \geq 0$, we compute:

$$-\beta c_\beta x^{\beta-1} < \beta c_\beta(x^{r-1} + x^{s-1}) < s \nu'(x^{r-1} + x^{s-1}) \leq r c_r x^{r-1} + s c_s x^{s-1}, \quad \forall x \geq 0.$$  

We conclude that $f(x)$ is strictly increasing on $(0,\infty)$. Since $f(0) = 0$, it is also positive on $(0,\infty)$.

We now claim that when $c_r, c_s > 0$, the function $f(x) = c_r x^r + c_s x^s + c_\beta x^\beta$ is also super-additive on $[0,\infty)$ when $-\nu' < c_\beta$. We may assume that $c_\beta \in (-\nu', 0)$ since otherwise the assertion is clear.

To show the claim, we first make some simplifications. Note that since $f(0) = 0$, a reformulation of superadditivity is that $\Delta_h f : [0,\infty)$ is minimized at 0 for all $h > 0$. Here $(\Delta_h f)(x) := f(x+h) - f(x)$.

In particular, $f$ is superadditive on $[0,\infty)$ if for all $h > 0$, the function $(\Delta_h f)(x)$ is nonincreasing for $x \in [0,\infty)$. Since $f$ is smooth on $(0,\infty)$, this latter condition is equivalent to saying that $\Delta_h(f')(x) \geq 0$ for all $x, h > 0$. In turn, this follows if $f''$ is nonnegative on $(0,\infty)$, by the Mean Value Theorem. Now note that if $x > 0$, then

$$f''(x) = x^{-2} \left( r(r-1)c_r x^{r'} + \beta(\beta-1)c_\beta x^{\beta} + s(s-1)c_s x^{s} \right) \geq x^{-2} \left( r(r-1)c_r x^{r'} - s(s-1)\nu' x^{s} + s(s-1)c_s x^{s} \right) \geq s(s-1)x^{-2} (\nu' x^{r-1} - \nu' x^{\beta} + \nu' x^{s}) = s(s-1)\nu' x^{-2}(x^r + x^s - x^\beta) \geq 0,$$

where we used the definition of $\nu'$, and also that $1 < r < \beta < s$. Therefore by the above analysis, $f$ is superadditive on $(0,\infty)$ if $c_\beta > -\nu'$. In the general case, one would set $\nu := \mu(B)^{-1} \nu'$.

**Proof of (2).** Suppose as in Equation (4.6) that

$$g(x) = c_r x^{r'} + c_s x^r + c_\beta x^\beta + c_s x^s + c_\beta x^\beta,$$

with $0 \leq r' < r < s < s' < \infty$ and $c_r, c_s, c_r, c_s > 0$. By Theorem (4.3)4, it is obvious that $x^\beta$ is multiplicatively convex on $[0,\infty)$ for all $\beta \geq 0$. Hence if $c_\beta \geq 0$, then $g(x)$ is multiplicatively convex by Theorem (4.3)1. Thus, suppose for the remainder of the proof that $c_\beta < 0$. We now use Theorem (4.3) to show that $g$ is multiplicatively convex on $[0,\infty)$ if $c_\beta \in (-\lambda,0)$ for some $\lambda > 0$. To do so, we need to compute $\Psi_g(x)$ (see Equation (4.4)) and obtain an expression for $\lambda$ using the previous part. The computation of $\Psi_g$ can be carried out in greater generality: suppose $T \subset \mathbb{R}$ is a
countable subset such that the addition map : \( T \times T \to \mathbb{R} \) has finite fibers. Now if \( g(x) = \sum_{t \in T} c_t x^t \) is defined for \( x \) in an open interval, then using the fact that \( g \) is a homogeneous linear polynomial in the \( c_t \) (and hence \( \Psi_g \) is homogeneous quadratic),

\[
\Psi_g(x) = \sum_{t \neq t' \in T} c_t c_{t'} (t - t')^2 x^{t + t' - 1}.
\]

Returning to the specific \( g \) above, \( \Psi_g(x) \) has lowest degree term \( c_r c_r' x^{r + r' - 1} \) and highest degree term \( c_s c_s' x^{s + s' - 1} \). Hence by the proof of the previous part, \( x \Psi_g(x) \), and hence \( \Psi_g \), are positive on \((0, \infty)\), if all “intermediate” negative coefficients are bounded below by a threshold, say \( \nu'' \). But these coefficients are precisely \( t_r c_\beta, s_r c_\beta, s_r' c_\beta, s_r'' c_\beta \). Finally, define

\[
\lambda := \max(c_r, c_s, c_r', c_s')^{-1} (s' - r')^{-2} \nu''.
\]

Now if \( -\lambda < c_\beta < 0 \), then a typical negative coefficient in \( \Psi_g(x) \) is of the form

\[
-c_\beta c_r (r - \beta)^2 \leq -c_\beta \max(c_r, c_s, c_r', c_s')(s' - r')^2 \leq \lambda \max(c_r, c_s, c_r', c_s')(s' - r')^2 \leq \nu'',
\]

which proves the result. \( \square \)

Using Theorem 4.6, we can now construct classes of polynomials with negative coefficients such that the polynomial and its derivatives are increasing, super-additive, or multiplicatively convex.

**Corollary 4.7.** Suppose \( p(x) = x^{m+1} \sum_{k=0}^n a_k x^k \) for some \( m, n \in \mathbb{N} \). Assume \( a_0, a_n > 0 \) and let \( I := \{0 < k < n : a_k < 0\} \).

1. There exists \( \nu > 0 \) such that if \( -\nu < a_k < \infty \) for all \( k \in I \), then \( p(x), p'(x), \ldots, p^{(m-1)}(x) \) are strictly increasing on \([0, \infty)\).
2. There exists \( \lambda > 0 \) such that if \( -\lambda < a_k < \infty \) for all \( k \in I \), then \( p(x), p'(x), \ldots, p^{(m-1)}(x) \) are super-additive on \([0, \infty)\).
3. Suppose \( n > 2 \) and \( a_1, a_{n-1} \) are also positive. Then there exists \( \eta > 0 \) such that if \( -\eta < a_k < \infty \) for all \( k \in I \), then \( p(x), p'(x), \ldots, p^{(m)}(x) \) are multiplicatively convex on \([0, \infty)\).

**Proof.** The first two parts follow by applying Theorem 4.6 (with \( h \equiv 0 \) or \( B = \emptyset \), and \( b_i \in \mathbb{N} \) for all \( i \)) to each of \( p, p', \ldots, p^{(m-1)} \), and considering the intersection of all such intervals. The third part follows by applying the theorem to each of \( p, p', \ldots, p^{(m)} \). \( \square \)

Using the above analysis, we can now prove Theorem B.

**Proof of Theorem B.** By Theorem A it suffices to construct an entire function \( f(z) = \sum_{n=0}^\infty a_n z^n \) such that (1) \( a_n \in [-1, 1] \), (2) the sequence \((a_n)_{n \geq 0}\) contains arbitrarily long strings of negative numbers, (3) \( f \) is nonnegative on \([0, \infty)\), and (4) \( f \) is multiplicatively convex and super-additive on \([0, \infty)\). To construct such a function, let \( q_n \geq n + 4 \) be a sequence of increasing integers and let \( r_n = \sum_{k=1}^n q_k \). By Corollary 4.7, for every \( n \geq 1 \) there exists a polynomial \( p_n(x) = x^{r_n} \sum_{k=0}^{n+3} a_{k, n} x^k \) satisfying properties (3) and (4), and such that \( p_n \) is increasing on \([0, \infty)\) and \( a_{k, n} < 0 \) for \( 2 \leq k \leq n + 1 \). Without loss of generality, we can also assume that the coefficients of \( p_n \) also belong to the interval \([-1, 1]\) for all \( n \geq 1 \). Now define

\[
f(z) := \sum_{n=1}^\infty \frac{p_n(z)}{(r_n + n + 3)!} \quad (z \in \mathbb{C}).
\]

Clearly, the function \( f \) is analytic on \( \mathbb{C} \) and satisfies all the required properties. This concludes the proof. \( \square \)
5. Bilinear forms of Schur powers of matrices according to a graph

Theorem [B] demonstrates that functions $f$ mapping $\mathbb{P}_{G_n}((0, \infty))$ into $\mathbb{P}_{G_n}$ are not necessarily absolutely monotonic, even if the family of graphs $\{G_n\}_{n \geq}$ has unbounded maximal degree. In this section, we prove our third main result by showing how a natural stronger hypothesis implies that $f$ is absolutely monotonic. We begin with some notation.

**Definition 5.1.** Given $A \in S_n$, denote by $Q_A$ the associated quadratic form $Q_A(x) := x^T A x$, with kernel $\ker Q_A := \{ \beta \in \mathbb{R}^n : Q_A(\beta) = 0 \}$. Also define $A^{\circ 0}$ to be the matrix with entries $(A^{\circ 0})_{ij} := 1 - \delta_{a_{ij}, 0}$ (where $\delta$ denotes the Kronecker delta function). For $k \geq 1$, define

$$N_k(A) := \bigcap_{m=0}^{k-1} \ker(Q_{A^m}) \cap \{ \beta \in \mathbb{R}^n : \beta^T A^k \beta > 0 \}.$$ 

When $k = 0$, we define $N_0(A) := \{ \beta \in \mathbb{R}^n : \beta^T A^{\circ 0} \beta > 0 \}$.

Notice that for a given nonzero matrix $A \in S_n$ and any $k \geq 1$, the set $N_k(A)$ is contained in $\ker Q_{A^{\circ 0}}$, and hence lives in a hypersurface of dimension strictly smaller than $n$. Thus $N_k(A)$ has zero n-dimensional Lebesgue measure.

Before proving Theorem [C] we recall the strategy of the proof of Theorem [24] provided by Vasudeva in [25, Theorem 6]. A fundamental ingredient in loc. cit. consists of constructing vectors belonging to the kernel of bilinear forms associated to the Schur powers of a matrix $A$. Using our notation, the first ingredient of the proof in loc. cit. is the following lemma.

**Lemma 5.2.** For every $n \geq 2$, there exists a positive semidefinite matrix $A$ such that $N_k(A) \neq \emptyset$ for $k = 1, \ldots, n - 1$.

**Proof.** Let $\alpha_1, \ldots, \alpha_n$ be $n$ distinct nonzero real numbers. Define $\alpha^{(k)} := (\alpha_1^k, \ldots, \alpha_n^k)^T$ for $k \geq 0$, and $A := \alpha^{(1)}(\alpha^{(1)})^T$. Note that the vectors $\alpha^{(0)}, \ldots, \alpha^{(n-1)}$ are linearly independent, so given $1 \leq k \leq n - 1$, there exists $\beta_k \in \mathbb{R}^n$ which is orthogonal to $\alpha^{(m)}$ for $m = 0, \ldots, k - 1$, but not to $\alpha^{(k)}$. For any $m \geq 0$, notice that $A^m \alpha^{(m)} = \alpha^{(m)}(\alpha^{(m)})^T$. Therefore $\beta_k \in \ker Q_{A^m}$ for $m = 1, \ldots, k - 1$, but $\beta_k \notin \ker Q_{A^k}$. Finally, we have $\beta_k^T A^k \beta_k = (\beta_k^T \alpha^{(k)})^2 > 0$. Thus $\beta_k \in N_k(A)$, showing that $k_G \geq n - 1$.

The rest of the proof of Theorem [24] goes as follows. Let $f : (0, \infty) \to \mathbb{R}$ be such that $f[A] \in \mathbb{P}_n$ for every $A \in \mathbb{P}_n((0, \infty))$. Consider the Taylor expansion of $f$ around $a > 0$:

$$f(a + t) = f(a) + f'(a)t + \cdots + f^{(k-1)}(a)\frac{t^{k-1}}{(k-1)!} + f^{(k)}(a + \xi t)\frac{(\xi t)^k}{k!}$$

for some $0 < \xi < 1$. Denoting by $1_{n \times n}$ the $n \times n$ matrix with every entry equal to 1, we obtain:

$$f[a1_{n \times n} + tA] = f(a)1_{n \times n} + f'(a)tA + \cdots + f^{(k-1)}(a)\frac{t^{k-1}}{(k-1)!} A^{\circ (k-1)} + (f^{(k)}(a + t\xi))_{ij}\frac{t^k}{k!} \circ A^k$$

for some $0 < \xi_{ij} < 1$. Since $f[a1_{n \times n} + tA] \in \mathbb{P}_n$ by hypothesis, we obtain for any $\beta \in N_k(A)$:

$$\beta^T f[a1_{n \times n} + tA] \beta = \beta^T \left( (f^{(k)}(a + t\xi))_{ij}\frac{t^k}{k!} \circ A^k \right) \beta \geq 0.$$  

(5.1)

Dividing by $t^k$ and letting $t \to 0^+$, it follows that $f^{(k)}(a) \geq 0$.

In light of Theorem [A], one can now ask if the above approach can be adapted to the case of general graphs. A first difficulty arises when trying to replace the matrix $1_{n \times n}$ in (5.1) by $A_G + \Id_{|G|}$, where $A_G$ denotes the adjacency matrix of a graph $G$. As shown by the following proposition, the matrix $A_G + \Id_{|G|} = (A_G + \Id_{|G|})^{\circ 0}$ is never positive semidefinite if $G$ is not a disconnected union of complete graphs.
Proposition 5.3. Given $A \in \mathbb{S}_n$, the following are equivalent:

1. $A^{00}$ is positive semidefinite.
2. There exists a permutation matrix $P$ such that $PA^{00}P^T = 0_{n_0 \times n_0} \oplus Id_{(n-n_0) \times (n-n_0)}$ for some $0 \leq n_0 \leq n$.

The proof is standard and resembles that of Proposition 3.3 and is therefore omitted. See also [18, Theorem 1.13] for more equivalent conditions.

A second major drawback in trying to adapt the proof of [25, Theorem 6] is provided by the following result, which shows that for large families of graphs $G$, the sets $N_k(A)$ can be empty for all matrices in $\mathbb{P}_G$.

Theorem 5.4. Let $G$ be a star graph with at least two vertices. Then $N_k(A)$ is empty for all $k > 2$ and all positive semidefinite $A \in \mathbb{P}_G$.

Proof. We will prove the following claim, which implies the assertion:

\begin{equation}
\ker Q_A \cap \ker Q_{A^0A} = \bigcap_{m \geq 1} \ker Q_{A^{m}}.
\end{equation}

To show the claim, suppose $A \in \mathbb{P}_G$ is as in the statement of Proposition 3.1 with $d \geq 1$. Then properties (1)-(3) in that result hold here. Now define $a_m, L_m$ as in (3.2). Then since $p_i \geq 0$ for all $i$ and $a_1 \geq 0$, hence

\[
\sum_{i>1: p_i \neq 0} \frac{a_i^2}{p_i^m} \leq \left( \sum_{i>1: p_i \neq 0} \frac{a_i^2}{p_i} \right)^m \leq p_1^m.
\]

This implies $a_m \geq 0 \forall m > 0$, so $L_m$ is a real matrix for all $m > 0$. Moreover, $A^{0m} = L_m L_m^T$ for all $m > 0$, so $A^{0m}$ is also positive semidefinite. Now if $Q_{A^{0m}}(\beta) = ||L_m^T \beta||^2 = 0$ for some $m > 0$, then $L_m^T \beta = 0$. Denoting $\beta = (\beta_1, \ldots, \beta_d^1)^T$, the condition $L_m^T \beta = 0$ translates into the following equivalent conditions for every $m > 0$:

\begin{equation}
Q_{A^{0m}}(\beta) = 0 \iff L_m^T \beta = 0 \iff (\beta_1 a_m = 0, \beta_1 a_1^m + \beta_i p_i^m = 0 \forall 2 \leq i \leq d + 1).
\end{equation}

(Note that we use the characterization (2) in the statement of Proposition 3.1) Now consider any vector $\beta \in \ker Q_A \cap \ker Q_{A^0A}$ such that $\beta_1 = 0$. Then by Equation (5.3) for $m = 1$, either $\beta_i$ or $p_i$ is zero for all $i > 1$. But then $\beta_1 a_1^m + \beta_i p_i^m = 0$ for all $m > 0$ and all $i > 1$. Moreover, $\beta_1 a_m = 0$ for all $m$. Hence by Equation (5.3), $Q_{A^{0m}}(\beta) = 0$ for all $m > 0$, as desired.

Next, assume that $\beta \in \ker Q_A \cap \ker Q_{A^0A}$ and $\beta_1 \neq 0$. Then Equation (5.3) holds for $m = 1, 2$. We now claim that $2 \leq i \leq d + 1$ fall into exactly one of the following three categories:

- Suppose $p_i = 0$ for some $2 \leq i \leq d + 1$. Then $\alpha_i = 0$ by Proposition 3.1 so $\beta_1 a_i^m + \beta_i p_i^m = 0 \forall m > 0$.
- Suppose $p_i \neq 0$ but $\alpha_i = 0$. Then $\beta_i = 0$ by Equation (5.3) for $m = 1$, so once again, $\beta_1 a_i^m + \beta_i p_i^m = 0$ for all $m > 0$.
- Suppose $p_i, \alpha_i \neq 0$. Then by Equation (5.3) for $m = 1, 2$, $\beta_i = -\beta_1 \alpha_i / p_i = -\beta_1 a_i^2 / p_i^2$. This implies that $\alpha_i = p_i \neq 0$, whence $\beta_i = -\beta_1$. Once again, this implies that $\beta_1 a_i^m + \beta_i p_i^m = 0$ for all $m > 0$.

Thus we see that the second part of the last equivalent assertion in Equation (5.3) holds in all three cases above, for all $m > 0$. It remains to prove that $a_m = 0$ for all $m > 0$ (since $\beta_1 \neq 0$). Now define $c_i := 0$ if $p_i = 0$, and $\alpha_i^2 / p_i$ otherwise. Then $a_m = p_1^m - \sum_{i=2}^{d+1} c_i^m$, so using $a_1 = 0 = a_2$ from Equation (5.3) implies:

\[
\sum_{i=2}^{d+1} c_i = p_1, \quad \sum_{i=2}^{d+1} c_i^2 = p_1^2 = \left( \sum_{i=2}^{d+1} c_i \right)^2.
\]
Since \( c_i \geq 0 \) for all \( i \), this system of equations has no solutions if even two \( c_i \) are positive. We thus conclude that \( c_i > 0 \) for at most one \( i \), say \( c_i = 0 \) if \( i \neq i_0 \). Then \( p_1 = c_{i_0} \) (since \( a_1 = 0 \)). Hence,

\[
a_m = p_1^m - \sum_{i=2}^{d+1} c_i^m = c_{i_0}^m - c_{i_0}^m - 0 = 0 \quad \forall m > 0,
\]
as desired. This proves the claim. \[
\]

### 5.1. Proof of Theorem [C]

Proposition 5.3 and Theorem 5.4 demonstrate that one faces major obstacles when trying to generalize the argument in [25, Theorem 6] to arbitrary graphs \( G \). New tools are required. In the rest of the paper, we carefully study bilinear forms associated to the Schur powers of matrices in \( \mathbb{P}_G \) for an arbitrary graph \( G \), and use this analysis to prove Theorem [C]. First, we introduce some notation.

**Definition 5.5.** Given a graph \( G \), let

\[
k_G := \max_{A \in \mathbb{S}_G} \max \{ k \geq 0 : N_m(A) \neq \emptyset \} \quad \text{for } m = 1, \ldots, k .
\]

Note also that for any pair of graphs \( G \) and \( H \),

\[
H \subseteq G \implies k_H \leq k_G.
\]

Theorem 5.6 below provides bounds for the constants \( k_G \), and will be crucially used in the proof of Theorem [C] as a replacement of Lemma 5.2 for a general graph \( G \). Recall that \( \Delta(G) \) denotes the maximum vertex degree of the graph \( G \).

**Theorem 5.6.** For all graphs \( G \) with at least one edge, we have

\[
\max(2, \Delta(G)) \leq k_G < |V(G)| + |E(G)| = \dim_{\mathbb{R}} S_G.
\]

**Proof.** We begin by proving the upper bound. Given a symmetric matrix \( A \), denote by \( \eta(A) \) the number of distinct nonzero entries of \( A \). We claim that for any symmetric \( A \),

\[
\eta(A) = \max_{k \geq 0} \ker Q_{A^{\otimes k}} = \bigcap_{k=0}^{\eta(A)-1} \ker Q_{A^{\otimes k}}.
\]

In particular, \( N_k(A) = \emptyset \quad \forall k \geq \eta(A) \). The \( \subseteq \) inclusion in Equation (5.7) is obvious. To prove the reverse inclusion, let \( A \in \mathbb{S}_n \), define \( d := \eta(A) \), and let \( \{ \alpha_1, \ldots, \alpha_d \} \) be the distinct nonzero entries of \( A \). Given a vector \( \beta = (\beta_1, \ldots, \beta_n) \), and \( 1 \leq l \leq d \), define:

\[
S_l := \{(i, j) : 1 \leq i, j \leq n, \alpha_{ij} = \alpha_l \}, \quad \beta'_l := \sum_{(i, j) \in S_l} \beta_i \beta_j.
\]

Then \( Q_{A^{\otimes k}}(\beta) = \sum_{i=1}^d \alpha_i^{k} \beta'_i \). Let \( B \) be the \( d \times d \) Vandermonde matrix whose \((i, j)\)th entry is \( \alpha_i^{j-1} \) for \( 1 \leq i, j \leq d \); then \( B \) is non-singular. Also define \( \beta' := (\beta'_1, \ldots, \beta'_d)^T \). Then \( \beta \in \ker Q_{A^{\otimes k}} \) for all \( 0 \leq k < d \) if and only if \( B \beta' = 0 \), if and only if \( \beta'_l = 0 \) for all \( l \). But then \( \beta \in \bigcap_{k \geq 0} \ker Q_{A^{\otimes k}} \). This proves the reverse inclusion, and hence Equation (5.7). To conclude the proof of the upper bound, note that if \( k \geq \eta(A) \), then by Equation (5.7),

\[
N_k(A) = \cap_{m=0}^{k-1} \ker(Q_{A^{\otimes m}}) \cap \{ \beta \in \mathbb{R}^n : \beta^T A^{\otimes k} \beta > 0 \} = \emptyset.
\]

Therefore \( k_G < \max_{A \in \mathbb{S}_G} \eta(A) = |V(G)| + |E(G)| \).

We now prove the lower bound for \( k_G \). To show that \( k_G \geq 2 \), it suffices by Equation (5.5) to show that \( k_{K_2} \geq 2 \). Hence, suppose \( V(G) = \{1, 2\} \) and \( E = \{(1, 2)\} \). Now fix positive numbers \( a \neq b > 0 \) and consider the matrix \( A_j = \begin{pmatrix} a & a+b \\ a+b & 2(3-j) \end{pmatrix} \). It is clear that \( \beta := (1, -1)^T \) is in \( N_j(A_j) \) for \( j = 1, 2 \). Hence \( k_G \geq k_{K_2} \geq 2 \).
Finally, we show that $k_G \geq \Delta(G)$. For ease of exposition, we divide this part of the proof into four steps.

**Step 1:** We begin by introducing the key matrix $A$. Without loss of generality, assume that $\deg_G 1 = \Delta(G) =: d > 0$, and $\{2, 3, \ldots, d + 1\}$ are incident to 1. Let $\alpha_i$ be distinct nonzero real numbers for $1 \leq i \leq d + 1$, and define

$$\alpha^{(k)} := \left(\frac{\alpha_1}{2}, \alpha_2, \ldots, \alpha_{d+1}, 0, \ldots, 0\right)^T \in \mathbb{R}^{\lvert G \rvert} \quad \forall k \geq 0,$$

(5.9)

$$A := e_1(\alpha^{(1)})^T + \alpha^{(1)}e_1^T \in \mathbb{S}_G([0, \infty)).$$

It follows from Equation (5.7) that $N_k(A)$ is empty if $k \geq d + 1$. We will show that this bound is sharp for generic $\alpha_i$. More precisely, we show in the remainder of the proof that $N_1(A), \ldots, N_d(A)$ are nonempty if the $\alpha_i$ are all nonzero and distinct and less than $\alpha_1$ for $i > 1$.

Since all $\alpha_i \neq 0$, the graph of $A^{(k)}$ is a star graph over $d + 1$ vertices for all $k \geq 0$, and satisfies $A^{(k)} \in \mathbb{S}_G$. Moreover, $A^{(k)}$ and $Q_{A^{(k)}}(\beta)$ can be easily computed for all $k \geq 0$:

$$A^{(k)} = e_1(\alpha^{(k)})^T + \alpha^{(k)}e_1^T \quad \Longrightarrow \quad Q_{A^{(k)}}(\beta) = 2(\beta^Te_1)(\beta^T\alpha^{(k)}) = 2b_1(\beta^T\alpha^{(k)}).$$

Thus, $\ker Q_{A^{(k)}} = \{e_1\}^\perp \cup \{\alpha^{(k)}\}^\perp$.

**Step 2:** Now define $V_k$ to be the span of $\alpha^{(0)}, \ldots, \alpha^{(k-1)}$. We then claim that the following two assertions hold:

$$\dim V_k = k, \quad \forall \ 0 \leq k \leq d + 1$$

(5.11)

$$e_1 \in V_k \iff k = d + 1.$$  

(In other words, the vectors $\alpha^{(0)}, \ldots, \alpha^{(d)}$ are linearly independent - and this continues to hold if we replace $\alpha^{(d)}$ by $e_1.$) The first assertion is immediate from Vandermonde determinant theory. To show the second assertion, consider the matrix whose columns are (the first $d + 1$ coordinates of) $e_1, \alpha^{(0)}, \ldots, \alpha^{(d-1)}$. Its determinant is equal to the minor obtained by deleting its first row and first column. This minor is exactly the determinant of the Vandermonde matrix whose columns are $\{(\alpha_1^k, \ldots, \alpha_{d+1}^k)^T : 0 \leq k \leq d - 1\}$. Hence it is nonzero, whence the second assertion follows.

**Step 3:** The next step is to produce $\beta_k \in N_k(A)$ for $k = 1, \ldots, d - 1$. We work in $V_{d+1}$ for the rest of this proof. Define $P_{V_k}$ to be the projection operator onto $V_k^\perp$. Now given $0 < k < d$, note that $\alpha^{(0)}, \ldots, \alpha^{(k)}, e_1$ are linearly independent from above. Therefore $P_{V_k}(\alpha^{(k)})$ and $P_{V_k}(e_1)$ are also linearly independent, so in particular, they have an angle of less than $180^\circ$ between them. Choose $\beta_k$ to be a positive scalar multiple of the unique angle bisector in the plane spanned by $P_{V_k}(\alpha^{(k)})$ and $P_{V_k}(e_1)$. More precisely, set $\beta_k := a_k + b_k$, where

$$a_k := \frac{P_{V_k}(\alpha^{(k)})}{\|P_{V_k}(\alpha^{(k)})\|}, \quad b_k := \frac{P_{V_k}(e_1)}{\|P_{V_k}(e_1)\|}.$$

Let $\gamma := \|P_{V_k}(\alpha^{(k)})\| \cdot \|P_{V_k}(e_1)\| > 0$. Since $a_k, b_k$ are unit vectors, it is clear that

$$Q_{A^{(k)}}(\beta_k) = 2(\beta^T_k e_1)(\beta^T_k \alpha^{(k)}) = 2(\beta^T_k P_{V_k}(e_1))(\beta^T_k P_{V_k}(\alpha^{(k)}))$$

$$= 2\gamma(a_k + b_k)^T(b_k + a_k)^T = 2\gamma(1 + (a_k, b_k))^2 > 0.$$  

The last inequality here is strict by the Cauchy-Schwartz inequality, since $a_k, b_k$ are not proportional from above. On the other hand, if $0 \leq m < k$, then $Q_{A^{(m)}}(\beta_k) = 2(\beta^T_k e_1)(\beta^T_k \alpha^{(m)}) = 0$, since $\alpha^{(m)} \in V_k$ and $\beta_k \in V_k^\perp$. We conclude that $\beta_k \in N_k(A)$ for $0 < k < d$. (In particular, using any set of distinct nonzero $\alpha_i$, we obtain that $k_G \geq \Delta(G) - 1$.)'
Step 4: Finally, we produce $\beta_d \in N_d(A)$. To do so, note that $V_d$ is a codimension one subspace in $V_{d+1}$, so $\dim V_d^\perp = 1$. Note that this uniquely determines $\beta_d \in V_d^\perp$ up to multiplying by a nonzero scalar $c \neq 0$; moreover, the sign of $Q_A(\beta_d)$ is independent of $c$.

Note that $\alpha^{(d)} \notin V_d$. Hence (5.10) implies the following: if $\pm \beta_d$ are the only two unit vectors in $V_d^\perp$, then $Q_A(\beta_d) = 2(\beta_d^T e_1)(\beta_d^T \alpha^{(d)})$. It is clear from the above remarks that $Q_A(\beta_d)$ is positive if and only if $e_1$ and $\alpha^{(d)}$ are on the “same side” of the hyperplane $V_d$ in $V_{d+1}$ (i.e., their inner products with $\beta_d$ have the same sign). In order to ensure this, one now needs a constraint on the $\alpha_i$. Thus, consider a matrix $A_d(X)$, whose columns are $\alpha^{(0)}, \ldots, \alpha^{(d-1)}, x$, where $x := (x_1, \ldots, x_{d+1})^T$ is a column vector of variables. It is clear that $\det A_d(x) = \sum_{i=1}^{d+1} c_i x_i$, for some scalars $c_i$. Moreover, $\det A_d(v) = 0$ if we replace $x$ by any vector $v \in V_d$, since the other columns form a basis of $V_d$.

In order that the two vectors $e_1, \alpha^{(d)} \in V_{d+1}$ lie on the same side of $V_d$ (in $V_{d+1}$), it is enough to ensure that $\det A_d(e_1)$, $\det A_d(\alpha^{(d)})$ have the same sign, i.e., $\det A_d(e_1) \cdot \det A_d(\alpha^{(d)}) > 0$. Now from the above remarks and the standard Vandermonde formula, we compute:

$$\det A_d(\alpha^{(d)}) = \prod_{1 \leq i < j \leq d+1} (\alpha_j - \alpha_i), \quad \det A_d(e_1) = (-1)^d \prod_{2 \leq i < j \leq d+1} (\alpha_j - \alpha_i).$$

Since all $\alpha_i$ are pairwise distinct, upon removing the perfect squares we obtain the condition needed to ensure that $e_1$ and $\alpha^{(d)}$ are on the same side of $V_d$; namely,

$$(-1)^d \prod_{j=2}^{d+1} (\alpha_j - \alpha_1) = \prod_{j=2}^{d+1} (\alpha_1 - \alpha_j) > 0.$$

This inequality holds if we choose $\alpha_1 > \max(\alpha_2, \ldots, \alpha_{d+1})$. Thus we have produced a matrix $A = e_1(\alpha^{(1)})^T + \alpha^{(1)} e_1^T \in S_{G_d}$ and a vector $\beta_d \in N_d(A)$, in addition to the vectors $\beta_k \in N_k(A)$ for $0 < k < d$ (constructed above for any nonzero distinct $\alpha_i$). This concludes the proof.

Remark 5.7. Note that the bounds in Theorem 5.6 are sharp for $G = K_2$.

We now proceed to prove Theorem C using Theorem 5.6.

Proof of Theorem C. Suppose $f$ is any function satisfying the hypotheses of the theorem. By considering the matrix $M = aI_{1 \times 1} \oplus 0_{(|G_1|-1) \times (|G_1|-1)}$ for $a \in I$, it follows from the hypotheses that $f(I) \subset [0, \infty)$. We next prove that $f$ is continuous on $I$. By Remark 2.7 and Theorem 2.6, $f$ is necessarily continuous and increasing on $(0, R)$, and so $f^+(0) := \lim_{x \to 0^+} f(x)$ exists. Since $\Delta(G_n) \to \infty$, choose $n$ such that $G_n$ contains $K_3$ or $A_3$ as an induced subgraph. Without loss of generality, assume that vertex 1 is connected to vertices 2, 3. To prove that $f$ is continuous at 0, first note that $tB(2, 1, 1) \oplus 0_{(n-3) \times (n-3)} \in P_{G}(I)$ for $t > 0$ small enough, where $B(2, 1, 1)$ was defined in Equation (3.3). Therefore $f(tB(2, 1, 1)) \in P_3$. Since $f$ is absolutely monotonic on $(0, R)$, it is nonnegative and increasing there, and so $f^+(0) := \lim_{x \to 0^+} f(x)$ exists. As a consequence,

$$\lim_{t \to 0^+} f[tB(2, 1, 1)] = \begin{pmatrix} f^+(0) & f^+(0) & f^+(0) \\ f^+(0) & f^+(0) & f(0) \\ f^+(0) & f(0) & f^+(0) \end{pmatrix} \in P_3.$$

Computing the determinant of the above matrix, we conclude that $f^+(0) = f(0)$, i.e., $f$ is continuous at 0. Therefore the function $f$ is now continuous on $I$.

Next suppose that $f \in C^\infty(I)$, and fix $k > 0$ and $a \in (0, R)$. We claim that $f^{(k)}(a) \geq 0$. To show the claim, choose $n \in \mathbb{N}$ such that $\Delta(G_n) \geq k$. By Theorem 5.6 there exists $A \in S_{G_n}$ and $\beta \in \mathbb{R}^n$ such that $\beta \in N_k(A)$. By the definition of $N_k(A)$, we have $\beta^T (aA^{(0)} + tA) \beta \geq 0$ for every $t > 0$ and thus, by hypothesis,

$$\beta^T f_{G_n}(A^{(0)} + tA) \beta \geq 0, \quad \forall 0 < t < \epsilon, \quad \text{where} \quad \epsilon := \min \left( \frac{a}{\max_{i,j} |a_{ij}|}, \frac{R - a}{\max_{i,j} |a_{ij}|} \right).$$
Note that \((a - \epsilon, a + \epsilon) \subset (0, R) \subset \text{dom}(f)\). Now expanding \(f\) in Taylor series around \(A\), we obtain:
\[
f_{G_n}[A^0 + tA] = \sum_{r=0}^{k-1} \frac{f^{(r)}(a)}{r!} (tA)^r + \left( f^{(k)}(a + \theta_{ij} t a_{ij}) \right) \frac{t^k}{k!},
\]
where \(0 < \theta_{ij} < 1\). In particular,
\[
\beta^T f_{G_n}[A^0 + tA] \beta = \sum_{i,j=1}^{n} \beta_i \beta_j f^{(k)}(a + \theta_{ij} t a_{ij}) \frac{t^k}{k!} a_{ij} \geq 0,
\]
since \(\beta \in N_k(A)\). Now divide by \(t^k/k!\) and let \(t \to 0^+\) to obtain: \(f^{(k)}(a) (\beta^T A^k \beta) \geq 0\). The claim follows since \(\beta^T A^k \beta > 0\) by hypothesis. Theorem 2.2 now implies that \(f\) is analytic and absolutely monotonic on \(I\). This shows the result when \(f \in C^\infty(I)\).

Finally, suppose \(f\) is continuous but not necessarily smooth on \(I\), and let \(0 < b < R\). For any probability distribution \(\phi \in C^\infty(\mathbb{R})\) with compact support in \((b/R, \infty)\), let
\[
f_\phi(x) := \int_{b/R} f(xy^{-1}) \phi(y) \frac{dy}{y}, \quad 0 < x < b.
\]
Then \(f_\phi \in C^\infty(0,b)\). Suppose \(\beta^T M \beta \geq 0\) for some \(M \in \mathcal{S}_{G_n}((0,b))\) and some \(n \geq 1\). Then,
\[
\beta^T (f_\phi)_{G_n}[M] \beta = \int_{b/R} \sum_{i,j=1}^{n} \beta_i \beta_j f(m_{ij} y^{-1}) \phi(y) \frac{dy}{y} = \int_{b/R} \beta^T f_{G_n}[y^{-1} M] \beta \phi(y) \frac{dy}{y}.
\]
Notice that the integrand is non-negative for every \(y > 0\). It thus follows that \(\beta^T (f_\phi)_{G_n}[M] \beta \geq 0\).

Now consider a sequence \(\phi_m \in C^\infty(\mathbb{R})\) of probability distributions with compact support in \((b/R, \infty)\) such that \(\phi_m\) converges weakly to \(\delta_1\), the Dirac measure at 1. Note that such a sequence can be constructed since \(b/R < 1\). By the above analysis in this proof, \(f_\phi_m\) is absolutely monotonic on \((0,b)\) for every \(m \geq 1\). Therefore by Theorem 2.2, the forward differences \(\Delta_h^{k} [f_\phi_m](x)\) of \(f_\phi_m\) are nonnegative for \(l \geq 0\) and all \(x\) and \(h\) such that \(0 \leq x < x + h < \cdots < x + lh < R\). Since \(f\) is continuous, \(f_{\phi_m}(x) \to f(x)\) for every \(x \in (0,b)\). Therefore \(\Delta_h^{k} [f](x) \geq 0\) for all such \(x\) and \(h\) as well. As a consequence, by Theorem 2.2, the function \(f\) is absolutely monotonic on \((0,b)\). Since this is true for every \(0 < b < R\), it follows that \(f\) is absolutely monotonic on \(I\). \(\square\)

**Remark 5.8.** In Theorem C, the assumptions only have to be verified for an appropriate sequence of matrices \((M_n)_{n \geq 1}\) such that \(M_n \in \mathcal{S}_{G_n}(I)\), and a sequence of vectors \(\beta_n, k\) such that \(\beta_n, k \in N_k(M_n)\) for \(1 \leq k \leq \Delta(G_n)\). Moreover, it also suffices to verify the hypotheses of Theorem C for matrices of the form \(aA^0 + tA\) for \(a, t > 0\) and where \(A\) has the form \((5.9)\).

**References**


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