AN ASYMPTOTIC EXPANSION OF THE DISTRIBUTION OF THE
"STUDENTIZED" CLASSIFICATION STATISTIC W

BY

T. W. ANDERSON

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THEODORE W. ANDERSON, PROJECT DIRECTOR

DEPARTMENT OF STATISTICS
STANFORD UNIVERSITY
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1. Introduction

A sample $x_1^{(1)}, \ldots, x_{N_1}^{(1)}$ is drawn from the normal distribution $N(\mu_1^{(1)}, \Sigma)$, and a sample $x_1^{(2)}, \ldots, x_{N_2}^{(2)}$ is drawn from $N(\mu_2^{(2)}, \Sigma)$. The $p$-component mean vectors $\mu_1^{(1)}$ and $\mu_2^{(2)}$ and the common covariance matrix $\Sigma$ are unknown; it is assumed that $\mu_1^{(1)} \neq \mu_2^{(2)}$ and $\Sigma$ is non-singular. Another observation $x$ is drawn. It is desired to classify this observation as coming from $N(\mu_1^{(1)}, \Sigma)$ or $N(\mu_2^{(2)}, \Sigma)$. [See T. W. Anderson (1951) or T. W. Anderson (1958), Chapter 6.]

The observation $x$ may be classified by means of the classification statistic

\[(1) \quad W = (\overline{x}^{(1)} - \overline{x}^{(2)})' \Sigma^{-1} [x - \frac{1}{2} (\overline{x}^{(1)} + \overline{x}^{(2)})],\]

\[(2) \quad \overline{x}^{(1)} = \frac{1}{N_1} \sum_{j=1}^{N_1} x_j^{(1)}, \quad \overline{x}^{(2)} = \frac{1}{N_2} \sum_{j=1}^{N_2} x_j^{(2)},\]

\[(3) \quad nS = \sum_{j=1}^{N_1} (x_j^{(1)} - \overline{x}^{(1)}) (x_j^{(1)} - \overline{x}^{(1)})' + \sum_{j=1}^{N_2} (x_j^{(2)} - \overline{x}^{(2)}) (x_j^{(2)} - \overline{x}^{(2)})',\]

and $n = N_1 + N_2 - 2$. The rule is to classify $x$ as coming from $N(\mu_1^{(1)}, \Sigma)$ if $W > c$ and from $N(\mu_2^{(2)}, \Sigma)$ if $W < c$, where $c$ may be a constant, particularly 0, or a function of $\overline{x}^{(1)}, \overline{x}^{(2)}$, and $S$.

The distribution of $W$ depends on the parameters $\mu_1^{(1)}, \mu_2^{(2)}$, and $\Sigma$ through the squared Mahalanobis distance

\[(4) \quad a = (\mu_1^{(1)} - \mu_2^{(2)})' \Sigma^{-1} (\mu_1^{(1)} - \mu_2^{(2)}),\]

which can be estimated by

\[(5) \quad a = (\overline{x}^{(1)} - \overline{x}^{(2)})' \Sigma^{-1} (\overline{x}^{(1)} - \overline{x}^{(2)}).\]
The limiting distribution of \( W \) as \( N_1 \to \infty \) and \( N_2 \to \infty \) is normal with variance \( \alpha \) and mean \( \frac{1}{2} \alpha \) if \( \bar{z} \) is from \( N(\mu_1, \Sigma) \) and mean \(- \frac{1}{2} \alpha \) if \( \bar{z} \) is from \( N(\mu_2, \Sigma) \). Bowker and Sitgreaves* (1961) for \( N_1 = N_2 \) and Okamoto (1963) [with correction, Okamoto (1968)] gave asymptotic expansions of the distribution of \( (W - \frac{1}{2} \alpha)/\sqrt{\alpha} \) for \( \bar{z} \) coming from \( N(\mu_1, \Sigma) \) and \( (W + \frac{1}{2} \alpha)/\sqrt{\alpha} \) for \( \bar{z} \) coming from \( N(\mu_2, \Sigma) \) to terms of order \( 1/N_1^2 \), \( 1/N_2^2 \), and \( 1/n^2 \) when \( N_1 \to \infty \), \( N_2 \to \infty \), and \( N_2/N_1 \to k \), a finite positive constant. In particular, \( \Pr[W \leq 0] \) was evaluated.

The statistician, who wants to classify \( \bar{z} \), may take \( c \) to be a constant, perhaps \( 0 \), and accept the pair of misclassification probabilities that result. The asymptotic expansion of the distribution of \( (W \pm \frac{1}{2} \alpha)/\sqrt{\alpha} \) gives approximate evaluations of these probabilities, which are functions of the unknown parameter \( \alpha \) as well as of \( c \).

On the other hand the statistician may want to determine the cut-off point \( c \) to adjust the probabilities of misclassification. Since the limiting distribution of \( (W-\alpha)/\sqrt{\alpha} \) and \( (W+\alpha)/\sqrt{\alpha} \) are \( N(0, 1) \) when \( \bar{z} \in \Sigma_1 \) and \( \bar{z} \in \Sigma_2 \), respectively, a first approximation to the pair of misclassification probabilities is \( \Phi(\frac{1}{2} \alpha + c\sqrt{\alpha}) \) and \( \Phi(- \frac{1}{2} \alpha + c\sqrt{\alpha}) \), where \( \Phi(a) \) is the cumulative distribution function of the standard normal variate. Since \( a \) is an estimate of \( \alpha \), one might base his choice of \( c \) on the fact that the limiting distribution of \( (W - \frac{1}{2} a)/\sqrt{a} \) and \( (W + \frac{1}{2} a)/\sqrt{a} \) are \( N(0, 1) \) when \( \bar{z} \in \Sigma_1 \) and \( \bar{z} \in \Sigma_2 \), respectively. In this paper we make asymptotic expansions of the distribution of \( (W - \frac{1}{2} a)/\sqrt{a} \) and \( (W + \frac{1}{2} a)/\sqrt{a} \) in these two cases, respectively.

*The coefficients \( a_{31} \) and \( a_{32} \) should be replaced by \(-a_{31} \) and \(-a_{32} \), respectively.
2. The Asymptotic Expansion

The statistics $\bar{x}_1$, $\bar{x}_2$, $\bar{x}_3$, and $\Sigma$ are independently distributed according to $N(\mu, \Sigma)$, $N(\mu_2, (1/N_1)\Sigma)$, $N(\mu_2, (1/N_2)\Sigma)$, and $W(\Sigma, n)$, respectively; here $\mu = \Sigma x$ and $W(\Sigma, n)$ denotes the Wishart distribution with $n$ degrees of freedom. We write

$$W - \frac{1}{2} a = (\bar{x}_1 - \bar{x}_2) \cdot \Sigma^{-1} (\bar{x}_1 - \bar{x}_2).$$

Then

$$\Pr \left\{ \frac{W - \frac{1}{2} a}{\sqrt{a}} \leq u \right\} = \Pr \left\{ (\bar{x}_1 - \bar{x}_2) \cdot \Sigma^{-1} (\bar{x}_1 - \bar{x}_2) \leq u \frac{1}{\sqrt{(\bar{x}_1 - \bar{x}_2) \cdot \Sigma^{-1} (\bar{x}_1 - \bar{x}_2)}} \right\}.$$ 

Since $\bar{x}_1$ has the distribution $N(\mu, \Sigma)$ independently of $\bar{x}_2$ and $\Sigma$, the conditional distribution of $(\bar{x}_1 - \bar{x}_2) \cdot \Sigma^{-1} (\bar{x}_1 - \bar{x}_2)$ is $N(0, (\bar{x}_1 - \bar{x}_2) \cdot \Sigma^{-1} (\bar{x}_1 - \bar{x}_2))$, and

$$r = \frac{(\bar{x}_1 - \bar{x}_2) \cdot \Sigma^{-1} (\bar{x}_1 - \bar{x}_2)}{\sqrt{(\bar{x}_1 - \bar{x}_2) \cdot \Sigma^{-1} (\bar{x}_1 - \bar{x}_2)}}$$

has the distribution $N(0, 1)$. Then (7) is

$$\Pr \left\{ \frac{W - \frac{1}{2} a}{\sqrt{a}} \leq u \right\} = \Pr \left\{ r \leq \frac{uv(\bar{x}_1 - \bar{x}_2) \cdot \Sigma^{-1} (\bar{x}_1 - \bar{x}_2) + (\bar{x}_1 - \bar{x}_2) \Sigma^{-1} (\bar{x}_1 - \bar{x}_2)}{\sqrt{(\bar{x}_1 - \bar{x}_2) \cdot \Sigma^{-1} (\bar{x}_1 - \bar{x}_2)}} \right\}$$

$$= \Phi \left[ \frac{uv(\bar{x}_1 - \bar{x}_2) \cdot \Sigma^{-1} (\bar{x}_1 - \bar{x}_2) + (\bar{x}_1 - \bar{x}_2) \Sigma^{-1} (\bar{x}_1 - \bar{x}_2)}{\sqrt{(\bar{x}_1 - \bar{x}_2) \cdot \Sigma^{-1} (\bar{x}_1 - \bar{x}_2)}} \right],$$

where the expectation is with respect to $\bar{x}_1$, $\bar{x}_2$, and $\Sigma$. 

3
The distribution of \( W \) and \( a \) is invariant with respect to the transformations 
\[
\begin{align*}
\hat{x}^* &= A \hat{x} + b, \\
\hat{x}^*_{(1)} &= A \hat{x}_{(1)} + b, \quad j = 1, \ldots, N_1, \text{ and} \\
\hat{x}^*_{(2)} &= A \hat{x}_{(2)} + b, \quad \text{where } A \text{ is nonsingular.}
\end{align*}
\]

The maximal invariant of these transformations is the distance \( \alpha \), given by (4). We can choose \( \hat{A} \) and \( \hat{b} \) to transform \( \hat{z} \) to \( \hat{z} = \mu_1 - \mu_2 \) to \( \delta = (\Delta, 0, \ldots, 0) \), where \( \Delta = \sqrt{\alpha} \), and \( \mu_1 \) to \( 0 \). We shall first treat the case where \( \mu = \mu_1 \).

The vectors \( \hat{x}^{(1)} \) and \( \hat{x}^{(2)} \) and the matrix \( S \) are distributed independently. The distribution of \( [\hat{x}^{(1)}, \hat{x}^{(2)}] \) is

\[
N\left[\begin{pmatrix} \delta \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{n}{N_1} + \frac{1}{n} & \frac{1}{n} \\ \frac{1}{n} & \frac{n}{N_1} \end{pmatrix}\right].
\]

Let \( \tilde{y}, \tilde{z} \) and \( \tilde{v} \) be defined by

\[
\begin{align*}
\tilde{x}^{(1)} - \tilde{x}^{(2)} &= \delta + \frac{1}{\sqrt{n}} \tilde{y}, \\
\tilde{x}^{(1)} &= \frac{1}{\sqrt{n}} \tilde{z},
\end{align*}
\]

\[
S = I + \frac{1}{\sqrt{n}} \tilde{v}.
\]

Then the joint distribution of \( (\tilde{y}', \tilde{z}') \) is

\[
N\left[\begin{pmatrix} \delta \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{n}{N_1} + \frac{1}{n} & \frac{n}{N_1} \\ \frac{n}{N_1} & \frac{n}{N_1} \end{pmatrix}\right].
\]
Then (9) is

\[
\Pr \left\{ \frac{W - \frac{1}{2} a}{\sqrt{a}} \leq u \right\} = \Phi \left[ \frac{u \sqrt{\frac{(\delta + \frac{1}{\sqrt{n}} Y)'}{(\delta + \frac{1}{\sqrt{n}} Y)^{-1} (\delta + \frac{1}{\sqrt{n}} Y)} + \frac{1}{\sqrt{n}} (\delta + \frac{1}{\sqrt{n}} Y)'}{(\delta + \frac{1}{\sqrt{n}} Y)^{-1} (\delta + \frac{1}{\sqrt{n}} Y)} \right] \]

We can write

\[
(I + \frac{1}{\sqrt{n}} Y)^{-1} = I - \frac{1}{\sqrt{n}} Y + \frac{1}{n} Y^2 - \frac{1}{n^{3/2}} Y^3 + \frac{1}{2 n} Y^4 - \frac{1}{n^{5/2}} Y^5 (I + \frac{1}{\sqrt{n}} Y)^{-1}
\]

\[
(I + \frac{1}{\sqrt{n}} Y)^{-2} = I - \frac{2}{\sqrt{n}} Y + \frac{3}{n} Y^2 - \frac{4}{n^{3/2}} Y^3 + \frac{5}{3 n} Y^4 - \frac{1}{n^{5/2}} (6 Y^5 + \frac{5}{\sqrt{n}} Y^6) (I + \frac{1}{\sqrt{n}} Y)^{-1}.
\]

Then (as Taylor series expansions) we have

\[
\left[ (\delta + \frac{1}{\sqrt{n}} Y)' (I + \frac{1}{\sqrt{n}} Y)^{-1} (\delta + \frac{1}{\sqrt{n}} Y) \right]^{1/2} = \left[ \delta' \delta + \frac{1}{\sqrt{n}} (2 \delta' Y - \delta' Y \delta) + \frac{1}{n} (\delta' Y^2 \delta + Y' Y - 2 \delta' Y Y) + r_{1n} (Y, Z, V) \right]^{1/2} = \Delta + \frac{1}{2 \Delta \sqrt{n}} (2 \delta' Y - \delta' Y \delta) + \frac{1}{n} \left[ \frac{1}{2 \Delta} (\delta' Y^2 \delta + Y' Y - 2 \delta' Y Y) \right. \]

\[
- \frac{1}{8 \Delta^3} (2 \delta' Y - \delta' Y \delta)^2 \left] + r_{2n} (Y, Z, V),
\]

5
(18) \[ \frac{1}{\sqrt{n}} \left( \delta + \frac{1}{\sqrt{n}} \gamma \right)'(1 + \frac{1}{\sqrt{n}} \gamma)^{-1} \zeta = \frac{1}{\sqrt{n}} \delta' \zeta + \frac{1}{n} (\gamma' \zeta - \delta' \zeta) + r_{3n}(Y, Z, V), \]

(19) \[ \left( \delta + \frac{1}{\sqrt{n}} \gamma \right)'(1 + \frac{1}{\sqrt{n}} \gamma)^{-2}(\delta + \frac{1}{\sqrt{n}} \gamma)^{-1/2} \]

\[= \left[ \delta' \delta + \frac{1}{\sqrt{n}} \left( 2 \delta' \gamma - 2 \delta' \gamma' \gamma \right) + \frac{1}{n} \left( 3 \delta' \gamma' \delta + \gamma' \gamma - 4 \delta' \gamma' \gamma \right) + r_{4n}(Y, Z, V) \right]^{1/2} \]

\[= \frac{1}{\Delta} - \frac{1}{\Delta^3 \sqrt{n}} \left( \delta' \gamma - \delta' \gamma' \gamma \right) - \frac{1}{n} \left[ \frac{1}{2\Delta^3} \left( 3 \delta' \gamma' \delta + \gamma' \gamma - 4 \delta' \gamma' \gamma \right) \right. \]

\[\left. - \frac{3}{2\Delta^3} \left( \delta' \gamma - \delta' \gamma' \gamma \right)^2 \right] + r_{5n}(Y, Z, V). \]

Here \( r_{jn}(Y, Z, V), j = 1, \ldots, 5 \) is a remainder term consisting of \( 1/n^{3/2} \) times a homogeneous polynomial (not depending on \( n \)) of degree 3 in the elements of \( Y, Z, \) and \( V \) plus \( 1/n^2 \) times a homogeneous polynomial of degree 4 plus a remainder term which is \( O(n^{-5/2}) \) for fixed \( Y, Z, \) and \( V. \)

The argument of \( \Phi() \) in (14) is the product of

(20) \[ u \Delta + \frac{1}{\sqrt{n}} \left[ \frac{u}{2\Delta} \left( 2 \delta' \gamma - \delta' \gamma' \gamma \right) + \delta' \zeta \right] + \frac{1}{n} \left[ \frac{u}{2\Delta} \left( \delta' \gamma' \delta + \gamma' \gamma - 2 \delta' \gamma' \gamma \right) \right. \]

\[- \frac{u}{8\Delta^3} \left( 3 \delta' \gamma - \delta' \gamma' \gamma \right)^2 + \frac{u}{8\Delta^3} \left( \delta' \gamma - \delta' \gamma' \gamma \right) + r_{6n}(Y, Z, V) \]

and (19), which is

(21) \[ u + \frac{1}{\sqrt{n}} \left( \frac{u}{2\Delta^2} \delta' \gamma - \frac{1}{\Delta^2} \delta' \zeta \right) + \frac{1}{n} \left[ \frac{u}{\Delta^2} \left( \delta' \gamma' \delta - \delta' \gamma' \gamma \right) + \frac{u}{\Delta^4} \left( \delta' \gamma' \delta + \gamma' \gamma - 2 \delta' \gamma' \gamma \right) \right. \]

\[+ \frac{1}{\Delta} \delta' \zeta - \frac{1}{\Delta^2} \delta' \zeta - \frac{1}{\Delta^3} \delta' \gamma' \delta + \frac{1}{\Delta^3} \delta' \gamma' \zeta + r_{7n}(Y, Z, V) \]

\[= u + \frac{1}{\sqrt{n}} C(Z, V) + \frac{1}{n} D(Y, Z, V) + r_{7n}(Y, Z, V), \]
say [as the definition of $C(Z, V)$ and $D(Y, Z, V)$] and $r_{6n} (Y, Z, V)$ and $r_{7n} (Y, Z, V)$ have the same properties as $r_{jn} (Y, Z, V)$, $j = 1, \ldots, 5$.

A Taylor series expansion of $\Phi(\ )$ in (14) gives

\[
(22) \quad \Phi[u + \frac{1}{\sqrt{n}} C(Z, V) + \frac{1}{n} D(Y, Z, V) + r_{7n} (Y, Z, V)] = \Phi(u) + \frac{1}{\sqrt{n}} C(Z, V) + \frac{1}{n} \left[D(Y, Z, V) - \frac{1}{2} u C^2(Z, V)\right] + \rho_8 (Y, Z, V) + \rho_9 (Y, Z, V) + \rho_{10n} (Y, Z, V),
\]

where $\rho_8 (Y, Z, V)$ is a homogeneous polynomial (not depending on $n$ but depending on $u$) of degree 3 in the elements of $Y$, $Z$, and $V$, $\rho_9 (Y, Z, V)$ is a polynomial of degree 4, and $\rho_{10n} (Y, Z, V)$ is a remainder term, which is $O(n^{-5/2})$ for fixed $Y$, $Z$, and $V$ (and $u$).

Let $J_n$ be the set of $Y$, $Z$, and $V$ such that $|y_i| < 2\sqrt{\log n}$, $|z_i| < 2\sqrt{\log n}$, $i = 1, \ldots, p$, and $|v_ij| < 2 \log n$, $i, j = 1, \ldots, p$.

As shown in the Appendix,

\[
(23) \quad \Pr[J_n] = 1 - o(n^{-2}).
\]

The difference between $\rho_8 \Phi(\ )$ and the integral of $\Phi(\ )$ times the density of $Y$, $Z$, and $V$ over $J_n$ is $o(n^{-2})$, because $0 < \Phi(\ ) < 1$. In $J_n$ each element of $Y$, $Z$, and $V$ divided by $\sqrt{n}$ is less than a constant times $(\log n/\sqrt{n})^5$. Hence

\[
(24) \quad \rho_{10n} (Y, Z, V) < \text{constant} \times \left(\frac{\log n}{\sqrt{n}}\right)^5,
\]

and the integral of this times the density of $Y$, $Z$, and $V$ over $J_n$ is $o(n^{-2})$. 

7
Since fourth-order absolute moments of \( y, z, \) and \( \bar{y} \) exist and are bounded, the integral of \( r_9(y, z, \bar{y}) \) times the density of \( y, z, \) and \( \bar{y} \) over \( J_n \) is bounded; hence, the contribution of this term (with the factor \( n^{-2} \)) is \( O(n^{-2}) \).

The differences between \( n^{-1/2} \xi C(z, \bar{y}), n^{-1} [\xi D(y, z, \bar{y}) - \frac{1}{2} uC^2(z, \bar{y})] \) and \( n^{-3/2} r_8(y, z, \bar{y}) \) and the integrals over \( J_n \) of \( n^{-1/2} C(z, \bar{y}), n^{-1} [D(y, z, \bar{y}) - \frac{1}{2} uC^2(z, \bar{y})] \) and \( n^{-3/2} r_8(y, z, \bar{y}) \) times the density of \( y, z, \) and \( \bar{y} \), respectively, are \( O(n^{-2}) \). Thus

\[
(25) \quad \text{Pr} \left\{ \frac{W - \frac{1}{2} a}{\sqrt{a}} \leq u \right\} = \phi(u) + \phi(u) \left\{ \frac{1}{\sqrt{n}} \xi C(z, \bar{y}) + \frac{1}{n} \left[ \xi D(y, z, \bar{y}) - \frac{u}{2} \xi C^2(z, \bar{y}) \right] \right\} + \frac{1}{n^{3/2}} \xi r_8(y, z, \bar{y}) + O(n^{-2})
\]

\[
= \phi(u) + \phi(u) \left\{ \frac{1}{\sqrt{n}} \xi C(z, \bar{y}) + \frac{1}{n} \left[ \xi D(y, z, \bar{y}) \right. \right.
\]

\[
- \frac{u}{2} \xi C^2(z, \bar{y}) \} + O(n^{-2}) .
\]

because the third-order moments of the elements of \( y, z, \) and \( \bar{y} \) are either 0 or \( O(n^{-2}) \).

Since \( C(z, \bar{y}) \) is linear and homogeneous, \( \xi C(z, \bar{y}) = 0 \). Since \( (y, z) \) and \( \bar{y} \) are independent

\[
(26) \quad \xi D(y, z, \bar{y}) = -\frac{u}{\Delta^2} \xi \bar{y}^2 \bar{y} + \frac{7}{8} \frac{u}{\Delta^4} \xi (\bar{y} \delta) \frac{\bar{y} \delta}{\Delta} \frac{\bar{y} \delta}{\Delta} + \frac{1}{\Delta} \xi \bar{y} \bar{z} - \frac{1}{3} \xi \bar{y} \bar{y} \bar{z} \bar{z} \]

\[
= -\frac{u}{\Delta^2} \Delta^2 (p+1) + \frac{7}{8} \frac{u}{\Delta^4} 2\Delta + \frac{1}{\Delta} \frac{n}{N_1} p - \frac{1}{3} \frac{\Delta^2 n}{N_1}
\]

\[
= -(p - \frac{3}{4})u + (p-1) \frac{n}{N_1} \frac{1}{\Delta}
\]
since

\begin{equation}
\tilde{\xi} \delta' V^2 \delta = \tilde{\xi} \delta' \overline{V V} \delta = \Delta^2 \xi \sum_{i=1}^{p} v_{11}^2 = \Delta^2 (\xi v_{11}^2 + \sum_{i=2}^{p} \xi v_{11}^2) = \Delta^2 (p+1),
\end{equation}

\begin{equation}
\tilde{\xi} (\delta' V \delta) = \Delta^4 \xi v_{11}^2 = 2 \Delta^4.
\end{equation}

We have

\begin{equation}
\tilde{\xi} C^2 (Z, V) = \frac{u^2}{4\Delta^4} \tilde{\xi} (\delta' V \delta)^2 + \frac{1}{\Delta^4} \tilde{\xi} \delta' Z Z' \delta
\end{equation}

\begin{equation}
= \frac{u^2}{4\Delta^4} \cdot 2\Delta^4 + \frac{1}{\Delta^4} \frac{n}{N_1} \Delta^2 = \frac{1}{2} u^2 + \frac{n}{N_1}.
\end{equation}

Thus

\begin{equation}
\tilde{\xi} D(Y, Z, V) = \frac{u}{2} \tilde{\xi} C^2 (Z, V) = (p-1) \frac{n}{N_1} \frac{1}{\Delta} - (p - \frac{3}{4} + \frac{1}{2} \frac{n}{N_1}) u - \frac{1}{4} u^3.
\end{equation}

Replacing \( n/N_1 \) by its limit \( 1+k \), we have

\begin{equation}
\Pr \left\{ \frac{W-a}{\sqrt{a}} \leq u \right\} = \phi(u) + \frac{1}{n} \phi(u) \left[ \frac{(p-1)}{\sqrt{a}} (1+k) - (p - \frac{3}{4} + \frac{1}{2} k) u - \frac{1}{4} u^3 \right] + O(n^{-2})
\end{equation}

when \( \tilde{\xi} x = \mu^{(1)} \). Interchanging \( N_1 \) and \( N_2 \) gives

\begin{equation}
\Pr \left\{ \frac{W + \frac{1}{2} a}{\sqrt{a}} \leq v \right\} = \phi(v) - \frac{1}{n} \phi(v) \left[ \frac{p - 1}{\sqrt{a}} (1 + \frac{1}{k}) + (p - \frac{1}{4} + \frac{1}{2k}) v + \frac{1}{4} v^3 \right]
\end{equation}

\begin{equation}
+ O(n^{-2}),
\end{equation}

when \( \tilde{\xi} x = \mu^{(2)} \).
3. Discussion

If \( N_1 = N_2 \) and costs of misclassification are equal, the minimax classification procedure is defined by the cut-off point 0 for \( W \); a cut-off point different from 0 will increase one probability of misclassification and decrease the other. The inequality \( W \leq 0 \) is equivalent to \((W - \frac{1}{2} \sqrt{a})/\sqrt{a} \leq -\frac{1}{2} \sqrt{a}\), and \(-\frac{1}{2} \sqrt{a}\) estimates \(-\frac{1}{2} \sqrt{\Delta}\) \(-\frac{1}{2} \Delta\). For most purposes, then, one is interested in \( u \leq 0 \). Then the correction term to \( \Phi(u) \) is nonpositive; use of the normal approximation alone tends to underestimate the probability of misclassification. The correction term decreases as the distance \( \Delta \) between the two populations increases if \( p > 1 \) and for nonpositive \( u \) the correction term increases with the number of coordinates \( p \) (for fixed \( \Delta \)).

The expansions of \( \Pr\{(W - \frac{1}{2} \sqrt{a})/\sqrt{a} \leq u|\mu=\mu(1)\} \) and \( \Pr\{(W + \frac{1}{2} \sqrt{a})/\sqrt{a} \leq u|\mu=\mu(2)\} \) given by Okamoto (1963) can be obtained by the method of this paper. It is interesting that the expansions for \((W + \frac{1}{2} \sqrt{a})/\sqrt{a}\) are much simpler than the expansions for \((W + \frac{1}{2} \sqrt{a})/\sqrt{a}\) as given by Okamoto. At \( \mu = -\frac{1}{2} \Delta = -\frac{1}{2} \sqrt{a} \) (corresponding to the cut-off point 0) the correction term of order \( 1/n \) to the probability for \((W + \frac{1}{2} \sqrt{a})/\sqrt{a}\) is about \( \frac{1}{2} \) as much as for \((W + \frac{1}{2} \sqrt{a})/\sqrt{a}\).

As indicated in the introduction, the statistician may want to use the evaluation of \( \Pr\{(W - \frac{1}{2} \sqrt{a})/\sqrt{a} \leq u\} \) in order to set the cut-off point \( c = u \sqrt{a} + \frac{1}{2} \sqrt{a} \) in order to obtain a specified probability of misclassification or at least approximate a specified probability. The crudest approximation is to take \( u \) so \( \Phi(u) \) is the specified probability.
This approximation, however, is not very good; the error of the approximation is evaluated above to order $1/n^{3/2}$. The error depends on the unknown parameter if $p > 1$. To get a better approximation let $\Phi(u + \Delta u)$ be the specified probability, where

$$\Delta u = -\frac{1}{n} \frac{(p-1)(1+k)}{\sqrt{a}} - (p - \frac{1}{4} + \frac{1}{2} k)u - \frac{1}{4} u^3.$$

Then the actual probability is the specified one with an error of order $n^{-3/2}$ (because $\sqrt{a}$ is $\sqrt{a}$ with an error of order $n^{-1/2}$).

For further discussion, see Anderson (1972).

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APPENDIX

To control the errors of approximation we define the set \( J_n \) by

\[
|y_j| < 2\sqrt{\log n}, \quad |z_j| < 2\sqrt{\log n}, \quad j = 1, \ldots, p, \quad \text{and} \quad |v_{ij}| < 2 \log n, \quad i, j = 1, \ldots, p.
\]

We want to show \( \Pr[J_n] = 1 - o(n^{-2}) \).

We have when \( \Sigma = I \)

\[
(A.1) \quad \Pr\{|y_j| > 2\sqrt{\log n}\} = \frac{2}{\sqrt{2\pi}} \int_{2\sqrt{\log n}}^{\infty} e^{-\frac{v}{2}} dv
\]

\[
< e^{-\frac{1}{2} (2\sqrt{\log n})^2} \frac{1}{\sqrt{2\pi}\log n}
\]

\[
= \frac{1}{n^2 \sqrt{2\pi}\log n}
\]

\[
= o(n^{-2})
\]

by use of Mill's ratio. Then

\[
(A.2) \quad \Pr\{|y_j| < 2\sqrt{\log n}, \quad |z_j| < 2 \log n, \quad j = 1, \ldots, p\}
\]

\[
= 1 - o(n^{-2}).
\]

Now consider \( \Psi = (v_{ij}) \). The moment generating function of \( n\Psi \)

when \( \Sigma = I \) is

\[
(A.3) \quad \phi_{n\Psi} = \phi_{\Psi} \mathbf{e}^{n\sum_{i=1}^{p} \theta_{ij} s_{ij}}
\]

\[
= |I - 2\diamond|^{-1/2} \mathbf{n}
\]

where \( \Theta = \Theta' \). We use the Tchebycheff-type inequality [Chernoff (1952), for example] for an arbitrary random variable \( X \) and \( \Theta > 0 \).
\[(A.4) \quad e^{-\theta a} \varphi e^{\theta X} = \varphi e^{\theta (X-a)} \geq \Pr \{X \geq a\}.
\]

Then

\[(A.5) \quad \Pr \{v_{ii} > 2 \log n\} = \Pr \{\sqrt{n} s_{ii} - \sqrt{n} > 2 \log n\}
= \Pr \{n s_{ii} > n + 2\sqrt{n} \log n\}
\leq (1-2\theta)^{-1/2} n \ e^{-\theta (n+2\sqrt{n} \log n)}
\]

for \(0 < \theta < \frac{1}{2}\). Let \(\theta = k/\sqrt{n}\), where \(k > 1\). For \(n > 4k^2\)

\[(A.6) \quad \Pr \{v_{ii} > 2 \log n\} \leq (1-2k/\sqrt{n})^{-1/2} n \ e^{-2k \log n-k\sqrt{n}}
\leq \text{constant} \times e^{-2k \log n}
= o(n^{-2k})
= o(n^{-2}).
\]

Similarly \(\Pr \{-v_{ii} > 2 \log n\} = o(n^{-2}).\) We have for \(i \neq j\)

\[(A.7) \quad \Pr \{v_{ij} > 2 \log n\} = \Pr \{\sqrt{n} s_{ij} > 2 \log n\}
= \Pr \{n s_{ij} > 2\sqrt{n} \log n\}
\leq e^{-\theta 2\sqrt{n} \log n} \ (1-\theta^2)^{-1/2} n
\]

for \(0 < \theta < \frac{1}{2}\). Let \(\theta = k/\sqrt{n}\), where \(k > 1\). For \(n > 4k^2\)

\[(A.8) \quad \Pr \{v_{ij} > 2 \log n\} \leq \text{constant} \times e^{-2k \log n}
= o(n^{-2}).
\]

Similarly \(\Pr \{-v_{ij} > 2 \log n\} = o(n^{-2}).\) Then

\[(A.9) \quad \Pr \{|v_{ij}| < 2 \log n, i, j = 1, \ldots, p\} = 1-o(n^{-2}).
\]
REFERENCES


An observation \( x \) is to be classified as coming from \( N(\mu^{(1)}, \Sigma) \) or \( N(\mu^{(2)}, \Sigma) \). The parameters, which are unknown, may be estimated by \( \bar{x}^{(1)} \), the mean of a sample of \( N_1 \) from the first population, \( \bar{x}^{(2)} \), the mean of a sample of \( N_2 \) from the second population, and \( \bar{x} \), the pooled sample covariance. The classification statistic \( W = (\bar{x}^{(1)} - \bar{x}^{(2)}) \Sigma^{-1} \) has a distribution which depends on \( \alpha = (\mu^{(1)} - \mu^{(2)})' \Sigma^{-1} (\mu^{1} - \mu^{(2)}) \) and its limiting distribution as \( N_1 \to \infty \) and \( N_2 \to \infty \) is \( N(1, \alpha, \alpha) \) or \( N(-\frac{1}{2} \alpha, \alpha) \). An asymptotic expansion of \( (W - \frac{1}{2} a)/\sqrt{a} \) and \( (W + \frac{1}{2} a)/\sqrt{a} \) is made to order \( n^{-3/2} \), where \( a = (\bar{x}^{(1)} - \bar{x}^{(2)}) \Sigma^{-1} (\bar{x}^{(1)} - \bar{x}^{(2)}) \) and \( n = N_1 + N_2 - 2 \).
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classification statistic
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