IDENTIFICATION OF PARAMETERS BY THE DISTRIBUTION OF A MAXIMUM RANDOM VARIABLE

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Identification of Parameters by the Distribution of a Maximum Random Variable

by

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1. Introduction and Summary.

Suppose $X_1, \ldots, X_n$ are independently distributed according to $N(\mu_1, \sigma_1^2), \ldots, N(\mu_n, \sigma_n^2)$, respectively. Does the distribution of the maximum of $X_1, \ldots, X_n$ uniquely determine $(\mu_1, \sigma_1^2), \ldots, (\mu_n, \sigma_n^2)$? In other terms, can more than one set of independent normal variables give rise to the same distribution of the random variable which is the maximum of the constituent random variables? Our paper answers this question; in fact, the relevant theorem answers this question about identification of parameters on the basis of the maximum of a set of random variables for a more general family of parent distributions.

Our attention was called to this problem by an inquiry from an econometrician (Professor Takeshi Amemiya, Department of Economics, Stanford University). In an econometric model $X_1$ is defined as the quantity of a commodity which consumers would be willing to purchase under specified circumstances including the price of a unit, and $X_2$ is the quantity that would be sold by the producers under certain conditions. A simple stochastic model specifies the demand and supply
schedules as

\[ x_1 = \alpha_1 + \beta_1 p + u_1, \]

\[ x_2 = \alpha_2 + \beta_2 p + u_2, \]

where \( p \) is the price of the commodity and \( u_1 \) and \( u_2 \) are independent (unobservable) normal random variables with means 0 and variances \( \sigma_1^2 \) and \( \sigma_2^2 \), respectively. If the price is set independently of the market (for example, by an outside agency or by custom), then the quantity actually sold by the producers to the consumers at a given price \( p \) is the minimum of \( x_1 \) and \( x_2 \) given by (1.1) and (1.2). Since the econometrician wishes to determine the demand and supply schedules, he raises the question of whether observations on the quantity passing from producers to consumers at various prices determine the parameters of the model: the intercepts \( \alpha_1 \) and \( \alpha_2 \), the slopes \( \beta_1 \) and \( \beta_2 \), and the variances \( \sigma_1^2 \) and \( \sigma_2^2 \). Since the normal distribution is symmetric, the question is mathematically equivalent to the question posed in terms of the maximum of \( x_1 \) and \( x_2 \). (In the more customary equilibrium model there is a third equation equating demand to supply, \( x_1 = x_2 \). This model, in which \( x_1 = x_2 \) and \( p \) are observable random variables, represents a market in which the price adjusts so that at that price the quantity \( x_1 \) that buyers demand is equal to the quantity \( x_2 \) that sellers produce.)

In this paper we have studied the question in greater generality by considering the maximum of an arbitrary number of random variables.
Intuitively, the approach is that if the variance of one of $X_1, \ldots, X_n$ is larger than the variance of the others the distribution (or density) of the maximum in the upper tail is similar to the distribution (or density) in the upper tail of the component with maximum variance. It is therefore convenient for us to prove identifiability on the basis of a general condition on the upper tails of the densities in a certain family (Section 2). As applications of this general theorem we answer the question posed in the first paragraph in the affirmative and also assure our econometrician that the parameters in the model are identified.

It is natural to ask in what way this property of identification carries over to multivariate cases. In the econometric example a set of consumers may consider purchasing a number of commodities and a set of producers may propose to sell these commodities; for each commodity the quantity actually sold from producer to consumer at a given price is the minimum of the amount desired at that price and the amount offered by the producers at that price.

We study the multivariate case within the framework of the normal distribution, the parameters of which are the means of the variables, the variances, and the correlations between the variables. The maximum of the vector variable is defined as the vector of the maxima of the respective elements of the constituent vector variables. Consideration of the marginal distribution of the maximum of a particular coordinate of the constituent vector variables determines the set of means and variances of that coordinate (by Theorem 2.1). However, the one-to-one correspondence between the means and variances of two different coordinates of the constituent vector variables is not available from study
of the several marginal distributions. Moreover, identification of the correlation coefficients in the distributions of the vector variable cannot be made from one-dimensional distributions.

In Section 3 we treat the multivariate normal distribution where all means are zero and all the correlations are nonnegative; in this case there is identification wherever the correlation is not zero. In Section 5 we treat the case of two vector random variables; then there is complete identification. In order to treat this case we develop a generalization of Mills' ratio, which is of interest in its own right. This is given in Section 4.
2. The Univariate Case.

Theorem 2.1. Let $\mathcal{F}$ be a family of probability density functions $f$ on $\mathbb{R}_+$ which are continuous and positive to the right of some point $A$ and such that if $f$ and $g$ are any two distinct members of $\mathcal{F}$ then $\lim_{x \to \infty} \{f(x)/g(x)\}$ exists and equals either $0$ or $\infty$. Let $X_1, \ldots, X_n$ be independent random variables with respective pdf's $f_1, \ldots, f_n$ in $\mathcal{F}$, and $Y_1, \ldots, Y_n$ be independent random variables with pdf's $g_1, \ldots, g_n$ in $\mathcal{F}$. If $\max\{X_1, \ldots, X_m\}$ and $\max\{Y_1, \ldots, Y_n\}$ have identical distributions, then $m = n$ and there exists a permutation $\{k_1, \ldots, k_m\}$ of $\{1, \ldots, m\}$ such that the pdf of $Y_i$ is $f_{k_i}$, $i = 1, \ldots, m$.

Proof. By definition

\begin{equation}
(2.1) \quad \Pr[\max\{X_1, \ldots, X_m\} \leq x] = \prod_{i=1}^{m} \Pr[X_i \leq x] = \prod_{i=1}^{m} F_i(x),
\end{equation}

where $F_i(x) = \int_{-\infty}^{x} f_i(u)du$, and similarly,

\begin{equation}
(2.2) \quad \Pr[\max\{Y_1, \ldots, Y_n\} \leq x] = \prod_{i=1}^{n} G_i(x),
\end{equation}

where $G_i(x) = \int_{-\infty}^{x} g_i(u)du$, and $g_1, \ldots, g_n \in \mathcal{F}$. We are given

\begin{equation}
(2.3) \quad \prod_{i=1}^{m} F_i(x) = \prod_{i=1}^{n} G_i(x), \quad -\infty < x < \infty.
\end{equation}

Hence,
(2.4) \[ \sum_{i=1}^{m} \ln F_i(x) = \sum_{i=1}^{n} \ln G_i(x). \]

Differentiating with respect to \( x \), we have for all \( x > A \)

(2.5) \[ \sum_{i=1}^{m} \frac{f_i(x)}{F_i(x)} = \sum_{i=1}^{n} \frac{g_i(x)}{G_i(x)}. \]

By changing notation we can rewrite (2.5) as

(2.6) \[ \sum_{i=1}^{m+n} a_i \frac{f_i(x)}{F_i(x)} = 0, \]

where \( a_i = 1, i = 1, \ldots, m \), and \( a_i = -1, i = m+1, \ldots, m+n \), and \( f_{m+i} = g_i, i = 1, 2, \ldots, n \).

There exists an \( f_1 \), say \( f_1 \), such that \( \lim_{x \to \infty} [f_i(x)/f_1(x)] \) is either 0 or 1 for \( i = 2, \ldots, m+n \). Let \( I = \{ i : f_i(x)/f_1(x) \to 1 \text{ as } x \to \infty \} \). Then dividing (2.6) by \( f_1(x) \) and letting \( x \to \infty \), we have \( \sum_{i \in I} a_i = 0 \), so that \( I \) contains an even number of elements, half of which are from \( \{1, \ldots, m\} \). Thus a certain number of \( f_i \) in (2.5) are identical and are identical to the same number of \( g_i \). Subtracting these identical terms from both sides of (2.5), we have a new equation of the same form but with fewer terms. The process can be iterated until each term on one side of (2.5) is matched with one term on the other. If \( m = n \), the proposition is proved. On the other hand if \( m \neq n \), say \( m < n \), we have \( n-m \) of the \( g_i \) such that \( g_i(x) = 0 \) for all \( x > A \), contrary to the assumption about \( \mathcal{G} \). Hence, \( m = n \). Q.E.D.
Example 2.1. The family of normal densities with arbitrary means and variances satisfies the assumptions of the theorem. In fact, if

\begin{equation}
\phi(x|\mu,\sigma) = \frac{1}{\sqrt{2\pi} \sigma} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right],
\end{equation}

then as \( x \to \infty \)

\begin{equation}
\frac{\phi(x|\mu_2,\sigma_2)}{\phi(x|\mu_1,\sigma_1)} \rightarrow \begin{cases} 
0 & \text{if } \sigma_1^2 > \sigma_2^2 \text{ or if } \sigma_1^2 = \sigma_2^2 \text{ and } \mu_1 > \mu_2, \\
\infty & \text{if } \sigma_1^2 < \sigma_2^2 \text{ or if } \sigma_1^2 = \sigma_2^2 \text{ and } \mu_1 < \mu_2, \\
1 & \text{if } \sigma_1^2 = \sigma_2^2 \text{ and } \mu_1 = \mu_2.
\end{cases}
\end{equation}

Example 2.2. Let

\begin{equation}
\phi(x|\alpha,\beta,\sigma,p) = \frac{1}{\sqrt{2\pi} \sigma} \exp\left[-\frac{(x-\alpha - \beta p)^2}{2\sigma^2}\right].
\end{equation}

Then as \( x \to \infty \)

\begin{equation}
\frac{\phi(x|\alpha_2,\beta_2,\sigma_2,p_2)}{\phi(x|\alpha_1,\beta_1,\sigma_1,p_1)} \rightarrow \begin{cases} 
0 & \text{if } \sigma_1^2 > \sigma_2^2 \text{ or if } \sigma_1^2 = \sigma_2^2 \text{ and } \alpha_1 + \beta_1 p_1 > \sigma_2 + \beta_2 p_2, \\
\infty & \text{if } \sigma_1^2 < \sigma_2^2 \text{ or if } \sigma_1^2 = \sigma_2^2 \text{ and } \alpha_1 + \beta_1 p_1 < \sigma_2 + \beta_2 p_2, \\
1 & \text{if } \sigma_1^2 = \sigma_2^2 \text{ and } \alpha_1 + \beta_1 p_1 = \sigma_2 + \beta_2 p_2.
\end{cases}
\end{equation}

The parameters are identified if at least two of \( X_1, \ldots, X_n \) have densities with the same pair \( \alpha, \beta \) and different values of \( p \). (In the
econometric "equilibrium" model \( X_1 = X_2 \) and \( p \) are random variables determined as the solution of (1.1) and (1.2); any joint normal density of these two variables is consistent with various sets of \( \beta_1, \beta_2, \sigma_1^2 \) and \( \sigma_2^2 \) and hence the parameters cannot be identified.)

**Example 2.3.** Let \( \phi_\lambda(x) = 0, \ x < 0, \ \phi_\lambda(x) = \lambda e^{-\lambda x}, \ x \geq 0. \) Then

\[
\frac{\phi_{\lambda_2}(x)}{\phi_{\lambda_1}(x)} \rightarrow \begin{cases} 
0 & \text{if } \lambda_1 < \lambda_2, \\
\infty & \text{if } \lambda_1 > \lambda_2, \\
1 & \text{if } \lambda_1 = \lambda_2.
\end{cases}
\]

(2.11)

There exist many families of pdf's which do not have the property postulated in the theorem but the members of which are identified by the distribution of the maximum random variable.

**Example 2.4.** Let

\[
f_a(x) = \begin{cases} 
e^{-\left(x-a\right)}, & x \geq a, \\
0, & x < a.
\end{cases}
\]

(2.12)

If \( f_i = f_{a_i}, \ i = 1, \ldots, m, \ g_i = f_{b_i}, \ i = 1, \ldots, n, \) then

\[
\Pr[\max\{X_i, \ i = 1, \ldots, m\} \leq \min\{a_i, \ i = 1, \ldots, m\}] = 0
\]

(2.13)

implies
(2.14) \[ \min\{a_1, \ldots, a_m\} = \min\{b_1, \ldots, b_n\}. \]

Further, the form of the cdf of the maximum for values of \( x \) between the smallest and second smallest distinct \( a_i \) tells us that the number of \( a_i = \min\{a_i\} \) is the same as the number of \( b_i = \min\{a_i\} \); and also that the second smallest \( a_i = \) the second smallest \( b_i \). Proceeding this way, we reach the conclusion that \( m = n \) and \( \{a_1, \ldots, a_m\} \) is a permutation of \( \{b_1, \ldots, b_m\} \).

**Example 2.5.** Let

(2.15) \[ f_a(x) = \frac{1}{2e} \cdot |x-a|. \]

In this case, if \( f_i = f_{a_i} \) and \( g_i = f_{b_i} \), then for all sufficiently large \( x \), equation (2.3) becomes

(2.16) \[ \frac{m}{\prod_{i=1}^{m}} \left( 1 - \frac{1}{2e} \cdot x + a_i \right) = \frac{n}{\prod_{i=1}^{n}} \left( 1 - \frac{1}{2e} \cdot x + b_i \right). \]

Without loss of generality, we may assume \( m \leq n \), write \( z = e^x \), and multiply both sides of (2.16) by \( z^n \), giving

(2.17) \[ z^{n-m} \prod_{i=1}^{m} \left( z - \frac{1}{2e} a_i \right) = \prod_{i=1}^{n} \left( z - \frac{1}{2e} b_i \right). \]

Equating the zeros of the two polynomials yields the conclusion that \( m = n \) and \( \{a_1, \ldots, a_m\} \) is a permutation of \( \{b_1, \ldots, b_m\} \).
There are also examples of a family of densities whose elements are not identified by the distribution of the maximum. For example, let

\[
 f_a(x) = \begin{cases} 
  ae^{ax}, & x \leq 0, \\
  0, & x > 0 
\end{cases}
\]

(2.18)

and \( \mathcal{F} = \{ f_a : a > 0 \} \). If \( f_i = f_{a_i}, i = 1, \ldots, m \), the cdf of \( \max\{X_i\} \) is

\[
 F_{\max}(x) = \begin{cases} 
  e^{a_1 \sum_{i=1}^m x}, & x \leq 0, \\
  1, & x > 0 
\end{cases}
\]

(2.19)

and the only inference from (1) is \( \sum_{i=1}^m a_i = \sum_{i=1}^n b_i \).
3. A Special Multivariate Normal Case.

Suppose the vector \( X_i = (X_{i1}, \ldots, X_{ip})' \) has the distribution \( N(\mu_i', \Sigma_i) \), \( i = 1, \ldots, n \). Let \( Y_j = \max(X_{j1}, \ldots, X_{jp}) \) and let \( \tilde{Y} = (Y_1, \ldots, Y_p)' \). Does the distribution of \( \tilde{Y} \) determine the parameters \( \mu_1', \Sigma_1, \ldots, \mu_n', \Sigma_n \)? Consideration of the marginal distribution shows that the distribution of \( Y_j \) determines the values of the pairs \( (\mu_{j1}, \sigma_{j1}^2), \ldots, (\mu_{jn}, \sigma_{jn}^2) \) for each \( j \). (The \( j, k \)th element of \( \Sigma_i \) is \( \sigma_{jk} \).) But, the marginal distributions do not lead to correspondences between the different values of \( j \). However, if bivariate distributions are identified, then the correspondences for \( j = 1, \ldots, p \) can be made. Accordingly we turn our attention to bivariate distributions.

We consider an arbitrary number \( n \) of bivariate normal distributions but restrict them so as to minimize the kind of tedious computations which we are not able to avoid in Section 5 in the detailed discussion of the case \( n = 2 \).

Theorem 3.1. If \( \Phi_1, \ldots, \Phi_n, F_1, \ldots, F_m \) are nonsingular bivariate normal cdf's with zero means and the correlations in \( \Phi_1, \ldots, \Phi_n \) are all nonnegative, then

\[
(3.1) \quad \prod_{i=1}^{n} \Phi_i(x, y) = \prod_{i=1}^{m} F_i(x, y)
\]

implies that \( m = n \), there are no \( F_i \) with negative correlations, the number of zero correlations is the same on both sides of (3.1), and the \( F_i \) with positive correlations can be identified one-for-one with the \( \Phi_i \) having positive correlations.
Proof. By letting one variable go to infinity in the identity (3.1), we see from the one-dimensional result of Example 2.1 that the set of x-variances on the l.h.s. of (3.1) is a permutation of that on the r.h.s.; and the same is true of the y-variances. Also, \( m = n \).
Therefore, if the x-variance, correlation and the y-variance in \( \phi_i \) are respectively \( \sigma_i^2, \rho_i, \tau_i^2 \), then we may assume, without loss of generality, that the corresponding parameters for \( F_i \) are \( \sigma_i^2, r_i, \tau_i^2 \), where \( (t_1, \ldots, t_n) \) is a permutation of \( (\tau_1, \ldots, \tau_n) \).

Taking logarithms in (3.1) and differentiating with respect to \( x \), we get

\[
\sum_{i=1}^{n} \frac{1}{\sigma_i} n\left(\frac{x - \mu_i}{\sigma_i}\right) \frac{1}{\sqrt{1 - \rho_i^2}} \phi_i(x, y) = \sum_{i=1}^{n} \frac{1}{\sigma_i} n\left(\frac{x - \mu_i}{\sigma_i}\right) \frac{1}{\sqrt{1 - r_i^2}} \phi_i(x, y),
\]

where \( n(\cdot) \) and \( N(\cdot) \) are, respectively, the one-dimensional standard normal pdf and cdf.

Now, let \( \sigma = \max\{\sigma_i, i = 1, \ldots, n\} \), \( I = \{i : \sigma_i = \sigma\} \),
\( I_0(\rho) = \{i : \rho_i = 0, i \in I\} \), \( I_0(r) = \{i : r_i = 0, i \in I\} \). On dividing (3.2) by \((1/\sigma)n(x/\sigma)\) and letting \( x \to \infty \), we get

\[
\sum_{i \in I_0(\rho)} 1 = \sum_{i \in I_0(r)} 1 + \sum_{i : i \in I, r_i < 0} 1/N(y/t_i).
\]

Since the last term is strictly monotone unless \( \{i : i \in I, r_i < 0\} \) is empty, we see that there is no \( r_i < 0 \) for \( i \in I \), and that
\( I_0(\rho) \) and \( I_0(\tau) \) contain the same number of points. Hence, the terms corresponding to \( (\sigma = \sigma_i, \rho_i = 0) \) and \( (\sigma = \sigma_i, r_i = 0) \) cancel one another out in (3.2).

Next, let \( \beta = \min\{\rho_i \tau_i : i \in I - I_0(\rho)\} \), and
\[
I_1(\rho) = \{i : \rho_i \tau_i = \beta, i \in I\}, \quad I_1(\tau) = \{i : r_i \tau_i = \beta, i \in I\}.
\]
In (3.2), let \( y = \beta x / \sigma + u \), divide by \( 1 / \sigma \cdot n(x / \sigma) \) and let \( x \to \infty \), holding \( u \) fixed. Then we obtain

\[
(3.4) \quad \sum_{i \in I_1(\rho)} N \left( \frac{u}{\tau_i \sqrt{1 - \rho_i^2}} \right) = \sum_{i \in I_1(\tau)} N \left( \frac{u}{\tau_i \sqrt{1 - \tau_i^2}} \right) + \sum_{i : r_i \tau_i < \beta, i \in I} 1.
\]

Letting \( u \to \infty \), we see that \( \{i : r_i \tau_i < \beta, i \in I\} \) is empty. Differentiating with respect to \( u \) in (3.4) we get

\[
(3.5) \quad \sum_{i \in I_1(\rho)} \frac{1}{\sqrt{\tau_i^2 - \beta^2}} \cdot n \left( \frac{u}{\sqrt{\tau_i^2 - \beta^2}} \right) = \sum_{i \in I_1(\tau)} \frac{1}{\sqrt{\tau_i^2 - \beta^2}} \cdot n \left( \frac{u}{\sqrt{\tau_i^2 - \beta^2}} \right).
\]

Let \( \tau = \max\{\tau_i, t_j : i \in I_1(\rho), j \in I_1(\tau)\} \), divide by \( n \left( u / \sqrt{\tau_i^2 - \beta^2} \right) \) and let \( u \to \infty \). We see that the number of \( \tau_i = \tau \) is the same as that of \( t_i = \tau \), and that for the corresponding \( \phi_i \) and \( F_i \), we have \( \rho_i = \beta / \tau = r_i \).

Eliminating the terms with \( \tau_i = t_i = \tau \) from (3.5), we can repeat the process with the remaining largest value of the \( y \)-variance; and continuing this way, we see that we have identified
\[
\{(\sigma_i, \rho_i, \tau_i), i \in I_1(\rho)\} \text{ with } \{(\sigma_i, r_i, t_i), i \in I_1(\tau)\} \text{ one-for-one.}
\]

Eliminating these terms from (3.2), we can now repeat the process with
the next larger value of $\rho_i \tau_i$, $i \in I - I_0(\rho)$, and continuing this way, we are able to identify \{($\sigma_i, \rho_i, \tau_i$), $i \in I - I_0(\rho)$\} with \{($\sigma_i, r_i, t_i$), $i \in I - I_0(r)$\} one-for-one. We are then left with an equation similar to (3.2) but with a smaller number of terms and with $\sigma_1 < \sigma$. Iterating the procedure successively, we are thus able to identify \{($\sigma_i, \rho_i, \tau_i$) : $\rho_i > 0$, $i = 1, \ldots, n$\} with \{($\sigma_i, r_i, t_i$) : $r_i > 0$, $i = 1, \ldots, n$\} one-for-one.

Finally, removing these identical terms from both sides of (3.1), we are left with products of one-dimensional normal cdfs in $x$ and $y$ with the same set of $x$-variances appearing on both sides and also the same set of $y$-variances, but there is no unique matching of $(x,y)$ pairs among these.

For the investigation in Section 5 we need a generalization of the univariate Mills' ratio

\[(4.1) \quad n(x) \left( \frac{1}{x} - \frac{1}{x^3} \right) < 1 - N(x) < n(x) \frac{1}{x}, \quad x > 0,\]

where \(n(\cdot)\) and \(N(\cdot)\) are the standard normal pdf and cdf. (See, e.g., Feller [1], p. 175.)

**Theorem 4.1.** Let

\[(4.2) \quad \Phi(x, y) = \int_{u > x}^{v > y} \phi(u, v) \, du \, dv,\]

where

\[(4.3) \quad \phi(u, v) = \frac{1}{{2\pi \sqrt{1-\rho^2}}} \exp\left[ -\frac{u^2 - 2\rho uv + v^2}{2(1-\rho^2)} \right].\]

If \(\rho > 0, A = (x-\rho y)\sqrt{1-\rho^2} > 0, B = (y-\rho x)\sqrt{1-\rho^2} > 0\), then

\[(4.4) \quad \phi(x, y) \frac{(1-\rho^2)^2}{(x-\rho y)(y-\rho x)} \left( 1 - \frac{1}{A^2} \right) \left( 1 - \frac{1}{B^2} \right) \leq \Phi(x, y) \leq \phi(x, y) \frac{(1-\rho^2)^2}{(x-\rho y)(y-\rho x)}.\]

If \(\rho < 0, A > 0, B > 0\), then

\[(4.5) \quad \phi(x, y) \frac{(1-\rho^2)^2}{(x-\rho y)(y-\rho x)} \left( 1 - \frac{1}{A^2} \right) \left( 1 - \frac{1}{B^2} \right) \leq \Phi(x, y) \leq \phi(x, y) \frac{(1-\rho^2)^2}{(x-\rho y)(y-\rho x)}.\]
Proof:

(4.6) \[ \phi(x,y) = \int_{s,t>0} \frac{1}{2\pi \sqrt{1-\rho^2}} \exp \left[ -\frac{(x+s)^2 - 2\rho(x+s)(y+t)+(y+t)^2}{2(1-\rho^2)} \right] ds \, dt \]

\[ = \phi(x,y) \int_{s,t>0} \exp \left[ -\frac{s^2 - 2\rho st + t^2}{2(1-\rho^2)} \right] \exp \left[ -\frac{s(x-y)+t(y-\rho x)}{(1-\rho^2)} \right] ds \, dt \]

\[ = \phi(x,y)(1-\rho^2) J(A,B) , \]

where

(4.7) \[ J(A,B) = \int_{u,v>0} \exp \left[ -\frac{u^2 - 2\rho uv + v^2}{2} \right] \exp \left[ -Au - Bv \right] \, du \, dv . \]

We have (for \( u \geq 0, v \geq 0 \))

(4.8) \[ (1-\rho)(u^2 + v^2) \leq u^2 - 2\rho uv + v^2 \leq u^2 + v^2 , \quad \rho > 0 , \]

(4.9) \[ u^2 - 2\rho uv + v^2 = u^2 + v^2 , \quad \rho = 0 , \]

(4.10) \[ u^2 + v^2 \leq u^2 - 2\rho uv + v^2 \leq (1-\rho)(u^2 + v^2) , \quad \rho < 0 . \]

Let

(4.11) \[ J_a = \int_{u,v>0} \exp \left[ -\frac{a^2(u^2 + v^2)}{2} \right] \exp(-Au - Bv) \, du \, dv . \]
Then for $a^2 = 1 - \rho$

$$J_1 \leq J(A,B) \leq J_a, \quad \rho > 0,$$

$$J(A,B) = J_1, \quad \rho = 0,$$

$$J_a \leq J(A,B) \leq J_1, \quad \rho < 0.$$

Note that $J_a$ is the product of two terms of the type

$$\int_{u>0} \exp\left[-\frac{a^2u^2 + 2cu}{2}\right] du,$$

where $c = A$ or $B$. The expression (4.15) equals

$$\exp\left[\frac{c^2}{2a^2}\right] \int_{u>0} \exp\left[-\frac{(au + \frac{c}{a})^2}{2}\right] du = \exp\left[\frac{c^2}{2a^2}\right] \frac{\sqrt{2\pi}}{a} \int_{v>0} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{v^2}{2}\right) dv.$$

If $c > 0$, this is bounded above and below, respectively, by $1/c$ and $(1/c)(1-a^2/c^2)$ according to (4.1). For $\rho = 0$ the theorem follows directly from (4.1) because $\bar{\Phi}(x,y) = \bar{\Phi}(x)\bar{\Phi}(y)$. For $\rho > 0$ and $\rho < 0$ the theorem follows from (4.6), (4.12), and (4.14). Q.E.D.

Remarks. Notice that the conditions $A > 0$, $B > 0$ for $x > 0$, $y > 0$ correspond to a point $(x,y)$ in the first quadrant that is above the regression line ($Y$ on $X$) and to the right of the other ($X$ on $Y$). If $\rho \leq 0$, then $A > 0$ and $B > 0$ are automatically satisfied if $x, y > 0$.

It is of interest to compare Theorem 2 with the bound of Harkness and Godambe ([2]). The principal term in our estimate (which is also our upper bound) is
\begin{equation}
\phi(x,y) \frac{(1-\rho^2)^2}{(x-\rho y)(y-\rho x)} ,
\end{equation}

subject to the assumption that \( x - \rho y, y - \rho x > 0 \) (which is a restriction only if \( \rho > 0 \)). Since (4.17) is symmetric in \((x,y)\), let \( y = cx \), with \( 1 \leq c < 1/\rho \) if \( \rho > 0 \). Then (4.17) is

\begin{equation}
\frac{1}{2\pi\sqrt{1-\rho^2}} \exp \left[ -\frac{x^2(1-2\rho c + c^2)}{2(1-\rho^2)} \right] \frac{(1-\rho^2)^2}{x^2(1-\rho c)(c-\rho)} .
\end{equation}

In the Harkness and Godambe estimates, the principal terms in the upper and lower bounds are, respectively,

\begin{equation}
\frac{(1+\rho)^2}{x^2} \frac{1}{2\pi\sqrt{1-\rho^2}} \exp \left[ -\frac{x^2(1-2\rho + 1)}{2(1-\rho^2)} \right] = \frac{1}{2\pi\sqrt{1-\rho^2}} \frac{(1+\rho)^2}{x^2} \exp \left[ -\frac{x^2(2(1-\rho))}{2(1-\rho^2)} \right] ,
\end{equation}

and

\begin{equation}
\frac{(1+\rho)^2}{c^2x^2} \frac{1}{2\pi\sqrt{1-\rho^2}} \exp \left[ -\frac{x^2c^2(1-2\rho + 1)}{2(1-\rho^2)} \right] = \frac{1}{2\pi\sqrt{1-\rho^2}} \frac{(1+\rho)^2}{c^2x^2} \exp \left[ -\frac{x^2c^2(2(1-\rho))}{2(1-\rho^2)} \right] .
\end{equation}

Now, (4.18) = (4.19) = (4.20) if \( c = 1 \), but for \( 1 < c < 1/\rho \),

(4.18)/(4.19) \to 0 and (4.20)/(4.18) \to 0 as \( x \to \infty \).

Finally, if \( \rho > 0 \) and \( y > x/\rho \), we can still get quite good bounds from (4.4) by using \( \tilde{\Phi}(x',y) \leq \tilde{\Phi}(x,y) \leq \tilde{\Phi}(x,y') \) where \( x < x' < y \) and \( x < y' < y \), which is better than using \( \tilde{\Phi}(y,y) \leq \tilde{\Phi}(x,y) \leq \tilde{\Phi}(x,x) \).

Since the denominator in the last factor in (4.18) is maximized by choosing \( c = (\rho + 1/\rho)/2 \), \( \tilde{\Phi}(x,y') \), with \( y' = [(\rho^2 + 1)/(2\rho)]x \), might
be a reasonably good upper bound for $\phi(x, y)$ when $y > x/\rho$, $\rho > 0$; similarly, $\bar{\phi}(x', y)$, with $x' = [2\rho/(\rho^2+1)]y$ might be a reasonably good upper bound. In the case of a general bivariate normal distribution, we have

\[(4.21) \quad \int_a^b \int_b^\infty \frac{1}{2\pi \sigma \tau \sqrt{1-\rho^2}} \exp \left[ \frac{(s-\mu)^2}{\sigma^2} - \frac{2\rho(s-\mu)(t-\nu)}{\sigma \tau} + \frac{(t-\nu)^2}{\tau^2} \right] \frac{dt}{\tau} ds \]

\[= \int_a^b \int_b^\infty \phi(u, v) dv du \]

\[= \bar{\phi} \left[ \frac{a-\mu}{\sigma}, \frac{b-\nu}{\tau} \right] \]

**Corollary 4.1.** Let (4.21) be denoted by $\bar{\phi}(a, b; \mu, \nu, \sigma, \tau, \rho)$, let the integrand be denoted by $\phi(s, t; \mu, \nu, \sigma, \tau, \rho)$, and let

\[(4.22) \quad A = \frac{a-\mu}{\sigma} - \frac{b-\nu}{\tau} \sqrt{1-\rho^2}, \quad B = \frac{b-\nu}{\tau} - \frac{a-\mu}{\sigma} \sqrt{1-\rho^2}. \]

If $\rho > 0$, $A > 0$, $B > 0$, then

\[(4.23) \quad \left(1 - \frac{1}{A^2}\right)\left(1 - \frac{1}{B^2}\right) \leq \frac{\bar{\phi}(a, b; \mu, \nu, \sigma, \tau, \rho)AB}{\sigma \tau \phi(a, b; \mu, \nu, \sigma, \tau, \rho)(1-\rho^2)} < 1. \]

If $\rho \leq 0$, $A > 0$, $B > 0$, then
(4.24) \( \left(1 - \frac{1-p}{A^2}\right) \left(1 - \frac{1-p}{B^2}\right) < \frac{\Phi(a,b; \mu,\nu,\sigma,\tau,p)AB}{\sigma \tau \Phi(a,b; \mu,\nu,\sigma,\tau,p)(1-p^2)} < 1 \). 

Corollary 4.2. Let \( y - \nu = c(x-\mu) \), where \( c \) is a positive constant. Then, as \( x \to \infty \), we have

\[
\Phi(x, cx; \mu, \nu, \sigma, \tau, p) (x-\mu)^2 (\tau - c \sigma)(c \sigma - \rho \tau) \to \frac{1}{\sigma^3 \tau^3} \Phi(x, cx; \mu, \nu, \sigma, \tau, p)(1-p^2)^2
\]

for all \( c > 0 \) if \( p \leq 0 \) and for \( c \in (\rho \tau / \sigma, \tau / [\rho \sigma]) \) if \( p > 0 \).
5. The General Case of Two Multivariate Normal Distributions.

As was explained in Section 3, the solution of identifiability for the multivariate normal case depends on the solution for the bivariate normal case. We treat the case of \( n = 2 \) in terms of the notation of Section 3.

**Theorem 5.1.** If \( \phi_1, \phi_2, F_1, F_2 \) are bivariate normal cdf's such that

\[
\phi_1(x,y) \phi_2(x,y) = F_1(x,y) F_2(x,y) ,
\]

then one of the following relations holds:

(i) \( \phi_1 = F_1 \) and \( \phi_2 = F_2 \), or

(ii) \( \phi_1 = F_2 \) and \( \phi_2 = F_1 \), or

(iii) \( \phi_i(x,y) = A_i(x) B_i(y) \), \( i = 1,2 \), and

\[
F_1(x,y) = A_1(x) B_2(y), F_2(x,y) = A_2(x) B_1(y) .
\]

**Proof.** We shall give a detailed proof under the assumption that the means are all zero. The proof for the general case of arbitrary means is not different in spirit, but only involves additional tedious computations of the same kind. As in Section 3, we shall assume the parameters to be \( (\sigma_i^2, \rho_i, \tau_i^2) \), \( i = 1,2 \), on the lhs of (5.1) and \( (\sigma_i^2, r_i, t_i^2) \), \( i = 1,2 \), on the rhs. From (5.1) we obtain
\[
\begin{align*}
\text{(5.2)} \quad g(x, y) &= \frac{\partial^2}{\partial x \partial y} [\phi_1(x, y) \phi_2(x, y)] \\
&= \phi_1 \phi_2 + \frac{\partial}{\partial x} \phi_1 \frac{\partial}{\partial y} \phi_2 + \frac{\partial}{\partial y} \phi_1 \frac{\partial}{\partial x} \phi_2 + \phi_1 \phi_2 \\
&= \phi_1(x, y) \phi_2(x, y) \\
&\quad + \frac{1}{\tau_1 \sigma_2} n\left(\frac{y}{\tau_1}\right) n\left(\frac{x}{\sigma_2}\right) N\left[\frac{x - \rho_1 y}{\sqrt{1 - \rho_1^2}} \right] N\left[\frac{y - \rho_2 x}{\sqrt{1 - \rho_2^2}} \right] \\
&\quad + \frac{1}{\sigma_1 \tau_2} n\left(\frac{x}{\sigma_1}\right) n\left(\frac{y}{\tau_2}\right) N\left[\frac{y - \rho_1 x}{\sqrt{1 - \rho_1^2}} \right] N\left[\frac{x - \rho_2 y}{\sqrt{1 - \rho_2^2}} \right] \\
&\quad + \phi_1(x, y) \phi_2(x, y),
\end{align*}
\]

where \(n(\cdot)\) and \(N(\cdot)\) are univariate standard normal pdf and cdf, respectively. Also,

\[
\text{(5.3)} \quad f(x, y) = \frac{\partial^2}{\partial x \partial y} [F_1(x, y) F_2(x, y)].
\]

I. First consider the simple case \(\sigma_1 = \sigma_2, \tau_1 = \tau_2\). By consideration of the marginal distributions we have \(\sigma_1 = \sigma_2 = s_1 = s_2\), \(\tau_1 = \tau_2 = t_1 = t_2\), so that the problem can be reduced to standard form by scale transformations on \(x\) and \(y\). In this case (5.2) and (5.3) give us
\begin{equation}
\tilde{g}_2(x,y) = \frac{1}{2\pi \sqrt{1-\rho_1^2}} \exp \left( -\frac{x^2-2\rho_1 xy+y^2}{2(1-\rho_1^2)} \right) \tilde{\phi}_2(x,y)
\end{equation}

\begin{align*}
+ n(x)n(y) N \begin{bmatrix} (x-\rho_1 y) \\ \sqrt{1-\rho_1^2} \end{bmatrix} N \begin{bmatrix} (y-\rho_2 x) \\ \sqrt{1-\rho_2^2} \end{bmatrix} \\
+ n(x)n(y) N \begin{bmatrix} (y-\rho_2 x) \\ \sqrt{1-\rho_2^2} \end{bmatrix} N \begin{bmatrix} (x-\rho_1 y) \\ \sqrt{1-\rho_1^2} \end{bmatrix} \\
+ \phi_1(x,y) \frac{1}{2\pi \sqrt{1-\rho_2^2}} \exp \left( -\frac{x^2-2\rho_2 xy+y^2}{2(1-\rho_2^2)} \right) \\
= g_r(x,y).
\end{align*}

On putting \( y = x \), we have

\begin{equation}
\frac{g_r(x,x)}{n^2(x)} = \frac{1}{\sqrt{1-\rho_1^2}} \exp \left( \frac{\rho_1}{1+\rho_1} x^2 \right) \tilde{\phi}_2(x,x) + 2N \left( \frac{\sqrt{1-\rho_1}}{1+\rho_1} x \right) N \left( \frac{1-\rho_2}{1+\rho_2} x \right) \\
+ \frac{1}{\sqrt{1-\rho_2^2}} \exp \left( \frac{\rho_2}{1+\rho_2} x^2 \right) \phi_1(x,x).
\end{equation}

Suppose \( \max(\rho_1, \rho_2, r_1, r_2) = \rho_1 > 0 \) and let \( r_1 \geq r_2 \). Then as \( x \to \infty \).
\[
\frac{g_\rho(x,x)}{n^2(x)} \exp\left(\frac{\rho_1}{1+\rho_1} x^2\right) = \begin{cases} 
\frac{1}{\sqrt{1-\rho_1^2}} & \text{if } \rho_1 > \rho_2 , \\
\frac{2}{\sqrt{1-\rho_1^2}} & \text{if } \rho_1 = \rho_2 . 
\end{cases}
\]

On the other hand

\[
\frac{g_r(x,x)}{n^2(x)} \exp\left(\frac{\rho_1}{1+\rho_1} x^2\right) = \begin{cases} 
0 & \text{if } \rho_1 > r_1 \geq r_2 , \\
\frac{1}{\sqrt{1-\rho_1^2}} & \text{if } \rho_1 = r_1 > r_2 , \\
\frac{2}{\sqrt{1-\rho_1^2}} & \text{if } \rho_1 = r_1 = r_2 . 
\end{cases}
\]

Hence, if \( \rho_1 = \rho_2 \), then \( r_1 = r_2 = \rho_1 = \rho_2 \) and \( \phi_1 = \phi_2 = F_1 = F_2 \).

If \( \rho_1 > \rho_2 \), then \( r_1 = \rho_1 \) and \( r_2 = \rho_2 \); thus \( \phi_1 = F_1 \) and \( \phi_2 = F_2 \).

Now suppose \( \max(\rho_1, \rho_2, r_1, r_2) = \rho_1 = 0 \). If \( \rho_2 = 0 \),
\( (5.5) \rightarrow 4 \); then \( g_r(x,x)/n^2(x) \rightarrow 4 \), which implies \( r_1 = r_2 = 0 \). If \( \rho_2 < 0 \), \( (5.5) \rightarrow 3 \); then \( g_r(x,x)/n^2(x) \rightarrow 3 \), which implies \( r_1 = 0 \geq r_2 = \rho_2 \).

Now suppose \( \max(\rho_1, \rho_2, r_1, r_2) = \rho_1 < 0 \). Then \( (5.5) \rightarrow 2 \).

Consider
\[
\frac{g_\rho(x,x)}{n^2(x)} - 2 = \frac{1}{\sqrt{1-\rho_1^2}} \exp\left(\frac{\rho_1}{1+\rho_1} x^2\right) \phi_2(x,x)
\]

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\[
\begin{align*}
&\frac{1}{\sqrt{1-\rho_2}} \exp\left(\frac{\rho_2}{1+\rho_2} x^2\right) \phi_1(x, x) \\
&- 2\Phi\left(\sqrt{\frac{1-\rho_1}{1+\rho_1}} x\right) - 2\Phi\left(\sqrt{\frac{1-\rho_2}{1+\rho_2}} x\right) \\
&+ 2\Phi\left(\sqrt{\frac{1-\rho_1}{1+\rho_1}} x\right) N\left(\sqrt{\frac{1-\rho_2}{1-\rho_2}} x\right).
\end{align*}
\]

By Mills' ratio

\[
(5.9) \quad \Phi\left(\sqrt{\frac{1-\rho_1}{1+\rho_1}} x\right) \sim \sqrt{\frac{1+\rho_1}{1-\rho_1}} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1-\rho_1}{2(1+\rho_1)} x^2\right).
\]

Then as \( x \to \infty \)

\[
(5.10) \quad \left[ \frac{g_\rho(x,x)}{n^2(x)} - 2 \right] \sqrt{1-\rho_1^2} \exp\left(\frac{-\rho_1}{1+\rho_1} x^2\right) \to \begin{cases} 
1 & \text{if } \rho_1 > \rho_2, \\
2 & \text{if } \rho_1 = \rho_2.
\end{cases}
\]

Similarly as \( x \to \infty \)

\[
(5.11) \quad \left[ \frac{g_\tau(x,x)}{n^2(x)} - 2 \right] \sqrt{1-\rho_1^2} \exp\left(\frac{-\rho_1}{1+\rho_1} x^2\right) \to \begin{cases} 
0 & \text{if } \rho_1 > \tau_1, \\
1 & \text{if } \tau_1 = \rho_1 > \rho_2, \\
2 & \text{if } \tau_1 = \tau_2 = \rho_1.
\end{cases}
\]

We obtain identification.

II. Having disposed of the case \( \sigma_1 = \sigma_2, \tau_1 = \tau_2 \), we now consider the situation where there is at least one inequality. Without loss of
generality, we may suppose $\sigma_1 > \sigma_2$. Then in Equation (5.2), if we keep $y$ fixed and let $x \to \infty$, we have

$$
(5.12) \quad \frac{g(x,y)\sigma_1}{n(x)} \rightarrow \begin{cases} 
0 & \text{if } \rho_1 > 0, \\
\frac{1}{\tau_1} n\left(\frac{y}{\tau_1}\right) N\left(\frac{y}{\tau_1}\right) + \frac{1}{\tau_2} n\left(\frac{y}{\tau_2}\right) N\left(\frac{y}{\tau_2}\right) & \text{if } \rho_1 = 0, \\
\frac{1}{\tau_2} n\left(\frac{y}{\tau_2}\right) & \text{if } \rho_1 < 0.
\end{cases}
$$

In the same way, from (5.3) we get

$$
(5.13) \quad \frac{f(x,y)\sigma_1}{n(x)} \rightarrow \begin{cases} 
0 & \text{if } \tau_1 > 0, \\
\frac{1}{\tau_1} n\left(\frac{x}{\tau_1}\right) N\left(\frac{x}{\tau_1}\right) + \frac{1}{\tau_2} n\left(\frac{x}{\tau_2}\right) N\left(\frac{x}{\tau_2}\right) & \text{if } \tau_1 = 0, \\
\frac{1}{\tau_2} n\left(\frac{x}{\tau_2}\right) & \text{if } \tau_1 < 0.
\end{cases}
$$

Consequently, from (5.1) we see that

(a) $\rho_1 > 0 \implies \tau_1 > 0$,

(b) $\rho_1 = 0 \implies \tau_1 = 0$,

(c) $\rho_1 < 0 \implies \tau_1 < 0$ and $\tau_2 = \tau_2$ (so that $\tau_1 = \tau_1$).

In Case (a), let $y = \gamma x$, where $\gamma = \rho_1 \tau_1 / \sigma_1$, and $x \to \infty$; then, noting that $1/\sigma^2 - 2\gamma/(\sigma \tau) + \gamma^2/\tau^2 = (1-\rho^2)/\sigma^2 + (\rho/\sigma - \gamma/\tau)^2$, from (5.2) we obtain

$$
(5.14) \quad \frac{g(x,y)}{n(x)} \rightarrow \frac{1}{\tau_1 2\pi (1-\rho_1^2)}.
$$
On the other hand, looking at the expression for (5.3) similar to that for (5.2), we observe that

\[(5.15) \quad f_1(x,y) = \frac{1}{\sigma_1 t_1 \sqrt{2\pi(1-r_1^2)}} \exp \left\{-\frac{(\frac{y}{t_1} - \frac{r_1}{\sigma_1})^2 x^2}{2(1-r_1^2)}\right\},\]

so that

\[(5.16) \quad g(x,y) = \begin{cases} 0 & \text{if} \quad \frac{y}{t_1} \neq \frac{r_1}{\sigma_1}, \\ \frac{1}{t_1 \sqrt{2\pi(1-r_1^2)}} & \text{if} \quad \frac{y}{t_1} = \frac{r_1}{\sigma_1}. \end{cases}\]

Hence, on account of (5.1), we conclude that

\[(5.17) \quad \frac{y}{t_1} = \frac{r_1}{\sigma_1} \quad \text{and} \quad t_1^2(1-r_1^2) = \tau_1^2(1-\rho_1^2).\]

This implies \(t_1 = \tau_1\) and \(r_1 = \rho_1\). So, in Case (a), \(\Phi_1 = F_1\) and hence \(\Phi_2 = F_2\).

Next, in Case (b), the original equation (5.1) becomes

\[(5.18) \quad N\left(\frac{x}{\sigma_1}\right) N\left(\frac{y}{t_1}\right) \Phi_2(x,y) = N\left(\frac{x}{\sigma_1}\right) N\left(\frac{y}{t_1}\right) F_2(x,y).\]

If we now remove the common factor \(N(x/\sigma_1)\) from both sides, differentiate with respect to \(x\) and remove the common factor \(1/\sigma_2 \cdot n(x/\sigma_2)\) from both sides, we are left with
(5.19) \[ N \left( \frac{y}{\tau_1} \right) \mathcal{N} \left[ \frac{y}{\tau_2} - \frac{\rho_2}{\sigma_2} \frac{x}{\sqrt{1-\rho_2^2}} \right] = N \left( \frac{y}{t_1} \right) \mathcal{N} \left[ \frac{y}{t_2} - \frac{r_2}{\sigma_2} \frac{x}{\sqrt{1-r_2^2}} \right]. \]

If \( \rho_2 = 0 \), the lhs of (5.19) is independent of \( x \), and hence \( r_2 = 0 \); in this case, both sides of (5.1) are products of univariate cdfs, and there is no unique matching of \((x,y)\) pairs. On the other hand, if \( \rho_2 \neq 0 \) then \( r_2 \neq 0 \); and differentiating (5.19) with respect to \( x \); we have

\[ (5.20) \quad N \left( \frac{y}{\tau_1} \right) \frac{\rho_2}{\sigma_2 \sqrt{1-\rho_2^2}} \mathcal{N} \left[ \frac{y}{\tau_2} - \frac{\rho_2}{\sigma_2} \frac{x}{\sqrt{1-\rho_2^2}} \right]
= N \left( \frac{y}{t_1} \right) \frac{r_2}{\sigma_2 \sqrt{1-r_2^2}} \mathcal{N} \left[ \frac{y}{t_2} - \frac{r_2}{\sigma_2} \frac{x}{\sqrt{1-r_2^2}} \right]. \]

Setting \( y = 0 \) yields \( \rho_2 = r_2 \), and with arbitrary \( y \) putting \( x = 0 \) gives \( t_2 = t_2 \). Hence, \( \Phi_1 = F_1 \) and \( \Phi_2 = F_2 \).

Finally, in Case(y), \( \rho_1 < 0, r_1 < 0, t_1 = t_1, i = 1,2 \). If, in (5.1), we take \( x,y < 0 \) and use the fact that \( \Phi(x,y) = \Phi(-x,-y) \) for a normal cdf, we obtain

\[ (5.21) \quad 2 \prod_{i=1}^{\frac{2}{\Phi_i}(x,y)} = 2 \prod_{i=1}^{\frac{2}{F_i}(x,y)}, \quad x,y > 0. \]

Now, if we set \( y = cx \) and let \( x \to \infty \), we see from Corollary 4.2 that the asymptotic relation (4.24) holds for \( \Phi_1 \) and \( F_1 \) for all \( c > 0 \).
and also holds for $\phi_2$ and $\mathbb{S}_2$ at least for all $c$ in an interval of positive length containing the point $\tau_2/\sigma_2$. Hence, we have

$$
(5.22) \quad \frac{2}{\Pi} \frac{\phi_i(x, cx)(1-\rho_1^2)^2}{x^2 \left( \frac{1}{\sigma_1} - \frac{c \rho_1}{\tau_1} \right) \left( \frac{c}{\tau_1} - \frac{\rho_1}{\sigma_1} \right)} \exp \left[ -\frac{1}{2} \frac{Q(c)}{x^2} \right] \to 1
$$

as $x \to \infty$, for all $c$ in an interval of positive length containing the point $\tau_2/\sigma_2$. But

$$
(5.23) \quad \frac{2}{\Pi} \frac{\phi_i(x, cx)}{\mathbb{S}_i(x, cx)} = \frac{2}{\Pi} \frac{(1-r_i^2)^{1/2}}{1-\rho_1^2} \exp \left[ -\frac{1}{2} \frac{Q(c)}{x^2} \right] ,
$$

where $Q(c)$ is a quadratic polynomial, and (5.22) implies that the rhs of (5.23) has a finite positive limit for all $c$ in an interval of positive length. This can happen only if $Q(c) = 0$; the limit is then

$$
\Pi_{i=1}^2 \frac{1}{(1-r_i^2)/(1-\rho_1^2)} .
$$

Thus we have

$$
(5.24) \quad \frac{2}{\Pi} \frac{1}{(\sigma_1^2 - \frac{2 \rho_1 c}{\tau_1}) (1-\rho_1^2)^{-1}} = \frac{2}{\Pi} \frac{1}{\sigma_1^2 - \frac{2r_i c}{\tau_1}} + \frac{c^2}{\tau_1^2} (1-r_i^2)^{-1} ,
$$

and

$$
(5.25) \quad \frac{2}{\Pi} (1-\rho_1^2)^{-3/2} \left( \frac{1}{\sigma_1} - \frac{c \rho_1}{\tau_1} \right) \left( \frac{c}{\tau_1} - \frac{\rho_1}{\sigma_1} \right) = \frac{2}{\Pi} (1-r_i^2)^{-3/2} \left( \frac{1}{\sigma_1} - \frac{c r_i}{\tau_1} \right) \left( \frac{c}{\tau_1} - \frac{r_i}{\sigma_1} \right) ,
$$

both relations holding for all $c$ in an interval of positive length.

Consequently, from (5.24) we obtain

$$
(5.26) \quad \frac{1}{2} \left( \frac{1}{c_1^2} - \frac{1}{1-\rho_1^2} \right) + \frac{1}{2} \left( \frac{1}{c_2^2} - \frac{1}{1-r_2^2} \right) = 0 ,
$$
\[
(5.27) \quad \frac{1}{\tau_1} \left( \frac{1}{1-\rho_1^2} - \frac{1}{1-r_1^2} \right) + \frac{1}{\tau_2} \left( \frac{1}{1-\rho_2^2} - \frac{1}{1-r_2^2} \right) = 0,
\]
\[
(5.28) \quad \frac{1}{c_1^2} \left( \frac{\rho_1}{1-\rho_1^2} - \frac{r_1}{1-r_1^2} \right) + \frac{1}{c_2^2} \left( \frac{\rho_2}{1-\rho_2^2} - \frac{r_2}{1-r_2^2} \right) = 0.
\]

If \( r_1 = \rho_1 \), then \( \phi_1 = F_1 \), and hence \( \phi_2 = F_2 \). So, it remains only to investigate the possibility \( r_1 \neq \rho_1 \); in this case, \((5.26) \Rightarrow r_2 \neq \rho_2\), and from \((5.26)\) and \((5.27)\) we have

\[
(5.29) \quad \frac{\tau_1}{c_1} = \frac{\tau_2}{c_2} = \tau, \text{ say}
\]

But from \((5.25)\) we know that the polynomials in \( c \) on the two sides of the equation have the same zeros. The zeros of the \( \text{lhs} \) are \( \{\tau/\rho_1, \tau \rho_1, 0\} \) if \( \rho_2 = 0 \) and \( \{\tau \rho_1, \tau/\rho_1, \tau \rho_2, \tau/\rho_2\} \) if \( \rho_2 \neq 0 \) and of the \( \text{rhs} \)
\( \{\tau/r_1, \tau r_1, 0\} \) if \( r_2 = 0 \) and \( \{\tau r_1, \tau/r_1, \tau r_2, \tau/r_2\} \) if \( r_2 \neq 0 \).

Hence, the assumption that \( r_1 \neq \rho_1 < 0 \) leads to the conclusion \( r_1 = \rho_2 \)
and \( r_2 = \rho_1 \). This, together with \((5.26)\), contradicts the assumption that
\( \sigma_1 \geq \sigma_2 \). Thus in Case (\( \gamma \)) also, we must have \( r_1 = \rho_1, r_2 = \rho_2 \), so that \( \phi_1 = F_1, \phi_2 = F_2 \). Q.E.D.

REFERENCES


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20. ABSTRACT

Under what conditions does the distribution of the maximum of a set of independent random variables uniquely determine the distributions of the component random variables? In the univariate case a sufficient condition is roughly that for every two distinct densities \( f(x) \) and \( g(x) \) in the family of possible densities \( f(x)/g(x) \to 0 \) or \( \infty \) as \( x \to \infty \). Hence, the distribution of \( \max X_i, i = 1, \ldots, n \), when \( X_i \) has the distribution \( N(\mu_i, \sigma_i^2) \) uniquely determines \( \mu_i, \sigma_i^2, i = 1, \ldots, n \) (except for indexing). The identifiability property has been proved for multivariate normal distributions for \( n = 2 \) and for every \( n \) when all correlations are positive; each component of the vector of maxima consists of the maximum of that component of the \( n \) constituent vectors. Inequalities for the probability in the upper right-hand quadrant of the bivariate normal distribution have been developed; these are generalizations of Mills' ratio.