Stanford University
Departments of Mathematics and Statistics

Probability Seminar

4pm, Monday, August 13, 2018
Sequoia Hall Room 200

Refreshments served at 3:30pm in the Lounge.

Speaker: Ross Pinsky, *Technion, Haifa, Israel*

Title: A natural probabilistic model on the integers and its relation to Dickman-type distributions and Buchstab's function

Abstract:

Let $\{p_j\}_{j=1}^{\infty}$ denote the set of prime numbers in increasing order, let $\Omega_N \subset \mathbb{N}$ denote the set of positive integers with no prime factor larger than p_N and let P_N denote the probability measure on Ω_N which gives to each $n \in \Omega_N$ a probability proportional to $1/n$. This measure is in fact the distribution of the random integer $I_N \in \Omega_N$ defined by $I_N = \prod_{j=1}^{N} p_j^{X_{p_j}}$, where $\{X_{p_j}\}_{j=1}^{\infty}$ are independent random variables and X_{p_j} is distributed as $\mathrm{Geom}(1 - 1/p_j)$. We show that $\frac{\log n}{\log N}$ under P_N converges weakly to the Dickman distribution. As a corollary, we recover a classical result from multiplicative number theory–*Mertens' formula*.

Let $D_{\mathrm{nat}}(A)$ denote the natural density of $A \subset \mathbb{N}$, if it exists, and let $D_{\text{log-indep}}(A) = \lim_{N\to\infty} P_N(A \cap \Omega_N)$ denote the density of A arising from $\{P_N\}_{N=1}^{\infty}$, if it exists. We show that the two densities coincide on a natural algebra of subsets of $\mathbb{N}$. We also show that they do not agree on the sets of $n^{1/s}$-*smooth numbers* $\{n \in \mathbb{N} : p^+(n) \le n^{1/s}\}$, $s > 1$, where $p^+(n)$ denotes the largest prime divisor of n. This last consideration concerns distributions involving the Dickman function. We also consider the sets of $n^{1/s}$-*rough numbers* $\{n \in \mathbb{N} : p^-(n) \ge n^{1/s}\}$, $s > 1$, where $p^-(n)$ denotes the smallest prime divisor of n. We show that as $N \to \infty$, the probabilities of these sets, under the uniform distribution on $[N] = \{1, \ldots, N\}$ and under the P_N-distribution on Ω_N, have the same asymptotic decay profile as functions of s, although their rates are necessarily different. This profile involves the Buchstab function.