ASYMPTOTIC NORMALITY OF LINEAR COMBINATIONS OF FUNCTIONS OF ORDER STATISTICS WITH CENSORED DATA

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JOAN M. SANDER

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DIVISION OF BIOSTATISTICS
STANFORD UNIVERSITY
STANFORD, CALIFORNIA
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1. Introduction.

The class of statistics which are linear combinations of functions of order statistics has been extensively studied in recent years, and much of the literature has concentrated on looking into conditions under which such statistics are asymptotically normally distributed. As indicated by Stigler (1974) and others, a major motivation for this research has been that linear functions of order statistics of the form

\[ T_N = \frac{1}{N} \sum_{i=1}^{N} c_{iN} h(X_{(i)}) \]  

(1.1)

for estimating location and scale parameters often exhibit desirable robustness qualities, and in addition these statistics are fairly easy to calculate. The recent papers by Stigler (1974) and Shorack (1974) as well as the references contained therein will introduce the interested reader to the vast literature on this subject.

By restricting the coefficients \( c_{iN} \) to have the form

\[ c_{iN} = J\left(\frac{i}{N}\right) \quad \text{or} \quad J\left(\frac{i}{N+1}\right), \]

where \( J(\cdot) \) is a well-behaved function on \([0,1]\), a smaller but nevertheless interesting class of statistics is obtained. In this case, one may express the statistic \( T_N \) as

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\[ T_N = \frac{1}{N} \sum_{i=1}^{N} J\left(\frac{i}{N}\right) \cdot h(X_{(i)}) \]

\[ = \int_{-\infty}^{\infty} h(x) \cdot J(F_N(x)) \cdot dF_N(x) , \quad (1.2) \]

where \( F_N(\cdot) \) is the usual right-continuous empirical c.d.f. based on the ordered observations \( X_{(1)} \leq X_{(2)} \leq \ldots \leq X_{(N)} \). Asymptotic normality of \( T_N \) properly normalized, will follow by imposing suitable conditions on \( h, J \), and the distribution of the observed random variables \( X_n, n=1, \ldots, N \). The integral representation (1.2) of \( T_N \) lends itself quite naturally to modification in the single sample problem with right censored data, which is the concern of this paper.

In the model to be considered here, the random variables of interest shall be denoted \( X_1^O, \ldots, X_N^O \). These are assumed to be a random sample from a single lifetime or survival distribution \( F^O \). Observation of the event of interest \( X_n^O \) (called a "death") may be prevented by the previous occurrence of some other event \( Y_n \) (called a "loss") for each \( n=1, \ldots, N \). That is, the \( X_n^O \) are censored on the right by the \( Y_n \), so that in reality one observes only the pairs \( (X_n, \delta_n) \), \( n=1, \ldots, N \), where

\[ X_n = \min(X_n^O, Y_n) \quad \text{and} \quad \delta_n = I\{X_n^O \leq Y_n\} . \quad (1.3) \]

For the single sample problem with right censored data, the product-limit estimator [cf., Kaplan and Meier (1958)] of the right tail of the underlying lifetime distribution \( F^O \) is defined by
\[ 1 - \hat{F}^O_N(t) = \prod_{r:X(r) \leq t} \left[ \frac{N-r}{N-r+1} \right] \hat{S}(r), \quad (1.4) \]

where \( X(1) \leq X(2) \leq \cdots \leq X(N) \) are the ordered observations (censored or uncensored), and where \( \delta(r) \) indicates whether \( X(r) \) is uncensored (\( \delta(r) = 1 \)) or censored (\( \delta(r) = 0 \)). Hence, the product in (1.4) is just over the uncensored observations less than or equal to \( t \). To avoid ambiguity, we remark that \( \hat{F}^O_N(t) = 0 \) for \( t < X(1) \), and define \( \hat{F}^O_N(t) = 1 \) for \( t \geq X(N) \). The convention in case of a tie is that uncensored observations are ranked ahead of censored observations with which they are tied.

The estimator (1.4) is a right continuous step-function which changes its value only by jumps at the uncensored points. The size of the jump at each point \( t \) will be denoted \( d\hat{S}^O_N(t) \). When no censoring is present, the Kaplan-Meier estimator \( \hat{F}^O_N \) reduces to the usual empirical distribution function which assigns weight \( 1/N \) to each observation. The reader is referred to Kaplan and Meier (1958) for a complete development of the estimator.

For the purpose of this discussion, it is of interest that (1.4) is the distribution, unrestricted as to form, which maximizes the likelihood of the observations. Moreover, \( \hat{F}^O_N(t) \) is an asymptotically consistent estimator of \( F^O(t) \), and is asymptotically unbiased for \( F^O(t) \). Under the assumptions of continuous distributions and random censorship, Breslow and Crowley (1974) have shown that \( \sqrt{N}(\hat{F}^O_N(t)-F^O(t)) \), as a function of \( t \), converges weakly to a Gaussian process.

In view of the above, for the single sample problem with right censored data, it seems natural to generalize the integral form of
(1.2) by replacing the empirical c.d.f. \( F_N \) with the Kaplan-Meier distribution function \( \hat{F}_N^0 \). Thus, we consider statistics of the form

\[
T_N = \int h(x) \cdot J(\hat{F}_N^0(x)) \cdot d\hat{F}_N^0(x)
\]

\[
= \sum_{i=1}^{N} h(X(i)) \cdot J(\hat{F}_N^0(X(i))) \cdot d\hat{F}_N^0(X(i)) \tag{1.5}
\]

and ask under what conditions (1.5) is asymptotically normally distributed.

We have been content to study a truncated form of this estimator in Theorems 1 and 2 which follow, since the censoring seems to preclude any serious attempt to study the general form (1.5). In addition to results on asymptotic normality, formulas for the asymptotic mean and variance are given.

The remainder of this paper is concerned with the problem of estimating the trimmed mean

\[
\mu(p) = \frac{1}{1-2p} \int_{F_0^{-1}(1-p)}^{F_0^{-1}(1-p)} x \ dF_0(x) , \tag{1.6}
\]

where \( 0 < p < \frac{1}{2} \), and where it is assumed that the \( p^{th} \) and \( (1-p)^{th} \) percentiles of \( F_0 \) are unique. In the absence of censoring, a standard approach is to use a statistic of the form (1.2) with \( h(x) = x \) and with \( J(\frac{i}{N}) \) replaced by \( \tilde{J}(\frac{i}{N+1}) \), where \( \tilde{J}(u) = (1-2p)^{-1} \) for \( p \leq u \leq 1-p \), \( \tilde{J}(u) = 0 \) otherwise. The resulting estimator of (1.6) would then have the form.
\[ T_N(p) = \frac{1}{N-2[pN]} \sum_{i=[pN]+1}^{N-[pN]} X(i). \]  

(1.7)

Stigler (1973) gives necessary and sufficient conditions for \( T_N(p) \) to be asymptotically normally distributed.

For the single sample problem with right censored data, an estimator of (1.6) based on the Kaplan-Meier quantile process is proposed. The estimator has the form

\[ \hat{F}_N^{-1}(p) = \frac{1}{1-2p} \int_p^{1-p} \hat{F}_N^{-1}(u)du, \]  

(1.8)

where \( \hat{F}_N^{-1}(\cdot) \) is the left continuous inverse of \( \hat{F}_N \). The asymptotic normality of (1.8) follows easily from previous results [cf., Sander (1975)], and a formula for the asymptotic variance of the estimator is given in Theorem 3 and its corollary.

As a final remark, we mention that a comparison of three estimators: the Kaplan-Meier mean estimator, the Kaplan-Meier median estimator, and the Kaplan-Meier p-trimmed mean estimator in the single sample problem with right censored data is being completed, and the results will be given in a separate report by this author.

2. **Statement of Results.**

Let \( X_1^O, \ldots, X_N^O \) denote the true survival times of \( N \) individuals. The random variables \( X_n^O \) are assumed to be independently, identically distributed according to the distribution \( F^O(t) = \Pr[X_n^O \leq t] \) with \( F^O(0) = 0 \). Let \( Y_1, \ldots, Y_n \) denote the censoring random variables.
Under the model of random censorship to be considered here, these are assumed to be independent of the \( X_n^0, n=1, \ldots, N \), and to be independently, identically distributed according to the distribution \( H(t) = \Pr[Y_n \leq t] \). Hence, the observed variables \( X_n = \min\{X_n^0, Y_n\} \), \( n=1, \ldots, N \), are independently distributed with common distribution \( F \) given by

\[
1 - F = (1-F^0)(1-H).
\]  
(2.1)

Let \( \hat{F}_n(t) \) be the Kaplan-Meier estimator of the distribution \( F^0(t) \) based on the pairs \( (X_n, \delta_n) \), \( n=1, \ldots, N \), as defined in (1.3) and (1.4). Let \( V_{[a,b]}(\varphi) \) denote the total variation of a function \( \varphi \) on the interval \([a,b] \).

Define

\[
\frac{T^1}{N} = \int_0^T x \cdot J(\hat{F}_n(x)) \cdot d\hat{F}_n(x),
\]

\[
\frac{1}{\mu_T} = \int_0^T x \cdot J(F^0(x)) \cdot dF^0(x). \quad (2.2)
\]

**Theorem 1.** Suppose \( F^0 \) and \( H \) are continuous distributions.

Let \( T < \infty \) be any number such that \( F(T) < 1 \). If \( J \) is a real-valued function defined on \([0,1]\) for which

(i) \( J' \) exists and is continuous on \([0,1]\), and

(ii) \( V_{[0,F^0(T)]}(J') < \infty\),

then, as \( N \to \infty \),

\[
\sqrt{N} \left( \frac{T^1}{N} - \frac{1}{\mu_T} \right) \overset{d}{\to} \int_0^T x \, dU(x), \quad (2.4)
\]

where \( U(x) \) is a Gaussian process on \([0,T]\) with mean zero and
covariance
\[
\text{Cov} (U(s), U(t)) = J(F(t)) J(F(s)) (1-F(s))(1-F(t)) \int_0^s \frac{dF(r)}{(1-F)(1-F)}.
\]
\[s \leq t. \tag{2.5}\]

**Corollary 1.** Under the conditions of Theorem 1, as \( N \to \infty \),
\[\sqrt{N} (T_N^1 - \mu_T^1) \] is asymptotically normally distributed with mean zero and variance \( \sigma^2 \), where
\[\sigma^2 = \int_0^T \int_0^T xy \text{dCov}(U(x), U(y)). \tag{2.6}\]

**Corollary 2.** Under the conditions of Theorem 1, and the additional assumption that \( J(F(T)) = 0 \) for the \( T \) specified there, it holds that
\[\int_0^T x \text{dU}(x) = \int_0^T U(x) \text{dx} \quad \text{a.s.}, \tag{2.7}\]
and therefore \( \sqrt{N} (T_N^1 - \mu_T^1) \) is asymptotically normally distributed with mean zero and variance
\[\int_0^T \int_0^T \text{Cov}(U(x), U(y)) \text{dx dy}. \tag{2.8}\]

The proof of Theorem 1 is deferred to a later section of this paper. Corollaries 1 and 2 follow directly from Theorem 1 and its proof. The next result is a slight generalization of Theorem 1.

**Theorem 2.** Assume the conditions of Theorem 1 hold. Suppose, in addition, that
(iii) $h$ is a continuous function on $[0,T]$ with

$$V_{[0,T]}(h) < \infty.$$  \hspace{1cm} (2.9)

Define

$$T_N^2 = \int_0^T h(x) \cdot J(\hat{F}_N^O(x)) \cdot \hat{F}_N^O(x),$$

$$\mu_T^2 = \int_0^T h(x) \cdot J(F^O(x)) \cdot dF^O(x).$$ \hspace{1cm} (2.10)

Then, as $N \to \infty$,

$$\sqrt{N} (T_N^2 - \mu_T^2) \overset{d}{\to} \int_0^T h(x) \cdot dU(x),$$ \hspace{1cm} (2.11)

where $U(x)$ is the Gaussian process defined in Theorem 1.

The proof of Theorem 2 exactly parallels that of Theorem 1 and will be omitted in this paper.

Remark. It would be desirable to dispense with the condition "$T < \infty$ such that $F(T) < 1$" in Theorem 1 and let $T \to \infty$ instead. Unfortunately, the censoring is most severe for large $T$ and therefore seems to preclude any attempt to strengthen the theorem in this manner. ||

Define the $p^{th}$ quantile of the Kaplan-Meier distribution function by

$$\hat{F}_N^{-1}(p) = \inf \{t : \hat{F}_N^O(t) \geq p\},$$ \hspace{1cm} (2.12)

where the infimum of the empty set is taken to be $+\infty$. 

---
For \( 0 < p < \frac{1}{2} \), let

\[
\hat{\mu}_N(p) = \frac{1}{1-2p} \int_p^{1-p} F_N^{-1}(u) \, du ,
\]

\[
\mu(p) = \frac{1}{1-2p} \int_p^{1-p} F^{-1}(u) \, du .
\]

(2.13)

Notice that the expression for \( \mu(p) \) in (2.13) is equivalent to (1.6) when \( F^0 \) is assumed to be continuous. The statistic \( \hat{\mu}_N(p) \) in (2.13) is proposed as an estimator of the \( p \)-trimmed mean \( \mu(p) \).

**Theorem 3.** (Trimmed mean) Suppose \( F^0 \) and \( H \) are continuous distributions. Let \( \alpha \) be any number such that \( 0 \leq \alpha < 1 \) and \( H\circ F^{-1}(\alpha) < 1 \). Let \( p \) be any number satisfying \( 1 - \alpha \leq p < \frac{1}{2} \). In addition, suppose that \( F^{-1} \) has a continuous, non-zero derivative at each \( u, \ 0 < u \leq 1-p \). Then, as \( N \to \infty \),

\[
\sqrt{N} (\hat{\mu}_N(p) - \mu(p)) \overset{D}{\to} \frac{1}{1-2p} \int_p^{1-p} \frac{1}{f^0[F^{-1}(u)]} X(u) \, du ,
\]

(2.14)

where \( f^0 \) is the continuous density corresponding to the distribution \( F^0 \), and where \( X(\cdot) \) is a mean zero Gaussian process on \([0,\alpha]\) with covariance

\[
\text{Cov}(X(s),X(t)) = (1-s)(1-t) \int_0^{\min(s,t)} \frac{1}{(1-w)^2(1-H\circ F^{-1}(w))} \, dw, \ s \leq t.
\]

(2.15)

**Corollary 3.** Under the conditions of Theorem 3, \( \sqrt{N} (\hat{\mu}_N(p) - \mu(p)) \) is asymptotically normally distributed with mean zero and variance \( \tau^2 \) given by
\[ t^2 = \frac{1}{(1-2p)^2} \int_p^{1-p} \int_p^{1-p} \text{Cov}(X(s), X(t)) \cdot \frac{1}{\mathcal{I}^{\alpha-1}(s)} \cdot \frac{1}{\mathcal{I}^{\alpha-1}(t)} \, ds \, dt. \]  

(2.16)

The proof of Theorem 3 is given in a later section of this paper. Additional comments about the estimator \( \hat{\mu}_N(p) \) are contained in that section.

3. Proof of Theorem 1.

To find the limiting distribution of \( \sqrt{N} \left( \frac{T_N}{N} - \mu_T \right) \), it is convenient to introduce the following decomposition [cf., Moore(1968)]:

\[ \sqrt{N} \left( \frac{T_N}{N} - \mu_T \right) = \sqrt{N} \left[ \int_0^T x J(F_N^O(x)) \, dF_N^O(x) - \int_0^T x J(F^O(x)) \, dF^O(x) \right] \]

\[ = \int_0^T x J^*(F^O(x)) \, Z_N(x) \, dF^O(x) + \int_0^T x J(F^O(x)) \, dZ_N(x) \]

\[ + \sqrt{N} \int_0^T x \left[ J(F_N^O(x)) - J(F^O(x)) \right] \, dF_N^O(x) \]

\[ - \int_0^T x J^*(F^O(x)) \, Z_N(x) \, dF^O(x) \]  

(3.1)

where \( Z_N(x) = \sqrt{N} \left( \frac{F_N^O(x)}{F^O(x)} - F^O(x) \right), \quad x \geq 0. \)

Since \( J^* \) exists on \([0,1]\), the Mean Value Theorem implies that

\[ J(F_N^O(x)) - J(F^O(x)) = (\frac{F_N^O(x)}{F^O(x)} - F^O(x)) J^*(V_N(x)) \]  

(3.2)
where \( V_N(x) = \theta F^o_N(x) + (1 - \theta) F^o(x) \), \( 0 < \theta < 1 \), and \( \theta \) depends on the particular realization of the \( X \)'s.

Substituting (3.2) into (3.1) and rearranging, we may rewrite

\[
\sqrt{N} \left( \frac{T}{N} - \frac{1}{T} \right) = M_{1N} + M_{2N} + M_{3N} \tag{3.3}
\]

where

\[
M_{1N} = \int_0^T x J'(F^o(x)) Z_N(x) \, dF^o(x) + \int_0^T x J(F^o(x)) \, dZ_N(x),
\]

\[
M_{2N} = \int_0^T x Z_N(x) [J'(V_N(x)) - J'(F^o(x))] \, d\tilde{F}^o_N(x),
\]

\[
M_{3N} = N^{-\frac{1}{2}} \int_0^T x Z_N(x) J'(F^o(x)) \, dZ_N(x).
\]

Because \( J \) and \( F^o \) are continuous and \( Z_N \) is piecewise continuous a.s., the term \( M_{1N} \) may be rewritten as

\[
M_{1N} = \int_0^T x \, d[J(F^o(x)) \, Z_N(x)]. \tag{3.4}
\]

The rest of the proof consists of showing that \( M_{2N} \) and \( M_{3N} \) converge to zero in probability as \( N \to \infty \), and that \( M_{1N} \) converges in distribution to the random integral given by the right hand side of (2.4).

First, notice that

\[
| M_{2N} | \leq \sup_{0 \leq x \leq T} |J'(V_N(x)) - J'(F^o(x))| \cdot \sup_{0 \leq x \leq T} |Z_N(x)| \cdot \int_0^T x \, d\tilde{F}^o_N(x). \tag{3.5}
\]

Now, \( Z_N(x) \) converges weakly to a Gaussian process \( Z(x) \) on \([0, T]\)
[cf., Breslow and Crowley (1974)], hence

\[ \sup_{0 < x \leq T} |Z_N(x)| \xrightarrow{P} \sup_{0 < x \leq T} |Z(x)| . \tag{3.6} \]

Also, \( V_N(x) \) satisfies \(|V_N(x) - F^O(x)| < |\hat{F}^O_N(x) - F^O(x)|\) for each \( x \), and since

\[ \sup_{0 < x \leq T} |\hat{F}^O_N(x) - F^O(x)| = N^{-\frac{1}{2}} \sup_{0 < x \leq T} |Z_N(x)| \xrightarrow{P} 0 , \tag{3.7} \]

it follows from the (uniform) continuity of \( J' \) on \([0,1]\) that

\[ \sup_{0 < x \leq T} |J'(V_N(x)) - J'(F^O(x))| \xrightarrow{P} 0 \tag{3.8} \]

also. Finally,

\[ \int_0^T x \, d\hat{F}^O_N(x) \leq T \int_0^T d\hat{F}^O_N(x) \leq T < \infty . \tag{3.9} \]

From (3.5) and (3.6) - (3.9), it follows that

\[ |M_{2N}| \xrightarrow{P} 0 \text{ as } N \to \infty . \tag{3.10} \]

**Lemma.** Let \( \varphi(\cdot) \) be continuous a.e. in \([0,T]\), and suppose \( \{ R_i \} \) is the (random) set of discontinuity points of \( \hat{F}_N^O(\cdot) \) in \([0,T]\). Then,

\[ \int_0^T \varphi(x) (\hat{F}_N^O(x) - F^O(x)) \, d(\hat{F}_N^O(x) - F^O(x)) \]

\[ = \frac{1}{2} \int_0^T \varphi(x) (\hat{F}_N^O(x) - F^O(x))^2 + \frac{1}{2} \sum_i \varphi(R_i) (d\hat{F}_N^O(R_i))^2 . \]
Proof. Recall that \( d\hat{F}_N^o(x) = \lim_{\epsilon \to 0^+} (\hat{F}_N^o(x) - \hat{F}_N^o(x-\epsilon)) = \hat{F}_N^o(x) - \hat{F}_N^o(x-) \), and since \( F^o(x) \) is continuous, \( F^o(x) = F^o(x-) \). Then, at a point of discontinuity of \( \hat{F}_N^o(x) \),

\[
d(\hat{F}_N^o(x) - F^o(x))^2 \equiv (\hat{F}_N^o(x) - F^o(x))^2 - (\hat{F}_N^o(x-) - F^o(x-))^2
\]

\[
= \left[ (\hat{F}_N^o(x) - \hat{F}_N^o(x-)) + (\hat{F}_N^o(x-) - F^o(x)) \right]^2 - (\hat{F}_N^o(x-) - F^o(x))^2
\]

\[
= (\hat{F}_N^o(x) - \hat{F}_N^o(x-))^2 + 2(\hat{F}_N^o(x) - \hat{F}_N^o(x-)) \cdot (\hat{F}_N^o(x-) - F^o(x)) + (\hat{F}_N^o(x-) - F^o(x))^2
\]

\[
= (\hat{F}_N^o(x) - \hat{F}_N^o(x-))^2 - 2(\hat{F}_N^o(x) - \hat{F}_N^o(x-))^2 + 2(\hat{F}_N^o(x) - \hat{F}_N^o(x-)) \cdot (\hat{F}_N^o(x) - F^o(x))
\]

\[
= - (d\hat{F}_N^o(x))^2 + 2(\hat{F}_N^o(x) - F^o(x)) \cdot d(\hat{F}_N^o(x) - F^o(x)) \quad (3.12)
\]

since

\[
d(\hat{F}_N^o(x) - F^o(x)) = (\hat{F}_N^o(x) - F^o(x)) - (\hat{F}_N^o(x-) - F^o(x-)) = \hat{F}_N^o(x) - \hat{F}_N^o(x-).
\]

Hence, by (3.12), at a point of discontinuity,

\[
(\hat{F}_N^o(x) - F^o(x)) \cdot d(\hat{F}_N^o(x) - F^o(x)) = \frac{1}{2} d(\hat{F}_N^o(x) - F^o(x))^2 + \frac{1}{2} (d\hat{F}_N^o(x))^2.
\quad (3.13)
\]

At a point of continuity,

\[
d(\hat{F}_N^o(x) - F^o(x))^2 = 2(\hat{F}_N^o(x) - F^o(x)) \cdot d(\hat{F}_N^o(x) - F^o(x)).
\quad (3.14)
\]

So, taking the integral over \([0,T]\) of \( \varphi(x) \) with respect to these expressions completes the proof of the lemma. ||
Apply this lemma to the expression \( M_{\frac{3N}{2}} \) with \( \varphi(x) = x J'(F^o(x)) \).

Then
\[
M_{\frac{3N}{2}} = N^{-\frac{3}{2}} \int_0^T x Z_N(x) J'(F^o(x)) \, dZ_N(x)
\]
\[
= \sqrt{N} \int_0^T x J'(F^o(x)) \left( \hat{F}_N^o(x) - F^o(x) \right) \, d(\hat{F}_N^o(x) - F^o(x))
\]
\[
= \frac{\sqrt{N}}{2} \int_0^T x J'(F^o(x)) \, d(\hat{F}_N^o(x) - F^o(x))^2
\]
\[
+ \frac{\sqrt{N}}{2} \sum_{i=1}^N R_i J'(F^o(R_i)) \left( d\hat{F}_N^o(R_i) \right)^2
\]
\[
= \frac{\sqrt{N}}{2} \int_0^T x J'(F^o(x)) \, d(\hat{F}_N^o(x) - F^o(x))^2
\]
\[
+ \frac{\sqrt{N}}{2} \sum_{i=1}^N X_i I_{[X_i < T]} J'(F^o(X_i)) \left( d\hat{F}_N^o(X_i) \right)^2,
\] (3.15)

where the last equality holds since \( d\hat{F}_N^o(X_i) = 0 \) unless \( X_i \) is a jump point of \( \hat{F}_N^o(\cdot) \). Hence, by (3.15) it holds that
\[
M_{\frac{3N}{2}} = M_{\frac{3N}{2}}^1 + M_{\frac{3N}{2}}^2
\] (3.16)

where
\[
M_{\frac{3N}{2}}^1 = \frac{1}{\sqrt{N}} \int_0^T x J'(F^o(x)) \, d(Z_N(x))^2,
\]
\[
M_{\frac{3N}{2}}^2 = \frac{1}{2N \sqrt{N}} \sum_{i=1}^N X_i I_{[X_i < T]} J'(F^o(X_i)) \left( d\hat{F}_N^o(X_i) \right)^2.
\]

Since the assumption \( V_{[0,F^o(T)]} (J') < \infty \) implies that the function \( x \mapsto x J'(F^o(x)) \) has bounded variation, we may apply integration by
parts to obtain

\[ M_{3N}^1 = \frac{1}{2\sqrt{N}} \int_0^T x^2 J'(F^0(x)) \, d(Z_N(x))^2 \]

\[ = \frac{1}{2\sqrt{N}} \left( xJ'(F^0(x)) \, Z_N^2(x) \right) \left[ T - \frac{1}{2\sqrt{N}} \int_0^T (Z_N(x))^2 \, d[xJ'(F^0(x))] \right] \]

\[ = \frac{1}{2\sqrt{N}} \int_0^T J'(F^0(T)) \, Z_N^2(T) - \frac{1}{2\sqrt{N}} \int_0^T (Z_N(x))^2 \, d[xJ'(F^0(x))] \, . \]

But since \( |J'(F^0(T))| < \infty \) by the continuity assumptions, and \( Z_N^2(T) \xrightarrow{P} \), it follows that the first term in (3.17) converges to zero in probability as \( N \to \infty \). In addition, because \( \left\{ \sup_{0 \leq x \leq T} |Z_N(x)| \right\}^2 \)

has a limit in distribution,

\[ \left| \frac{1}{2\sqrt{N}} \int_0^T (Z_N(x))^2 \, d[xJ'(F^0(x))] \right| \]

\[ \leq \frac{1}{2\sqrt{N}} \left( \sup_{0 \leq x \leq T} |Z_N(x)| \right)^2 \left| \int_0^T d[xJ'(F^0(x))] \right| \]

\[ \leq \frac{1}{2\sqrt{N}} \left( \sup_{0 \leq x \leq T} |Z_N(x)| \right)^2 \nu([0,T])(xJ'(F^0(x))) \]  

\( \xrightarrow{P} 0 \) as \( N \to \infty \). \hspace{1cm} (3.18)

Consequently,

\[ |M_{3N}^1| \xrightarrow{P} 0 \quad \text{as} \quad N \to \infty \, . \]  

(3.19)

Now look at the other term \( M_{3N}^2 \) in expression (3.16). Applying Lemmas 1 and 2 of this author's earlier technical report [cf., Sander (1975)], for \( X_i \leq T \) we know that
\[ 0 \leq N \frac{dF_N}{N} (X_1) \leq \frac{1}{1 - H(T)} + o_p(1). \tag{3.20} \]

Therefore,

\[ |M_{3N}^2| \leq \frac{1}{2N\sqrt{N}} \sum_{i=1}^{N} |X_i| \{ X_i \leq T \} J'(F^0(X_i)) \sum_{i=1}^{N} (N \frac{dF_N}{N}(X_i))^2 I\{X_i \leq T\} \]

\[ \leq \frac{1}{2N\sqrt{N}} T \max_{0 \leq x \leq F(T)} |J'(x)| \sum_{i=1}^{N} \left( N \frac{dF_N}{N}(X_i) \right)^2 I\{X_i \leq T\} \]

\[ \leq \frac{T}{2N\sqrt{N}} \max_{0 \leq x \leq F(T)} |J'(x)| \sum_{i=1}^{N} \left( \frac{1}{1 - H(T)} + o_p(1) \right)^2 \]

\[ = \frac{T}{2N\sqrt{N}} \max_{0 \leq x \leq F(T)} |J'(x)| \left[ \left( \frac{1}{1 - H(T)} \right)^2 + o_p(1) \right] \tag{3.21} \]

\[ \xrightarrow{P} 0 \quad \text{as } N \to \infty, \]

since by assumption \( J' \) is bounded on \([0, F(T)]\) and \( H(T) < 1 \).

Hence, by (3.16), (3.19), and (3.21) we may conclude that

\[ |M_{3N}| \xrightarrow{P} 0 \quad \text{as } N \to \infty. \tag{3.22} \]

Finally, consider \( M_{1N} \) as given in (3.4). Integration by parts gives

\[ M_{1N} = \int_{0}^{T} x \, d[J(F^0(x))] Z_N(x) \]

(continued)
\[ x \oint_0^T J(F^0(x)) Z_N(x) \, dx - \oint_0^T J(F^0(x)) Z_N(x) \, dx \]
\[ = T \oint_0^T J(F^0(x)) Z_N(T) - \oint_0^T J(F^0(x)) Z_N(x) \, dx \quad (3.23) \]

Breslow and Crowley (1974) have shown that \[ Z_N(x) = \frac{1}{\sqrt{N}} \left( F^0_N(x) - F^0(x) \right) \]
converges weakly to \( Z(x) \), \( 0 \leq x \leq T \), where \( Z(x) \) is a mean zero Gaussian process with covariance

\[ \text{Cov}(Z(s), Z(t)) = (1-F^0(s))(1-F^0(t)) \oint_0^s \frac{dF^0}{(1-F)(1-F^0)} , \quad s \leq t. \quad (3.24) \]

Thus, by the continuous mapping theorem [cf., Billingsley (1968)],

\[ T \oint_0^T J(F^0(x)) Z_N(T) \xrightarrow{D} T \oint_0^T J(F^0(x)) Z(T), \]

and

\[ \oint_0^T J(F^0(x)) Z_N(x) \, dx \xrightarrow{D} \oint_0^T J(F^0(x)) Z(x) \, dx. \quad (3.25) \]

Let \( U(x) = J(F^0(x)) Z(x) \), a well-defined Gaussian process on \([0, T]\), and define the integral

\[ \oint_0^T x \, dU(x) = x \oint_0^T U(x) \, dx - \oint_0^T U(x) \, dx \]
\[ = T U(T) - \oint_0^T U(x) \, dx \quad (3.26) \]

Then, by (3.23) - (3.26), we have that \( M_{LN} \) converges in distribution to the integral given by the right hand side of (2.4). By (3.3),
(3.10) and (3.22), this completes the proof of Theorem 1.

Corollaries 1 and 2 follow directly from Theorem 1 and its proof.

4. Proof of Theorem 3.

The conditions assumed in Theorem 3 guarantee [cf., Sander (1975)]
that \( \sqrt{N} \left( \hat{F}_N^{-1}(u) - F^{-1}(u) \right) \), considered as a stochastic process in
\( u \), converges weakly to the process \( f^{-1}[f^{-1}(u)]^{-1}(X(u)), 0 \leq u \leq \alpha \),
where \( X(\cdot) \) is a mean zero Gaussian process on \([0, \alpha]\) with
covariance

\[
\text{Cov} (X(s), X(t)) = (1-s)(1-t) \int_0^s \frac{1}{(1-w)^2(1-He^{-1}(w))} \, dw, \quad s \leq t.
\]  

Hence, by the continuous mapping theorem [cf., Billingsley (1968)],

\[
\int_0^{1-p} \sqrt{N} \left( \hat{F}_N^{-1}(u) - F^{-1}(u) \right) \, du \overset{p}{\to} \int_0^{1-p} \frac{1}{f^{-1}[f^{-1}(u)]} X(u) \, du
\]

as \( N \to \infty \). Multiplication of both sides of (4.2) by \((1-2p)^{-1}\)
completes the proof of Theorem 3.

Corollary 3 follows immediately from Theorem 3.

Remark. At first, in view of formula (1.6), one might try to use the statistic

\[
\hat{\mu}_N(p) = \int_{\hat{F}_N^{-1}(1-p)}^{\hat{F}_N^{-1}(1-p)} x \, d\hat{F}_N(x)
\]

rather than the proposed \( \hat{\mu}_N(p) \) as an estimator of \( \mu(p) \).
In general, \( \hat{\mu}_{N}(p) \) is not equal to \( \hat{\mu}_{N}(p) \) for finite \( N \), but the two statistics are asymptotically equivalent. To be more precise, consider the following figure.

Then,

\[
\hat{\mu}_{N}^{-1}(p) = \int_{p}^{\hat{F}_{N}^{-1}(p)} x \, d\hat{F}_{N}^{-1}(x) = \hat{F}_{N}^{-1}(p) \ (r+s) \\
+ \sum_{X_{i} \in (a_{p}, b_{p})} X_{i} \, d\hat{F}_{N}(X_{i}) + \hat{F}_{N}^{-1}(1-p) \ (u+v),
\]

while

\[
\hat{\mu}_{N}(p) = \int_{p}^{1-p} \hat{F}_{N}^{-1}(u) \, du = \hat{F}_{N}^{-1}(p) \ (r) \\
+ \sum_{X_{i} \in (a_{p}, b_{p})} X_{i} \, d\hat{F}_{N}(X_{i}) + \hat{F}_{N}^{-1}(1-p) \ (u).
\]
Therefore, from (4.4) and (4.5),

\[
0 \leq \hat{\mu}_N(p) - \hat{\mu}_N(p) = \hat{F}_N^{-1}(p)(s) + \hat{F}_N^{-1}(1-p)(v).
\]  

(4.6)

But, since the size of the jumps of \( \hat{F}_N(x) \) is non-decreasing as \( x \) increases, the right hand side of (4.6) is

\[
\leq 2 \hat{F}_N^{-1}(1-p)(u+v)
\]

\[
= 2 \hat{F}_N^{-1}(1-p)[d\hat{F}_N(\hat{F}_N^{-1}(1-p))]
\]

\[
= \frac{2}{N} \hat{F}_N^{-1}(1-p)[N d\hat{F}_N(\hat{F}_N^{-1}(1-p))]
\]

\[
= \frac{2}{N} \left[ \hat{F}_N^{-1}(1-p) + o_p(1) \right] \left[ \frac{1}{1-H(\hat{F}_N^{-1}(1-p)+o_p(1))} \right] + o_p(1),
\]  

(4.7)

which converges to zero in probability as \( N \rightarrow \infty \). From this it follows that \( \hat{\mu}_N(p) \) and \( \hat{\mu}_N(p) \) have the same asymptotic distribution.
REFERENCES


