TESTING SURVIVAL UNDER RIGHT CENSORING AND LEFT TRUNCATION

BY

JOHN HYDE

TECHNICAL REPORT NO. 18
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John Hyde
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1. Introduction.

In some areas of life testing one is not able to observe a subject's entire lifetime. The subject may leave the study, the subject may survive to the closing date, or the subject may enter the study at some time after its lifetime has started.

Kaplan and Meier [1] have shown how to estimate the life curve in these situations. In this paper a method is presented for testing the hypothesis that the lifetime is distributed according to some predetermined distribution function $F$.

To describe the situation precisely, let $X$ be the random variable representing lifetime. Let random variables $\nu$ and $\lambda$ be the times of entry and exit, where $\nu < \lambda$ with probability one. It will be assumed that $\nu$ and $\lambda$ are independent of $X$. The quantities actually observed are $\nu, \lambda^* = \min(X, \lambda)$ and $\delta$, where $\delta = 1$ if $\nu < X \leq \lambda$ and $\delta = 0$ if $\lambda < X$. Nothing is observed if $X \leq \nu$. In effect, observations are made from the distribution of $(\nu, \lambda^*, \delta)$ conditioned on the event $\nu < X$.

Two cases are considered. In the discrete case, discussed in section 2, it is assume that $\nu, X$ and $\lambda$ take on only the values $t_0 < t_1 < t_2 \cdots$. In section 3 $X$ is assumed to have a continuous distribution function but the distribution of $(\nu, \lambda)$ is arbitrary. The approaches to these two cases are similar. First the probability space
is given a particular structure. Martingale theory is used to find a
function of \( F \) whose expected deviation from observed mortality is zero.
An unbiased estimator of the variance of the deviation is derived and
used as a normalizing factor.

In section 4 confidence intervals are found for multiples of the
hazard rate. Section 5 treats modifications of the test statistic to
improve power against specified alternatives. Comparisons with a
statistic used by Turnbull, Brown and Hu are made in section 6. The
test is also compared with conventional methods in the uncensored case.
An application is worked out in section 7.

2. The Discrete Case.

This case assumes the random variables \( \nu, X \) and \( \lambda \) can take only
the values \( t_0 < t_1 < t_2 \ldots \). This is relevant when the subjects can
be observed only at particular times. Since the time scale is unimportant,
let \( t_1 = i \).

Call the time interval from \( i-1 \) to \( i \) the \( i \)-th period. Let
\( p_i = P[X=i|X > i-1] \). Thus \( p_i \) is the probability of dying by the end
of the \( i \)-th period, given survival through the beginning of the \( i \)-th
period.

Construct observations as follows: First choose \( \nu \) and \( \lambda \) from
the distribution of \( (\nu, \lambda)|\nu < X \). The subject then enters the study at
time \( \nu \) (i.e., at the beginning of period \( \nu + 1 \)). Flip a coin with success
probability \( 1 - p_{\nu+1} \). If a failure occurs, the subject is observed dead at
time \( \nu + 1 \), so \( X = \nu + 1 \). If a success occurs, the subject moves into period
and a second, independent, coin is flipped which has probability
\(1-p_{\nu+2}\) of success. Continue this process until either a failure occurs
in the \(i\)-th period, in which case \(X = i, \delta = 1\) and \(\lambda^* = \lambda\); or else the
subject survives through the end of period \(\lambda\), in which case \(\lambda^* = \lambda\)
and \(\delta = 0\). A straightforward exercise shows that this construction
gives the correct distribution for \((\nu, \lambda^*, \delta)\).

In what follows it is convenient to assume that the coins for all
of the periods \(i = 1, 2, \ldots\) are flipped before \(\nu\) is chosen. Once \(\nu\)
is determined ignore all coin flips for any periods before period \(\nu+1,\)
even if there are failures among them. \(X\) is then the number of the
first period \(> \nu\) in which a coin flip produces a failure. This assumption,
which is convenient for theoretical reasons, in no may changes the distri-
bution of \((\nu, \lambda^*, \delta)\).

Let the random variables \(Y_i\) correspond to the coin flips by setting
\(Y_i = -p_i\) if a success is observed (the subject remains alive) and setting
\(Y_i = 1-p_i\) if a failure occurs. Then \(E[Y_i] = 0\). The \(Y_i\) are independent
of each other and of \((\nu, \lambda)\).

Define the process \(Z_n = \sum_{i=1}^{\nu} Y_i\). Observe that \(\{Z_n\}\) forms a
martingale with respect to the \(\sigma\)-algebras \(B_n = B(\nu, Y_1, \ldots, Y_n)\). The
inclusion of \(B(\nu)\) in all of the \(\sigma\)-algebras makes \(X\) a stopping time.
By the martingale stopping theorem,

\[ E[Z_{\min(X,n)}] = E[Z_1] = 0 \]

for any finite \(n \geq 1\). Use the independence of \((\nu, \lambda)\) and \(\{Z_n\}\) to
conclude
\[ E[Z_\nu] = E[E[Z_\nu | \nu]] = E[0] = 0 \]

and

\[ E[Z_{\lambda*}] = E[E[Z_{\min(X, \lambda)} | \lambda]] = E[0] = 0 . \]

Combine these results to see that

\[ 0 = E[Z_{\lambda*} - Z_\nu] = E[ \sum_{i=\nu+1}^{\lambda*} Y_i] \]
or

\[ E[\delta - \sum_{i=\nu+1}^{\lambda*} p_i] = 0 . \]

In a sense, summing the \( p_i \)'s over the periods of observation results in an estimate of the probability of death.

An estimator of the variance can be obtained by a similar device.

Consider the slightly more complicated process defined by

\[ W_n = Z_n^2 - \sum_{i=1}^{n} p_i (1-p_i) . \]

This is also a martingale:

\[ E[W_{n+1} | B_n] = E[Z_{n+1}^2 - \sum_{i=1}^{n+1} p_i (1-p_i) | B_n] \]

\[ = E[Y_{n+1}^2 + 2Y_{n+1}Z_n + Z_n^2 - \sum_{i=1}^{n+1} p_i (1-p_i) | B_n] \]

\[ = p_{n+1}(1-p_{n+1}) + 2Z_n \cdot E[Y_{n+1} | B_n] + Z_n^2 - \sum_{i=1}^{n+1} p_i (1-p_i) \]
\[ Z_n^2 - \sum_{i=1}^{n} p_i (1-p_i) = W_n. \]

Therefore \( E[W_n] = E[W_1] = E[Y_1^2] - p_1 (1-p_1) = 0 \), so that \( E[W_\nu] = 0 \) and \( E[W_{\lambda \nu}] = 0 \).

Now expand the expression for variance,

\[
\text{Var}[Z_{\lambda \nu} - Z_{\nu}] = E[(Z_{\lambda \nu} - Z_{\nu})^2]
\]

\[
= E[Z_{\lambda \nu}^2 - 2Z_{\lambda \nu}Z_{\nu} + Z_{\nu}^2]
\]

\[
= E[Z_{\lambda \nu}^2 - Z_{\nu}^2 - 2Z_{\nu}(Z_{\lambda \nu} - Z_{\nu})].
\]

The third term goes away since, by independence of \( \{Y_i\} \) and \( \nu \),

\[
E[Z_{\nu}(Z_{\lambda \nu} - Z_{\nu})] = E[E[Z_{\nu}(Z_{\lambda \nu} - Z_{\nu})|\nu]]
\]

\[= E[E[\sum_{i=\nu+1}^{\lambda \nu} Y_i (\sum_{i=\nu+1}^{\lambda \nu} Y_i)|\nu]]
\]

\[= E[0] = 0.\]

Apply the martingale stopping theorem to \( W_\nu \) and \( W_{\lambda \nu} \) to conclude

\[
\text{Var}[Z_{\lambda \nu} - Z_{\nu}] = E[\sum_{i=\nu+1}^{\lambda \nu} p_i (1-p_i)].
\]
Denote this quantity by $\sigma^2$. If $E[\lambda^*] < \infty$ then $\sigma^2$ will be finite.

Now let $(v_j, \lambda_j^*, \delta_j)$ $j=1, \ldots, n$ be an i.i.d. sample. By the central limit theorem

$$\frac{\sum_{j=1}^{n} [\delta_j - \sum_{i=v_j+1}^{\lambda_j^*} p_i]}{\sqrt{n} \sigma} \xrightarrow{d} \text{N}(0,1) \quad n \to \infty$$

Use the strong law of large numbers to see that

$$\sqrt{n} \sum_{j=1}^{n} [\delta_j - \sum_{i=v_j+1}^{\lambda_j^*} p_i]/n \xrightarrow{a.s.} \sigma \quad n \to \infty$$

Put the pieces together to get

$$\frac{\sum_{j=1}^{n} [\delta_j - \sum_{i=v_j+1}^{\lambda_j^*} p_i]}{\sqrt{n} \sum_{j=1}^{n} [\sum_{i=v_j+1}^{\lambda_j^*} p_i]} \xrightarrow{d} \text{N}(0,1), \quad (1)$$

provided that $\sigma^2 < \infty$. A sufficient condition for (1) is that $E[\lambda^*] < \infty$.

The expression in (1) is the test statistic. Observe that $\sum_{j=1}^{n} \delta_j$ is simply the number of deaths observed, while $\sum_{j=1}^{n} \sum_{i=v_j+1}^{\lambda_j^*} p_i$ can be viewed as an estimator of the number of deaths based on the time spent in the study. The statistic is the normalized difference between observed mortality and expected mortality.
This can be generalized to situations in which subjects may return to the study after leaving. For example, define entry times \( \nu^{(1)}, \nu^{(2)} \), and exit times \( \lambda^{(1)}, \lambda^{(2)} \) so that \( \nu^{(1)} < \lambda^{(1)} < \nu^{(2)} < \lambda^{(2)} \) and so that \( (\nu^{(1)}, \lambda^{(1)}, \nu^{(2)}, \lambda^{(2)}) \) is independent of \( \{Z_n\} \). Employ the methods of this section to see that each subject contributes the term
\[
8 - \left[ \sum_{i=\nu^{(1)}+1}^{\lambda^{(1)}} p_i + \sum_{i=\nu^{(2)}+1}^{\lambda^{(2)}} p_i \right]
\]
to the numerator of the statistic. Note that the covariance of the two summations is zero to conclude that each subject contributes the term
\[
\sum_{i=\nu^{(1)}+1}^{\lambda^{(1)}} p_i (1-p_i) + \sum_{i=\nu^{(2)}+1}^{\lambda^{(2)}} p_i (1-p_i)
\]
to the variance.

3. **The Continuous Case.**

When the distribution function \( F \) is continuous coin flipping is inappropriate, so the martingale must be generated in a different way. As before, choose \( \nu \) and \( \lambda \) from the distribution of \( (\nu, \lambda) | \nu < X \).

Generate \( X_1 \) from the distribution \( F \). If \( X_1 > \nu \) then set \( X = X_1 \).

If \( X_1 \leq \nu \) continue to generate i.i.d. observations \( X_i \) until the first \( i \) is reached for which \( X_i > \nu \). Say that \( i(\nu) = i \) and \( X = X_i(\nu) \), then \( (\nu, X, \lambda) \) will have the correct distribution. This can be checked by writing the distribution functions as integrals of conditional probabilities.

In analogy with the discrete case it is convenient to prepare the sequence \( \{X_i\} \) before choosing \( \nu \). Because of the way the sequence is used, it is only necessary to consider the sequence \( \{Y_i\} \) of record values. An observation is a record if its value is strictly greater than the previous record. If \( X_i \) is not a record, \( Y_i \) is set equal to the record
up to that time; if \( X_i \) is a record, \( Y_i = X_i \). In this framework \( X \)
has the value of the first record greater than \( \nu \), also \((\nu, X, \lambda)\)
\[ = (\nu, Y_i(\nu), \lambda). \]

Define the record counting process by \( N_t = 1 \) if and only if
\[ Y_i \leq t < Y_{i+1}. \]
Then \( \delta = N_{\lambda^*} - N_{\nu} \), i.e. a death is observed when the
record \( Y_i(\nu) \) occurs before \( \lambda \). From the theory of stochastic processes
[2] \( N_t \) is a non-homogeneous Poisson process having mean \( -\ln(1-F(t)) \).
Note that if a hazard rate \( h \) exists, then \( \int_0^t h(u) \, du = -\ln(1-F(t)) \).

Create a martingale from \( N_t \) by letting \( B_t = B(\nu, N_s, s \leq t) \)
and \( Z_t = N_t + \ln(1-F(t)) \). Take \( s < t \) and use the property of
independent increments to show that \( [Z_t] \) is a martingale:

\[
E[Z_t | B_s] = E[N_t + \ln(1-F(t)) | B_s]
\]
\[ = \ln(1-F(t)) + E[N_s + N_t - N_s | B_s] \]
\[ = \ln(1-F(t)) + N_s + E[N_t - N_s | B_s] \]
\[ = \ln(1-F(t)) + N_s - \ln(1-F(t)) + \ln(1-F(s)) \]
\[ = N_s + \ln(1-F(s)) \]
\[ = Z_s. \]

Observe that \( X \) is a stopping time for \( \{N_t\} \). Use the independence
of \((\nu, \lambda)\) and \( \{N_t\} \) as in the discrete case to see \( E[N_\nu] = 0 \) and
\( E[N_{\lambda^*}] = 0 \). Thus
\[ E[Z_{\nu^*} - Z_{\nu}] = 0, \]

or

\[ E[N_{\nu^*} - N_{\nu} + \ln(1-F(\lambda^*)) - \ln(1-F(\nu))] = 0, \]

so

\[ E[\delta + \ln(1-F(\lambda^*)) - \ln(1-F(\nu))] = 0. \]

The observation that \( \text{Var}[Z_t] = \text{Var}[N_t] = -\ln(1-F(t)) \) motivates defining the martingale \( W_t = Z_t^2 + \ln(1-F(t)) \). An application of the martingale stopping theorem to this process results in establishing that

\[ \text{Var}[Z_{\lambda^*} - Z_{\nu}] = E[-\ln(1-F(\lambda^*)) + \ln(1-F(\nu))] . \]

The arguments for this and the fact that \( W_t \) is a martingale parallel those used in the discrete case. Use the above results to obtain

\[
\frac{\sum_{j=1}^{n} (\delta_j - [\ln(1-F(\nu_j)) - \ln(1-F(\lambda_j^*)))])}{\sqrt{\sum_{j=1}^{n} [\ln(1-F(\nu_j)) - \ln(1-F(\lambda_j^*)))]}} \xrightarrow{n \to \infty} \mathcal{N}(0,1)
\]

for an i.i.d. sequence \( \{(\nu_j, \lambda_j^*, \delta_j)\} \) when \( \text{Var}[Z_{\lambda^*} - Z_{\nu}] \) is finite.

If a hazard rate \( h \) exists, the statistic can be written as

\[
\frac{\sum_{j=1}^{n} (\delta_j - \int_{\nu_j}^{\lambda_j^*} h(z)dz)}{\sqrt{\int_{\nu_j}^{\lambda_j^*} h(z)dz}} ,
\]

which more closely resembles the version derived in section 2. Note that
h(z) need not be bounded, so it may require more than the condition 
E[λ* < ∞] to insure finiteness of the variance.

This can also be generalized to the situation described at the end of section 2. The modifications are analogous to those needed in the discrete case.

4. **Approximate Confidence Sets for Multiples of the Hazard Rate.**

A natural companion of the testing problem is the determination of confidence sets. One simple case occurs when we consider alternatives 
F_c for the continuous case where h_c(z) = ch(z). (More generally, F_c satisfies -ln(1-F_c(z)) = -c·ln(1-F(z)) when a hazard rate does not exist.)

Let \( d = \sum_{j=1}^{n} e_j \) and \( e = \sum_{j=1}^{n} [-\ln(1-F(\lambda^*_j)) + \ln(1-F(v_j))] \). The test statistic for \( F_c \) as the null hypothesis is

\[
\frac{d - c \cdot e}{\sqrt{c \cdot e}}.
\]

Note that this is monotonically decreasing in \( c \). The problem of finding confidence sets boils down to solving for \( c \) in

\[
\frac{d - c \cdot e}{\sqrt{c \cdot e}} = \phi^{-1}(\alpha).
\]

where \( \phi^{-1}(\alpha) \) is some percentile of the normal distribution. This only requires the quadratic formula. Use the positive solution for \( \sqrt{c} \) to obtain

\[
c = \left( \frac{\phi^{-1}(\alpha) \pm \sqrt{\phi^{-1}(\alpha)^2 - 4d}}{4e} \right)^2.
\]
Observe from this that an approximate median unbiased estimator of $c$ is simply $d/e$.

5. Weighting and Power Considerations.

5.1 The Absolutely Continuous Case.

Consider the continuous case first, and assume that a hazard rate $h(z)$ exists. In section 3 we employed the process $N_t = \int_0^t d\mu$ where $\mu$ was Poisson random measure with cumulative mean measure $m(t) = -\ln(1-F(t))$. As a generalization, consider the process $\tilde{N}_t = \int_0^t w d\mu$ where $w(z)$ is some weighting function. Work with simple functions and take limits to see that

$$E[\tilde{N}_t] = \int_0^t w dm = \int_0^t w(z)h(z) dz,$$

and

$$\text{Var}[\tilde{N}_t] = \int_0^t w^2 dm = \int_0^t w^2(z)h(z) dz.$$

Also observe $\tilde{N}_{*} - \tilde{N}_{\nu} = w(\lambda^*)s$.

Now modify the derivation in section 3 by using the martingales $\tilde{Z}_t = \tilde{N}_t - E[\tilde{N}_t]$ and $\tilde{W}_t = \tilde{Z}_t^2 - \text{Var}[\tilde{N}_t]$ to conclude

$$\frac{\sum_{j=1}^{n} [w(\lambda^*_j)\delta_j - \int_{\nu_j}^{\lambda^*_j} w(z)h(z)dz]}{\sqrt{\sum_{j=1}^{n} \int_{\nu_j}^{\lambda^*_j} w^2(z)h(z)dz}} \xrightarrow{n \to \infty} N(0,1).$$

provided $\text{Var}[\tilde{N}_{*} - \tilde{N}_{\nu}] < \infty$. 
Attention is naturally drawn to the question of how to choose the weighting function \( w(z) \) so as to give good power in testing a hazard rate \( h_0 \) against an alternative \( h_1 \). One approach is simply to compute the power and choose \( w(z) \) to maximize the expression obtained, but this leads to difficulties. However, a reasonable approximate solution can easily be obtained using a heuristic approach.

Let \( T = \sum_{j=1}^{n} w(\lambda_j^*) \delta_j \). Under \( h_0 \), and conditional on \( \{v_j\} \) and \( \{\lambda_j^*\} \), \( T \) has mean \( \frac{n}{\int v_j} \sum_{j=1}^{n} \lambda_j^* w(z)h_0(z)dz \) and variance \( \frac{n}{\int v_j} \sum_{j=1}^{n} \lambda_j^* w^2(z)h_0(z)dz \), while under \( h_1 \), \( T \) has conditional mean \( \frac{n}{\int v_j} \sum_{j=1}^{n} \lambda_j^* w(z)h_1(z)dz \) and variance \( \frac{n}{\int v_j} \sum_{j=1}^{n} \lambda_j^* w^2(z)h_1(z)dz \). Good power should be obtained when these distributions are "well separated." One measure of separation is the coefficient of variation of the distribution of the difference. This quantity is

\[
\frac{n}{\int v_j} \sum_{j=1}^{n} \lambda_j^* w(z)\frac{h_1(z)-h_0(z)}{h_0(z)+h_1(z)}dz \]

which can be rewritten as

\[
\frac{n}{\int v_j} \sum_{j=1}^{n} \lambda_j^* w(z)\frac{h_1(z)-h_0(z)}{h_0(z)+h_1(z)}dz \]

This is essentially the problem of maximizing the inner product of \( w(z) \) with \( \frac{h_1(z)-h_0(z)}{h_0(z)+h_1(z)} \), subject to a constraint on the norm

\[
\|w\| = \frac{n}{\int v_j} \sum_{j=1}^{\infty} w^2(z)h_0(z) + h_1(z)I[v_j < z \leq \lambda^*_j]dz \]

Here \( I \) is the indicator function. The decision to maximize means the sign of \( w \) will be correct.
for use in the one-sided test in which \( h_0 \) is rejected for large values of the test statistic. Minimizing would only change the sign of the solution. One solution is

\[
 w(z) = \frac{(h_1 - h_0)}{(h_0 + h_1)}.
\]

This measures the size of the difference in hazard rates compared to the average hazard rate. In the case of proportional hazard rates, \( w \) becomes a constant function.

Since the solution is the same for any values of \( \nu \) and \( \lambda^* \), it is reasonable to expect the solution to provide good power unconditionally. Although this choice of \( w(z) \) may not be an optimal one, it seems to be a sensible one and has the pleasant feature that it is very easy to compute from the hazard rates.

5.2 The Discrete Case.

In the derivations in section 2 replace \( Y_i \) with \( w_i Y_i \) to get the asymptotic normality of the weighted statistic:

\[
\sum_{j=1}^{n} \left( \lambda^*_j \delta_j \sum_{i=v_j} \left( w_i p_i \right) \right)
\frac{1}{\sqrt{\sum_{j=1}^{n} \lambda^*_j \sum_{i=v_j} w_i^2 p_i (1-p_i)}} \overset{d}{\rightarrow} N(0,1).
\]

The problem of choosing optimal weights is still difficult. However, the analogue of the heuristic analysis yields a tractable problem of maximizing
\[ \sum_{j=1}^{n} \sum_{i=v_j}^{\lambda_j^*} w_i (p_i^0 - p_i^1) / \sqrt{\sum_{j=1}^{n} \sum_{i=v_j}^{\lambda_j^*} w_i^2 (p_i^0 (1-p_i^0) + p_i^1 (1-p_i^1))} . \]

A solution is

\[ w_i = (p_i^0 - p_i^1) / (p_i^0 (1-p_i^0) + p_i^1 (1-p_i^1)) . \]

For proportional probabilities \((p_i^1 = c p_i^0)\) the weights simplify to

\[ w_i = 1 / (1 - (c-1)p_i^0) . \]

6. **Comparisons with Other Procedures.**

6.1 **The Actuarial Prediction Test.**

In analyzing heart transplant data, Turnbull, Brown and Hu [3] considered a test statistic which can be written as

\[ \sum_{j=1}^{n} \left[ (1-S_j) - q_j \right] / \sqrt{\sum_{j=1}^{n} q_j (1-q_j)} \]

where \( q_j = (1-F(\lambda_j)) / (1-F(v_j)) \), a conditional probability of survival.

Observe that this test uses only the censoring time \( \lambda_j \) which does not incorporate information from \( X_j \), as does \( \lambda_j^* \).

If the \( q_j \) are close to 1 the statistic can be made to look more like the hazard rate statistic. Use the approximations \( \ln(q_i) \approx q_i - 1 \).
and \( q_1(1-q_1) \approx q_1 \) to obtain

\[
\frac{\sum_{j=1}^{n} [ -s_j + (1-q_j) ]}{\sqrt{\sum_{j=1}^{n} q_j (1-q_j)}} \approx \frac{\sum_{j=1}^{n} s_j + \ln(q_j)}{\sqrt{\sum_{j=1}^{n} -\ln(q_j)}} = -\frac{\sum_{j=1}^{n} [s_j - \ln[(1-F(v_j))/(1-F(\lambda_j))]}}{\sqrt{\sum_{j=1}^{n} \ln[(1-F(v_j))/(1-F(\lambda_j))]}},
\]

which is the negative of the statistic of section 3 with \( \lambda_j \) replacing \( \lambda_j^* \).

6.2 No Censoring.

In there is no censoring, the only times which are observed are \( v_j \) and \( X_j; s = 1 \) for each subject. Transform the data by letting \( U_j = -\ln(1-F(X_j))/(1-F(v_j)) \). The one-sided test of section 3 is

\[
\text{reject } H_0 \text{ if } \frac{n - \sum_{j=1}^{n} U_j}{\sqrt{\sum_{j=1}^{n} U_j}} > k.
\]

Write the condition as

\[
n - \sum_{j=1}^{n} U_j - k \sqrt{\sum_{j=1}^{n} U_j} > 0.
\]

It is easy to check that the quadratic polynomial in \( \sqrt{\sum_{j=1}^{n} U_j} \) has exactly one positive root. Thus the test can be expressed as

\[
\text{reject } H_0 \text{ if } \sum_{j=1}^{n} U_j < k'.
\]
On the other hand, \( U_j \) has an exponential distribution with mean \( 1/\lambda = 1 \) under \( H_0 \). A UMP test against \( 1/\lambda < 1 \) has critical region of the form above (note that the parameter \( \lambda \) equals the hazard rate). The hazard rate test gives critical regions which are UMP for hypotheses about \( \lambda \), but the size will be right only asymptotically.

Note that the MLE of \( \lambda \) is \( \frac{1}{\bar{U}} \), which is the same as the median unbiased estimator of section 4.

6.3 No Truncation.

If \( \nu \) is constant the problem can be modified so that \( \nu = 0 \) and \( F(0) = 0 \). In this way only the censoring time needs to be dealt with. Transform the observations by

\[
U_j = -\ln(1-F(\lambda^*)) = \min\{-\ln(1-F(X)), -\ln(1-F(\lambda^*))\}.
\]

Under the null hypothesis \( \{U_j\} \) is a random sample from a censored exponential distribution with mean 1. For constant \( \lambda \), the MLE of the parameter is \( \frac{1}{n} \sum_{j=1}^{n} U_j \). This agrees with the asymptotic median unbiased estimator of section 4.

7. An Example.

Table 7.1 shows survival data for male members of a retirement community in Palo Alto, California. The times \( \nu+1 \) and \( \lambda \) are ages in months. There were 97 men; 46 died, 46 survived to the closing date of the study and 5 actually withdrew from the community. To provide an illustration of the hazard rate test, assume that the reasons for
entering and leaving the community were unrelated to a person's health relative to others of the same age.

The 1958 Commissioners Standard Ordinary Mortality Table for Male Lives was used to generate the distribution \( F \) to age 100 for the hazard rate test. This table was used because more recent tables from the Public Health Service stop at age 85. The table provides values of \( F \) only at intervals of one year, so linear interpolation was used to compute \( F \), and hence \( p_i \), for each month. Since the table stops after age 100, the data must be artificially censored at 1200 to use the hazard rate test. This does not affect the data in table 7.1, but it does mean that \( \lambda \) is bounded by 1200 in this problem. The boundedness of \( \lambda \) means \( E[\lambda^*] < \infty \) and justifies the conclusions of section 2.

The expected number of deaths was

\[
\sum_{j=1}^{97} \sum_{i=v_j + 1}^{\lambda^*_j} p_i \approx 72.2
\]

while the estimated variance was

\[
\sum_{j=1}^{97} \sum_{i=v_j + 1}^{\lambda^*_j} p_i (1-p_i) \approx 68.3
\]

The hazard rate statistic is then

\[
\frac{46 - 72.2}{\sqrt{68.3}} \approx -3.17
\]
which indicates the null hypothesis should be rejected in favor of a
smaller hazard rate.

If the discrete version is viewed as an approximation for the
continuous case, and if it is assumed that the actual hazard rate is
a multiple, c, of the hazard rate of F, then the ideas of section
4 can be applied. A 90% confidence interval for c is (.487, .807)
and the approximate median unbiased estimate of c is .637.
Table 7.1: Survival Data for Males in a Retirement Community

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