LIMIT DISTRIBUTIONS FOR STOCHASTIC INTEGRALS

BY

J. SETHURAMAN

TECHNICAL REPORT NO. 13
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1. Introduction.

Let \( T \) be a subset of the real line. Let \((\Omega, \mathcal{F}, P)\) be a probability space. Let \((x_n(t,\omega), t \in T)\) be real valued stochastic processes, \(n = 0, 1, \ldots\). Several notions of convergence of a stochastic process are current in the literature. We shall now describe one of the most fruitful and common notions, namely that of weak convergence. Suppose that \( x_n(\cdot, \omega) \) belongs to some function space \( X \) of functions on \( T \).

Let \( T \) be a topology on \( X \) and \( \mathcal{B} \) the Baire \( \sigma \)-field in \( X \) w.r.t. \( T \). Let \( x_n(\cdot, \omega) \) be \( \mathcal{B} \)-measurable and \( \mu_n \) be its induced distribution. \( \mu_n \) is said to converge weakly to \( \mu_0 \) (written in symbols as

\[
\mu_n \rightarrow \mu_0 \text{ or } \mathcal{L}(x_n(\cdot, \omega)) \rightarrow \mathcal{L}(x_0(\cdot, \omega))
\]

if for each bounded \( T \)-continuous function \( h \) on \( X \),

\[
\int h(x) \mu_n(dx) \rightarrow \int h(x) \mu_0(dx).
\]  

(1)

To make our ideas more concrete let us assume that \( T \) is a separable and complete metric topology on \( X \). The importance of the concept of weak convergence lies in the following equivalence:

\[
\mu_n \rightarrow \mu_0 \text{ is equivalent to}
\]

\[
\mathcal{L}(f(x_n(\cdot, \omega))) \rightarrow \mathcal{L}(f(x_0(\cdot, \omega)))(\text{i.e. } \mu_n^{f^{-1}} \rightarrow \mu_0^{f^{-1}})
\]  

(2)
for each real valued function $f$ on $X$ which is continuous with probability 1 with respect to $\mu_0$. In this case it is enough that $f$ be defined on an open set containing the support of $\mu_0$. (The convergence in (2) is weak convergence of distributions in a finite dimensional Euclidean space.)

Consequently, it happens that in all applications one finds limit distributions for random variables of the form $f(x_n (\cdot, \omega))$ or of the form $f_n (x_n (\cdot, \omega))$ where $f, f_n$ are functions defined on $X$. The purpose of this paper is to study limit distributions of stochastic integrals of stochastic processes, which cannot always be expressed in the above simple form. Stated this way, the problem is very general and admits of no simple solution. This general problem has been stated in Skorohod (1957). We shall study a special class of stochastic integrals, namely $L_2$-stochastic integrals and obtain some theorems regarding limit distributions (Section 3). In Section 4 we present some applications of the theorems of Section 3. We present the background and necessary preliminaries in Section 2.

2. Background.

Let $(x(t, \omega), t \in T)$ be a stochastic process. Consider the family $\mathcal{A}(x)$ of random variables $a(x) = a(x; t, \xi)$ defined by

$$a(x; t, \xi) (\omega) = \sum_{i=1}^{k} c_i x(t_i, \omega) ,$$

$$t = (t_1, \ldots, t_k), \xi = (c_1, \ldots, c_k) .$$
The random variables in $\mathbb{A}(x)$ will be called the primitive stochastic integrals of the process $x$. A general stochastic integral is a linear function of $x$. Thus a $N$-stochastic integral of $x$ will be a limit in $N$-sense of a sequence in $\mathbb{A}(x)$. Here one can take $N$ to be convergence with probability one, or convergence in probability or convergence in quadratic mean etc. This last sense is called the $L_2$-sense and we shall now describe $L_2$-stochastic integrals in detail.

Assume that for $t, s \in T$,

$$f \in L_2(T, \mathbb{B}, \mathbb{P}) = \mathbb{K}(t, s) < \infty.$$

For each $a(x) = a(x; t, q) \in \mathbb{A}(x)$,

$$0 \leq \int a^2(x)(\omega) \mathbb{P}(d\omega) = \sum_{i=1}^{k} \sum_{j=1}^{k} c_i c_j K(t_i, t_j) < \infty.$$

Thus $\mathbb{A}(x) \subset L_2(\Omega, \mathbb{P}, \mathbb{P})$, the Hilbert space of random variables on $\Omega$ with finite second moments. $L_2(x)$, the space of $L_2$-sense integrals of $x$ is defined to be the closure of $\mathbb{A}(x)$ in $L_2(\Omega, \mathbb{P})$. $L_2(x)$ is a Hilbert space, and if $\xi \in L_2(x)$, the norm of $\xi$ is

$$[\int_\Omega^2(\omega) \mathbb{P}(d\omega)]^{1/2}.$$

There is a canonical isometry between the Hilbert space, $\mathbb{K}$, called the reproducing kernel Hilbert space (RKHS) of $K$ and $L_2(x)$. Let $\mathbb{A}_x$ be the space of functions $a(K; t, q)$ on $T$ defined by

$$a(K; t, q)(s) = \sum_{i=1}^{k} c_i K(s, t_i),$$

$$t = (t_1, \ldots, t_k), \; q = (c_1, \ldots, c_k).$$

The norm of this function is defined to be
\[ \|a(K, t, z)\|_K = \left( \sum_{i=1}^{k} \sum_{j=1}^{k} c_i c_j K(t_i, t_j) \right)^{1/2}. \]

The smallest complete inner product (defined by the above norm) space, \( H_K \), containing \( H_K \) is called the RKHS of \( K \). \( H_K \) is a Hilbert space and is also defined by

(i) \( K(\cdot, t) \in H_K \) for each \( t \in T \)

(ii) \( (K(\cdot, t), K(\cdot, t'))_K = K(t, t') \)

(iii) If \( f \in H_K \), \( (f, (K(\cdot, t))_K = f(t), t \in T. \)

(iv) A function \( f \) on \( T \) belongs to \( H_K \) if and only if there
exists a constant \( C \) such that for any \( z = (t_1, \ldots, t_k), z = (c_1, \ldots, c_k), \)
\[ \left| \sum_{i=1}^{k} c_i f(t_i) \right|^2 \leq C \left[ \sum_{i=1}^{k} \sum_{j=1}^{k} c_i c_j K(t_i, t_j) \right]. \]

The smallest such \( C \) is equal to \( \|f\|_K^2 \).

Consider the canonical \( 1-1 \), norm preserving mapping \( \psi \) between \( H_K \) and \( A(\cdot) \) defined by
\[ \psi(a(K, t, z)) = a(x, t, z). \]

It is now obvious that \( \psi \) can be extended to a \( 1-1 \) isometry between \( H_K \) and \( L_2(x) \). Hence any \( L_2 \)-stochastic integral of \( x \) may be written as \( \psi f \) where \( f \in H_K \).

Let \( K_1 \) and \( K_2 \) be two functions on \( T \times T \) which are covariance kernels, namely,
\[
\sum_{i=1}^{k} \sum_{j=1}^{k} c_i c_j K_p(t_i, t_j) \geq 0, \quad p = 1, 2
\]

for all \( \mathbf{t} = (t_1, \ldots, t_k) \), \( \mathbf{c} = (c_1, \ldots, c_k) \). \( K_1 \) is said to be dominated by \( K_2 \) (\( K_1 \ll K_2 \)) if \( K_2 - K_1 \) is a covariance kernel. When \( K_1 \ll A K_2 \), we can verify from (iv) that \( \mathbf{f} \in H_{K_1} \) implies \( \mathbf{f} \in H_{K_2} \) and \( \| \mathbf{f} \|^2_{K_2} \leq A \| \mathbf{f} \|^2_{K_1} \).

Let \( \eta_{K_1, K_2} \) be this mapping that takes \( H_{K_1} \) into \( H_{K_2} \).

Let \( A_1 K_2 \ll K_1 \ll A_2 K_2 \) then \( \eta_{K_1, K_2} \) is a 1-1 continuous map from \( H_{K_1} \) onto \( H_{K_2} \). Let \( K_0 \) be a covariance kernel on \( T \times T \). Let

\[
T_n = \{ t_{n0}, \ldots, t_{nk_n} \}
\]

be a finite subset of \( T \) that increases and becomes dense in \( T \). Let the restriction of \( K_0 \) to \( T_n \times T_n \) be \( M_n \), namely let

\[
M_n(t_{ni}, t_{nj}) = K_0(t_{ni}, t_{nj}) \quad i, j = 0, \ldots, k_n.
\]

For simplicity we assume that the matrix \( (M_n(t_{ni}, t_{nj}), i, j = 0, \ldots, k_n) \) is non-singular and denote its inverse by \( M_n^{-1} \). The RKHS, \( H_{M_n} \), of \( M_n \) is the space of functions on \( T_n \). Let \( \mathbf{a} = (a(t_{n0}), \ldots, a(t_{nk_n})) \in H_{M_n} \). Its norm is given by

\[
\| \mathbf{a} \|^2_{M_n} = \mathbf{a} M_n^{-1} \mathbf{a}'.
\]

To any \( \mathbf{a} \) in \( H_{M_n} \) we can find a unique vector \( \mathbf{b} \) such that

\[
a(t_{nj}) = \sum_{i=0}^{k_n} b_i M_n(t_{nj}, t_{ni}).
\]

(3)

In this case \( \| \mathbf{a} \|^2_{M_n} = \sum b_i b_j M_n(t_{ni}, t_{nj}) \). Also if \( n' > n \) the
mapping \( \tau_{n',n} \):

\[
\begin{align*}
\alpha &= \left( \sum_{i=0}^{k_n} b_{i,n}(t_{n0}, t_{ni}), \ldots, \sum_{i=0}^{k_n} b_{i,n}(t_{nk_n}, t_{ni}) \right) \\
\rightarrow \tau_{n',n}; \quad \alpha = \left( \sum_{i=0}^{k_n} b_{i,n}(t_{n'o}, t_{ni}), \ldots, \sum_{i=0}^{k_n} b_{i,n}(t_{n'k_n}, t_{ni}) \right)
\end{align*}
\]

maps \( H_{M_n} \) into \( H_{M_{n'}} \) and preserves norm. In a similar way, the mapping \( \tau_{n,0} \)

\[
\alpha \rightarrow (\tau_{n,0}; \alpha)(s) = \sum_{i=0}^{k_n} b_{i,0}(s, t_{ni})
\]

maps \( H_{M_n} \) into \( H_{K_0} \) and preserves norm. Conversely, let \( f \in H_{K_0} \). Let

\[
\xi_n = (f(t_{n0}), \ldots, f(t_{nk_n})).
\]

Then \( \xi_n \in H_{M_n} \) and \( \|\xi_n\|_{M_n}^2 \rightarrow \|f\|_{K_0}^2 \) and, indeed \( \|\tau_{n,0} \xi_n - f\|_{K_0}^2 \rightarrow 0 \)
as \( n \to \infty \). Now let \( \{y_n^*(t_{ni}, \omega), i=0, \ldots, k_n\} \) be a random variable on \( \mathbb{R}^{k_n+1} \) such that

\[
|y_n^*(t_{ni}, \omega)y_n^*(t_{nj}, \omega)P(d\omega)| = M_n(t_{ni}, t_{nj})
\]

We then have an isometry \( \psi_n \) between \( H_{M_n} \) and \( L_2(y_n^*) \). Thus let \( \alpha \in H_{M_n} \). Then \( \psi_n \alpha = \sum_{i=0}^{k_n} b_i y_n^*(t_{ni}, \omega) \) where the \( b_i \)'s are uniquely determined from \( \alpha \) as described in (3). All the ideas presented in this paragraph are well known and easily proved as in Aronzon (1950), Parzen (1961), etc. We shall adhere to the notation of this paragraph in Theorem 3.
Weak convergence of random variables on the real line can be introduced by the separable complete metric \( L \) defined by Prohorov (1956) on the space of random variables. If \( \xi, \eta \) are two random variables with \( P(|\xi(\omega) - \eta(\omega)| \geq \varepsilon) \leq \delta \) then this metric \( L \) satisfies

\[
L(\xi, \eta) \leq \max(\varepsilon, \delta),
\]

e.g., Prohorov (1956). Thus if \( (\xi(\omega) - \eta(\omega))^2 P(d\omega) = \alpha \)

then \( L(\xi, \eta) \leq \alpha^{1/2} \).

3. Main Theorems.

Let \( (y_n(t, \omega)) \, n=0,1,\ldots \) be a sequence of stochastic processes with covariance kernels \( K_n, \, n=0,1,\ldots \). Let

\[
\mathcal{L}(y_n(t_1, \omega), \ldots, y_n(t_k, \omega)) \rightarrow \mathcal{L}(y_0(t_1, \omega), \ldots, y_0(t_k, \omega))
\]

weakly for each finite collection \( \{t_1, \ldots, t_k\} \) in \( T \). Then

(4) implies that

\[
\mathcal{L}\left( \sum_{i=0}^{m} a_i y_n(t_i, \omega) \right) \rightarrow \mathcal{L}\left( \sum_{i=0}^{m} a_i y_0(t_i, \omega) \right)
\]

for any finite collection \( (a_0, \ldots, a_m) \) and finite set \( \{t_0, \ldots, t_m\} \) in \( T \).

Theorem 1. Let (4) hold and \( K = K_n \). Let \( H_K = H_{K_n} \). Let \( \psi_n \) be the canonical mapping from \( H_{K_n} \) into \( L_2(y_n) \).

(i) Let \( f \in H_K \). Then

\[
\mathcal{L}(\psi_n f) \rightarrow \mathcal{L}(\psi f).
\]

(ii) Let \( f_n \in H_K, \, n=0,\ldots \) and \( \|f_n - f_0\|_K \rightarrow 0 \). Then

\[
\mathcal{L}(\psi_n f_n) \rightarrow \mathcal{L}(\psi f_0).
\]
Proof. Choose and fix \( \epsilon > 0 \). There exists a finite collection \( \{t_1, \ldots, t_k\} \) and constants \( \{c_1, \ldots, c_k\} \) such that

\[
\|a(K, t, c) - f\|_K \leq \epsilon^3.
\]

This means that \( \psi_n(a(K, t, c)) = a(y_n, t, c) \) and

\[
f(a(y_n, t, c))(\omega) - \psi_n f(\omega) \leq \epsilon^3
\]

and \( L(\psi_n f, \psi_n(a(K, t, c))) < \epsilon \) for \( n = 0, 1, \ldots \). Thus

\[
(5) \quad L(\psi_n f, \psi_o f) \leq L(\psi_n f, \psi_n(a(K, t, c))) + L(\psi_o f, \psi_o(a(K, t, c))) + L(\psi_n(a(K, t, c)), \psi_o(a(K, t, c)))< \epsilon.
\]

The first and the last terms on the right hand side of (5) are less than \( \epsilon \). The middle term goes to 0 in view of (4). This establishes (i). Now let \( f_n \in K \) and \( \|f_n - f_o\|_K \rightarrow 0 \). Choose \( n_o \rightarrow \) for \( n \geq n_o \), \( \|f_n - f_o\|_K \leq \epsilon^3 \). Then

\[
L(\psi_n f_n, \psi_o f_o) \leq L(\psi_n f_n, \psi_n f_o) + L(\psi_n f_o, \psi_o f_o) < \epsilon.
\]

The first term on the right hand side is less than \( \epsilon \) for \( n \geq n_o \). The second term goes to zero from (i). This proves (ii).

Remark. The above is a theorem on convergence in distribution of stochastic integrals. Random variables of the form \( \psi_n f_n \) are defined only with respect to \( y_n \) and in general cannot be extended uniquely on to the domain of another process \( y_{n+1} \).
In Theorem 1 we imposed the restriction that \( K_n = K \). We remove that restriction in two different ways in Theorems 2 and 3 below. It seems plausible that more far reaching extensions are possible.

**Theorem 2.** Let (4) hold. Let \( K \) be a covariance kernel such that

\[
K \ll A_n K_n \quad n=0,1,\ldots
\]

where

\[
\sup_n A_n = A < \infty.
\]

(i) Let \( f \in H_K \). Let \( g_n = \eta_{\eta,K_n} f \) where \( \eta_{\eta,K_n} \) is the canonical mapping from \( H_K \) into \( H_{K_n} \). Then

\[
\mathcal{I}(\psi_n g_n) \rightarrow \mathcal{I}(\psi_0 g_0).
\]

(ii) Let \( f \in H_K \) and \( \|f - f_0\|_K \rightarrow 0 \), \( g_n = \eta_{\eta,K_n} f_n \). Then

\[
\mathcal{I}(\psi_n g_n) \rightarrow \mathcal{I}(\psi_0 g_0).
\]

**Proof.** The proof of this theorem follows by adapting the proof of Theorem 1 and by using the inequality

\[
\|\eta_{\eta,K_n} h\|_{K_n}^2 \leq A \|h\|_K^2 \quad \text{for} \quad n=0,1,\ldots.
\]

**Theorem 3.** Let \( T_n = \{t_{n0}, t_{n1}, \ldots, t_{nk_n}\} \) be a finite subset of \( T \) that increases and gets dense in \( T \). Let \( M_n \) be the covariance kernel on \( T_n \times T_n \) defined by

\[
M_n(t_{ni}, t_{nj}) = K_n(t_{ni}, t_{nj}) \quad i,j=0,\ldots,k_n.
\]
where \( K_n \) is the covariance kernel of the process \( \{ y_n(t, \omega) \} \), \( n=1,2, \ldots \).

Assume that

\[
M_n(t_{ni}, t_{nj}) = K_o(t_{ni}, t_{nj}), \quad i,j=0, \ldots, k_n
\]

and that the matrix \( (M_n(t_{ni}, t_{nj}), i,j=0, \ldots, k_n) \) is non-singular.

Let \( y_n^*(t_{ni}, \omega) = y_n(t_{ni}, \omega), \quad i=0, \ldots, k_n \). There is an isometry \( \psi_n \) between \( H_{M_n} \) and \( L_2(y_n^*) \).

(i) Let \( f \in H_{K_o} \), \( \sim f_n = (f(t_{n0}), \ldots, f(t_{nk_n})) \in H_{M_n} \). Then

\[
L(\psi \sim f_n) \rightarrow L(\psi f)
\]

(ii) Let \( h \in H_{K_o} \), \( \| h - h_o \|_{K_o} \rightarrow 0 \), \( h_n^* = (h_n(t_{n0}), \ldots, h_n(t_{nk_n})) \).

Then

\[
L(\psi h_n) \rightarrow L(\psi h_o).
\]

**Proof.** The proof of this theorem is similar to the proofs of Theorems 1 and 2. As an illustration we shall prove (i) only. Let \( f \in H_{K_o} \).

Then \( \| \xi_{n,0} \sim f - f \|_{K_o}^2 \rightarrow 0 \) as \( n \rightarrow \infty \), as can be found in the discussion in Section 2. Thus given \( \epsilon > 0 \), we can find \( m \) such that

\[
\| \xi_{n,0} \sim f - f \|_{K_o}^2 < \epsilon^3 \quad \text{for} \quad n \geq m
\]

and

\[
\| \xi_{n,0} \sim f - \xi_{n',0} \sim f \|_{K_o}^2 = \| \xi_{n,n'} \sim f - \xi_{n',n} \sim f \|_{M_{n'}}^2 \leq \epsilon^3 \quad \text{for} \quad m \leq n \leq n'.
\]
Now, for \( n \geq m \)

\[
L(\psi_{\frac{n}{r^m+n}}, \psi_{\frac{m}{o^m}}) \leq L(\psi_{\frac{n}{r^m+n}}, \psi_{\frac{m}{r^m+m}}) + L(\psi_{\frac{n}{o^m}}, \psi_{\frac{o^m}{o^m}})
\]

\[
+ L(\psi_{\frac{m}{r^m+n}}, \psi_{\frac{o^m}{o^m}}, \psi_{\frac{o^m}{o^m}}).
\]

\[
\int \left( \psi_{\frac{n}{r^m+n}}(\omega) - \psi_{\frac{m}{r^m+m}}(\omega) \right)^2 P(d\omega) = \| \psi_{\frac{n}{r^m+n}} - \psi_{\frac{m}{r^m+m}} \|_{\mathbb{L}^2} \leq \epsilon^2
\]

and

\[
\int \left( \psi_{\frac{n}{o^m}}, \psi_{\frac{m}{o^m}}(\omega) \right)^2 P(d\omega) = \| \psi_{\frac{n}{o^m}}, \psi_{\frac{m}{o^m}} \|_{K_0} \leq \epsilon^2
\]

from (6) and (7). Thus the first and second terms of the right hand side of (8) are less than \( \epsilon \). The middle term tends to zero in virtue of (4). This proves (1).

4. **Applications.**

We now give two simple applications wherein the second application is more startling.

Let \( X_1(\omega), X_2(\omega), \ldots \) be independent real valued random variables with a common distribution function \( F(x) \). Let \( F_n(x, \omega) \) be the empirical distribution function of \( X_1(\omega), \ldots, X_n(\omega) \). Let \( y_n(t, \omega) = \sqrt{n} \left( F_n(t, \omega) - F(t) \right), \quad -\infty < t < \infty \). It is easy to see that, for \( t < s \)

\[
K(t, s) = \int y_n(t, \omega) y_n(s, \omega) P(d\omega) = F(t)(1 - F(s)) \quad \text{for} \quad n=1,2,\ldots .
\]
Let \( \{y_0(t, \omega)\} \) be a Gaussian process with covariance kernel \( F(t)(1-F(s)) \), \( t \leq s \). It is also easy to verify that

\[
\mathcal{L}(y_n(t_1, \omega), \ldots, y_n(t_k, \omega)) \rightarrow \mathcal{L}(y_0(t_1, \omega), \ldots, y_0(t_k, \omega)).
\]

Thus all the conditions of Theorem 1 are met. Therefore if \( f \in \mathcal{H}_K \) then

\[
\mathcal{L}(\psi_n f) \rightarrow \mathcal{L}(\psi f).
\]

However, on closer inspection this reduces to the standard central limit theorem namely,

\[
\mathcal{L} \left( \sqrt{n} \left( \int g(x) F_n(dx, \omega) - \int g(x) F(dx) \right) \right) \\
\rightarrow N(0, \int g^2(x) F(dx))
\]

where \( N(0, \sigma^2) \) denotes the normal distribution with mean \( 0 \) and variance \( \sigma^2 \).

Let \( X_1(\omega), X_2(\omega), \ldots \) be independent and identically distributed as the uniform distribution on \([0,1]\). Let \( X_{n1}(\omega), \ldots, X_{nn}(\omega) \) be the order statistics of \( X_1(\omega), \ldots, X_n(\omega) \). Let \( X_{n0}(\omega) = 0, X_{nn}(\omega) = 1 \).

Define

\[
y_n(t, \omega) = X_{ni}(\omega) + \frac{t - \frac{i}{n+1}}{\frac{1}{n+1}} (X_{ni+1}(\omega) - X_{ni}(\omega)), \frac{i}{n+1} \leq t \leq \frac{i+1}{n+1}.
\]

Here \( T = [0,1] \). Let \( T_n = \{0, \frac{1}{n+1}, \ldots, \frac{n+1}{n+1}\} \). Let \( \{y_0(t, \omega)\} \) be a Gaussian process with covariance kernel \( K(t,s) = t(1-s), 0 \leq t \leq s \leq 1 \).
It is easy to see that

\[ \mathcal{L}(y_n(t_1, \omega), \ldots, y_n(t_k, \omega)) \rightarrow \mathcal{L}(y_o(t_1, \omega), \ldots, y_o(t_k, \omega)) \]

for every finite collection \( \{t_1, \ldots, t_k\} \). Also

\[ \int y_n(\frac{i}{n+1}, \omega)y_n(\frac{j}{n+1}, \omega)P(d\omega) = K_n(\frac{i}{n+1}, \frac{j}{n+1}) \]
\[ = \frac{1}{n+1}(1 - \frac{j}{n+1}) = K(\frac{i}{n+1}, \frac{j}{n+1}), \quad 0 \leq i \leq j \leq n+1. \]

Let \( r \) be an integer and \( (n+1) \) tend to infinity in multiples of \( r \). Then the set \( \{\frac{0}{n+1}, \frac{1}{n+1}, \frac{n+1}{n+1}\} \) is increasing and becomes dense in \([0,1]\). Thus all the conditions of Theorem 3 are met.

Let

\[ z_n(\omega) = \sum_{i=0}^{n+1} b_{ni} y_n(\frac{i}{n+1}, \omega). \]

Then \( \mathcal{L}(z_n(\omega)) \rightarrow N(0, \sigma^2) \) if \( \{f_n\} \) is a Cauchy sequence in \( H_K \) where

\[ f_n(s) = \sum_{i=0}^{n+1} b_{ni} K(s, \frac{i}{n+1}) \]

and in this case

\[ \sigma^2 = \lim_n \sum_{i=0}^{n+1} \sum_{i=0}^{n+1} b_{ni}(\frac{i}{n+1})(1 - \frac{j}{n+1}). \]
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