A PERSISTENCY PROBLEM CONNECTED WITH A POINT PROCESS

BY

G. ELFVING

TECHNICAL REPORT NO. 16
APRIL 20, 1966

PREPARED UNDER THE AUSPICES
OF
NATIONAL SCIENCE FOUNDATION GRANT GP-5705

DEPARTMENT OF STATISTICS
STANFORD UNIVERSITY
STANFORD, CALIFORNIA
A PERSISTENCY PROBLEM CONNECTED WITH A POINT PROCESS

By

G. Elfving

TECHNICAL REPORT NO. 16
April 20, 1966

PREPARED UNDER THE AUSPICES
OF
NATIONAL SCIENCE FOUNDATION GRANT GP-5705

DEPARTMENT OF STATISTICS
STANFORD UNIVERSITY
STANFORD, CALIFORNIA
A Persistency Problem Connected with a Point Process

By

G. Elfving

1. Introduction.

Imagine a man owning a commodity, e.g., a house, which is for sale. Offers at varying amounts are coming in every now and then. The longer he postpones selling the more he looses because of deterioration, interest losses, or the like. At each offer he must decide whether to accept it or wait for a better one. (A more picturesque example would be that of a girl scrutinizing successive suitors.)

We shall study the following idealized model of the situation sketched above. Consider a point process on $t \geq 0$, characterized by a positive and continuous intensity function $p(t)$; the probability of an event (an offer) in an interval $(t, t+dt)$ is $p(t)dt$, and events taking place in disjoint time intervals are independent. Let $\tau_1, \tau_2, \ldots$ be the time points of the successive events. With these events are associated independent positive random variables $Y_1, Y_2, \ldots$ (the offered amounts) with a common d.f. $f(y)$ and finite mean $\mu$. Furthermore there is given a positive non-increasing discount function $r(t)$ with $r(0) = 1$.

We shall investigate decision rules of the following form. Let $y = y(t)$ be a non-negative at least piece-wise continuous function to be called the critical curve. The first offer $Y_n$ for which $Y_n \geq y(\tau_n)$, is accepted, the process stops, and its value to the decision maker is recorded as $Y_n r(\tau_n)$; if no offer is ever accepted, the value is considered to be zero. We shall denote the accepted offer (or zero) by $\bar{Y}$ and the
corresponding time point by $\tilde{T}$. It seems reasonable to require that $y(\cdot)$ should maximize $E \tilde{Y}(\tilde{T})$.

The time constancy of the offer distribution $f(y)$ may seem unrealistic with regard to the applications suggested; however, the average effect of deterioration etc. may be incorporated in the discount factor $r(t)$.

Problems similar to the described one have been treated by many authors. A fundamental type is that where the decision maker is faced with a sequence of independent and equally distributed observations; after each observation he either has to select the largest value hitherto appeared, or wait for a larger one, each additional observation being at a certain cost $c$. This problem was treated by Chow and Robbins [2], and generalized by MacQueen and Miller [5]. A considerably harder version arises when the sampling distribution is unknown, c.f. Yahav [6] and DeGroot [3]. Investment policies somewhat reminding of our problem are surveyed in McCall [4]. A nice systematic account of the whole stopping problem is given by Breiman [1].

The problem treated below is in a way more elementary than those mentioned above, in that:

(a) Past offers are no more available. As a consequence, there is nothing like a "state of the process".
(b) All data describing the process are assumed to be known. On the other hand, the following features should be noted:
(c) The offers constitute a random point process.
(d) A fairly general time dependence is built in, in form of the intensity and discount functions.
Summary. An integral equation (2.4) and a differential equation (3.1) for the critical curve are derived. Under the convergence assumption (3.2), the existence and uniqueness of an optimal decision rule is established, and several illustrations are given. In the divergence case, various possibilities are discussed.

2. An integral equation for the critical curve.

Our first task is to derive, for any given $y(\cdot)$, an expression for the expected value $I_y = E\tilde{Y}(\tilde{T})$; more generally, we shall derive $I_y(t) = E[\tilde{Y}(\tilde{T})|\tilde{T} > t]$.

Let us denote

\begin{equation}
G(y) = \int_y^\infty f(y)dy, \quad H(y) = \int_y^\infty yf(y)dy.
\end{equation}

Applying the usual differential argument, we find that the probability for no acceptance during $(\sigma, \sigma+d\sigma)$ given $\tilde{T} > \sigma$ is

\begin{equation}
1-p(\sigma)d\sigma \cdot G(y(\sigma)) = e^{-p(\sigma)G(y(\sigma))}d\sigma
\end{equation}

and hence

\begin{equation}
P[\tilde{T} > \tau|\tilde{T} > t] = \exp\left[-\int_0^\tau p(\sigma)G(y(\sigma))d\sigma\right].
\end{equation}

The probability of acceptance during $(\tau, \tau+d\tau)$, conditionally on $\tilde{T} > \tau$, is $p(\tau)G(y(\tau))d\tau$; the expected value of $\tilde{Y}(\tilde{T})$, conditionally on $\tilde{T}e(\tau, \tau+d\tau)$, is $r(\tau)H(y(\tau))/G(y(\tau))$. The contribution to $E[\tilde{Y}(\tilde{T})|\tilde{T} > t]$ of the interval $(\tau, \tau+d\tau)$ being obviously

\begin{equation}
P[\tilde{T} > \tau|\tilde{T} > t] P[\tilde{T}e(\tau, \tau+d\tau)|\tilde{T} > \tau] E[\tilde{Y}(\tilde{T})|\tilde{T}e(\tau, \tau+d\tau)],
\end{equation}

we find
(2.3) \[ I_y(t) = \int_t^\infty p(\tau)r(\tau)H(y(\tau))\exp[-\int_t^\tau p(\sigma)G(y(\sigma))d\sigma]d\tau. \]

This expression holds true for any \( y(\cdot) \); in particular, it yields \( I_y = I_y(0) \).

For an optimal \( y(\cdot) \) it seems intuitively obvious, and can be justified by a calculus of variation argument, that we should have
\[ r(t)y(t) = I_y(t); \]
i.e., the discounted value of a borderline offer should equal the expected value of the process if continued. In view of (2.3) we thus have

**Theorem 2.1.** The optimal critical curve \( y(\cdot) \) must satisfy the integral equation

(2.4) \[ r(t)y(t) = \int_t^\infty p(\tau)r(\tau)H(y(\tau))\exp[-\int_t^\tau p(\sigma)G(y(\sigma))d\sigma]d\tau. \]

We note from the theorem that the optimal \( y(t) \) — if it exists — is differentiable along with \( r(t) \).

3. **A differential equation for the critical curve.**

Differentiating (2.4) with respect to \( t \) we get

\[ \frac{d(ry)}{dt} = -p(t)r(t)H(y(t)) + \int_t^\infty p(\tau)r(\tau)H(y(\tau))\exp[-\int_t^\tau p(\sigma)G(y(\sigma))d\sigma]d\tau \cdot p(t)G(y(t)). \]

Since the integral is the r.h.s. of (2.4), and hence equals \( r(t)y(t) \), we have

(3.1) \[ \frac{d(ry)}{dt} = -p(t)r(t)[H(y(t)) - y(t)G(y(t))]. \]
The differential equation (3.1) has infinitely many solutions. It turns out that only one of these will in general do as solution of (2.4). The connection between the two equations can be made precise under a fairly reasonable general condition, viz.,

$$\int_0^\infty p(t)r(t)\,dt < \infty.$$ 

This assumption will be upheld throughout most of this paper; at the end we shall briefly discuss the divergence case.

**Theorem 3.1.** Assuming that (3.2) holds, necessary and sufficient conditions for a function $y(t) > 0$ $(0 \leq t < \infty)$ to fulfill (2.4) are that (a) $y$ be a solution of (3.1), and (b) $r(t) y(t) \to 0$ as $t \to \infty$.

**Proof.** The necessity of (a) has been shown above. The necessity of (b) follows from (2.4), noticing that $\exp[.] \leq 1$, $H(y) = \int_y^\infty yf(y)\,dy \leq \mu$, and using (3.2).

Conversely, assume that $y(t)$ fulfills (a) and (b), and define $y_1(t)$ by

$$r(t)y_1(t) = \int_t^\infty p(\tau)r(\tau)H(y(\tau))\exp[-\int_t^\tau p(\sigma)G(y(\sigma))\,d\sigma]\,d\tau.$$ 

We wish to prove that actually $y_1 = y$.

We first note that (3.3) implies $r y_1 \to 0$ $(t \to \infty)$ exactly as (2.4) implies $r y \to 0$.

Differentiating (3.3) we find, analogously to (3.1),

$$\frac{d(r y_1)}{dt} = -p(t)r(t)[H(y(t))-y_1(t)G(y(t))]$$

(note the factor $y_1$ pro $y$ on the r.h.s.). Subtracting (3.1) from (3.4) we get
\[ (3.5) \quad \frac{d[r(y_1-y)]}{dt} = p(t)G(y(t)) \cdot r(y_1-y). \]

Assume (antithesis) that \( y_1 \neq y \) at some point \( t_0 \), and let \( t_1 \leq \infty \) be the first point after \( t_0 \) in which \( r(y_1-y) = 0 \) (at least \( t_1 = \infty \) will do, since \( ry \to 0, \ ry_1 \to 0 \) as \( t \to \infty \)). Integrating the differential equation (3.5) with respect to \( r(y_1-y) \) we find for any \( t \) in \( (t_0, t_1) \)

\[ (3.6) \quad \log[r(t)|y_1(t)-y(t)|] = \log[r(t_0)|y_1(t_0)-y(t_0)|] + \int_{t_0}^{t} p(s)G(y(s))ds. \]

For \( t \to t_1 \), the l.h.s. goes to \( -\infty \) whereas the r.h.s. remains bounded or goes to \( +\infty \). Hence the antithesis \( y_1 \neq y \) must be rejected, and the proof is complete.

4. Existence and uniqueness.

We have now reduced our problem to the finding of a solution of (3.1) fulfilling \( ry \to 0 \). To ensure the existence and uniqueness of such a solution, we must begin by studying the function

\[ (4.1) \quad \varphi(y) = H(y) - yG(y) \]

appearing on the r.h.s. of (3.1).

Lemma 4.1. The function \( \varphi(y) \) in (4.1) is nonnegative and decreases from \( u \) to 0 as \( y \) increases from 0 to \( \infty \).

Proof. According to (2.1) we have, for \( y \geq 0 \),

\[ (4.2) \quad \varphi(y) = \int_{y}^{\infty} v \ d(-G(v)) - yG(y) = \int_{y}^{\infty} G(v)dv \geq 0, \]
\[(4.3) \quad \varphi'(y) = -G(y) \leq 0 \quad ; \quad \varphi(0) = \mathcal{H}(0) = \mu; \quad \varphi(\infty) = 0.\]

Actually, \(\varphi(y)\) becomes zero already at the upper bound of the distribution \(f(y)\), if this bound is finite.

It will be convenient to consider functions \(y(t)\) others than those which are always nonnegative. Since \(\varphi(y)\) has hitherto been defined for \(y \geq 0\) only, we shall extend the domain of this function by putting

\[(4.4) \quad \varphi(y) = 2\mu - \varphi(-y) , \quad y \leq 0.\]

Since \(\varphi(0) = \mu, \varphi'(0) = -G(0) = -1\), the extended function and its derivative are seen to be continuous at \(y = 0\), and we have

\[(4.4') \quad \varphi'(y) = \varphi'(-y) = -G(-y) , \quad y \leq 0.\]

As a consequence of \((4.4-4.4')\),

\[(4.5) \quad 0 \leq \varphi(y) \leq 2\mu , \quad -1 \leq \varphi'(y) \leq 0 , \text{ all } y.\]

For the following analytic developments it is also convenient to adopt the well-known transformation of time scale,

\[(4.6) \quad u = \int_{0}^{t} p(\tau) d\tau\]

which maps the positive \(t\)-axis \(1-1\) onto the finite or infinite interval \(0 \leq u < U = \int_{0}^{\infty} p(\tau) d\tau\). With respect to the new time variable, the intensity is, of course, \(1\); the other functions of \(t\) are transformed into corresponding functions of \(u\): \(\bar{F}(u) = r[t(u)], \bar{Y}(u) = y[t(u)].\)

For the sake of simplicity we shall, however, drop the bars. In particular, our differential equation \((3.1)\) (simplified by notation \((4.1)\)) becomes

\[\]
\( (4.7) \quad \frac{d(ry)}{du} = -r(u)\varphi[y(u)]; \)

the convergence condition (3.2) adopted throughout Sections 3-6 becomes

\( (4.8) \quad R = \int_0^U r(u)du < \infty. \)

and is trivially fulfilled when \( U < \infty. \) For the sake of brevity, we shall adopt the notation

\( (4.9) \quad z(u) = r(u)y(u) \)

which transforms (4.7) into

\( (4.10) \quad \frac{dz}{du} = -r \frac{\varphi(z)}{r}. \)

We are now in a position to prove the existence theorem for the solutions of (4.7) and (4.10). The corresponding solutions of (3.1) are then obtained by a transformation back to the \( t \)-axis.

**Theorem 4.1.** To any real \( \eta \), there is precisely one solution \( z(u) \) \((0 \leq u \leq U)\) of (4.10) satisfying \( z(0) = \eta \), and hence precisely one solution \( y(u) \) \((0 \leq u \leq U)\) of (4.7) with \( y(0) = \eta \).

**Proof.** Using the classical approximation procedure, we put

\( (4.11) \quad z_0(u) = \eta \)

\( z_n(u) = \eta - \int_0^u r(\tau) \frac{z_{n-1}(\tau)}{r(\tau)} d\tau. \)

The functions \( z_n(u) \) exist and are continuous for all \( u < U. \) Subtracting consecutive equations (4.11) we get to begin with, using (4.5),

\( (4.12) \quad |z_1 - z_0| \leq \int_0^u r(\tau)\varphi[\frac{\eta}{r(\tau)}] d\tau \leq 2\mu u \)
and generally, using the middle value theorem and (4.5)

\[(4.13) \quad |z_n - z_{n-1}| \leq \int_0^n |z_{n-1} - z_{n-2}| d\tau \quad (n=2,3,\ldots) .\]

By consecutive substitution, (4.12-4.13) yield

\[(4.14) \quad |z_n - z_{n-1}| \leq 2u \frac{u^n}{n!} .\]

Hence the functions \(z_n(u)\) converge to a continuous limiting function \(z(u)\); letting \(n \to \infty\) in (4.11) and noting that \(\varphi(y)\) is continuous, we find that

\[(4.15) \quad z(u) = \eta - \int_0^u r(\tau) \varphi\left[\frac{z(\tau)}{r(\tau)}\right] d\tau\]

from which (4.10) follows by derivation.

As to uniqueness, let \(z(u)\) and \(z^*(u)\) be two solutions of (4.10) satisfying \(z(0) = z^*(0) = \eta\). Then

\[(4.16) \quad \frac{d(z^*-z)}{du} = -(z^*-z)\varphi'(\xi_u)\]

where \(\xi_u\) is a value between \(z/r\) and \(z^*/r\). If \(z^* \neq z\) at some point \(u_0\), and hence in some closed interval \([u,u_0]\), it follows from (4.16) that

\[(4.17) \quad \log|z^*(u_0) - z(u_0)| = \log|z^*(u) - z(u)| - \int_u^{u_0} \varphi'(\xi_\tau) d\tau.\]

This provides a contradiction when we let \(u\) tend to the nearest point to the left of \(u_0\) where \(z^* = z\) (0 at least is such a point). This completes the proof of Theorem 4.1.
5. The relevant solution.

According to Theorem 3.1, what we need is a solution of (4.7) fulfilling \( r(u) \gamma(u) \to 0 \) \( (u \to U) \) or, in other words, a solution of (4.10) with \( z(u) \to 0 \) \( (u \to U) \).

Let \( z(u, \eta) \) be the solution mentioned in Theorem 4.1. From (4.15), (4.5), and (4.8), it is obvious that \( z(u, \eta) \) is non-increasing with respect to \( u \), and bounded from below by \( \eta - 2\mu R \). Hence

\[
Z(\eta) = \lim_{u \to U} z(u, \eta)
\]

exists and is finite. Clearly, \( Z(0) < 0 \), \( Z(2\mu R) \geq 0 \). To ensure the existence of precisely one solution with \( Z(\eta) = 0 \), we must make sure that \( Z(\eta) \) is strictly increasing and continuous.

Let \( \eta, \eta^* \) be two initial values \( (\eta^* > \eta) \) and let \( z(u), z^*(u) \) be the corresponding solutions of (4.10). Precisely as in the uniqueness proof of the previous section it is seen that \( z(u) < z^*(u) \) for all \( u < U \). Furthermore, (4.16) holds, and since \( -1 \leq \varphi' \leq 0 \), we conclude that

\[
0 \leq \log \frac{z^*-z}{\eta^*-\eta} \leq u,
\]

hence

\[
\eta^*-\eta \leq z^*-z \leq (\eta^*-\eta)e^u.
\]

The first inequality shows, when \( u \to U \), that \( Z(\eta^*) - Z(\eta) \geq \eta^*-\eta \) and hence \( Z(\eta) \) is strictly increasing.

To show the continuity of \( Z(\eta) \), take \( 0 < u_1 < u \) and write

\[
0 \leq z^*(u) - z(u) = [z^*(u_1) - z(u_1)] + [z^*(u) - z^*(u_1)] - [z(u) - z(u_1)]
\]
By (4.15), the two last brackets may be written as

\[- \int_{u_1}^{u} r(\tau) \varphi \frac{z^*(\tau)}{r(\tau)} d\tau \quad \text{and} \quad - \int_{u_1}^{u} r(\tau) \varphi \frac{z^*(\tau)}{r(\tau)} d\tau.\]

Since \( \varphi \) is non-increasing, the latter dominates the former, and is itself dominated by \( 2u \int_{u_1}^{\infty} r(\tau) d\tau. \) Hence, combining (5.2) (taken at \( u = u_1 \)) and the facts just mentioned, we have

\[(5.3) \quad 0 \leq z^*(u) - z(u) \leq (\eta^*-\eta)u_1 + 2u \int_{u_1}^{\infty} r(\tau) d\tau.\]

By the convergence of the last integral, we can, for any \( \epsilon > 0, \) take \( u_1 = u_\epsilon \) such that the last term on the r.h.s. in (5.3) is \( \leq \frac{\epsilon}{2} \) and then \( \delta_\epsilon \) so that the first term is \( \leq \frac{\epsilon}{2} \) when \( \eta^*-\eta \leq \delta_\epsilon. \) Then

\[0 \leq z^*(u) - z(u) \leq \epsilon \quad \text{when} \quad u > u_\epsilon, \quad 0 \leq \eta^*-\eta \leq \delta_\epsilon. \]

Letting finally \( u \to U, \) we conclude that \( Z(\eta) \) is (uniformly) continuous.

The monotonicity and continuity of \( Z(\eta) \) imply that (4.10) has exactly one solution with \( Z(U) = 0. \) Retracing our steps, through the transformations (4.9), (4.6), and Theorems 4.1 and 3.1, we end up with

**Theorem 5.1.** Under the convergence assumption (3.2), the integral equation (2.4) has precisely one solution, viz., that uniquely determined solution of the differential equation (3.1) which satisfies \( ry \to 0 \) as \( t \to \infty. \)

6. **Examples and special cases.**

1° Consider first the simple example \( p(t) = 1, \quad r(t) = e^{-\alpha t}. \)

Since now \( u = t, \) we may write (4.7) as

\[\frac{dy}{dt} - \alpha y = -\varphi(y).\]
This equation has a constant solution \( y = y_o \) where \( y_o \) is the unique root of the equation \( \phi y = \phi(y) \). Since \( y_o e^{-\alpha t} \to 0 \) as \( t \to \infty \), we have found the critical curve, a horizontal line.

2° Next, let us illustrate the case where \( \int p(t)dt < \infty \), i.e., where the expected number of offers is finite. Write

\[
(6.1) \quad v = U - u = \int_t^\infty p(\tau) d\tau .
\]

Assume, in particular, that after the transformation (6.1), \( r \) is of form \( r = v^\beta (\beta > 0) \). This will, for instance, be the case if on the original time scale both \( p \) and \( r \) are exponential, say

\[
p = \lambda e^{-\lambda t}, \quad r = e^{-\alpha t}
\]

and hence \( v = e^{-\lambda t}, \quad r = r(v) = v^\beta \) with \( \beta = \alpha/\lambda \). Noticing the change of direction of the time axis, we may now write (4.7) as

\[
(6.2) \quad \frac{dv}{dy} + \frac{\beta}{2} y = \phi(y) .
\]

If \( \phi(y) \) has a Taylor expansion, c.f. (4.3),

\[
(6.3) \quad \phi(y) = \mu - y + a_2 y^2 + \cdots
\]

around \( y = 0 \), we may try for \( y \) an expansion

\[
(6.4) \quad y = b_1 v + b_2 v^2 + \cdots .
\]

Identifying terms on both sides of (6.2) we find

\[
(6.5) \quad b_1 = \frac{\mu}{\beta + 1}, \quad b_2 = \frac{-\mu}{(\beta + 1)(\beta + 2)} , \cdots .
\]

In the case of exponential intensity and discount rate, we have on the original time scale
\[ y = \frac{\mu}{\beta+1} e^{-\lambda t} - \frac{\mu}{(\beta+1)(\beta+2)} e^{-2\lambda t} + \ldots . \]

We notice that the asymptotical decrease rate of the critical curve depends solely on the decrease rate \( \lambda \) of the intensity; the discount rate and the offer distribution enter only in the coefficient of the leading term.

\( Y^0 \) Preserving still the assumption \( U = \int_0^\infty p(t) dt < \infty \), let us examine the case \( r = 1 \) of no discounting. Again, introduce \( v = U - u = \int_t^\infty p(\tau) d\tau \) as time variable. Since, for the relevant solution, \( y = 0 \) at \( v = 0 \), eq. (4.7) may be integrated to give

\[ \int_0^y \frac{dy}{\phi(y)} = v = \int_t^\infty p(\tau) d\tau . \]

The integral on the l.h.s. diverges as \( y \) approaches the upper bound \( y_2 \) of the \( Y \)-distribution: if \( y_2 < \infty \), this is seen from \( \phi(y) = \phi(y) - \phi(y_2) = G(x)(y_2 - y) \leq y_2 - y \); if \( y_2 = \infty \), the divergence is obvious. Thus there exists a number \( \eta \) for which

\[ \int_0^\eta \frac{dy}{\phi(y)} = \int_0^\infty p(\tau) d\tau , \]

and to any \( t \) in \( (0, \infty) \) there is a unique solution of (6.7) with value in \( (0, \eta) \). This solution \( y = y(t) \) is the critical curve. For instance, if \( p(t) = \lambda e^{-\lambda t} \) and \( f(y) = \theta e^{-\theta y} \) then eq. (6.7) yields the critical curve

\[ y = \theta^{-1} \log(1 + e^{-\lambda t}) \sim \frac{1}{\theta} e^{-\lambda t} , \]

in accordance with the expansion (6.6) under \( 2^0 \) above.
Let us now assume that the intensity integral diverges, and that
the time-scale has been adjusted so as to make \( p(t) = 1, 0 \leq t < \infty \).
Assume furthermore that \( r(t) \) is a step function, with steps at
\( t_1 < t_2 < \ldots \),

(6.9) \( r(t) = r_k \quad \text{in} \quad [t_k, t_{k+1}) \), \( k = 0, 1, 2, \ldots ; \quad t_0 = 0, \quad r_0 = 1 \).

Since, for the relevant solution, \( r_y \) is always continuous, \( y \) will
have steps at \( t_1, t_2, \ldots \); in the intervening intervals, the differential
equation (4.7) is elementarily integrable. The integral function

(6.10) \( F(y) = \int_0^y \frac{dy}{\varphi(y)} \)

increases monotonically from \(-\infty\) to \(+\infty\) when \( y \) increases from \(-\infty\)
to the upper bound \( y_2 \) of the Y-distribution. Integrating (4.7)
between \( t_{k-1} \) and \( t \) we have

(6.11) \( F(y) - F(y_{k-1}^+) = - (t - t_{k-1}) \quad (t_{k-1} < t < t_k) \),

which determines the continuous decrement of \( y(t) \) within any interval
\((t_{k-1}, t_k)\). The jump rates are determined by the fact that \( z(t) = r(t)y(t) \)
is continuous, hence \( r_k y_k^+ = r_{k-1} y_{k-1}^- \) and consequently

(6.12) \( \frac{y_k^+}{y_k^-} = \frac{r_{k-1}}{r_k} > 1 \).

The initial value \( \eta \) has to be determined so that \( r_k y_k^+ \to 0 \).

Consider for illustration the case \( t_k = k \), \( r_k = q^k \) \((0 < q < 1)\),
cf. Example 1 above. Equations (6.11-6.12) yield the recurrence
relation \( F(q y_k^+) - F(y_{k-1}^+) = -1 \), which has a constant solution \( y_k^+ = \eta \)
fulfilling the equation \( F(q \eta) - F(\eta) = -1 \). This is the relevant solution.
since \( r_k \eta \to 0 \). Figure 1 shows the critical curve when \( t_k = k \), 
\[
r_k = \left(\frac{1}{2}\right)^k, \quad f(y) = e^{-y}.
\]

![Graph of the critical curve](image)

Figure 1

In the light of our original selling problem, the discontinuous solution \( y(t) \) is exactly what one would expect: it pays to sell cheap before the discount rate changes; after than, there is again time to wait for better offers.

7. **The divergence case.**

The case where \( \int r(t) \, dt \) diverges (assuming still for simplicity that \( p(t) = 1 \)) presents anomalies which require a separate discussion. For brevity, we shall use the notations

\[
(7.1) \quad r_\infty = \lim_{t \to \infty} r(t) ; \quad Y^* = \sup_{t} p r(t) Y ; \quad M = \sup_{t} I_y
\]

where \( I_y = E^{y} r(T) \) is the expected value of the process when using the critical curve \( y = y(t) \), cf. Section 2.
It turns out that $M$ may be infinite, and our main interest will be in finding out under what circumstances this happens. The question of the existence and determination of an optimal critical curve presents more difficulties than in the convergence case, and will not be generally solved.

Some straightforward cases.

1° If $Y^*$ is finite, it is obvious that $\mathbb{E} Y_r(T) \leq Y^*$ for any critical curve $y(t)$, and hence $M \leq Y^*$. In particular, if $r(t) = 1$, the critical curve $y = Y^* - \varepsilon$ will yield $I_y \geq Y^* - \varepsilon$, since with probability one there will sooner or later appear an offer exceeding $Y^* - \varepsilon$; thus in this case $M = Y^*$. If $r(t)$ decreases but $r_\infty > 0$, the same policy will yield $I_y \geq r_\infty (Y^* - \varepsilon)$, and hence $r_\infty Y^* \leq M \leq Y^*$.

2° If $Y^*$ is infinite and $r_\infty > 0$, the same argument as above shows that $M = \infty$. As a matter of fact, take $Y_o$ arbitrarily large, and consider the policy $y(t) = Y_o$. Again, there will almost surely appear an offer exceeding $Y_o$, and hence $\mathbb{E} Y_r(T) \geq Y_o r_\infty$. Since $Y_o$ was arbitrary, this implies $M = \infty$.

There remains the case $Y^* = \infty, \ r_\infty = 0$, in which we have not been able to find any simple criterion for the finiteness of $M$. However, the question is in principle decidable by means of the truncation procedure described below. The procedure provides, at the same time, a means for approximating the optimum value $M$ of the process, whether it be finite or infinite.

**Truncation.**

Consider the same persistency process as before, but truncated at $t = T$; that is, if no offer is accepted before time $T$, the process
stops and its value is zero. For this truncated process, all the considerations of Sections 2-5 may be carried through without making any convergence assumptions. The content of Theorems 2.1, 3.1, 4.1, and 5.1 may then (with \( p(t) = 1 \)) be condensed into the following:

**Theorem 7.1.** Under truncation at \( t = T \), the persistency problem has precisely one optimal solution \( y(t) \), fulfilling the integral equation

\[
(7.2) \quad r(t)y(t) = \int_t^T r(\tau)H(y(\tau)) \exp[-\int_t^T G(y(\sigma)) d\sigma] d\tau,
\]

and the differential equation and initial condition

\[
(7.3) \quad \frac{d[r(t)y(t)]}{dt} = -r(t)\varphi[y(t)] ; \quad y(0) = 0.
\]

We shall denote the value of the truncated process for an arbitrary critical curve \( y(t) \) by \( I_y^T \), and the maximum value provided by \( y(t) \), i.e., the r.h.s. of (7.2) taken at \( t = 0 \) by \( I_T \); hence \( I_y^T \leq I_T \).

Denote

\[
(7.4) \quad M^* = \sup_T I_T
\]

We shall prove

**Theorem 7.2.** \( M = M^* \).

**Proof.** For any \( T \), the critical curve

\[
y^*(t) = y(t) , \quad t \leq T ; \quad y^*(t) = 0 , \quad t > T
\]

yields \( I_{y^*}^T \geq I_T \); thus, for an appropriate \( T \), we will have \( I_{y^*}^T > M^*-\varepsilon \), and hence \( M = \sup_y I_y \geq M^* \). On the other hand, assume (antithesis) that \( M > M^* (< \infty) \). We then could find a \( y(t) \) such that
\( M^* < I_{\frac{T}{y}} \) and thence a truncation at \( T \) such that \( M^* < I_{\frac{T}{y}} \). But
\[ I_{\frac{T}{y}} \leq I_T \leq \sup I_T = M^* \]. This leads to the contradiction \( M^* < M^* \).

Theorem 7.2 shows that the maximum value of the persistency process (be it finite or infinite) can be approximated by means of solutions to truncated problems, i.e., by solutions of (7.3).

We shall conclude our study by exhibiting three examples. The first shows how truncation works in the case \( Y^* = \infty \), \( r(t) \equiv 1 \), where we know that \( M = \infty \). The second and third example show that in the case \( Y^* = \infty \), \( r(t) \to 0 \), the value of the process may be either finite or infinite.

**Example 7.1.** Let \( r(t) \equiv 1 \) and take for \( Y \) the standard exponential distribution:

\[
(7.5) \quad \phi(y) = e^{-y}, \ G(y) = -\phi'(y) = e^{-y}, \ H(y) = yG(y) + \phi(y) = (y+1)e^{-y}.
\]

The differential equation (7.3) becomes

\[
(7.6) \quad \frac{dy}{dt} = -e^{-y}; \quad y(T) = 0
\]

which yields \( y_T(t) = \log(1+T-t) \). Inserting this on the r.h.s. of (7.2) taken at \( t = 0 \), we get the truncated value

\[
I_T = \int_0^T [1+\log(1+T-\tau)](1+T-\tau)^{-1}\exp[-\int_0^T (1+T-\sigma)^{-1}d\sigma]d\tau
\]

\[
(7.7)
\]

\[
= (1+T)^{-1} \int_0^T [1+\log(1+T-\tau)]d\tau = \log(1+T).
\]

The result confirms that \( M = \sup I_T = \infty \).
Example 7.2. We now take the distribution of $Y$ as in (7.5) but let $r(t) = (1+t)^{-\delta}$, $0 < \delta \leq 1$ (for $\delta > 1$, the result is obvious). We wish to show that the truncated value $I_Y^{T}$ is bounded independently of $T$ and $y$; hence $M$ is finite.

We shall not use the truncation procedure as applied to the discounted process, since the differential equation (7.3) becomes complex. Instead, denote for an arbitrary critical curve,

$$s_y(t) = H(y(t)) \exp[-\int_0^t G(y(\sigma)) d\sigma],$$

(7.8)

$$S_y(t) = \int_0^t s_y(\tau) d\tau,$$

and write the truncated value of the process as

(7.9) $$I_Y^{T} = \int_0^T r(\tau) s_y(\tau) d\tau = r(T) S_y(T) + \int_0^T |r'(\tau)| S_y(\tau) d\tau.$$  

The function $S_y(t)$ is the value over $(0,t)$ of a non-discounted process with the given $Y$-distribution. The maximum value of such a process was calculated in Example 7.1 and found to be $\log(1+t)$. Hence, for any $y$ and $T$,

(7.10) $$I_Y^{T} \leq (1+T)^{-\delta} \log(1+T) + \delta \int_0^T (1+\tau)^{-\delta-1} \log(1+\tau) - 1 d\tau.$$

It is easily checked that both terms on the r.h.s. are bounded with respect to $T$. Hence $M < \infty$.

Example 7.3. To exhibit an example where $M = \infty$ in spite of $r(t) \to 0$, take $r(t) = (1+t)^{-1/4}$ and
\[(7.11) \quad \varphi(y) = 2(1+y)^{-1/2}, \quad g(y) = (1+y)^{-3/2}, \]
\[H(y) = (2+3y)(1+y)^{-3/2} \geq 2(1+y)^{-1/2}.\]

Since we only wish to show that \( I_y = \infty \) for some \( y \), we need not bother about optimal truncated solutions; actually, the simple policy \( y(t) = t \) yields
\[
\int_0^T g(y(\sigma))d\sigma = \int_0^T (1+\sigma)^{-3/2} d\sigma = 2[1-(1+t)^{-1/2}] < 2,
\]
\[H(y(\tau)) \geq 2(1+t)^{-1/2},\]

and hence
\[
I_y^T \geq \int_0^T (1+t)^{-1/4} \cdot 2(1+t)^{-1/2} \cdot e^{-2}d\tau = 2e^{-2} \int_0^T (1+t)^{-3/4}d\tau.
\]

The integral diverges, and hence not only is \( M = \infty \) but even \( I_y = \infty \); i.e., the infinite value of the process is actually attained, e.g., by means of the policy \( y(t) = t \).
REFERENCES


