A NOTE ON RISK AND MAXIMAL REGULAR GENERALIZED
SUBMARTINGALES IN STOPPING PROBLEMS

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In a recent paper Haggstrom [1] rearranged and amplified on the excellent work of Snell [2] on optimal stopping problems as a preliminary to generalizing the framework to deal with problems in control and experimental design. The purpose of this note is to point out that one of Haggstrom's results can be derived under weaker conditions by using the results of Snell.

Let \( \{Z_n^r, F_n; n \geq 1\} \) be a stochastic process on a probability space \((\Omega, F, P)\) with points \(\omega\). A stopping variable (sv) is a random variable (rv) \(t\) with values in \(\{1, 2, \ldots, \infty\}\) such that \(t < \infty\) a.e. and \(t=n\) \(\in F_n\) for each \(n\). For any such sv \(t\), a rv \(Z_t\) is defined by

\[
Z_t(\omega) = Z_n(\omega) \quad \text{if} \quad t(\omega) = n \quad \text{and} \quad Z_t(\omega) = \infty \quad \text{if} \quad t(\omega) = \infty.
\]

The optimal stopping problem consists of finding \(t\) to minimize the risk \(E(Z_t)\).

Random variables are defined in the extended sense, i.e., they can take on the values \(\pm \infty\) and \(\infty\). We regard \(E(X)\) as defined as long as either the positive or negative part of \(X\) has finite expectation. This extension is usually referred to by the term generalized. The reader is referred to Snell [2] or Haggstrom [1] for detailed discussion of the technical terms used in this note.

Snell [1] obtained results characterizing the solution of the optimal stopping problem in terms of \(\{Y_n^r, F_n; n \geq 1\}\), the maximal regular
generalized submartingale relative to \( \{Z_n, F_n, n \geq 1\} \). In particular, as part of his Theorem 3.6, he proved that

\[
E(Y_n) = \inf_{t \in T_n} E(Z_t)
\]

where \( T_n \) is the class of \( sv \) for which \( t \geq n \) a.e. The proof of this theorem used the hypothesis that

\[
E(\inf_{n \geq 1} Z_n) > -\infty.
\]

Haggstrom [1] introduced, and developed his results in terms of,

\[
X_n = \text{ess inf}_{t \in T_n} E(Z_t|F_n)
\]

which represents the optimal risk at stage \( n \) to the player who, for one reason or another, has not stopped previously. The relation between the results of Snell and Haggstrom, and in fact the motivation of Snell, is clarified by Haggstrom's Theorem 3.5 which states

\[
X_n = Y_n \text{ a.e. for } n=1,2,\ldots
\]

under (2) and the extra condition that \( E(|Z_n|) < \infty \) for each \( n \). Haggstrom pointed out that this extra condition can be relaxed to require only that for each \( n \) there exist a \( sv \) in \( T_n \) such that \( E(Z_t) < \infty \).

He referred to the fact that Y.S. Chow proved (4) using \( E(|Z_n|) < \infty \) for each \( n \) and \( E(\sup_{n \geq 1} Z_n) < \infty \).

The object of this note is to prove

\[
E(Y_n|F_r) = \text{ess inf}_{t \in T_n} E(Z_t|F_r) \text{ a.e. for } n \geq r
\]

as a corollary of Snell's theorem using only condition (2). Equation (4) follows by setting \( r = n \).
To this end let $B$ be an arbitrary set of $F_r$. For $n \geq r$, let

$$Z^n = Z, \quad Y^n = Y_n \quad \text{on } B$$

$$Z^n = 0, \quad Y^n = 0 \quad \text{on } B^c.$$  

Then $(Z^n, F_n, n \geq r)$ is a stochastic process and $(Y^n, F_n, n \geq r)$ is the maximal regular generalized submartingale relative to it. Applying Snell's result

$$E(Y^n) = \inf_{t \in T_n} E(Z_t^n).$$

Hence, with $X_B$ the characteristic function of $B$,  

$$E[X_B E(Y^n | F_r)] = \inf_{t \in T_n} E[X_B E(Z_t^n | F_r)].$$

Suppose now that there is a set of positive measure on which

$$E(Y^n | F_r) > \inf_{t \in T_n} E(Z_t^n | F_r).$$

Then there is a subset of positive measure on which $E(Y^n | F_r)$ is bounded away from $-\infty$ and $\inf_{t \in T_n} E(Z_t^n | F_r)$ is bounded away from $+\infty$. Hence there exist $t \in T_n$ and $B \subset F_r$ such that $B$ has positive measure, $E(Y^n | F_r) > E(Z_t^n | F_r)$, $E(Y^n | F_r)$ is bounded away from $-\infty$, and $E(Z_t^n | F_r)$ is bounded away from $+\infty$ a.e. on $B$. Then

$$E[X_B E(Y^n | F_r)] > E[X_B E(Z_t^n | F_r)]$$

which contradicts (8). Similarly, assuming $E(Y^n | F_r) < \inf_{t \in T_n} E(Z_t^n | F_r)$ on a set of positive measure implies the existence of a set of positive measure $B \subset F_r$ such that for all $t \in T_n$, $E(Y^n | F_r)$ is bounded away from $+\infty$, $E(Z_t^n | F_r)$ is bounded away from $-\infty$, and $E(Y^n | F_r) < E(Z_t^n | F_r)$ a.e. on $B$. This also leads to a contradiction of (8). Hence equation (5) holds and (4) follows as a special case.
Condition (2) may be interpreted as stating that even the player who can see the future cannot achieve unbounded expected gain. One may conjecture whether it suffices to assume that the ordinary player has bounded expectation. More precisely, does $E(X_n) > -\infty$ imply $X_n = Y_n$ a.e.?

REFERENCES
