ON MOMENTS OF THE MAXIMUM OF NORMED PARTIAL SUMS

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DAVID SIEGMUND

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1. Introduction and Summary.

Let $X, X_1, X_2, \ldots$ be independent random variables with $E(X_n) = 0$ $(n \geq 1)$, and put $S_n = X_1 + \ldots + X_n$ $(n \geq 1)$. Marcinkiewicz and Zygmund [5] and Wiener [8] have shown that if the $X$'s have a common distribution, then

\begin{equation}
E\left( \sup_n \left| \frac{S_n}{n} \right| \right) < \infty
\end{equation}

provided that

\begin{equation}
E[|X|U(|X|)] < \infty,
\end{equation}

where we have put $U(x) = \max(1, \log x)$ ($U_2(x) = U(U(x))$, etc.).

Burkholder [2] has extended this result by showing that (1), (2), and

\begin{equation}
E\left( \sup_n \frac{X_n}{n} \right) < \infty,
\end{equation}

are equivalent. More recently, motivated by certain optimal stopping problems, Teicher [7] and Bickel [1] under various assumptions on the distributions of $X_1, X_2, \ldots$ have shown that

\begin{equation}
E\left( \sup_n c_n \left| S_n \right|^{\alpha} \right) < \infty
\end{equation}
for certain sequences \((c_n)\) and positive constants \(\alpha\). The interesting special case

\[(5) \quad c_n = (nU_2(n))^{-\alpha/2}\]

is not covered by the results of these authors.

This note gives a method which seems suitable for proving statements like (4) in a variety of cases. The method involves modifications of standard techniques used in the study of the law of the iterated logarithm. In particular, for each \(\alpha = 1, 2, \ldots\) we are able to establish necessary and sufficient conditions for (4) when the \(X\)'s are identically distributed and the sequence \((c_n)\) satisfies (5). In Section 2 we state and prove one such theorem. Section 3 is devoted to explaining in somewhat more detail the scope of our results and their relation to the previously mentioned literature.

2. **A Maximal Theorem.**

**Theorem 1.** Let \(X, X_1, X_2, \ldots\) be independent, identically distributed random variables with \(EX = 0\). The following statements are equivalent:

\[(6) \quad E\left\{\frac{X^2 U(|X|)}{U_2(|X|)}\right\} < \infty;\]

\[(7) \quad E\left\{\sup_n (nU_2(n))^{-1} \frac{S_n^2}{n}\right\} < \infty;\]
\[ E(\sup_n (nU_2(n))^{-1}X_n^2) < \infty. \]

**Proof.** We shall show that (6) \(\Rightarrow\) (7) \(\Rightarrow\) (8) \(\Rightarrow\) (6). Suppose initially that the distribution of \(X\) is symmetric and \(EX^2 = 1\). Put

\[ c_n = (nU_2(n))^{-1}, \quad b_n = n^{1/2} (U_2(n))^{-1/2} \quad (n \geq 1), \]

and define

\[ X'_n = X_n I[|X_n| \leq b_n], \quad X''_n = X_n - X'_n; \]

\[ S'_n = \sum_{k=1}^{n} X'_k, \quad S''_n = S_n - S'_n. \]

To prove (7) it suffices to show

\[ E(\sup_n c_n |S'_n|^2) < \infty \]

and

\[ E(\sup_n c_n |S''_n|^2) < \infty. \]

Now

\[ E(\sup_n c_n |S''_n|^2) \leq E\left( \sum_{k=1}^{\infty} c_k |X''_k|^2 \right)^2 \]

\[ \leq \sum_{k=1}^{\infty} c_k E|X''_k|^2 + 2\left( \sum_{k=1}^{\infty} c_k^{1/2} E|X''_k| \right)^2, \]

and from (6)
\[ \sum_{k=1}^{\infty} c_k E|X_k''|^2 = \sum_{k=1}^{\infty} c_k \sum_{j=k}^{\infty} \int_{b_j < |x| \leq b_{j+1}} x^2 \]
\[ = \sum_{j=1}^{\infty} \sum_{k=1}^{j} c_k \int_{b_j < |x| \leq b_{j+1}} x^2 \leq \text{const.} \sum_{j=1}^{\infty} \frac{U(j)}{U_2(j)} \int_{b_j < |x| \leq b_{j+1}} x^2 \]
\[ \leq \text{const.} E\left\{ x^2 \frac{U(|x|)}{U_2(|x|)} \right\} < \infty. \]

Similarly
\[ \sum_{k=1}^{\infty} c_k^{1/2} E|X_k| \leq \text{const.} E X^2 < \infty, \]

and (11) follows. To prove (10) it suffices to show that

(12) \[ \int_{x_0}^{\infty} u P(\sup n^k |S_n'| > u) du < \infty \]

for some \( x_0 > 0 \). For each \( k = 0, \ldots, \) let \( n_k \) be the largest integer \( \leq \frac{\sigma}{\bar{c}^k} \). Writing \( e_n = c_n^{1/2} \), we have by Levy's inequality

\[ P(\sup n^k |S_n'| > u) \leq \sum_{k=0}^{\infty} P(e_n \sup_{n_k \leq n < n_{k+1}} |S_n'| > u) \]
\[ \leq 4 \sum_{k=0}^{\infty} P(e_{n_k} S_{n_{k+1}}' > u). \]

We now use the fact that if \( |Z| \leq b \), then for any \( t > 0 \) for which \( tb \leq 1 \)
\[ E(\exp(tZ)) \leq \exp(tEZ + t^2EZ^2), \]

and Chebyshev's inequality

\[ (P[S_n > x] \leq \exp(-tx) \prod_1^n E(\exp(tX'_k)), \ t > 0) \]

to obtain

\[ \log P[S_{n_{k+1}} > e^{u_{n_k}}] \leq -te^{-1}u + t^2n_{k+1} \]

\[ (0 < t \leq b^{-1}, k=0,1,\ldots) . \]

Setting \( t = b^{-1} \), we have

\[ \log P[S_{n_{k+1}} > e^{u_{n_k}}] \leq -K_1(u-K_2)U_2(n_{k+1}), \]

where \( K_1, K_2, \ldots \) denote constants, the exact values of which are of no interest. Taking \( x_o \) to satisfy \( K_1(x_o - K_2) \geq 2 \), we have from (12)-(14)

\[ \int_{x_o}^{\infty} u P(\sup_n e_n | S_n | > u) du \leq K_3 \sum_{k=1}^{\infty} \int_{x_o}^{\infty} u \exp(-K_1(u-K_2)U_2(n_k)) du \]

\[ \leq K_3 \sum_{k=1}^{\infty} \int_{x_o}^{\infty} u \exp(-K_1(u-K_2)\log k)\exp[-K_1(u-K_2)U_2(3)] du \]

\[ \leq K_3 \sum_{k=1}^{\infty} k^{-2} \int_{x_o}^{\infty} u \exp[-K_1(u-K_2)U_2(3)] du < \infty . \]
This proves that (6) \(\Rightarrow\) (7) for symmetrically distributed \(X\). In general, let \(X_1^{(s)}, X_2^{(s)}, \ldots\) be i.i.d. and independent of \(X_1, X_2, \ldots\) with

\[
P(X_1^{(s)} \leq x) = P(X \leq x) \quad (-\infty < x < \infty).
\]

Let \(S_n^{(s)} = \sum_{k=1}^{n} X_k^{(s)}\). Then (see Loève [3], p. 263, or Bickel [1])

\[
E[\sup_n c_n |S_n^{(s)}|^2] = E[\sup_n c_n |S_n - E(S_n^{(s)})|X_1, X_2, \ldots, X_n] \leq E[\sup_n c_n E[|S_n - S_n^{(s)}|^2|X_1, X_2, \ldots, X_n]] \leq E[E[\sup_n c_n |S_n - S_n^{(s)}|^2|X_1, X_2, \ldots, X_n]] = E[\sup_n c_n |S_n - S_n^{(s)}|^2] < \infty
\]

by our previous result.

To show that (7) \(\Rightarrow\) (8) we merely observe that

\[
c_n X_n^2 = c_n (S_n - S_{n-1})^2 \leq 2(c_n S_n^2 + c_{n-1} S_{n-1}^2).
\]

Suppose now that (8) is satisfied. Then

\[
\sum_{k=1}^{\infty} P(\sup_n c_n X_n^2 > k) < \infty,
\]

or equivalently
\[ \prod_{k=1}^{\infty} \prod_{n=1}^{\infty} F(c_{n,k}) > 0, \]

where we have let \( F \) denote the distribution function of \( X^2 \) and have assumed, as we may by a change of scale, that \( F(1) > 0 \). Hence

\[ \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \log F(c_{n,k}) > -\infty, \]

and since \( \log F(x) \sim -(1-F(x)) \) as \( x \to \infty \), we have

\[ \int_1^{\infty} \int_1^{\infty} (1-F(xU_2(x)y))dydx \leq \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} (1-F(c_{n,k})) < \infty. \]

Setting \( u = xU_2(x)y \), we have

\[ \int_1^{\infty} \int_1^{\infty} (1-F(u))du \frac{dx}{xU_2(x)} < \infty. \]

If we let \( \varphi \) denote the inverse of the function \( x \to xU_2(x) \), we have by Fubini

(15) \[ \int_1^{\infty} \int_1^{\infty} (1-F(u))du \frac{dx}{xU_2(x)} < \infty. \]

Since \( \varphi(u) \sim \frac{u}{U_2(u)} \) (\( u \to \infty \)) and \( \int_1^{t} \frac{dx}{xU_2(x)} \sim \frac{U(t)}{U_2(t)} \) (\( t \to \infty \)), it follows that (15) is equivalent to

\[ \int_1^{\infty} \frac{U(u)}{U_2(u)} (1-F(u))du < \infty, \]

which in turn is equivalent to (6).
3. Remarks.

Relatively straightforward modifications of the proof of Theorem 1 lead to various other results, a few of which are summarized below.

Let $X, X_1, X_2, \ldots$ be independent random variables with $EX_n = 0$ ($n \geq 1$).

(16) If the $X$'s are identically distributed, $\alpha = 1$, and $(c_n)$ satisfies (5), then (4) is equivalent to

$$E(X^2) < \infty.$$  

(17) If the $X$'s are identically distributed, $\alpha = 3, 4, \ldots$, and $(c_n)$ satisfies (5), then (4) is equivalent to

$$E|x|^\alpha < \infty.$$  

(18) If the $X$'s are identically distributed, $\alpha = 2$, $c_n = (nU(n))^{-1}$, then (4) is equivalent to

$$E(X^2U_2(|X|)) < \infty.$$  

(19) If the $X$'s are identically distributed and $c_n = (nU_2(n))^{-1/2}$, then

$$E(\exp(t \sup_n c_n |S_n|)) < \infty$$

for some $t > 0$ if and only if

$$E(\exp(t |X|)) < \infty$$

for some $t > 0$.  

8
The result (18) improves on Teicher's theorem [7] in the sense that with the sequence \((c_n)\) of (18) Teicher requires that

\[
E[x^2U(|X|)] < \infty
\]

to insure (4). In this regard note that even (6) is weaker than (20). Moreover, our methods apply in the non-identically distributed case, whereas Teicher's, which depend on the Wiener ergodic theorem [8], do not. (19) in part generalizes a result of Freedman [4].

It is interesting to compare our results with those of Marcinkiewicz and Zygmund [5] in the special case \(\alpha = 2\). For future reference we state the elementary

\[
\text{Lemma (Marcinkiewicz and Zygmund). If } x_1, x_2, \ldots \text{ is any sequence of real numbers and } a_1, a_2, \ldots \text{ a non-increasing sequence of positive numbers, then}
\]

\[
\sup_n |a_n \sum_1^n x_k| \leq 2 \sup \sum_1^n a_k x_k.
\]

The proof, which is omitted, is similar to that of the closely related Kronecker lemma. If \(c_n \downarrow \) and \(\sum_1^\infty c_n E x_n^2 < \infty\), to prove (4) it suffices by (21) to prove

\[
E[\sup_n \sum_1^n c_k^{1/2} x_k^2] \leq \text{const.} \sum_1^\infty c_k E x_k^2,
\]

which is what Marcinkiewicz and Zygmund do (see their Theorems 1 and 7). (In the case \(\alpha = 2\), Bickel's method likewise proves (20).) Moreover, when applicable, this idea leads to elegant proofs. For example, if
are independent and symmetrically distributed, then with \( e_k = c_k^{1/2} \) we have by Levy's inequality

\[
E\left[ \max_{1 \leq k \leq n} \left| \sum_{1}^{k} e_k X_k \right| \right] = \int_{0}^{\infty} P\left( \max_{1 \leq k \leq n} \left| \sum_{1}^{k} e_k X_k \right| > u^{1/2} \right) du
\]

\[
< 2 \int_{0}^{\infty} P\left( \left| \sum_{1}^{n} e_k X_k \right| > u^{1/2} \right) du = 2 \sum_{1}^{n} c_k E X_k^2 ,
\]

from which (22) follows by monotone convergence. Symmetrization as in the proof of Theorem 1 proves (22) in general. Truncation and a similar calculation provide an easy proof that \( (2) \Rightarrow (1) \) in the identically distributed case. However, under the assumptions of, say, (18) the right hand side of (20) is \(+\infty\), and in fact

\[
(23) \quad P\left( \sup_{n} [U_2(n)]^{-1/2} \left| \sum_{1}^{n} e_k X_k \right| = +\infty \right) = 1 .
\]

To prove (23) observe that by the Lindeberg-Feller theorem (some calculation is required to verify the Lindeberg condition)

\[
[U_2(n)]^{-1/2} \sum_{1}^{n} e_k X_k
\]

covers in law to the standard normal random variable; (23) follows by the Kolmogorov 0-1 law (see, e.g., [5]). Thus the method of Marcinkiewicz and Zygmund does not without essential modification prove (18).
REFERENCES


