ON SOME PROBLEMS IN OPERATIONS RESEARCH

BY

ESTER SAMUEL

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1. **Introduction and Summary.**

Consider an airplane which is at a distance $t \geq 0$ from its destination, (the distance being measured in time-units) and has $n$ anti-aircraft missiles. The enemy sends its aircrafts according to a Poisson process with known parameter, which by redefining the time-scale may, without loss of generality, be taken to be one. When meeting an enemy aircraft, the plane can use, simultaneously, some of its missiles, and it is assumed that these are independent random variables, with known probability $1 > l - q > 0$ of hitting. If no missile hits the enemy aircraft, the airplane is shot down with known probability $1 > 1 - u > 0$. The case $u = 0$ is of particular interest. (We have excluded the values $q = 0$, $q = 1$ and $u = 1$ since these yield trivial solutions.)

Let $k(n, t)$ denote the optimal number of missiles to be used, when meeting an enemy aircraft at time $t$ from the destination, and when a total of $n$ missiles are at hand. Optimality here refers to a strategy which maximizes the probability of reaching the destination. It is intuitively clear that the function $k(\cdot, \cdot)$ has the two following properties:

(A) For fixed $n \geq 1$ and any $0 < t < s$, $k(n, t) > k(n, s)$.

(B) For fixed $t \geq 0$ and any $1 < m < n$, $k(n, t) > k(m, t)$.

The problem of proving (A) and (B) was first posed to me by Professor H. Chernoff. It is considered in [1], where (B) is stated as an unsolved problem, and (A) is proved, for $u = 0$, under the assumption that (B) holds. (See [1], Theorem 2.2.1 (a).) In spite of the intuitive plausibility
of (A) and (B) the present author has found no proof of (B). Here we give a quite simple and direct proof of (A), without any assumptions about (B). In fact, the proof is so simple that it can be appreciated only by those who themselves have made futile attempts to prove it. In the course of our proof we also obtain additional results which in [1] were obtained only under the assumption of the validity of (B).

Our proof yields (A) under more general models than the one described above. Sufficient conditions for (A) to hold are given in Section 5. The model described above is considered in Section 3 and is extended in Section 6. In Section 7 we give another concrete example, where (A) holds.

2. A Lemma.

In our proof we shall use the following.

**Lemma:** Let \( f(i,t) \) be positive continuous functions of \( t > 0 \), \( i = 1, \ldots, n \), and such that \( f(i,t)/f(i+1,t) \) are monotone increasing functions of \( t \), \( i = 1, \ldots, n-1 \). Let

\[
(2.1) \quad k(t) = \min\{ j \mid \max_{i=1,\ldots,n} f(i,t) = f(j,t) \}.
\]

Then \( k(t) \) is nonincreasing and right-continuous.

**Proof:** To see that \( k(t) \) is nonincreasing, assume the contrary, i.e. assume that there exist \( t_1 < t_2 \) and \( j < r \) such that \( k(t_1) = j < r = k(t_2) \). Then \( f(j,t_1)/f(r,t_1) > 1 \) whereas \( f(j,t_2)/f(r,t_2) < 1 \). But this contradicts the
fact that \( f(j,t)/f(r,t) = \prod_{i=j}^{r-1} f(i,t)/f(i+1,t) \) is increasing. The right-continuity follows from the continuity of \( f(i,t) \).

Clearly the conclusion remains valid if \( f(i,t)/f(i+1,t) \) are monotone decreasing, and \( \max \) in (2.1) is replaced by \( \min \).

3. The Model.

Let \( H(n,t) \) denote the optimal probability of survival, when meeting an enemy aircraft at time \( t \) from the destination, and \( n \) missiles are at hand. Clearly

\[
(3.1) \quad H(0,t) = u e^{-t} \sum_{i=0}^{\infty} \frac{1}{i!} = u e^{-t(1-u)} , \quad t > 0 .
\]

since the number of remaining encounters with enemy aircrafts has a Poisson distribution with parameter \( t \), and the enemy must fail to shoot the plane down at each encounter. For \( n > 1 \) let \( G_n(j,t) \) denote the probability of survival when meeting the enemy at time \( t \) and using \( j \) of the available \( n \) missiles, \( 1 \leq j \leq n \), and then proceeding optimally. To abbreviate notation write

\[
(3.2) \quad a(j) = 1 - (1-u)q^j .
\]

Then for \( t > 0, 1 \leq j \leq n \)

\[
(3.3) \quad G_n(j,t) = a(j) \int_0^t H(n-j,t-s)e^{-s}ds e^{-t} = a(j) e^{-t} \int_0^t H(n-j,v)e^{-v}dv + 1 .
\]

If we define \( H(n,t) = 1 \) for \( t < 0 \), then (3.3) can also be written as

\[
(3.4) \quad G_n(j,t) = a(j)E(H(n-j,t-X))
\]

where \( X \) is a random variable with exponential distribution. Notice that
$C_n(j,t)$ is defined by means of $H(r,t)$ with $r<n$. Notice also that any allocation strategy which at some stage, when missiles are still at hand, does not allocate any missile when an enemy aircraft is encountered, can be improved upon, simply by moving to the next stage of that strategy. This follows directly from the assumption that the enemy sends its aircrafts according to a Poisson process, and from the independence assumptions of the model. Thus clearly the optimal strategy must allocate at least one missile, if $n \geq 1$. Hence $G$ and $H$ can be defined inductively by (3.1), (3.3) and

\begin{equation}
H(n,t) = \max_{j=1,\ldots,n} G_n(j,t), \quad t \geq 0,
\end{equation}

and the optimal strategy can be defined by

\begin{equation}
k(n,t) = \min\{j \mid \max_{i=1,\ldots,n} G_n(i,t) = G_n(j,t)\}.
\end{equation}

(The definition of an optimal strategy is not quite unique, and (3.6) is a reasonable version.) For $j=1,2,\ldots$ set $b(j) = a(j+1)/a(j)$ and notice that by (3.2), and simple arithmetic one has

\begin{equation}
b(1)>b(2)>\ldots>1.
\end{equation}

Also, from (3.4)

\begin{equation}
G_n(j,t) = b(j-1)G_{n-1}(j-1,t) \quad \text{for } j=2,\ldots,n.
\end{equation}

This yield at once

\begin{equation}
k(n+1,t) \leq k(n,t)+1.
\end{equation}
Clearly (3.9) is correct for \( k(n+1,t) = 1 \). For \( k(n+1,t) > 1 \) we have for \( j = 1, \ldots, n \)

\[
G_n(j, t) = G_n(j+1, t)/b(j) \leq G_{n+1}(k(n+1,t), t)/b(j) = G_n(k(n+1,t)-1,t)b(k(n+1,t)-1)/b(j). \tag{3.10}
\]

Now by (3.7) for all \( j < k(n+1,t) - 1 \), \( b(k(n+1,t)-1)/b(j) < 1 \) and hence the maximum over the left hand side of (3.10) cannot be achieved for \( j < k(n+1,t) - 1 \), i.e. (3.9) holds. Notice also that for \( n > 2 \)

\[
H(n,t)/H(n-1,t) > 1, \quad t > 0, \tag{3.11}
\]

since \( H(n,t) > G_n(k(n-1,t)+1,t) = H(n-1,t)b(k(n-1,t)) > H(n-1,t) \).

Clearly the (unconditional) probability of survival if at time \( t \) one has \( n \) missiles at hand, is given by \( H^A(n,t) = EH(n,t-X) \).


Differentiation of (3.3) yields, for \( t > 0 \)

\[
\frac{d}{dt} G_n(j, t) = a(j)H(n-j, t) - G_n(j, t) \tag{4.1}
\]

and hence, for \( j = 1, \ldots, n-1 \)

\[
\frac{d}{dt} \left( \frac{G_n(j, t)}{G_n(j+1, t)} \right) = \frac{a(j)H(n-j, t)G_n(j+1, t) - a(j+1)H(n-j-1, t)G_n(j, t)}{G_n(j+1, t)^2}. \tag{4.2}
\]

The value of (4.2) is therefore positive if and only if

\[
H(n-j, t)/H(n-j-1, t) > b(j)G_n(j, t)/G_n(j+1, t). \tag{4.3}
\]
We shall prove (A) by induction. (A) is trivially true for \( n = 1 \), since \( k(1,t) \equiv 1 \). Notice that if the value of (4.2) is positive for some \( n \) and \( j = 1, \ldots, n-1 \), then it follows directly by the Lemma that (A) holds for that value of \( n \). Hence we shall prove (A) by induction, through proving by induction that (4.2) is positive for all \( n \) and \( j = 1, \ldots, n-1 \). Using (3.1) and (3.3) we get by direct verification, for \( u \neq 0 \)

\[
G_2(2,t) = a(2)e^{-t(1-u)} \quad , \quad G_2(1,t) = a(1)e^{-t\frac{a(1)}{u}(e^{tu} - 1) + 1}
\]

(4.4) \[
\frac{G_2(1,t)}{G_2(2,t)} = \frac{(1)(1-e^{-tu}) + e^{-tu}}{b(1)}
\]

\[
dt{\frac{G_2(1,t)}{G_2(2,t)}} = \frac{a(1) - u}{b(1)}
\]

and it is easily verified that for \( u = 0 \) one gets the corresponding limits of (4.4) as \( u \to 0 \). Substituting the value of \( a(1) \) given by (3.2) we get

\[
\frac{d}{dt}{\frac{G_2(1,t)}{G_2(2,t)}} = (1-q)(1-u)e^{-tu}/b(1) > 0 \quad , \quad \text{so (4.2) is positive for } n = 2.
\]

Now assume that

(4.5) \[
\frac{d}{dt}{\frac{G_m(j,t)}{G_m(j+1,t)}} > 0 \quad \text{for } j = 1, \ldots, m-1
\]

for all \( 1 \leq m \leq n \), \( n > 3 \). We shall show that (4.5) holds for \( m = n \). Notice that by (3.8) we have \( G_n(j,t)/G_n(j+1,t) = c_j G_{n-1}(j-1,t)/G_{n-1}(j,t) \) with \( c_j = b(j-1)/b(j) > 0 \) for \( j = 2, \ldots, n-1 \), and hence (4.5) holds for \( m = n \) and \( j = 2, \ldots, n-1 \) by the induction hypothesis. Thus it remains only to show that (4.5) holds for \( m = n \) and \( j = 1 \), which by (4.3) is equivalent to showing that

(4.6) \[
H(n-1,t)/H(n-2,t) > b(1)G_n(1,t)/G_n(2,t)
\]

We shall verify below that \( H(n-1,t)/H(n-2,t) \) is a nondecreasing
function of $t$. From this (4.6) follows quite easily, since this yields

$$
(4.7) \quad \frac{H(n-1,t)}{H(n-2,t)} > \frac{H(n-1,t-X)}{H(n-2,t-X)}
$$

where $X$ is exponentially distributed, and the inequality in (4.7) is strict whenever $X > t$, by (3.11) and the definition $H(n,s) = 1$ for $s < 0$. Thus, using (3.4) and taking expectations on both sides of the inequality

$$
H(n-1,t)H(n-2,t-X) > H(n-2,t)H(n-1,t-X)
$$

we get

$$
(4.8) \quad H(n-1,t)G_n(2,t)/a(2) > H(n-2,t)G_n(1,t)/a(1)
$$

which is equivalent to (4.6).

To prove that $H(n-1,t)/H(n-2,t)$ is nondecreasing, notice that $k(1,t) = 1$ and that by the induction hypothesis it follows from the Lemma that for $1 \leq m \leq n$

$k(m,t)$ is constant on left closed, right open intervals. Thus $k(n-1,t)$ and $k(n-2,t)$ are constant on such intervals. Suppose that in some interval

$k(n-1,t) = j$ and $k(n-2,t) = r$. Then by (3.9) $1 \leq j \leq r+1$, and hence in that interval

$$
H(n-1,t) = G_{n-1}(j,t)/G_{n-2}(r,t) = \frac{G_{n-1}(j,t)b(r)}{G_{n-1}(r+1,t)}
$$

which is constant if $j=r+1$, and otherwise, i.e. for $j \neq r+1$, is increasing, since

$$
G_{n-1}(j,t)/G_{n-1}(r+1,t)
$$

can be written as

$$
\prod_{i=1}^{r} G_{n-1}(i,t)/G_{n-1}(i+1,t),
$$

and each of the terms in the product is increasing by the induction hypothesis.

This completes the proof of (4.5) for $m=n$, and hence of (A).
Remark: Notice that it follows directly from (4.5) for \( j=1 \), and use of (3.8) that \( G_n(j,t)/G_{n-1}(1,t) \) is an increasing function of \( t \). This fact is proved in [1] only under the assumption that (B) holds. (See [1], Theorem 2.2.1 (c) for \( u=0 \), and Theorem 4.1.6 for \( 0<u<1 \).)

5. Generalizations of the Validity of (A).

(a). It should be noticed that nowhere in our proofs, except when obtaining (3.7) and verifying that (4.5) holds for \( m=2 \), have we used the specific values of \( a(j) \) and \( H(0,t) \), given in (3.2) and (3.1), respectively. Thus whenever the problem under consideration is such that (3.3), (3.5) and (3.7) hold for some constants \( a(j) \), the results (3.9) and (3.11) are valid. Furthermore, if \( H(0,t) \) is such that (4.5) can be verified directly for \( m=2 \), \( j=1 \) then (4.5) holds for all \( m \), and thus also (A).

We shall use this generalization in the next section.

(b). There exists an exactly dual version of the previous remark. If (3.3) holds for some constants \( a(j) \) and

\[
(5.1) \quad H(n,t) = \min_{j=1,...,n} G_n(j,t)
\]

and \( b(j)=a(j+1)/a(j) \) satisfies

\[
(5.2) \quad b(1)<b(2)<...<1
\]

and \( k(n,t) \) is defined by

\[
(5.3) \quad k(n,t) = \min\{j| \min_{i=1,...,n} G_n(i,t) = G_n(j,t)\}
\]
then (3.9) holds and (3.11) holds with reversed inequality.

Furthermore if \( H(0,t) \) is such that

\[
(5.4) \quad \frac{d}{dt}(G_m(j,t) / G_m(j+1,t)) < 0 \quad \text{for } j=1,\ldots,m-1
\]

can be verified directly for \( m=2, j=1 \), then (5.4) is true generally, and (A) follows from the dual version of the Lemma. Also \( H(n,t)/H(n-1,t) \) is a non-increasing function of \( t \).

We shall use this dual version in Section 7.

(c). If the Poisson process under consideration is nonstationary with parameter \( m(t) \), the above results still remain valid. See [1]. \( m(t) \) should then be substituted for \( t \) in the functions \( G, H, k \).

6. **Extension of the Previous Example.**

Consider the example described in Section 1, but assume that the enemy aircrafts are of two kinds; some need to be hit by two missiles in order to be put out of action whereas the others need to be hit by one missile only. We assume that each enemy aircraft has probability \( w \) and \( 1-w \) of being of either kind, independently of the other aircrafts, and independently of the other variables considered. Our previous example is obtained when \( w=0 \). Notice that now it is by no means obvious that (A) should hold.

Clearly (3.1) is still valid. Notice that \( a(j) \) given in (3.2) denotes just the probability that the airplane is not shot down instantaneously, if it uses
j missiles. Thus here (3.2) must be replaced by

\[ (6.1) \quad a(j) = 1 - (1-u)(q^{j+1}w(1-q)q^{j-1}) \]

and (3.3), (3.5) and (3.6) are still valid. (In the argument preceding (3.5) the words "improved upon" should be taken in a weak sense whenever \( w=1 \). We must check the validity of (3.7). It is easily seen that \( b(j)>1 \) all \( j \), with no restriction on the parameters. Some algebra yields that for \( j \geq 2 \)

\( b(j)<b(j-1) \) if and only if

\[ (6.2) \quad q^{-j}[q+w(j(1-q)-(1+q))]>-(1-u)w^2. \]

Since the left hand side of (6.2) is an increasing function of \( j \), (3.7) is valid if and only if

\[ (6.3) \quad q^{-2}[q+w(1-3q)]>-(1-u)w^2. \]

Thus if the left hand side of (6.3) is positive, (6.3) holds for all values of \( u \) and hence (3.7) holds. For \( q<\frac{1}{2} \) (6.3) is no restriction, but for \( q>\frac{1}{2} \) the l.h.s of (6.3) is positive if and only if \( w<q/(3q-1) \). To compute

\[ \frac{d}{dt}(G_2(1,t)/G_2(2,t)) \]

we recall that (4.4) was derived by use of (3.1) only, without using any specific values of \( a(j) \). Substituting the value given by (6.1) we get by simple algebra

\[ \frac{d}{dt}(G_2(1,t)/G_2(2,t)) = (1-u)(1-q)(1-w)\text{e}^{-tu}/b(1), \]

and this is positive for \( w<1 \). Thus for \( w<1 \) the induction hypothesis holds, and hence if \( u, w, q \) are such that (6.3) holds, (A) follows.

Clearly the example given can be generalized in various other ways.
7. A Related Problem.

Consider a ship lost at sea, or an expedition lost in the desert, which can survive $t \geq 0$ time units without being discovered. It has $n$ flares at its disposal. Airplanes frequent the area according to a Poisson process, which again will, for simplicity, be taken to have parameter one. When seeing an airplane, the ship can use, simultaneously, some of its flares, and these are assumed to be independent random variables with known probability $0 \leq 1-q \leq 1$ of being detectable. It is easily seen that if the airplanes are assumed to detect, with probability one, a detectable flare, then the strategy which maximizes the probability of being detected uses all flares at hand when seeing an airplane, and the problem is trivial. We therefore assume that each plane, independently of the others and of the flares, has a known, fixed probability $0 \leq u \leq 1$ of detecting flares. Let $k(n,t)$ denote the optimal number of flares to be used when seeing an airplane at time $t$ and $n$ flares are at hand, when the goal is to maximize the probability of being detected. Here, and in the sequel, time $t$ stands for the remaining survival time. We shall show that (A) holds.

Let $h(n,t)$ denote the optimal probability of being detected when seeing an airplane at time $t$. Clearly $h(0,t)=0$. Let $g_n(j,t)$ denote the probability of detection when seeing an airplane at time $t$ and using $j$ of the available $n$ flares ($1 \leq j \leq n$), and then proceeding optimally. Thus for $t \geq 0$,

\begin{equation}
(7.1) \quad g_n(j,t) = u(1-q^j) + [1-u(1-q^j)] \int_0^t h(n-j,t-s) e^{-s} ds.
\end{equation}

An argument similar to that of Section 3 yields that an optimal strategy necessarily uses at least one flare, if there are flares at hand and an airplane is seen. Thus
\[(7.2) \quad h(n,t) = \max_{j=1,\ldots,n} g_n(j,t), \quad t>0\]

and

\[(7.3) \quad k(n,t) = \min\{j\mid \max_{i=1,\ldots,n} g_n(i,t) = g_n(j,t)\}.

Set

\[(7.4) \quad H(n,t) = 1 - h(n,t), \quad G_n(j,t) = 1 - g_n(j,t), \quad \]

and

\[(7.5) \quad a(j) = 1 - u(1-q^j). \]

Then simple arithmetic shows that (3.3) holds. Clearly (7.2), (7.3) and (7.4) yield (5.1) and (5.3), and (5.2) is easily checked. Now simple computations yield

\[\frac{d}{dt}(G_2(1,t)/G_2(2,t)) = -u(1-q)e^{-t}/b(1)<0\]

and thus we are in the situation described in Section 5 (b), and (A) holds.

**REFERENCES**