AN APPROXIMATE CHARACTERIZATION OF OPTIMAL STOPPING BOUNDARIES

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Summary.

An identity (18) is deduced for the loss function of a stopping process. This identity is particularly well adapted to the case of an optimal stopping process, in that it then provides a set of relations (22) characterizing the optimal stopping boundary, which do not require simultaneous determination of the loss function in the continuation region. The loss function for this optimal boundary is then approximately determined by relation (21). These relations depend on approximations of the "no-overshoot" type, but a bound on the magnitude of the terms neglected is provided by condition (13), which can be regarded as a general version of Chernoff's "tangency condition" at an optimal boundary.

1. Introduction.

Consider a process \( \{X_t\} \) in discrete time whose elements take values in p-dimensional Euclidean space \( \mathbb{R}^p \). The process can be "terminated" at any time, and costs \( c(x) \) and \( K(x) \) are associated with the decisions of continuation and termination if the value of the current \( X_t \) is \( x \). These costs are additive, in that, if the sequence of actual values is \( \{x_t\} \) and the process is terminated at time \( \tau \),
then the cost associated with the segment of the process from the time \( \tau \) up to termination is \( \sum_{u=\tau}^{\tau} c(x_u) + K(x_\tau) \). The aim is now to determine a stopping rule which minimizes the expected value of this cost.

Let \( F_s(x_t, x_{t-1}, \ldots) \) be the minimal expected future cost from time \( \tau \), conditional on \( X_u = x_u \) \( (u \leq t) \) and with the restriction that at most \( s \) further moves are permitted, so that the process must be stopped by time \( t+s \) at the latest. Then it follows from general results in dynamic programming that this cost is in fact a function of \( x_t \) and \( s \) alone, say \( F_s(x_t) \), and obeys the equations

\[
(1) \quad F_0(x) = K(x),
\]

\[
F_s(x) = \min[K(x), c(x) + E[F_{s-1}(X_{t+1})|X_t = x]], \quad (s > 0).
\]

It is convenient to suppress the \( x \)-argument, and write these equations in the more compact form

\[
(2) \quad F_0 = K,
\]

\[
F_s = \min[K, c + TF_{s-1}], \quad (s > 0)
\]

where the averaging operator \( T \) is implicitly defined in (1).

The sequence \( \{F_s\} \) is monotone decreasing, and so has a limit \( F \) which will, under appropriate regularity conditions, satisfy the equation
(3) \[ F = \min[K, c + TF] \].

\( F \) is the minimal loss which can be incurred if indefinite continuation is permitted. The optimal decision rule can be given the form: stop if the current \( x \) value falls in the set \( D \) specified as the set of \( x \) such that

(4) \[ K \leq c + TF \].

One continues if \( x \) falls in the complementary set (the continuation set, \( C \)) specified by

(5) \[ K > c + TF \].

We are thus making the convention that \( D \) is to be taken as closed. The \( C/D \) boundary, the decision boundary, will be denoted by \( B \).

If we subtract from both \( F \) and \( K \) any solution \( J \) of

(6) \[ J = c + TJ \]

then relation (3) can be standardized to the form

(7) \[ F = \min[K, TF] \].

That is, the continuation cost \( c(x) \) can be normalized to zero (see
Whittle, 1964); we shall henceforth work with this normalized form of the problem. The loss function $F$ then obeys

\begin{equation}
F = K, \quad (x \in D)
\end{equation}

\begin{equation}
F = TF, \quad (x \in C).
\end{equation}

These relations can be regarded as a homogeneous linear equation system (9) constrained by boundary conditions (8). Relations (8), (9) do not imply (7), and hold in fact for the loss function $F$ associated with a prescribed stopping set $D$ which is not necessarily optimal.

One sees from (7) that the minimal loss function will have the properties additional to (8), (9):

\begin{equation}
F \leq TF, \quad (x \in D)
\end{equation}

\begin{equation}
F \leq K, \quad (x \in C).
\end{equation}

These imply loosely that, not merely is $F$ equal to $K$ in $D$, but that $F-K$ grows only slowly as $x$ moves into $C$ from $D$. One has to impose rather specific conditions on the operator $T$ and the function $K$ before this statement can be made explicit, but there is a simple general inequality which makes the point. We see from (10), (11) and (8) that

\begin{equation}
0 \leq T(K-F) \leq TK - K, \quad (x \in D).
\end{equation}
From this it follows that

\[(13) \quad |T(K-F)| \leq |TK-K| \quad (x \in D)\]

which is the desired inequality.

So, suppose for example that

\[(14) \quad T = e^{\Delta \Omega}\]

where $\Delta$ is a small scalar, and $\Omega$ is a diffusion operator (i.e., a second-order differential operator of elliptic type). This would be the case if our random walk were the $\Delta$-skeleton of a diffusion process.

Then, if $K$ is differentiable in the neighborhood of the decision boundary, $TK-K$ is $o(\Delta^{\frac{1}{2}})$. On the other hand, one finds that $T(K-F)$ will have a value of order $\Delta^{\frac{1}{2}}$ exactly unless $K-F$ possesses a derivative; it then follows from (12) that $K-F$ possesses a derivative in $D$. This is a trivial statement for $x$ in the interior of $D$, where $K-F$ is identically zero. However, for $x$ on the decision boundary $B$ it implies that the derivative is zero into $C$. That is, the derivatives of $F$ and $K$ are equal on the continuation side of the decision boundary; a known result due to Chernoff (1961).

Relation (13) can be regarded as a general expression of this tangency condition.
2. A representation of the loss function in terms of boundary values.

We shall denote the integral with respect to p-dimensional Lebesgue measure of a function \( f(x) \) on a set \( A \) of \( \mathbb{R}^p \) simply by \( \int f \). If no set is indicated, then the whole space is intended.

An adjoint \( T' \) to the operator \( T \) is then defined implicitly by

\[
\int T'g = \int gTf
\]

Denote the restriction of a function \( f \) to a set \( A \) by \( f_A \), so that

\[
f_A = \begin{cases} 
  f & x \in A \\
  0 & x \notin A 
\end{cases}
\]

Suppose now that \( G, H \) are a pair of functions related by

\[
G = H + T'G \quad (x \in \mathbb{C})
\]

Then

\[
\int_{HF} = \int (G - T'G)F \]

\[
= \int [G_F - GTF_F] 
\]

\[
= \int [G_F (F - TF_F) - G_D TF_F]
\]
\[ = \int [G_C (F - TF_D^D) - G_D^D TF^D_C] \]
\[ = \int [G_C TK_D - G_D^D TK^D_C] . \]

We have appealed to relations (8) and (9) at various points in this calculation, but not required that D be the optimal stopping region. Relation (18) is usefully expressed as

\[ (19) \quad \int_{HF_C} = \int [G_C TK_D - G_D^D TK^D_C] \]
\[ + \int G_D^D T(K-F)_C . \]

The function G is essentially a Green's function for the problem F - TF in C, but for unspecified boundary conditions outside C. Formula (19) is then a general form of Green's formula.

Let S(x) denote the set of values of \( X_{k-1} \) which have positive probability, given that \( X_t = x \). Then the only values of x which contribute to an integral such as \( \int G_C TK_D \) are those for which \( x \in C \) and S(x) intersects D. Loosely speaking, such values lie within an x-increment of the decision boundary. If one considers a situation in which x-increments are small (as is necessary if the familiar "no-overshoot" approximations are to be appropriate) then the integrals in the right-hand number of (19) are essentially boundary integrals, although expressed as volume integrals.

We have not yet invoked optimality. If D is optimal,
then not only is the difference $K-F$ zero on the decision boundary (as is implied by (8)) but also it grows only slowly as $x$ enters $C$ from $D$. In fact, it follows from (13) that
\begin{equation}
|\int_{D_1} G_D T(K-F)| \leq \int_{D_1} |G(TK-K)| + \int_{D_2} |G_T(K-F)|
\end{equation}
where $D_1 + D_2 = D$. By choosing the partition $(D_1, D_2)$ appropriately, so that $D_1$ constitutes the part of $D$ close to the boundary, one can show that if $K$ is differentiable this bound will be $\mathcal{O}(\Delta^3)$ for $T$ of the form (14). On the other hand, the first integral in the right-hand member of (19) will be of order $\Delta^{\frac{3}{2}}$. This illustrates the fact that the second integral will be negligible relative to the first in the "no-overshoot" approximation, so that (19) reduces in this approximation to
\begin{equation}
\int_{H_F} G \sim \int (G_C TK_D - G_D TK_C).
\end{equation}
The advantage of this relation is that $F$ does not appear in the right-hand member at all, so that appropriate choice of $G$ yields explicit expressions for $F$. For example, suppose one sets $H=0$ in (17); the resulting equation
\[ G = T'G \]
would have many $G$ solutions. For each of these we obtain a relation (21), now of the form
(22) \[ \int [G_C \cdot TK_D - G_D \cdot TK_C] \sim 0 \]

in which \( F \) does not appear at all. These constitute constraints on the form of \( D \), which one hopes will determine the optimal \( D \) completely, at least in a no-overshoot approximation. The advantage of this boundary characterization is that it does not invoke simultaneous solution of a functional equation in the interior of \( D \), as free boundary problems normally do. For example, in the Chernoff treatment of the diffusion problem one would solve \( TF=F \) in \( C \) subject to \( F=K \) on \( B \), and then use the additional boundary condition \( F_x=K_x \) on \( B \) to determine the optimal \( B \). Here \( F_x \) denotes the vector of first differentials of \( F \) with respect to \( x \).

If one sets \( H \) equal to a delta function \( \delta(x-x) \) in (17), then relation (21) provides an approximate solution for \( F(x) \).

3. **Particular cases and examples.**

Set \( H(x) = \delta(x-x) \) in (17), and choose the solution of the resulting equation which satisfies \( G=0 \) in \( D \); denote this by \( G^{(\xi)}(x) \). Then, for a subset \( A \) of \( C \), \( \int_A G^{(\xi)}(x) \) is nothing but the expected number of visits to \( A \) made by the process \( \{X_{\xi}\} \) before absorption in \( D \), if the process starts from a point \( x \) of \( C \). That is, in \( C \) \( G^{(\xi)} \) is the expected density of visits before absorption, with initial condition \( x=x \). Thus \( \pi \cdot G^{(\xi)} \) will, in \( D \), give the absorption probability density. Since \( G^{(\xi)}=0 \), relation (21) reduces simply to
\[ F(\xi) = \int G(\xi) T_D \]

which is just the representation of \( F \) as an average of terminal loss \( K \) over \( D \) with respect to the probability density \( T'G(\xi) \) at termination. This is a rather trivial result, valid whether or not \( D \) is optimal, but it is, at least, an important special case of relation (21).

However, one's principal interest would be to exploit relation (22) to determine, approximately at least, the optimal stopping set \( D \), and then use (21) to determine the associated \( F \). In fact, it seems to do so readily only for the well-known one-dimensional cases, soluble by other methods, although one may hope that it will ultimately prove useful in higher dimensional cases.

As an example, consider the case where the process consists of a simple random walk on the integers, so that \( F(x) \) is defined for integral \( x \), and \( T \) is defined by

\[ Tf(x) = pf(x+1) + qf(x) + rf(x-1) \]

say. Suppose that the optimal \( C \) is the interval \( m < x < n \), so that the two points \( x = m, n \) constitute the boundary \( B \). Then relation (22) amounts to

\[ rG(m+1) K(m) + pG(n-1) K(n) \sim pG(m) K(m+1) + rG(n) K(n-1) \]
This is to hold for any \( G \) satisfying \( G=T^*G \); this equation has just two fundamental solutions:

\[
G(x) = 1, \quad \left( \frac{P}{T} \right)^X.
\]

Demanding condition (25) for these two choices of \( G \), we deduce two determining equations for \( m \) and \( n \) (only approximately soluble in integers). These equations are identical with those that would be obtained by solving \( F=TF \) in \( C \), and demanding that \( K-F \) and its "derivative" be zero at \( x=m, n. \)

As follows from the remarks after (20), relations (21) and (22) will be exact for case (14) in the limit \( \Delta \downarrow 0 \). That is, they are exact for the diffusion process, which is the only true no-overshoot process.

REFERENCES
