OPTIMAL CONTROL OF DIFFUSION PROCESSES
WITH DISCONTINUOUS COEFFICIENTS

by

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1. INTRODUCTION AND SUMMARY

The problem in this paper is concerned with arbitrary state continuous-time Markov decision processes. The system we consider is described by a special but important class of time-homogeneous stochastic functional-differential equations. The drift coefficients in the equations are considered controlled and the control policy is a measurable function of the present state of the system. Some cost functionals associated with controls are considered and our problem is to find optimal controls which minimize these cost functionals. Although some results for computing optimal controls are presented, this paper mainly considers the basic and qualitative aspect of the problem, i.e., the question of existence of optimal controls is discussed. After formulating the problem in Section 2 we discuss the existence problem of solutions to stochastic differential equations. Since the controls are only assumed to be measurable, the coefficients in the equations do not necessarily satisfy a Lipschitz condition in the state variable which is required to assure the unique existence of solutions in the classical theorems on stochastic differential equations. In Section 3 the construction of a solution to a stochastic functional-differential equation, which is due to Girsanov, is briefly stated. In Section 4 some preliminary facts are provided. The main result is a continuity dependence of solutions on the drift coefficients. That is to say, it is shown that, if a sequence of drift coefficients converges weakly to a drift coefficient, then the corresponding solutions of the stochastic functional-differential equation converge to the solution corresponding
to the limiting coefficient. As a result, the continuity dependence of some cost functionals on the drift coefficients are also proved. In Section 5 we derive some theorems on the existence of optimal stochastic controls as an application of the results obtained in Section 4. Two cost functionals are considered. One is the so-called discounted cost functional over the infinite time interval. The other is the interval (over a random time) of a cost rate which depends only on the sample path of the process. In Section 6 the minimization problem for a third cost functional, the long term average cost, is considered. For this problem we obtain an existence theorem under strongly restricted assumptions. In Subsection 6.1 Ito's differentiation rule is extended to Sobolev space and applied to derive a sufficiency condition for optimality. In Subsection 6.2 an existence theorem is proved using these results. In Section 7 a computational method to obtain an approximate optimal control is developed for the long term average cost criterion problem for cases where only a finite number of actions are available. The problem is discretized in time and then converted to a finite state Markovian decision problem with finite actions by an appropriate bounding and reflection of the process. Linear programming is applicable to this converted problem. Computations have not been carried out. The extent of the computations required with this method may limit its usefulness.

There are several papers which treat theorems on the existence for stochastic optimal controls. For example, Kushner [10]
gave an existence theorem for processes of diffusion type where uniformly Lipshitz closed loop controls and a cost, the integral of a cost rate over a random time, were considered. Benes [2] gave a fairly complete existence theorem where closed loop measurable controls were considered for the cost of the integral (over a finite fixed time interval) of a cost rate. His result is deeper than the results given here in the sense that the information available to the controller is all the past history of the processes. It seems, however, that his method is not applicable to the cost functionals considered in this paper.
2. PROBLEM STATEMENT

We consider a control system described by a stochastic differential equation of the form:

\[
x_1(t) = x_1 + \int_0^t g_1(x(s))\,ds
\]
\[
x_2(t) = x_2 + \int_0^t g_2(x(s))\,ds + \int_0^t f(x(s), u(x(s)))\,ds + \int_0^t \sigma(x(s))\,dw(s)
\]

where \( x(t) = (x_1(t), x_2(t)) \) is the \( n \)-dimensional vector process, \( w(t) \) is a standard \( n_2 \)-dimensional Brownian motion in a probability space \( (\Omega, \mathcal{F}, P) \), and \( x_1(t) \) and \( x_2(t) \) are \( n_1 \) and \( n_2 \) dimensional vector processes respectively. Let \( \mathbb{R}^n \) denote the \( n \)-dimensional Euclidian space. We assume the following:

(A1) \( g_i(x), i = 1, 2, x \in \mathbb{R}^n \) satisfy boundedness and Lipshitz condition of the types

\[
|g_i(x)|^2 \leq K(1 + |x|^2)
\]
\[
|g_i(x) - g_i(y)| \leq K|x - y|,
\]

where \( |x| \) denotes the Euclidian norm \( |x|^2 = \sum_{i=1}^n x_i^2 \), \( K \) represents a generic constant, and two \( K \)'s are not necessarily the same. The same convention for this constant symbol will be used in the sequel.

(A2) \( \sigma(x) \) is a bounded function and satisfies a Lipshitz condition
\[ |\sigma(x) - \sigma(y)| \leq K|x - y| \]  

(3)

where \(|\sigma(x)|\) denotes the Euclidean norm \(|\sigma(x)|^2 = \sum \sigma_{ij}^2(x)\). Also \(\sigma(x)\) has a uniformly bounded inverse \(\sigma^{-1}(x)\).

\((A3)\) A control vector \(u(x)\) is a measurable function from \(\mathbb{R}^n\) to \(\mathbb{R}^m\).

\((A4)\) \(f(\cdot, \cdot)\) is a bounded measurable function.

We can associate the control \(u(x)\) with some cost functionals of the following types:

\[(I)\]
\[Q_u = E \int_0^\tau L(x(s), u(x(s)))ds \quad (4)\]

where \(\tau\), a stopping time associated with the process, is the first time \(x(t)\) exists from a bounded open region \(G\).

\[(II)\]
\[Q_u = E \int_0^\infty e^{-\alpha s} L(x(s), u(x(s)))ds, \quad \alpha > 0 \quad (5)\]

\[(III)\]
\[Q_u = \lim_{T \to \infty} \frac{1}{T} E \int_0^T L(x(s), u(x(s)))ds \quad . \quad (6)\]

Our purpose is to present theorems for the existence of optimal stochastic controls for these criteria. Preliminary facts will be given in Section 4 and they will be used systematically for our purpose. Later, additional assumptions on \(f(\cdot, \cdot)\) and \(L(\cdot, \cdot)\) will be made, but, for the moment the assumptions \((A1-4)\) are sufficient.
3. THE SOLUTION OF THE STOCHASTIC DIFFERENTIAL EQUATION (1)

Since \( f(\cdot, \cdot) \) does not necessarily satisfy a Lipshitz condition, the existence of the solution of (1) is not assured by classical theorems \([6]\). Under the above assumptions, however, we can construct a solution of (1) by the following method of Girsanov \([8]\). By Ito's theorems, Assumptions (A1-2) imply that there is a unique solution to the control free equation

\[
\begin{align*}
x_1(t) &= x_1 + \int_0^t g_1(x(s)) \, ds \\
x_2(t) &= x_2 + \int_0^t g_2(x(s)) \, ds + \int_0^t \sigma(x(s)) \, dw(s)
\end{align*}
\]

(7)

Let \( y(t) \) be the solution of (7) and define \( \xi_{t_0}^t(f) \) by

\[
\xi_{t_0}^t(f) = \int_{t_0}^t f(y(s), u(y(s)))' \left( \sigma(y(s))' \right)^{-1} \, dw(s) \\
- \frac{1}{2} \int_{t_0}^t f(y(s), u(y(s)))' \left( \sigma(y(s))' \right)^{-1} \left( \sigma(y(s)) \right)^{-1} \times f(y(s), u(y(s))) \, ds,
\]

(8)

a measure \( \tilde{P} \) by

\[
\tilde{P}(d\omega) = \exp[\xi_{t_0}^t(f)] \, P(d\omega)
\]

and a stochastic process \( \tilde{\omega}(t) \) for the time interval \([0, t]\) by
\[ \tilde{\omega}(t) = w(t) - \int_{0}^{t} \sigma(y(s))^{-1} f(y(s), u(y(s))) ds. \]

Girsanov showed the following [2].

1. Under the assumptions (Al-4), \( \tilde{\mathbb{F}} \) is a probability measure on \( (\Omega, \mathcal{F}) \).

2. \( \tilde{\omega}(t) \) is a Brownian motion on \( (\Omega, \mathcal{F}, \tilde{\mathbb{P}}) \), and the process \( y(t) \) is a solution of

\[
\begin{align*}
    x_1(t) &= x_1 + \int_{0}^{t} g_1(x(s)) ds \\
    x_2(t) &= x_2 + \int_{0}^{t} g_2(x(s)) ds + \int_{0}^{t} f(x(s), u(x(s))) ds \\
             &\quad + \int_{0}^{t} \sigma(x(s)) d\tilde{\omega}(s),
\end{align*}
\]

i.e., the solution of (7), \( y(t) \), when considered with respect to \( (\Omega, \mathcal{F}, \tilde{\mathbb{P}}) \), is a solution of (1) with the Brownian motion \( \tilde{\omega}(t) \) replacing \( w(t) \).

Hereafter when we say the solution of the stochastic differential equation (1), we always assume that it is constructed by the above method. The following fact (see [8]) will often be used later:

\[
E_x[\exp(\alpha_0 t f)] \leq \exp\left[\frac{\alpha^2}{2} tK\right] \quad (9)
\]
where $|f(x)| \leq K$ and $\alpha > 1$. The notation $E_x$ represents the expectation with respect to processes associated with stochastic differential equations where the initial point is $x(0) = x$. Occasionally $E_x^P$ will be used to indicate the underlying probability measure $P$.

**Example (Satellite control problem).** Let $x_1(t)$ be the position of a satellite in one dimension. Then the motion of the satellite is expressed as follows:

$$dx_1(t) = x_2(t)dt$$

$$dx_2(t) = -u(t)dt + \sigma(t, x(t), u(t))dw(t)$$

where $x_2(t)$ is the velocity of the satellite, $u(t)$ is the (controlled) acceleration, $x(t) = (x_1(t), x_2(t))$, and $w(t)$ is a standard Brownian motion representing random forces operating on the satellite. If $u(t)$ takes the form $u(t) = u(x(t))$, then it is a feedback type control and, if it takes only a finite number of values (actions), then it is a so called "bang-bang type control" and this is a typical example of a discontinuous control. Thus, if $u(t) = u(x(t))$ and $u(\cdot)$ is bounded and measurable, and $\sigma(t, x(t), u(t)) = \sigma(x(t))$, then the satellite motion equation above belongs to our system (1).

We can associate several cost functionals with the above system. For example, let $\tau$ be the stopping time which is the first time the satellite hits some object. Then
\[ Q_u = E \int_0^\tau L(x(s), u(x(s))) ds \]

expresses the accumulated cost for the satellite until the hitting time. If, for instance, \( L(\cdot, \cdot) = 1 \), then \( Q_u \) is the time until hitting some object, and the problem of minimizing the cost is called the "time optimal control problem". Or we can consider the following cost functionals:

\[ Q_u = \lim_{T \to \infty} \frac{1}{T} E \int_0^T (x_1^2(s) + ku^2(x(s))) ds \]

or

\[ Q_u = \lim_{T \to \infty} \frac{1}{T} E \int_0^T (x_1^2(s) + k|u(x(s))|^2) ds \]

In the first example if \( u(x(t)) \) is specified and \( \tau \) is to be chosen from several stopping rules so as to minimize the cost functionals, then the problem becomes an "optimal stopping time problem" and one example can be found in [4].
4. INTERMEDIATE RESULTS

In this section we present preliminary facts which will be used for the theorems on the existence of optimal stochastic controls. To develop such facts we need the existence of the probability density function for the process \( x(t) \) of equation (1).

We assume (A5).

(A5) The process \( y(t) \) of the solution of the equation (7) with initial point \( x(0) = x \) has a density \( p(t,x,y) \) such that, in each compact \( x \) set and in each compact \( t \) set, there is an \( M < \infty \) so that

\[
\int p(t,x,y)^2 \, dy \leq M.
\]

Also \( p(t,x,y) \) satisfies that

\[
\int_{B_r} p(t,x,y) \, dy \to 1 \quad \text{as} \quad r \to \infty
\]

uniformly in any compact \( t \) and \( x \) set, where \( B_r = \{x; |x| \leq r, x \in \mathbb{R}^n\} \).

**Definition 1.** Let \( \Omega \) be a domain in \( \mathbb{R}^n \). We define \( L_2(\Omega) \) to be the space of measurable functions \( f(x) \) defined on \( \Omega \) such that

\[
\int_{\Omega} f(x)^2 \, dx < \infty \quad \text{with the norm} \quad \|f\|_{L_2(\Omega)}^2 = \int_{\Omega} f(x)^2 \, dx.
\]

**Definition 2.** Let \( f_n(x) \in L_2(\Omega), \ n = 1, 2, \ldots \) and \( f(x) \in L_2(\Omega) \). We say that \( f_n(x) \) converges weakly to \( f(x) \) in \( L_2(\Omega) \) if, for all \( \varphi(x) \in L_2(\Omega) \),
\[
\int_{\Omega} f_n(x) \varphi(x) \, dx \to \int_{\Omega} f(x) \varphi(x) \, dx \quad \text{as } n \to \infty
\]

holds.

**Definition 3.** We denote \( f_n(x) \Rightarrow f(x) \) if \( f_n(x) \) and \( f(x) \) are uniformly bounded measurable functions defined on \( \mathbb{R}^n \) and \( f_n(x) \) converges weakly to \( f(x) \) on any bounded domain \( \Omega \) in \( L^2(\Omega) \). If \( f_n(x) \) and \( f(x) \) are vector functions, \( f_n(x) \Rightarrow f(x) \) means \( f_{n_i}(x) \Rightarrow f_i(x) \) for each \( i^{th} \) component.

The following Lemma 1 and Lemma 2 are extensions of the result obtained in [5].

**Lemma 1.** Suppose \( h_n(x) \Rightarrow h(1) \). Then under (A1-2) and (A3), for arbitrary \( T > 0 \),

\[
\lim_{n \to \infty} \mathbb{E} \left[ \int_0^T (h_n(y(s)) - h(y(s))) \, ds \right]^2 = 0
\]

where \( y(t) \) is the solution of (7).

**Proof.** Let \( g_n(x) = h_n(x) - h(x) \). Then

\[
\mathbb{E} \left[ \int_0^T (h_n(y(s)) - h(y(s))) \, ds \right]^2
\]

\[
= \mathbb{E} \left[ \int_0^T g_n(y(s)) \, ds \right]^2
\]

\[
= \int_0^T \int_0^T \mathbb{E} g_n(y(s)) g_n(y(u)) \, ds \, du
\]

\[
= \int \int \mathbb{E} g_n(y(s)) g_n(y(u)) \, ds \, du.
\]
Let \( s < u \). Then

\[
E_x g_n(y(s)) g_n(y(u))
\]

\[
= \int_{\mathbb{R}^n} g_n(z) \left( \int_{\mathbb{R}^n} g_n(y) p(u-s, z, y)dy \right) p(s, x, z)dz,
\]

and

\[
\int_{\mathbb{R}^n} g_n(y) p(u-s, z, y)dy
\]

\[
= \int_{B_r} g_n(y) p(u-s, z, y)dy + \int_{B_r^c} g_n(y) p(u-s, z, y)dy.
\]

Since \( |g_n(y)| \leq K \), assumption (A5) implies that the second term goes to zero as \( r \to \infty \). The first term goes to zero as \( n \to \infty \) by (A5) and the weak convergence assumption. Thus

\[
E_x g_n(y(s)) g_n(y(u)) \to 0 \quad \text{as } n \to \infty.
\]

From this we get the conclusion.

Q.E.D.

Consider a stochastic differential equation

\[
x_1(t) = x_1 + \int_0^t g_2(x(s))ds
\]

\[
x_2(t) = x_2 + \int_0^t g_2(x(s))ds + \int_0^t f_n(x(s))ds + \int_0^t a(x(s))dw(s)
\]

where \( f_n(x) \) is a bounded measurable function.
Lemma 2. Assume (A1-2) for $g_i(x)$ and $\sigma(x)$, uniform boundedness for $f_n(x)$ in (10). Furthermore we assume (A5). We make the same assumptions as in Lemma 1 for $h_n(x)$ and $h(x)$. Then, for $T > 0$,

$$\lim_{n \to \infty} E_x \left[ \int_0^T (h_n(x_n(s)) - h(x_n(s))) ds \right]^2 = 0$$

where $x_n(t)$ is the solution of (10) corresponding to $f_n(x)$.

Proof. Define $\zeta_0^T(f_n)$ by

$$\zeta_0^T(f_n) = \int_0^T f_n(y(s))' (\sigma(y(s))')^{-1} dw(s)$$

$$- \frac{1}{2} \int_0^T f_n(y(s))' (\sigma(y(s))')^{-1} \sigma(y(s))^{-1} f_n(y(s)) ds.$$

Then

$$E_x \left[ \int_0^T (h_n(x_n(s)) - h(x_n(s))) ds \right]^2$$

$$= E_x \left[ \int_0^T (h_n(y(s)) - h(y(s))) ds \right]^2 \exp(\zeta_0^T(f_n))$$

$$\leq \left( E_x \left[ \int_0^T (h_n(y(s)) - h(y(s))) ds \right]^2 \right)^{1/2} \left( E_x \exp(2\zeta_0^T(f_n)) \right)^{1/2}$$

$$\leq K E_x \left[ \int_0^T (h_n(y(s)) - h(y(s))) ds \right]^2.$$

The last inequality comes from $E_x \exp(2\zeta_0^T(f_n)) \leq \exp(TK) \leq K$ and the boundedness of $h_n(x)$ and $h(x)$.

Now by Lemma 1, the last term goes to zero as $n \to \infty$. Q.E.D.
Before stating Lemma 3, we introduce a metric topology on $\Omega$. A particular realization $y(t, \omega) (0 \leq t < \infty)$ of the solution of (7) can be identified with $\omega$ and hence $\Omega$ can be identified with $C(0, \infty)$ -- the set of all continuous functions on $[0, \infty)$. The space $\Omega$ will be viewed as a complete separable metric space where the relevant topology $\rho$ is that of uniform convergence on compact subsets of $[0, \infty)$, i.e., $\omega_n \to \omega$ as $n \to \infty$ in $\rho$ implies that for any $T > 0$, $\sup_{0 < t < T} |y(t, \omega_n) - y(t, \omega)| \to 0$ as $n \to \infty$. $\mathcal{G}$ is the Borel $\sigma$-field of $\Omega$ and can be identified as the smallest $\sigma$-field with respect to which $y(t, \omega)$ is measurable for $0 \leq y < \infty$.

**Lemma 3.** Assume (A1-2) for $g_i(x)$, uniform boundedness for $f_n(x)$ in (10). Furthermore we assume (A5). Suppose $f_n(x) \Rightarrow f(x)$. Then the probability measure $P_n$ of the process $x_n(t)$ of (10) converges to the probability measure $P_0$ of the process $x(t)$ -- the solution of

$$
\begin{align*}
  x_1(t) &= x_1 + \int_0^t g_1(x(s)) \, ds \\
  x_2(t) &= x_2 + \int_0^t g_2(x(s)) \, ds + \int_0^t f(x(s)) \, ds + \int_0^t \sigma(x(s)) \, dw(s)
\end{align*}
$$

in the sense that, for every $A \in \mathcal{G}$, $P_n(A) \to P_0(A)$ as $n \to \infty$.

**Proof.** This result was indicated in Varadhan and Stroock [15] when the system is non-degenerate, i.e., $n_1 = 0$ and $g_2(x) = 0$. The method of proof is similar to that used in the proof of Theorem 9.1 in [15]. Let us define probability measures by
\[ P_n(d\omega) = \exp[\int_0^t f_n] P(d\omega) \]
\[ P_0(d\omega) = \exp[\int_0^t f] P(d\omega) \]

Then in [15] it is shown that \( P_n \) and \( P_0 \) are the unique probability measures with respect to which

\[ X_n(t) = h(y(t)) - \int_0^t \mathcal{L}_n h(y(u))du \]

and

\[ X_0(t) = h(y(t)) - \int_0^t \mathcal{L}_0 h(u)du \]

are martingales respectively, where \( y(t) \) is the process of (7), \( h(\cdot) \) is an arbitrary bounded function with two bounded continuous derivatives, and \( \mathcal{L}_n(\mathcal{L}_0) \) is the differential operator defined by

\[ \mathcal{L}_n h(x) = \sum_{i,j} a_{ij}(x) \frac{\partial^2 h(x)}{\partial x_i \partial x_j} + \sum_i f_n(x) \frac{\partial h(x)}{\partial x_i} \]

\[ (a_{ij}(x)) = \frac{1}{2} \sigma(x) \sigma'(x) \]

\[ \left( \mathcal{L}_0 h(x) = \sum_{i,j} a_{ij}(x) \frac{\partial^2 h(x)}{\partial x_i \partial x_j} + \sum_i f(x) \frac{\partial h(x)}{\partial x_i} \right). \]

Now we first prove that \( \{P_n\} \) has a weakly convergent subsequence. Since, for \( t_1 \leq t_2 \leq T, T \) arbitrary,
\[ E\!
olimits_{x}^{n}\{|y(t_{2}) - y(t_{1})|^{3}\} = \int_{\Omega} \{ |y(t_{2}) - y(t_{1})|^{3}\} \, dp_{n}(\omega) \]

\[ = E\!
olimits_{x}\{ |y(t_{2}) - y(t_{1})|^{3} \, \exp(\xi_{0}(f_{n})) \} \]

\[ \leq \{ E\!
olimits_{x}\{ |y(t_{2}) - y(t_{1})|^{3} \}^{4/3} \}^{3/4} \{ E\!
olimits_{x} \exp(4\xi_{0}(f_{n})) \}^{1/4} \]

\[ = \{ E\!
olimits_{x}\{ |y(t_{2}) - y(t_{1})|^{4} \}^{3/4} \}^{3/4} \{ E\!
olimits_{x} \exp(4\xi_{0}(f_{n})) \}^{1/4} \]

\[ \leq H(t_{2} - t_{1})^{3/2} \cdot (\exp(6TK))^{1/4} \quad (12) \]

The last inequality follows from Corollary 2 of [6, p. 400] and from equation (10), and \( H \) is independent of \( n \).

Now from (12) \( \{P_{n}\} \) is weakly compact with respect to the topology \( \rho \) on \( \Omega \) (see [15, Theorem 2.3]). Thus there exists a subsequence, indexed by \( n' \), of \( \{P_{n}\} \) such that \( P_{n'} \) converges weakly to a probability measure \( P' \). Next we shall show that \( P' \) must be the (unique) probability measure \( P_{0} \) with respect to which \( X_{0}(t) \) is a martingale. This will imply that every subsequence of \( P_{n} \) has a subsequence which converges weakly to \( P_{0} \) and hence \( P_{n} \Rightarrow P_{0} \) and finally that \( P_{n}(A) \to P_{0}(A) \) for every \( A \in \mathcal{G} \).

To establish this desired property of \( P' \), one must show that

\[ E\!
olimits_{x}^{p'}[\eta(\omega) \, X_{0}(t)] = E\!
olimits_{x}^{p'}[\eta(\omega) \, X_{0}(s)], \quad 0 \leq s \leq t \quad (13) \]

for any bounded \( \mathcal{M}_{s} \)-measurable function \( \eta(\omega) \) where \( \mathcal{M}_{s} \) is the \( \sigma \)-field generated by \( \{y(u), 0 \leq u \leq s\} \). As stated by Varadhan
and Stroock \[15, p. 376\], it suffices to establish (13) for \( \eta(\omega) \) continuous in \( \omega \). But, since \( X_n'(t) \) is a martingale with respect to the probability measure \( P_n' \),

\[
E_x^n'[\eta(\omega) X_n'(t)] = E_x^n'[\eta(\omega) X_n'(s)]
\]

and taking the limit as \( n \to \infty \) we obtain (13) by first establishing

\[
\lim_{n' \to \infty} E_x^n'[\eta(\omega) X_n'(t)] = E_x'[\eta(\omega) X_0(t)] . \tag{14}
\]

To derive (14), we note that since \( \eta(\omega) \) and \( h(y(t, \omega)) \) are bounded and continuous in \( \omega \) with respect to the topology \( \rho \), \( P_n' \Rightarrow P' \) implies

\[
\lim_{n' \to \infty} E_x^n'[\eta(\omega) h(y(t, \omega))] = E_x'[\eta(\omega) h(y(t, \omega))] .
\]

Similarly

\[
\lim_{n' \to \infty} E_x^n'\left[\eta(\omega) \int_0^t a_i(y(u, \omega)) \frac{\partial^2 h(y(u, \omega))}{\partial x_i \partial x_j} \, du \right]
\]

\[
= E_x'[\eta(\omega) \int_0^t a_i(y(u, \omega)) \frac{\partial^2 h(y(u, \omega))}{\partial x_i \partial x_j} \, du] .
\]

We need to show that

\[
\lim_{n' \to \infty} E_x^n'\left[\eta(\omega) \int_0^t f_n'(y(u, \omega)) \frac{\partial h(y(u, \omega))}{\partial x_i} \, du \right]
\]

\[
= E_x'[\eta(\omega) \int_0^t f(y(u, \omega)) \frac{\partial h(y(u, \omega))}{\partial x_i} \, du] .
\]
Since \( f_n(x) \rightarrow f(x) \),

\[
f_n(x) \frac{\partial h(s)}{\partial x_i} \rightarrow f(x) \frac{\partial h(x)}{\partial x_i},
\]

and it is sufficient to show that, if \( k_n'(x) \rightarrow 0 \), then

\[
\lim_{n' \to \infty} \mathbb{E}_{X}^{P_{n'}} \left[ \eta(\omega) \int_{0}^{t} k_n'(y(u,\omega))du \right] = 0.
\]

But this is clear from the fact that

\[
\left| \mathbb{E}_{X}^{P_{n'}} \left[ \eta(\omega) \int_{0}^{t} k_n'(y(u,\omega))du \right] \right|
\]

\[
\leq (\mathbb{E}_{X}^{P_{n'}} \eta(\omega)^2)^{1/2} \left( \mathbb{E}_{X}^{P_{n'}} \left[ \int_{0}^{t} k_n'(y(u,\omega))du \right]^2 \right)^{1/2}
\]

\[
\rightarrow 0 \quad \text{as } n' \rightarrow \infty \quad \text{by Lemma 2}.
\]

From these results we get (14), and this implies that

\[
P_n \rightarrow P_0.
\]

Finally we must show that, for any \( A \in \mathcal{S} \), \( P_n(A) \rightarrow P_0(A) \) as \( n \rightarrow \infty \). For this, it is sufficient to show that, for any bounded \( \mathcal{S} \)-measurable function \( \eta(\omega) \),

\[
\lim_{n \to \infty} \mathbb{E}_{X}^{P_n}[\eta] = \mathbb{E}_{X}^{P_0}[\eta].
\]

Let \( \{\eta_m(\omega), \, m = 1, 2, \ldots\} \) be bounded continuous (with respect to the topology \( \rho \)) functions such that \( \mathbb{E}_{X}(\eta_m(\omega) - \eta(\omega))^2 \rightarrow 0 \) as \( m \rightarrow \infty \). Then
\[ \begin{align*}
\mathbb{P}_n[\eta] & - \mathbb{P}_0[\eta] \\
& = \mathbb{E}_X^n[\eta_m + \eta - \eta_m] - \mathbb{E}_X^n[\eta_m + \eta - \eta_m] \\
& = \mathbb{E}_X^n[\eta_m] - \mathbb{E}_X^n[\eta_m] + \mathbb{E}_X^n[\eta - \eta_m] - \mathbb{E}_X^n[\eta - \eta_m].
\end{align*} \]

Since
\[ |\mathbb{E}_X^n[\eta - \eta_m]| \leq (\mathbb{E}_X^n[\eta - \eta_m]^2 \mathbb{E}_X \exp(2t_0^T(f_n)))^{1/2} \]
\[ \leq (\exp tK \mathbb{E}_X^n[\eta - \eta_m]^2)^{1/2} \to 0 \]
as \(m \to \infty\) uniformly in \(n\). Similarly
\[ |\mathbb{E}_X^n[\eta - \eta_m]| \to 0 \]
as \(m \to \infty\).

For a fixed \(m\), since \(\mathbb{P}_n \Rightarrow \mathbb{P}_0\),
\[ \mathbb{E}_X^n[\eta_m] \to \mathbb{E}_X[\eta_m] \]
as \(n \to \infty\).

Thus,
\[ \mathbb{E}_X^n[\eta] \to \mathbb{E}_X[\eta] \]
as \(n \to \infty\),
and we have proved that \(\mathbb{P}_n(A) \to \mathbb{P}_0(A)\) for every \(A \in \mathcal{G}\). Q.E.D.
**Lemma 4.** Let \( h(x) \) be a bounded measurable function. Under the assumptions of Lemma 3,

\[
\lim_{n \to \infty} E_x h(x_n(s)) = E_x h(x(s))
\]

where \( x_n \) and \( x \) were defined in Lemma 3.

**Proof.** Let \( |h(x)| \leq K \). For an arbitrary positive integer \( N \), define

\[
\gamma_i = \left( \frac{-N}{N} \leq h(x) \leq \frac{N}{N} K \right), \quad i = -N, \ldots, N.
\]

Then

\[
|E_x h(x_n(s)) - E_x h(x(s))| = \left| \int_{R^n} h(y) P_n(s, x, dy) - \int_{R^n} h(y) P(s, x, dy) \right|
\]

\[
\leq \left| \int_{R^n} h(y) P_n(s, x, dy) - \sum_{-N}^{N} \frac{i}{N} K P_n(s, x, \gamma_i) \right|
\]

\[
+ \left| \int_{R^n} h(y) P(s, x, dy) - \sum_{-N}^{N} \frac{i}{N} K P(s, x, \gamma_i) \right|
\]

\[
+ \left| \sum_{-N}^{N} \frac{i}{N} K (P_n(s, x, \gamma_i) - P(s, x, \gamma_i)) \right|
\]

\[
\leq \frac{K}{N} + \frac{K}{N} + K \sum_{-N}^{N} \left| P_n(s, x, \gamma_i) - P(s, x, \gamma_i) \right|.
\]
For an arbitrary \( \varepsilon > 0 \), first select \( N \) sufficiently large so that \( 2K/N \leq \varepsilon \). Then let \( n \to \infty \) and by Lemma 3 the last term goes to zero as \( n \to \infty \). Hence the result follows. Q.E.D.

**Lemma 5.** Assume \( h_n(x) \to h(x) \) in addition to the assumptions of Lemma 3. Then, for \( T > 0 \),

\[
\lim_{n \to \infty} \int_0^T h_n(x_n(s))ds = \int_0^T h(x(s))ds
\]

where \( x_n(t) \) and \( x(t) \) were defined in Lemma 3.

**Proof.**

\[
\left| \int_0^T (h_n(x_n(s))ds - h(x(s)))ds \right|
\]

\[
= \left| \int_0^T [h_n(x_n(s)) - h(x_n(s)) + h(x_n(s)) - h(x(s))]ds \right|
\]

\[
\leq \left| \int_0^T [h_n(x_n(s)) - h(x_n(s))]ds \right|
\]

\[
+ \left| \int_0^T [h(x_n(s)) - h(x(s))]ds \right|
\]

\[
\to 0 \quad \text{as } n \to \infty
\]

by Lemma 2 and Lemma 4. Q.E.D.
Corollary 1. For $\alpha > 0$ and $T > 0$,

$$\lim_{n \to \infty} E \left[ \int_0^T e^{-\alpha s} h_n(x_n(s)) \, ds \right] = E \left[ \int_0^T e^{-\alpha s} h(x(s)) \, ds \right].$$

Proof. From the proof of Lemma 1, it is clear that

$$\lim_{n \to \infty} E \left[ \int_0^T e^{-\alpha s} (h_n(y(s)) - h(y(s))) \, ds \right] = 0.$$

Hence

$$\lim_{n \to \infty} E \left[ \int_0^T e^{-\alpha s} (h_n(x_n(s)) - h(x_n(s))) \, ds \right] = 0.$$

Thus, by a slight modification of the proofs of Lemmas 2, 4, and 5, the result follows.

Q.E.D.

Lemma 6. Under the conditions of Lemma 5, if $\alpha > 0$,

$$\lim_{n \to \infty} E \left[ \int_0^\infty e^{-\alpha s} h_n(x_n(s)) \, ds \right] = E \left[ \int_0^\infty e^{-\alpha s} h(x(s)) \, ds \right].$$

Proof. Since $h_n(x)$ and $h(x)$ are uniformly bounded, it follows that for an arbitrary $\epsilon > 0$, we can take $T$ sufficiently large so that

$$|E \left[ \int_T^\infty e^{-\alpha s} h_n(x_n(s)) \, ds \right]| \leq \epsilon.$$
uniformly in $n$ and

$$|E_x \int_T^\infty e^{-Qs} h(x(s))ds| \leq \varepsilon.$$ 

Combining these facts with Corollary 1, we obtain the result. Q.E.D.

**Definition 4.** Let $G$ be a bounded open set and $x \in G$. Define

$$\tau_n = \inf \{t | x_n(t) \in G^c\}$$

$$\tau = \inf \{t | x(t) \in G^c\}$$

and

$$\tilde{\tau} = \inf \{t | y(t) \in G^c\}$$

where $x_n(t)$ and $x(t)$ were defined in Lemma 3, and $y(t)$ is the process of ($\gamma$).

**Lemma 7.** Let $h(x)$ be a bounded measurable function. Then

$$\lim_{n \to \infty} E_x \int_0^{\tau_n} h(x_n(s))ds = E_x \int_0^\tau h(x(s))ds.$$ 

**Proof.** Let $T > 0$ be arbitrary. Define, for an arbitrary stopping time $\eta$, $I_\eta(s) = 1$ if $s \leq \eta$, $I_\eta(s) = 0$ if $s > \eta$. Then

$$E_x \int_0^{\tau_n} h(x_n(s))ds = E_x \int_0^{\tau_n} h(x_n(s))ds + E_x \int_{\tau_n}^{\tau_n^T} h(x_n(s))ds$$

$$= E_x \int_0^T h(x_n(s))I_{\tau_n}(s)ds + E_x \int_{\tau_n}^{\tau_n^T} h(x_n(s))ds.$$ 

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where \( \tau_n^T = \min(\tau_n, T) \). Since \( E_x h(x_n(s, \omega)) I_{\tau_n}^{\omega} = E_x^n h(y(s, \omega)) I_{\tau_n}^{\omega} \), \( P_n(\cdot) \to P_0(\cdot) \) by Lemma 3, and \( h(y(s, \omega)) I_{\tau_n}^{\omega} \) is bounded and measurable, by the same argument as in Lemma 4

\[
\lim_{n \to \infty} E_x^n h(y(s, \omega)) I_{\tau_n}^{\omega} = E_x h(x(s, \omega)) I_{\tau_n}^{\omega} = E_x h(x(s, \omega)) I_{\tau_n}^{\omega}
\]

and this implies

\[
E_x \int_0^T h(x(s)) I_{\tau_n}^{\omega} ds \to E_x \int_0^T h(x(s)) I_{\tau_n}^{\omega} ds \quad \text{as } n \to \infty.
\]

Next we show that \( E_x(\tau_n - \tau_n^T) \to 0 \) as \( T \to \infty \) uniformly in \( n \).

First we show that \( E_x \tau_n^2 \leq K \) (uniformly bounded). Note that

\[
E_x \tau_n^2 = E_x \tau_n^2 \exp(\frac{T_0}{2} f_n) \cdot
\]

By Lemma 2 of Girsanov [2], the above value is independent of \( T \).

Thus we fix \( T = T_0 > 0 \). Then

\[
E_x \tau_n^2 = E_x \tau_n^2 \exp(\frac{T_0}{2} f_n) \leq (E_x \tau_n^2 \exp(2T_0 f_n))^{1/2}.
\]

Since \( E_x \exp(2T_0 f_n) \leq \exp(T_0 K) \), where \( |f_n(\cdot)| \leq K \), and \( \tau \) has finite moments (see [6, p. 434] for the one-dimensional case)

\[
E_x \tau_n^2 \leq K \quad \text{(uniformly bounded)}.
\]
Now

\[ E_x(\tau_n - \tau_n^T) = \int \chi_{\{\tau_n > T\}}(\omega) (\tau_n(\omega) - T) \, dp(\omega) \leq \int \tau_n(\omega) \, dp(\omega) \]

\[ = \int \chi_{\{\tau_n > T\}}(\omega) \tau_n(\omega) \, dp(\omega) \leq (E_x \chi_{\{\tau_n > T\}}(\omega))^{1/2} (E_x \tau_n^2(\omega))^{1/2} \]

\[ \leq K (E_x \chi_{\{\tau_n > T\}}(\omega))^{1/2} \]

\[ = KP(\omega: \tau_n > T)^{1/2} \]

\[ \leq K \left( \frac{E_x \tau_n}{T} \right)^{1/2} \leq K \left( \frac{K'}{T} \right)^{1/2} \rightarrow 0 \quad \text{as} \quad T \rightarrow \infty \]

uniformly in \( n \), where \( E_x \tau_n \leq K' \) and \( \chi_{\{\tau_n > T\}}(\omega) \) denotes the characteristic function of the set \( \{\tau_n > T\} \). Thus

\[ \left| E_x \int_{\tau_n^T}^{\tau_n} h(x_n(s)) \, ds \right| \leq KE_x(T_n - \tau_n^T) \rightarrow 0 \quad \text{as} \quad T \rightarrow \infty \]

uniformly in \( n \).

From these results, the conclusion follows. Q.E.D.
5. EXISTENCE THEOREMS FOR OPTIMAL STOCHASTIC CONTROLS (I)

In this section we present some theorems on the existence of optimal stochastic controls. First we consider the system (1) with objective function (5). We impose the following assumptions on $u(\cdot), f(\cdot,\cdot)$ and $L(\cdot,\cdot)$.

(A6) The control $u(x)$ takes values on a compact set $U \subseteq \mathbb{R}^m$. Let $\mathcal{U}$ be the set of all controls which satisfy (A3) and (A6).

(A7) $f(\cdot,\cdot)$ is continuous in the second argument.

(A8) $L(\cdot,\cdot)$ is a bounded real valued measurable function on $\mathbb{R}^n \times \mathbb{R}^m$, continuous in the second argument.

(A9)

$$
\begin{pmatrix}
L(x,U) \\
f(x,U)
\end{pmatrix} = \begin{pmatrix}
L(x,u) \\
f(x,u)
\end{pmatrix}, \quad u \in U
$$

is a convex set for each $x \in \mathbb{R}^n$.

Remark 1. Essentially, Assumption (A9) disqualifies from consideration problems where the control $u(x)$ is restricted to a finite number of discrete values. However, it permits consideration of control functions $u(\cdot)$ which are discontinuous. A proof of the existence of an optimal control under assumption (A9) can be useful.
even for problems where \( u(x) \) is restricted to a finite number of discrete values if it can be supplemented by an additional result showing that the optimal control only assumes values in a suitable finite set. That type of result is obtained in Section 7 for a discrete approximation to a continuous time control problem.

We consider the problem of selecting \( u \in \mathcal{U} \) to minimize

\[
Q_u = E_x \int_0^\infty e^{-\alpha s} L(x(s), u(x(s))) ds, \quad \alpha > 0
\]

under the system (1). Let \( Q = \inf_{u \in \mathcal{U}} Q_u \).

**Theorem 1.** Under the Assumptions (A1-9), there exists an optimal control \( \bar{u} \in \mathcal{U} \) which attains the value \( Q \).

**Proof.** 1) Let \( \{u_n(x)\} \) be a minimizing sequence of controls, i.e.,

\[
\lim_{n \to \infty} Q_{u_n} = Q.
\]

Let

\[
v_{n}(x) = \begin{pmatrix} L(x, u_n(x)) \\ f(x, u_n(x)) \end{pmatrix} = v(x, u_n(x))
\]

We prove that there exists a subsequence of \( v_n(x) \), indexed by \( n \), and a bounded measurable function \( v(x) \) such that \( v_n(x) \Rightarrow v(x) \).

Let us prove this fact only for the first component, i.e., \( L(x, u_n(x)) \). Let \( \Omega_i, i = 1, 2, \ldots \) be bounded domains such that
\( \Omega_i \subset \Omega_{i+1}, \, i = 1, 2, \ldots \) and \( \Omega_i \to \mathbb{R}^n \). Since \( |L(x, u_n(x))| \leq K \),
\[ ||L(x, u_n(x))||_{L_2(\Omega_i)} \leq K \] (\( K \) is a generic constant). Thus there exists a subsequence \( \{L(x, u^1_n(x))\} \) of \( \{L(x, u_n(x))\} \) and a measurable function \( \tilde{L}_1(x) \) such that \( L(x, u^1_n(x)) \) converges weakly to \( \tilde{L}_1(x) \) in \( L_2(\Omega_1) \) and \( |\tilde{L}_1(x)| \leq K \). Next take \( \Omega_2 \).

Since \( ||L(x, u^1_n(x))||_{L_2(\Omega_2)} \leq K_2 \), there exists a subsequence \( \{L(x, u^2_n(x))\} \) of \( \{L(x, u^1_n(x))\} \) and a measurable function \( \tilde{L}_2(x) \) such that \( L(x, u^2_n(x)) \) converges weakly to \( \tilde{L}_2(x) \) in \( L_2(\Omega_2) \) and \( |\tilde{L}_2(x)| \leq K \). Note that we can construct \( \tilde{L}_2(x) \) so that \( \tilde{L}_2(x) = \tilde{L}_1(x) \) in \( \Omega_1 \). Continuing this process, we get a subsequence \( \{L(x, u^2_n(x))\} \) and a measurable function \( \tilde{L}_k(x) \) such that \( L(x, u^k_n(x)) \) converges weakly to \( \tilde{L}_k(x) \) in \( L_2(\Omega_k) \). Define \( h_n(x) = L(x, u_n(x)) \) and \( \tilde{L}(x) = \tilde{L}_k(x) \) where \( x \in \Omega_k \). Note that the definition of \( \tilde{L}(x) \) is consistent and \( |\tilde{L}(x)| \leq K \). It is clear that \( \tilde{L}(x) \) is measurable.

Now let \( \Omega \) be an arbitrary bounded domain. Then there exists a \( \Omega_k \) such that \( \Omega \subset \Omega_k \) and \( h_n(x) \) converges weakly to \( \tilde{L}(x) \) in \( L_2(\Omega_k) \). Next take an arbitrary \( \varphi(x) \in L_2(\Omega) \) and define \( \tilde{\varphi}(x) = \varphi(x) \) on \( \Omega \), \( \tilde{\varphi}(x) = 0 \) on \( \Omega_k - \Omega \). Then

\[
\int_{\Omega} h_n(x) \varphi(x) dx = \int_{\Omega_k} h_n(x) \tilde{\varphi}(x) dx \to \int_{\Omega_k} \tilde{L}(x) \tilde{\varphi}(x) dx \to \int_{\Omega_k} \tilde{L}(x) \tilde{\varphi}(x) dx
\]

\[= \int_{\Omega} \tilde{L}(x) \varphi(x) dx \quad \text{as} \ n \to \infty . \]

This means that \( h_n(x) \) converges weakly to \( \tilde{L}(x) \) in \( L_2(\Omega) \). The above argument can be extended to the other components of \( v_n(x) \).
In the following part we confine ourselves to the subsequence for which \( v_n(x) \) converges weakly to \( v(x) \).

2) We will show that there exists a control \( \bar{u} \in \mathcal{U} \) such that

\[
v(x) = \begin{pmatrix}
L(x, \bar{u}(x)) \\
\tilde{f}(x, \tilde{u}(x))
\end{pmatrix}.
\]

The method follows Roxin [14] and Kushner [10]. Let

\[
v(x, U) = \left( \begin{pmatrix} L(x, u) \\ \tilde{f}(x, u) \end{pmatrix}, u \in U \right)
\]

and take an arbitrary bounded domain \( \Omega \) and an arbitrary measurable set \( A \subset \Omega \). Then, for any \( y \in \mathbb{R}^{1+d} \) \((d = n_2)\) is the dimension of the vector function \( f(\cdot, \cdot) \),

\[
\int \sup_{A, u \in U} y' v(x, u) dx \geq \lim_{n \to \infty} \int y' v(x, u_n(x)) dx
\]

\[
= \lim_{n \to \infty} \int y' v(x, u_n(x)) X_A(x) dx
\]

\[
= \int y' v(x) X_A(x) dx = \int y' v(x) dx
\]

\[
= \lim_{n \to \infty} \int y' v(x, u_n(x)) dx \geq \int \inf_{A, u \in U} y' v(x, u) dx
\]

where \( X_A(\cdot) \) is the characteristic function of the set \( A \). Thus
\[ \sup_{u \in U} y'v(x,u) \geq y'v(x) \geq \inf_{u \in U} y'v(x,u) \quad \text{a.e. } x \in \Omega \]

But since \( \Omega \) is an arbitrary bounded domain in \( \mathbb{R}^n \), for each \( y \in \mathbb{R}^{1+d} \)

\[ \sup_{u \in U} y'v(x,u) \geq y'v(x) \geq \inf_{u \in U} y'v(x,u) \] \quad (15)

for almost all \( x \in \mathbb{R}^n \). We shall make use of the fact that a closed convex set can be determined by the \( \sup \) and \( \inf \) of a dense set of linear functions; i.e., \( z_0 \) is an element of a closed convex set \( S \subset \mathbb{R}^{1+d} \) if and only if \( \inf_{z \in S} y_1'z \leq y_1'z \leq \sup_{z \in S} y_1'z \) for a countable set \( \{y_1, y_2, \ldots\} \) which is dense on the unit sphere in \( \mathbb{R}^{1+d} \). Let \( \{y_1, y_2, \ldots\} \) be such a dense set. For each \( y_i \), let \( E_i \) be the exceptional set in which (15) does not hold. Then for each \( x \notin \bigcup_{i=1}^{\infty} E_i \), (15) holds for all \( y \) in \( \{y_1, y_2, \ldots\} \). Hence for each \( x \notin \bigcup_{i=1}^{\infty} E_i \), \( v(x) \in v(x,U) \) by the closedness and convexity of \( v(x,U) \) (closure comes from the continuity of \( v(x,\cdot) \) on a compact set). By redefining \( v(x) \) on the set of zero measure, \( \bigcup_{i=1}^{\infty} E_i \), we can induce it to satisfy \( v(x) \in v(x,U) \) for all \( x \in \mathbb{R}^n \) without losing the property that \( v_n(x) \Rightarrow v(x) \). Then by Lemma 5 of Benè̆s [2], there exists \( \tilde{u} \in \mathcal{U} \) such that

\[ v(x, \tilde{u}(x)) = \begin{pmatrix} L(x, \tilde{u}(x)) \\ \tilde{f}(x, \tilde{u}(x)) \end{pmatrix} \]

(The result in 2) is also demonstrated with a different approach in the appendix at the end of this section.)
3) Now \( \tilde{u} \in \mathcal{U} \) is an optimal control. In fact, in 1) we have proved that \( v_n(x) = v(x, u_n(x)) \) converges weakly to \( v(x) = v(x, \tilde{u}(x)) \) in \( L^2(\Omega) \) for an arbitrary bounded domain \( \Omega \).

Thus by Lemma 6, \( \lim_{n \to \infty} Q_n u_n = E \int_0^\infty e^{-\xi(t)} L(\bar{x}(s), \tilde{u}(-\bar{x}(s)))ds \), where \( \bar{x}(t) \) corresponds to \( \tilde{u} \in \mathcal{U} \). On the other hand \( Q_n u_n \to Q = \inf_{u \in \mathcal{U}} Q_u \).

Thus

\[
\inf_{u \in \mathcal{U}} Q_u = E \int_0^\infty e^{-\xi(t)} L(\bar{x}(s), \tilde{u}(-\bar{x}(s)))ds
\]

and this shows that \( \tilde{u} \in \mathcal{U} \) is optimal. Q.E.D.

For this cost criterion (4), we consider a little more restricted one, i.e., we consider the problem of selecting \( u \in \mathcal{U} \) to minimize

\[
Q_u = E \int_0^\tau L(x(s))ds ,
\]

where \( x(s) \) is the process corresponding to \( u \in \mathcal{U} \) and \( \tau = \inf(t \mid x(t) \in G^c, G: \text{a bounded open set}) \). Note that the cost depends only on the sample path of the process. This restriction would not be required if Lemma 7 could be generalized to replace \( h(x_n(s)) \) by \( h_n(x_n(s)) \).

**Theorem 2.** Under (A1-9), there exists an optimal control \( \tilde{u} \) in \( \mathcal{U} \) which attains the value

\[
Q = \inf_{u \in \mathcal{U}} E \int_0^\tau L(x(s))ds .
\]
Proof. As in the proof of Theorem 1, there exists a minimizing sequence of controls \( u_n \in \mathcal{U} \) and a control \( \tilde{u} \in \mathcal{U} \) such that \( f(x, u_n(x)) \) converges weakly to \( f(x, \tilde{u}(x)) \) on each bounded domain \( \Omega \) in \( L_2(\Omega) \). Then by Lemma 7,
\[
\lim_{n \to \infty} E_x \int_0^{\tau_n} L(x_n(s))ds = E_x \int_0^\tau L(\tilde{x}(s))ds
\]
where \( \tau_n \) and \( \tau \) are the exit times from \( G \) of the process \( x_n(t) \) and \( \tilde{x}(t) \) corresponding to \( u_n \) and \( \tilde{u} \). Since
\[
Q = \lim_{n \to \infty} E_x \int_0^{\tau_n} L(x_n(s))ds
\]
\( \tilde{u} \in \mathcal{U} \) is an optimal control. Q.E.D.
APPENDIX

We will show that, for an arbitrary bounded domain \( \Omega \), there exists a sequence \( \{v_{n'}(x)\} \) such that \( v_{n'}(x) \) converges strongly to \( v(x) \) in \( L_2(\Omega) \). We have shown in 1) that the subsequence \( v_n(x) \) converges weakly to \( v(x) \) on \( \Omega \) in \( L_2(\Omega) \). We now see that \( \{v(x, u(x)), u \in \mathcal{U}\} \) is convex. In fact, let \( \lambda \in [0,1] \) and \( u, \tilde{u} \in \mathcal{U} \). Then, for each \( x \),

\[
\lambda v(x, u(x)) + (1-\lambda) v(x, \tilde{u}(x)) \in v(x,u)
\]

by the convexity of \( v(x,u) \), where \( v(x,v) = \{v(x,\xi), \xi \in U\} \) and

\[
v(x,\xi) = \begin{pmatrix} L(x,\xi) \\ f(x,\xi) \end{pmatrix}
\]

Since \( v(\cdot, \cdot) \) is continuous in the second argument and measurable in the first argument, and since \( U \) is compact, by Lemma 5 of Benes [2], there exists a measurable function \( w(x) : \mathbb{R}^n \to U \) such that

\[
\lambda v(x, u(x)) + (1-\lambda) v(x, \tilde{u}(x)) = v(x, w(x))
\]

Thus \( \{v(x, u(x)), u \in \mathcal{U}\} \) is convex.
From the properties of weak $L_2$ convergence it is known that there exists a subsequence $v_n(x)$, indexed by $n$, such that 
\[ m^{-1} \sum_{1}^{m} v_n(x) \] converges strongly to $v(x)$ in $L_2(\Omega)$. But since 
\[ \{v(x, u(x)), u \in \mathcal{U}\} \] is convex, there exists a function $u'_m(x) \in \mathcal{U}$ such that 
\[ m^{-1} \sum_{1}^{m} v_n(x) = v(x, u'_m(x)) = v'_m(x). \]

Thus $v(x, u'_m(x))$ converges strongly to $v(x)$ in $L_2(\Omega)$.

We show that there exists a control $\bar{u} \in \mathcal{U}$ such that $v(x) = v(x, \bar{u}(x))$. We have shown that $v_m'(x)$ converges strongly to $v(x)$ in $L_2(\Omega)$. Thus there exists a subsequence of $\{v_m'(x)\}$, indexed by $m'$, such that $v_m'(x)$ converges to $v(x)$ almost everywhere in $\Omega$. Since $v(x, U)$ is closed for each $x$, $v(x) \in v(x, U)$ for almost every $x$ in $\Omega$. But $\Omega$ is an arbitrary bounded domain, so $v(x) \in v(x, U)$ for almost every $x$ in $\mathbb{R}^n$. By redefining $v(x)$ on the exceptional set of Lebesgue measure zero appropriately, we can induce it to satisfy $v(x) \in u(x, U)$ for all $x$ in $\mathbb{R}^n$. Now by Lemma 5 of Beneš [2], there exists a control $\bar{u} \in \mathcal{U}$ such that 
\[ v(x) = v(x, \bar{u}(x)). \]
6. EXISTENCE THEOREM FOR OPTIMAL STOCHASTIC CONTROLS (II)

We consider the problem of minimizing the criterion (6) for the system described by (1). For this problem we obtain a theorem for the existence of optimal controls under quite strong assumptions. In Subsection 6.1, an extension of Ito's differentiation rule to Sobolev space is described. It is needed for our existence theorem. It is also used to derive a sufficiency condition for optimality in terms of a "potential loss" function.

6.1. Extended Ito's Formula and Sufficiency Conditions for Optimality.

An extended Ito's formula which is developed here is a little modification of the result obtained in Rishel [12]. The development is almost the same as in Rishel [12]. We consider the system (1) and assume (A1) - (A5),

**Definition 1 (Sobolev space).** Let \( L_0^\beta(\mathbb{R}^n) = \{f(x), x \in \mathbb{R}^n | \int_{\mathbb{R}^n} f(x) dx < \infty \} \), \( f(x) \), a measurable function. Define \( W_0^2(\mathbb{R}^n) \) by

\[
W_0^2(\mathbb{R}^n) = \{ u(x) | u \in L_0^\beta(\mathbb{R}^n), D^k u \in L_0^\beta(\mathbb{R}^n), |k| \leq 2 \}
\]

where \( k = (k_1, \ldots, k_n) \), \( k_i \) non-negative integer, \( |k| = k_1 + k_2 + \cdots + k_n \), and

\[
D^k u(x) = \frac{\partial^{|k|} u(x)}{\partial x_1^{k_1} \partial x_2^{k_2} \cdots \partial x_n^{k_n}}
\]
expresses weak derivative or derivative in the sense of distribution.

Define the norm in $W_2^\beta(R^n)$ by

$$
\|u\|_{W_2^\beta(R^n)} = \left(\sum_{k=2}^{\infty} \int_{R^n} \left| \partial^k u(x) \right|^\beta \, dx \right)^{1/\beta}.
$$

**Definition 2.** Associated with the system (1) we define the differential operator $\mathcal{L}_u$ for $v(x) \in W_2^\beta(R^n)$ as

$$
\mathcal{L}_u v(x) = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 v(x)}{\partial x_i \partial x_j} + \sum_{i=1}^{n_1} g_{1i}(x) \frac{\partial v(x)}{\partial x_i}
$$

$$
+ \sum_{i=n_1+1}^{n} g_{2i}(x) \frac{\partial v(x)}{\partial x_i} + \sum_{i=n_1+1}^{n} f_i(x, u(x)) \frac{\partial v(x)}{\partial x_i}
$$

where $n_1$ is the dimension of the $x_1(t)$ vector in (1) and $a_{ij}(x)$ is an element of $\frac{1}{2} \bar{\sigma}(x) \bar{\sigma}'(x)$ where $\bar{\sigma}(x)$ is the $n \times n$ matrix of the form

$$
\bar{\sigma}(x) = \begin{pmatrix}
0 & 0 \\
0 & \sigma(x)
\end{pmatrix}
$$

Under (Al-5) the process $x(t)$ of (1) has a density $q(t, x, y)$ which satisfies

$$
\int q(t, x, y)^\beta \, dy \leq M
$$

for some $\beta > 1$ in any compact $t$ and $x$ set (see Rischel [3]).
In this section we assume the boundedness of \( g_i(x), i = 1, 2 \) in (1).

**Lemma 8.** For the system (1) assume (A1-5). Let \( \gamma = \beta(\beta - 1)^{-1} \). Then, for any \( \phi(x) \in W^2(\mathbb{R}^n) \),

\[
E_x \phi(x(t)) = \tilde{\phi}(x) + E_x \int_0^t \mathcal{L}_u \phi(x(s))ds
\]

for some finite-valued function \( \tilde{\phi}(x) \) independent of \( t \).

**Proof.** Let \( \phi(x) \in W^2(\mathbb{R}^n) \). Then by the property of \( W^2(\mathbb{R}^n) \), there exists a sequence \( \{\phi_n(x)\} \) whose elements are twice continuously differentiable with compact support and \( \phi_n(x) \to \phi(x) \) as \( n \to \infty \) in \( W^2(\mathbb{R}^n) \). By applying Ito's differential rule to \( \phi_n(x) \),

\[
\phi_n(x(t)) = \phi_n(x) + \int_0^t \mathcal{L}_u \phi_n(x(s)) ds + \int_0^t \phi_{nx}(x(s)) \omega(x(s)) dw(s)
\]

Since \( \phi_n(x) \) has a compact support

\[
E_x \phi_n(x(t)) = \phi_n(x) + E_x \int_0^t \mathcal{L}_u \phi_n(x(s)) ds \quad . \tag{16}
\]

Now

\[
|E_x \phi_n(x(t)) - E_x \phi(x(t))| \leq E_x |\phi_n(x(t)) - \phi(x(t))|
\]

\[
= \int_{\mathbb{R}^n} |\phi_n(y) - \phi(y)| q(t, x, y) dy
\]

\[
\leq (\int_{\mathbb{R}^n} |\phi_n(y) - \phi(y)|^\gamma dy)^{1/\gamma} (\int_{\mathbb{R}^n} q(t, x, y)^\beta dy)^{1/\beta}
\]

\[
\to 0 \quad \text{as } n \to \infty .
\]
\[
\left| E_x \int_0^t \mathcal{L}_u \rho_n(x(s)) ds - E_x \int_0^t \mathcal{L}_u \rho(x(s)) ds \right|
\]

\[
\leq \int_0^t \int_{\mathbb{R}^n} \left\{ \sum_{i,j} a_{ij}(y) \left( \frac{\partial^2 \rho_n(y)}{\partial x_i \partial x_j} - \frac{\partial^2 \rho(y)}{\partial x_i \partial x_j} \right) + \sum_{i=1}^n g_{1i}(y) \left( \frac{\partial \rho_n(y)}{\partial x_i} - \frac{\partial \rho(y)}{\partial x_i} \right) \right. \\
+ \sum_{i=n_1+1}^n g_{2i}(y) \left( \frac{\partial \rho_n(y)}{\partial x_i} - \frac{\partial \rho(y)}{\partial x_i} \right) \\
+ \sum_{i=n_1+1}^n f_i(y, u(y)) \left( \frac{\partial \rho_n(y)}{\partial x_i} - \frac{\partial \rho(y)}{\partial x_i} \right) \right\} q(s,x,y) dy \ ds 
\]

Since \( a_{ij}(y) \) is bounded, the first term

\[
\leq K \sum_{i,j} \int_0^t \left( \int_{\mathbb{R}^n} \frac{\partial^2 \rho_n(y)}{\partial x_i \partial x_j} - \frac{\partial^2 \rho(y)}{\partial x_i \partial x_j} \right)^\gamma dy \right)^{1/\gamma} \\
\left( \int_{\mathbb{R}^n} q(s,x,y)^\beta dy \right)^{1/\beta} \ ds \to 0 
\]

as \( n \to \infty \) using \((A_5)\) and the fact that \( \rho_n(x) \to \rho(x) \) in \( W_1^2(\mathbb{R}^n) \).
Similarly from the boundedness of \( g_1(x) \) and \( f_1(x, u(x)) \) and the fact that \( \phi_n(x) \to \phi(x) \) in \( W_2^2(R^n) \), it follows that all of the remaining terms go to zero as \( n \to \infty \). Applying (16) we have

\[
\phi_n(x) = E_x \phi_n(x(t)) - E_x \int_0^t \mathcal{L}_u \phi_n(x(s))ds \\
\to E_x \phi(x(t)) - E_x \int_0^t \mathcal{L}_u \phi(x(s))ds .
\]

Hence \( \phi_n(x) \) has a finite valued limit \( \tilde{\phi}(x) \), which is of course independent of \( t \), and our lemma follows. Q.E.D.

**Remark 1.** Rishel [12] proved that, for any continuous bounded \( \phi(x) \in W_2^2(R^n) \),

\[
E_x \phi(x(t)) = \phi(x) + E_x \int_0^t \mathcal{L}_u \phi(x(s))ds .
\]

The following lemma is a minor modification of Lemma 8 which establishes the extended Itô rule to a class of \( \phi \) which overlaps those satisfying the conditions of Lemma 8.

**Definition 3.** Let \( \Omega \) be an arbitrary domain in \( R^n \). Then \( u \in W_2^{1,\text{loc}}(\Omega) \) if, for any bounded domain \( \Omega' \) in \( \Omega \), \( u \in W_2^2(\Omega') \).

**Lemma 2.** For the system (1) assume (A1-5). Then, for any bounded \( \phi(x) \in W_2^{1,\text{loc}}(R^n) \) which satisfies \( |\phi(x)| + |\phi_x(x)| + |\phi_{xx}(x)| \leq K \),
\[ E_x \varphi(x(t)) = \bar{\varphi}(x) + E_x \int_0^t \mathcal{L}_u \varphi(x(s)) \, ds \]

for some finite-valued function \( \bar{\varphi}(x) \) independent of \( t \). Here

\[ \varphi_x(x) = \left( \frac{\partial \varphi(x)}{\partial x_i} \right) \quad \text{and} \quad \varphi_{xx}(x) = \left( \frac{\partial^2 \varphi(x)}{\partial x_i \partial x_j} \right). \]

\textbf{Proof.} For an arbitrary \( \varepsilon > 0 \), choose the sphere \( B_r = \{ x : |x|^2 \leq r \} \) so that

\[ \int_{B_r^c} p(s,x,y) \, dy \leq \varepsilon \]

uniformly in \( s \leq t \), where \( p(s,x,y) \) is the density function of the process \( y(t) \) of (7).

Let \( \varphi(x) \in W^{1,2}_\text{loc}(\mathbb{R}^n) \). Then by a property of \( W^{1,2}_\text{loc}(\mathbb{R}^n) \), (see Corollary of Theorem 1.11 of Agmon [1]), there exists a sequence \( \{ \varphi_n(x) \} \) whose elements are twice continuously differentiable with compact support and \( \varphi_n(x) \rightarrow \varphi(x) \) as \( n \rightarrow \infty \) in \( W^2_2(B_r) \). Since

\[ |\varphi(x) + \varphi_x(x)| + |\varphi_{xx}(x)| \leq K \]

by the assumption, \( \varphi_n(x) \) can be taken so that \( |\varphi_n(x)| + |\varphi_{nx}(x)| + |\varphi_{nxx}(x)| \leq K \). By applying Ito's differentiation rule to \( \varphi_n(x) \),

\[ \varphi_n(x(t)) = \varphi_n(x) + \int_0^t \mathcal{L}_u \varphi_n(x(s)) \, ds + \int_0^t \varphi_{nx}(x(s)) \, \bar{\varphi}(x(s)) \, dw(s). \]

Since \( \varphi_n(x) \) has a compact support,

\[ E_x \varphi_n(x(t)) = \varphi_n(x) + E_x \int_0^t \mathcal{L}_u \varphi_n(x(s)) \, ds \]. (17)
Now

\[ |E_x \phi_n(x(t)) - E_x \phi(x(t))| \leq E_x |\phi_n(x(t)) - \phi(x(t))| \]

\[ = E_x |\phi_n(y(t)) - \phi(y(t))| \exp(\int_0^t (s) \, ds) \]

\[ \leq (E_x |\phi_n(y(t)) - \phi(y(t))|^2 E_x \exp(2 \int_0^t (s) \, ds))^{1/2} \]

\[ \leq (Kt)^{1/2} (E_x |\phi_n(y(t)) - \phi(y(t))|^2)^{1/2} \]

and

\[ E_x |\phi_n(y(t)) - \phi(y(t))|^2 \]

\[ = \int_{B_r} (\phi_n(y) - \phi(y))^2 p(t,x,y) \, dy + \int_{B_r^c} (\phi_n(y) - \phi(y))^2 p(t,x,y) \, dy \]

\[ \leq \left( \int_{B_r} (\phi_n(y) - \phi(y))^4 \, dy \int_{B_r} p(t,x,y)^2 \, dy \right)^{1/2} \]

\[ + K \int_{B_r^c} p(t,x,y) \, dy \]

Since \( \phi_n(y) \) and \( \phi(y) \) are uniformly bounded, the first term is less than

\[ \left( K \int_{B_r} (\phi_n(y) - \phi(y))^2 \, dy \int_{B_r} p(t,x,y)^2 \, dy \right)^{1/2} \]

and this term goes to zero as \( n \to \infty \) by (A5) and the fact that

\( \phi_n(y) \to \phi(y) \) in \( W^2_2(B_r) \). The second term is less than \( eK \) uniformly in \( n \). Thus
\[
\lim_{n \to \infty} E_x \varphi_n(x(t)) = E_x \varphi(x(t)) .
\]

Next we estimate

\[
\left| E_x \int_0^t \mathcal{L}_u \varphi_n(x(s))ds - E_x \int_0^t \mathcal{L}_u \varphi(x(s))ds \right|
\leq \left| \int_0^t E_x \left\{ \sum a_{ij}(x(s)) \left( \frac{\partial^2 \varphi_n(x(s))}{\partial x_i \partial x_j} - \frac{\partial^2 \varphi(x(s))}{\partial x_i \partial x_j} \right) \right. \\
+ \sum_{i=1}^{n_1} g_{1i}(x(s)) \left( \frac{\partial \varphi_n(x(s))}{\partial x_i} - \frac{\partial \varphi(x(s))}{\partial x_i} \right) \\
+ \sum_{i=n_1+1}^{n} g_{2i}(x(s)) \left( \frac{\partial \varphi_n(x(s))}{\partial x_i} - \frac{\partial \varphi(x(s))}{\partial x_i} \right) \\
+ \sum_{i=n_1+1}^{n} f_i(x(s), u(x(s))) \left( \frac{\partial \varphi_n(x(s))}{\partial x_i} - \frac{\partial \varphi(x(s))}{\partial x_i} \right) \right] ds \right|
\]

(18)

Then, since \( a_{ij}(x) \) is bounded and \( |\varphi_{xx}(x)| \leq K \),

\[
E_x \left\{ \sum a_{ij}(x(s)) \left( \frac{\partial^2 \varphi_n(x(s))}{\partial x_i \partial x_j} - \frac{\partial^2 \varphi(x(s))}{\partial x_i \partial x_j} \right) \right\}
\leq K \sum E_x \left[ \left( \frac{\partial^2 \varphi_n(y(s))}{\partial x_i \partial x_j} - \frac{\partial^2 \varphi(y(s))}{\partial x_i \partial x_j} \right) \exp(t_0^t(f)) \right]
\to 0 \quad \text{as } n \to \infty
\]

just as in the preceding case.

Similarly from the boundedness of \( g_i(x) \) and \( f_i(x, u(x)) \) and the fact that \( \varphi_n(x) \to \varphi(x) \) in \( W^2_{2}(B_r) \), it follows that all of the remaining terms in (18) go to zero as \( n \to \infty \). It follows that
\[ \mathbb{E}_x \int_0^t \mathcal{L}_u \hat{\varphi}_n(x(s))ds \rightarrow \mathbb{E}_x \int_0^t \mathcal{L}_u \hat{\varphi}(x(s))ds \]

Hence, applying (17),

\[ \hat{\varphi}_n(x) = \mathbb{E}_x \hat{\varphi}_n(x(t)) - \mathbb{E}_x \int_0^t \mathcal{L}_u \hat{\varphi}_n(x(s))ds \rightarrow \mathbb{E}_x \hat{\varphi}(x(t)) - \mathbb{E}_x \int_0^t \mathcal{L}_u \hat{\varphi}(x(s))ds \]

Thus the left hand side, which does not involve \( t \), converges to some limit \( \hat{\varphi}(x) \) and the desired result has been established. Q.E.D.

Using Lemma 9 we obtain a relation between the cost functional (6) and a partial differential equation connected to the differential operator \( \mathcal{L}_u \).

**Lemma 10.** Suppose there exists a measurable function \( u(x) \), a number \( \lambda \) and a real-valued function \( v(x) \) defined on \( \mathbb{R}^n \) with the properties

(i) \( V(x) \in W_2^{\text{loc}}(\mathbb{R}^n) \) and \( |V(x)| + |V_x(x)| + |V_{xx}(x)| \leq K \)

(ii) \( \lambda = \mathcal{L}_u V(x) + L(x, u(x)) \quad \forall x \in \mathbb{R}^n \).

Assume (A1) - (A5) for (1). Then

\[ \lambda = \lim_{T \to \infty} \frac{1}{T} \mathbb{E}_x \int_0^T L(x(s), u(x(s)))ds \]

where \( x(t) \) is the process of (1) corresponding to \( u \in \mathcal{U} \).
**Proof.** From (ii)

\[
\lambda T = E_x \int_0^T \mathcal{L}_u V(x(s))ds + E_x \int_0^T L(x(s), u(x(s)))ds \quad . \quad (19)
\]

Since \( V(x) \in W^{1,0}_2(\mathbb{I}^n) \), by Lemma 9, there exists a function \( \widetilde{V}(\cdot) < \infty \) such that

\[
E_x V(x(T)) = \widetilde{V}(x) + E_x \int_0^T \mathcal{L}_u V(x(s))ds \quad .
\]

Thus (19) can be written as

\[
\lambda T = E_x V(x(T)) - \widetilde{V}(x) + E_x \int_0^T L(x(s), u(x(s)))ds ,
\]

and

\[
\lambda = \frac{1}{T} E_x V(x(T)) - \frac{1}{T} \widetilde{V}(x) + \frac{1}{T} E_x \int_0^T L(x(s), u(x(s)))ds .
\]

Noting that \( V(x) \) is bounded and \( \widetilde{V}(x) \) is finite, we obtain, by letting \( T \to \infty \),

\[
\lambda = \lim_{T \to \infty} \frac{1}{T} E_x \int_0^T L(x(s), u(x(s)))ds \quad . \quad \text{Q.E.D.}
\]

For the space \( W^{2,0}_\gamma(\mathbb{I}^n) \), we have the following analogue of Lemma 10.
Lemma 11. Suppose there exists a measurable function \( u(x) \), a number \( \lambda \) and a real-valued function \( V(x) \) defined on \( \mathbb{R}^n \) with the properties

(i) \( V(x) \in W^2_\gamma(\mathbb{R}^n) \) and \( V(x) \) is bounded
(ii) \( \lambda = L_u V(x) + L(x, u(x)) \quad \forall x \in \mathbb{R}^n. \)

Assume (A1-5) for (1). Then

\[
\lambda = \lim_{T \to \infty} \frac{1}{T} \mathbb{E}_x \int_0^T L(x(s), u(x(s))) ds
\]

where \( x(t) \) is the process of (1) corresponding to \( u \in \mathcal{U} \).

Proof. The proof is the same as in Lemma 10 except that Lemma 8 is used instead of Lemma 9. Q.E.D.

Remark 2. In Lemma 11, the boundedness condition on \( V(x) \) is used only to show that \( \mathbb{E}_x V(x(T))/T \to 0 \). This condition can be weakened. In fact if the process \( x(t) \) corresponding to \( u \) has a stationary measure \( \mu \) such that for any measurable function \( f(x) \),

\[
\lim_{t \to \infty} \mathbb{E}_x f(x(t)) = \int f(x) \mu(dx)
\]

and

\[
\int |x|^2 \mu(dx) < \infty
\]

and

\[
V(x) \leq K(1 + |x|^2)
\]

then Lemma 11 holds.
**Remark 3.** It is easy to see that if there exists a number \( \lambda \) and \( V(x) \) which satisfy (i) and (ii) in Lemma 11 for a given measurable \( u \), then \( \lambda \) is unique. In fact \( \lambda \) measures the average loss per unit time for the control \( u \) and is independent of the initial point \( x \). Furthermore if \( V(x) \) is continuous, then we shall see that with an extra condition

\[
L_T(x) = E_x \int_0^T L(x(s), u(x(s))) ds,
\]

the accumulated loss over the time interval \([0, T]\), is related to \( V(x) \) in the following way:

\[
L_T(x) = \lambda T + V(x) - V_0 + o(1)
\]  \hspace{1cm} (20)

where \( V_0 \) is a constant. If \( V(x) \) satisfying (ii) is continuous and bounded, Remark 1 implies that

\[
E_x V(x(T)) = V(x) + E_x \int_0^T \mathcal{L}_u V(x(s)) ds
\]

\[
= V(x) + E_x \int_0^T (\lambda - L(x(s), u(x(s)))) ds
\]

\[
= V(x) + \lambda T - E_x \int_0^T L(x(s), u(x(s))) ds
\]

\[
= V(x) + \lambda T - L_T(x)
\]

Thus if the process \( x(t) \) corresponding to the control \( u \) has an invariant probability measure \( \mu \) such that
\[
\lim_{t \to \infty} \mathbb{E}_x V(x(t)) = \int_{\mathbb{R}^n} V(x) \mu(dx) = V_0
\]
then (20) follows.

\(V(x)\) can be interpreted as a "potential" cost associated with starting at the initial point \(x\).

**Lemma 12.** (Sufficient condition for optimality). Suppose (A1-5) are satisfied and there is a measurable control \(\tilde{u}(x)\), a real number \(\lambda\), and a real-valued function \(V(x)\) defined on \(\mathbb{R}^n\) with the following properties:

(i) \(V(x) \in W^2_2(\mathbb{R}^n)\) and \(V(x)\) is bounded,

(ii) \(\lambda = \mathcal{L}^u_x V(x) + L(x, \tilde{u}(x)), \quad \forall \ x \in \mathbb{R}^n\)

\(\lambda \leq \mathcal{L}^0_x V(x) + L(x, u), \quad \forall (x, u) \in \mathbb{R}^n \times \mathbb{R}^m\)

where \(\mathcal{L}^0_x V(x)\) in the last equation is defined as

\[
\mathcal{L}^0_x V(x) = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 V(x)}{\partial x_i \partial x_j} + \sum_{i=1}^{n} g_{i1}(x) \frac{\partial V(x)}{\partial x_i} + \sum_{i=n_1+1}^{n} g_{i2}(x) \frac{\partial V(x)}{\partial x_i} + \sum_{i=n_1+1}^{n} f(x, u) \frac{\partial V(x)}{\partial x_i}.
\]

Then \(\tilde{u}(x)\) is optimal.

**Proof.** By Lemma 11, we have

\[
\lambda = \lim_{T \to \infty} \frac{1}{T} \mathbb{E}_x \int_0^T L(x(t), \tilde{u}(x(t))) dt.
\]
From the second relation, for any control \( u(x) \),

\[
\lambda \leq \mathcal{L}_u V(x) + L(x, u(x))
\]

Integrating from 0 to \( T \) and dividing by \( T \), we have

\[
\lambda \leq \lim_{T \to \infty} \frac{1}{T} \mathbb{E}_x \int_0^T L(x(t), u(x(t))) dt
\]

and the conclusion follows. \( \text{Q.E.D.} \)

Remark 4. The boundedness condition on \( V(x) \) in Lemma 12 can be weakened since Lemma 11 can be extended to treat some cases of unbounded \( V(x) \) using Remark 2.

6.2. Existence Theorem for Optimal Controls.

We consider the system (1) with the criterion functional (6).

Recall that \( \mathcal{U} \) is the set of measurable controls which take values on a compact set \( U \subset \mathbb{R}^m \). Let \( \Phi \) be the subset of \( \mathcal{U} \) with the following property (P):

(P) for each \( u \in \Phi \), there exists a function \( V_u(x) \) and a constant \( \lambda^u \) such that

\[
\mathcal{L}_u V_u(x) + L(x, u(x)) = \lambda^u, \quad \forall x \in \mathbb{R}^n
\]

\[
|V_u(x)| + |V_{ux}(x)| + |V_{uxx}(x)| \leq K
\]

and
where \( K \) is independent of \( u \).

**Remark 5.** The conditions of property \((P)\) are partially satisfied in the following case. Suppose the system is non-degenerate, i.e., \( n = 0 \) and \( g_2(x) = 0 \). Let \( u \in \mathcal{U} \) be a control which satisfies the following condition \((*)\):

\( (*) \) There exists sequences \( \{f^n_i(x)\} \) and \( \{L^n_i(x)\} \) such that

\[
\lim_{n \to \infty} f^n_i(x) = f_i(x, u(x)), \quad \lim_{n \to \infty} L^n_i(x) = L(x, u(x)), \quad f^n_i(x) \text{ and } L^n_i(x) \text{ satisfy a Lipshitz condition and } |f^n_i(x)| \leq K_1 \text{ and } |L^n_i(x)| \leq K_2.
\]

Also the process \( x_n(t) \) corresponding to the drift coefficient \( f^n_i(x) \) has an invariant probability measure \( \mu_n \) such that

\[
|P_n(t, c, \Gamma) - \mu_n(\Gamma)| \leq R(t)
\]

for any Borel set \( \Gamma \) in \( \mathbb{R}^n \) where \( P_n(t, x, \Gamma) \) is the probability distribution of the process \( x_n(t) \) and \( R(t) \) satisfies the following conditions: (i) \( R(t) \) depends only on \( K_1 \), (ii) there exists an increasing positive function \( a(t) \) such that \( \int_0^\infty a(t) R(t) \, dt < \infty \) and \( \int_0^\infty \frac{1}{a(t)} \, dt < \infty \).

Under the assumption \((*)\), there exists a number \( \lambda \) and \( \psi(x) \in W^{1,0}(\mathbb{R}^n) \) such that
\[ \mathcal{L} \, V(x) + L(x, u(x)) = \lambda \]

and \(|V(x)| \leq K\) where \(K\) depends only on \(R(t)\). (For an outline of the proof, see [9, p. 504].)

**Theorem 3.** Assume (A1-9). If \(\Phi\) is not empty, then there exists a control \(\bar{u} \in \Phi\) which attains the value

\[
Q = \inf_{u \in \Phi} \lim_{T \to \infty} \frac{1}{T} E \int_0^T L(x(s), u(x(s))) ds .
\]

**Proof.** Let \(\{u_n(x)\}\) be a minimizing sequence in \(\Phi\) with corresponding \(\lambda^n\) and \(V_n\), i.e.,

\[
\lim_{n \to \infty} Q u_n = Q . \tag{21}
\]

By Lemma 10, \(Q u_n = \lambda^n\). Hence (21) means

\[
\lim_{n \to \infty} \lambda^n = Q .
\]

By the definition of \(\Phi\),

\[
\mathcal{L} u_n V_n(x) + L(x, u_n(x)) = \lambda^n \tag{22}
\]

\[
|V_n(x)| + |v_{nx}(x)| + |v_{nxx}(x)| \leq K .
\]
Hence \( \|V_n(x)\|_{W^2_2(\Omega)}^2 \leq K \) for an arbitrary bounded domain \( \Omega \). So by
the same argument as in 1) of the Proof of Theorem 1, there exists
a subsequence, indexed by \( n \), of \( V_n(x) \) which converges weakly to
\( V(x) \in W^{1,\infty}(\mathbb{R}^n) \) on each bounded domain \( \Omega \) in \( W^2_2(\Omega) \) and
\[ |V(x)| + |V_x(x)| + |V_{xx}(x)| \leq K. \]
In the Proof of Theorem 1, we have shown that there exists a control \( \bar{u} \in \mathcal{U} \) and a subsequence, indexed
by \( n \), such that \( f(x, u_n(x)) \) and \( L(x, u_n(x)) \) converges weakly to
\( f(x, \bar{u}(x)) \) and \( L(x, \bar{u}(x)) \) respectively on each bounded domain \( \Omega \)
in \( L^2_2(\Omega) \). We shall show that
\[
\mathcal{L} \bar{u} V(x) + L(x, \bar{u}(x)) = Q . \tag{23}
\]

Rewriting (22),
\[
\sum a_{ij}(x) \frac{\partial^2 V_n(x)}{\partial x_i \partial x_j} + \sum f_i(x, u_n(x)) \frac{\partial V_n(x)}{\partial x_i} + \sum g_{1i}(x) \frac{\partial V_n(x)}{\partial x_i}
+ \sum g_{2i}(x) \frac{\partial n(x)}{\partial x_i} + L(x, u_n(x)) = \lambda^n . \tag{24}
\]

First, for an arbitrary infinitely differentiable function \( \phi(x) \)
with compact support \( A \), we show that
\[
\int_{\mathbb{R}^n} a_{ij}(x) \frac{\partial^2 V_m(x)}{\partial x_i \partial x_j} \phi(x) \, dx \rightarrow \int_{\mathbb{R}^n} a_{ij}(x) \frac{\partial^2 V(x)}{\partial x_i \partial x_j} \phi(x) \, dx
\]
as \( m \to \infty \) . \tag{25}
To see this, we note that, for an arbitrary \( \psi(x) \in L^2(A) \) with compact support, the mapping

\[
f(x) \in W^2_2(A) \rightarrow \int_A \nabla^\alpha f(x) \cdot \psi(x) \, dx , \quad |\alpha| \leq 2
\]

is a continuous linear functional on \( W^2_2(A) \). So by Riesz's representation theorem, there exists a function \( \theta(x) \in W^2_2(A) \) such that

\[
\int_A \nabla^\alpha f(x) \psi(x) \, dx = (f(x), \theta(x))_{W^2_2(A)}
\]

where

\[
(f(x), \theta(x))_{W^2_2(A)} = \sum_{|\alpha| \leq 2} \int_A \nabla^\alpha f(x) \cdot \nabla^\alpha \theta(x) \, dx.
\]

Now let \( \psi(x) = a_{ij}(x) \varphi(x) \in L^2(A) \). Then

\[
\lim_{m \to \infty} \int_{\mathbb{R}^n} \nabla^\alpha V_m(x) \psi(x) \, dx = \lim_{m \to \infty} (V_m(x), \theta(x))_{W^2_2(A)}
\]

\[
= (V(x), \theta(x))_{W^2_2(A)}
\]

\[
= \int_{\mathbb{R}^n} \nabla^\alpha V(x) \psi(x) \, dx.
\]

This yields (25). Next we show that there exists a subsequence, indexed by \( m \), of \( V_n(x) \) such that
\[ \int_{\mathbb{R}^n} f_i(x, u_m(x)) \frac{\partial V_m(x)}{\partial x_i} \varphi(x) dx \to \int_{\mathbb{R}^n} f_i(x, \bar{u}(x)) \frac{\partial V(x)}{\partial x_i} \varphi(x) dx. \quad (26) \]

The left hand side can be written as

\[
\int_{\mathbb{R}^n} f_i(x, u_m(x)) \frac{\partial V(x)}{\partial x_i} \varphi(x) dx = \int_{\mathbb{R}^n} f_i(x, u_m(x)) \frac{\partial V(x)}{\partial x_i} \varphi(x) dx
\]

\[
+ \int_{\mathbb{R}^n} f_i(x, u_m(x)) \left( \frac{\partial V_m(x)}{\partial x_i} - \frac{\partial V(x)}{\partial x_i} \right) \varphi(x) dx.
\]

Since \( f_i(x, u_m(x)) \) converges weakly to \( f_i(x, \bar{u}(x)) \) on each bounded domain \( \Omega \) in \( L_2(\Omega) \), the first term goes to

\[
\int_{\mathbb{R}^n} f_i(x, \bar{u}(x)) \frac{\partial V(x)}{\partial x_i} \varphi(x) dx.
\]

For the second term, since \( f_i(x, u_m(x)) \) is bounded,

\[
\left| \int_{\mathbb{R}^n} f_i(x, u_m(x)) \left( \frac{\partial V_m(x)}{\partial x_i} - \frac{\partial V(x)}{\partial x_i} \right) \varphi(x) dx \right|
\]

\[
\leq K \left( \int_{\mathbb{R}^n} \left( \frac{\partial V_m(x)}{\partial x_i} - \frac{\partial V(x)}{\partial x_i} \right)^2 \varphi(x) dx \int \varphi^2(x) dx \right)^{1/2}
\]

where \( \mathcal{A} = \text{support of } \varphi(x) \). Now by an extension of Rellich's selection theorem (see Remark 6 which follows this proof),

\[
\int_{\mathcal{A}} \left| \frac{\partial V_m(x)}{\partial x_i} - \frac{\partial V(x)}{\partial x_i} \right|^2 \varphi(x) dx \to 0 \quad \text{as } m \to \infty.
\]

Thus we have shown (26).
Similarly we can show

\[ \int_{\mathbb{R}^n} g_{k1}(x) \frac{\partial V_m(x)}{\partial x_i} \varphi(x) dx \to \int_{\mathbb{R}^n} g_{k1}(x) \frac{\partial V(x)}{\partial x_i} \varphi(x) dx \]

as \( m \to \infty, \ k = 1, 2 \).

Combining these results with (24), we get (25). Also at the same time we showed that \( \tilde{u} \in \Phi \). Now from (25) and Lemma 10,

\[ Q = \lim_{T \to \infty} \frac{1}{T} \mathbb{E} \int_0^T L(x(s), \tilde{u}(x(s))) ds \]

and \( \tilde{u} \) is optimal.

Q.E.D.

Remark 6. The following extension of Rellich's selection theorem appears in [11, Corollary of Theorem 3.7].

"Let \( \Omega \) be an arbitrary domain. Then the identity operator \( W^{1, \text{loc}}_2(\Omega) \to W^{1, \text{loc}}_1(\Omega) \) is a compact operator, i.e., any weakly convergent sequence in \( W^{1, \text{loc}}_2(\Omega) \) is a strongly convergent sequence in \( W^{1, \text{loc}}_1(\Omega) \)."
7. COMPUTATION OF OPTIMAL CONTROLS FOR THE AVERAGE COST PROBLEM

We consider the problem (P) of minimizing the average cost criterion (6) under a general system:

\[ x(x) = x + \int_0^t f(x(s), u(t))ds + \int_0^t c(x(s)) dw(s) \]  

We assume that the control \( u(t) \) is a function of the present state of the system and takes only a finite number of actions \( a_1, a_2, \ldots \). The approach we will take is heuristic, and so we do not discuss the problem of the existence of solutions to the general system (27). First we discretize the system (27) as follows:

For a small time increment \( \Delta \), let \( x^*(k) = x(k\Delta) \). Then (27) will be approximated by

\[ x^*(k) = x^*(k-1) + f(x^*(k-1), u((k-1)\Delta)) + \sigma(x^*(k-1)) \cdot \sqrt{\Delta} \]  

where \( v_k \) is a random vector \((\alpha_1, \alpha_2, \ldots, \alpha_n)\) and each \( \alpha_i \) independently takes the values \(+1\) with probability \( 1/2 \). If we assume \( f_i(\cdot, \cdot), i = 1, \ldots, n \) takes on only integer values and also assume \( \sigma_i(\cdot) \) takes a value of the form \( m\sqrt{\Delta} \) (\( m \) integer), the state expressed in (28) takes only a mesh point of the form

\[ x^*_i(k) = m\Delta \]  

(\( m \) integer), \( i = 1, 2, \ldots, n \). Given a control \( u(x) \), it is possible to calculate the transition probabilities of the states from (28) and the original problem is approximated by the problem (P*) of minimizing the average cost.
\[ A_u = \lim_{n \to \infty} \frac{1}{n+1} \mathbb{E}_{x} \sum_{i=0}^{n} l(x^*(i), u(x^*(i))) \]

where \( x^*(i) \) is determined by (28).

Next we modify the problem to restrain \( x^* \) to a bounded region. We consider a region \( D_m = \{ x; x \in \mathbb{R}^n, |x_i| \leq m \Delta t, i = 1, 2, \ldots, n \} \). Associated with an action \( a \), we define the transition probabilities \( p_a(\cdot, \cdot) \) on the states in \( D_m \) as follows. Let \( \tilde{p}_a(\cdot, \cdot) \) denote the transition probabilities which are determined from (28) when the action \( a \) is taken. Let \( D_m = \{ s_1, s_2, \ldots, s_N \}, N = (2m)^n \). Then define \( p_a(\cdot, \cdot) \) by

\[
p_a(s_i, s_j) = \begin{cases} 
\tilde{p}_a(s_i, s_j) & \text{if } s_i \neq s_j \\
\tilde{p}_a(s_i, s_i) + (1 - \sum_{j=1}^{N} \tilde{p}_a(s_i, s_j)) & \text{if } s_i = s_j 
\end{cases}
\]

Clearly

\[
\sum_{j=1}^{N} p_a(s_i, s_j) = 1
\]

is satisfied. Associated with the transition probabilities \( p_a(\cdot, \cdot) \), each stationary control function \( u(\cdot) \) leads to a Markov chain \( x^* \) with corresponding transition probabilities and we consider the problem \((p^{**})\) of minimizing
\[ \bar{A}_u = \lim_{n \to \infty} (n+1)^{-1} \mathbb{E}_x \sum_{i=0}^{n} L(x^*(i), u(x^*(i))) \]

If \( m \) is sufficiently large, then \( D_m \) is sufficiently large and the problem (p**) should approximate the problem (p*) and hence the original problem (p).

For the problem (p**), the following linear programming approach [13] is possible. We make the simplifying assumption that all controls give rise to an irreducible Markov chain. For convenience we introduce randomized controls, and we let \( R_i(a) \) denote the probability of taking action \( a \) when the system is in state \( s_i \).

Let \( Z_i, i' = 1, \ldots, N \) be the vector of stationary probabilities corresponding to the randomized control \( R_i(a) \). Then, letting

\[ Z_i(a) = Z_i \cdot R_i(a) \]

it follows that the average cost is

\[ \sum_{i=1}^{N} \sum_{a} L(s_i, a) Z_i(a) \]

subject to the restrictions

\[ \sum_{a} Z_i(a) = \sum_{j=1}^{N} \sum_{a} Z_j(a) P_a(j, i) \]

\[ \sum_{i=1}^{N} \sum_{a} Z_i(a) = 1 \]

\[ Z_i(a) \geq 0 \]
where \( \sum_a \) represents the sum over all actions \( a_1, a_2, \ldots \).

The problem then reduces to a linear program of minimizing \( (29) \) subject to \( (30), (31) \) and \( (32) \). It can be proved [13] that the minimal average cost can be achieved by a nonrandomized control. It was also proved in [16] that the solution of the problem \( (29-32) \) obtained by the simplex algorithm has the property that for each \( i \) there is at most one \( a \) for which \( Z_i(a) > 0 \). Thus, under the simplifying assumption that any control give rise to an irreducible Markov chain, the simplex algorithm always yields a nonrandomized control.

In this section we proposed a linear programming approach to obtain an approximate optimal solution to the minimization problem of the long-term average cost criterion with finite number of actions. As is well known, the computation time of linear programming depends on the number of the constraints. Since the number of the constraints in the problem \( (p^{**}) \) is the number of meshpoints in \( D_m \), the number of the constraints in \( (p^{**}) \) rapidly becomes quite large as the dimension of the state space \( \mathbb{R}^n \) increases. As a matter of fact if \( n \geq 3 \), from the viewpoint of computational time, the approach by \( (p^{**}) \) will be infeasible without some more effective computational device.

For the computational aspect of the long term average cost problems considered here, several approaches were tried. For example, in [7], Gimon considered an optimal control of inventory with a long term average cost criterion and, by finding specific solutions to a partial differential equation which is similar to the one given in
Lemma 12, he obtained approximate optimal policies and bounds on the optimal average cost. Whether his method is applicable to the satellite control problem in Section 3 is an unsolved problem.

In [3], a computational method for obtaining a suboptimal control is discussed.
REFERENCES


