ON MODELS OF ECONOMIC GROWTH WITH RANDOM ELEMENTS

BY

T. DAR, A. RAZIN, J. YAHAV

TECHNICAL REPORT NO. 46
NOVEMBER 6, 1972

PREPARED UNDER THE AUSPICES
OF
NATIONAL SCIENCE FOUNDATION GRANT GP-27550

DEPARTMENT OF STATISTICS
STANFORD UNIVERSITY
STANFORD, CALIFORNIA
ON MODELS OF ECONOMIC GROWTH WITH RANDOM ELEMENTS

by

T. Dar, A. Razin, J. Yahav

TECHNICAL REPORT NO. 46
November 6, 1972

PREPARED UNDER THE AUSPICES
OF

NATIONAL SCIENCE FOUNDATION GRANT GP-27550

DEPARTMENT OF STATISTICS
STANFORD UNIVERSITY
STANFORD, CALIFORNIA
ON MODELS OF ECONOMIC GROWTH WITH RANDOM ELEMENTS

by

T. Dar, A. Razin, J. Yahav

0. Introduction

The stochastic dynamic programming model of Economic Growth in the one-commodity case was considered by several authors: Phelps [4], Levhari and Srinivasan [3] (L & S for short), Samuelson [5], Brock and Mirman [1].

In this paper we elaborate on some aspects of this model that did not get much attention by previous authors.

We use the set-up given by L & S [3], namely:

At each period \( t \), an individual (or a planner) has the option of either consuming all his capital, \( k_t \), or investing part of it. Denoting the consumption at time \( t \) by \( c_t \), investment would be \( k_t - c_t \) (we assume \( 0 \leq c_t \leq k_t \) throughout this paper). This investment will result in his capital in the next period \( k_{t+1} \) becoming \( f_{r_{t+1}}(k_t - c_t) \) where for any given \( r_t \), \( f_{r_t}(\cdot) \) would be called the production function. We take \( r_t \) to be random. The planner has an instantaneous utility function \( U(\cdot) \), where the argument is the consumption at time \( t \).

The objective of the planner is to maximize

\[
E\left( \sum_{t=0}^{\infty} \beta^t U(c_t) \right)
\]
where $\beta$ is the subjective discount factor (we assume $0 < \beta < 1$) and $E(\cdot)$ is the expectation operator.

We assume throughout this paper unless otherwise stated that $f_t(\cdot)$ and $U(\cdot)$ are twice differentiable and that

$$
(0.2) \quad U'(\cdot) > 0, \quad U''(\cdot) < 0 \quad \lim_{x \to 0} U'(x) = \infty
$$

$$
(0.3) \quad f'_r(\cdot > 0, \quad f''_r(\cdot) \leq 0 \quad \text{for each } r_t \text{ a.s.}
$$

$$
(0.4) \quad \{r_t : t = 1, 2, \ldots\} \text{ are independent and identically distributed elements (for short i.i.d.)}
$$

It can be shown that under these assumptions a stationary solution exists and is given by a consumption function that for each $k$ strictly satisfies the constraints, namely

$$
(0.5) \quad 0 < c(k) < k .
$$

In Section 1 we consider the limiting behavior of $k_t$ in two special cases which are not covered in Brock and Mirman [1]. The cases are

(i) $$
(0.6) \quad f_r(x) = rx, \quad U(x) = \frac{x^{1-\alpha}}{1-\alpha}, \quad \alpha > 0, \quad r \geq 0
$$

(ii) $$
(0.7) \quad U(x) = \log x, \quad f_r(x) = rx^\alpha, \quad 0 < \alpha \leq 1, \quad 0 \leq r .
$$

In these cases the solution is $c(k) = \lambda k$ and we are able to express $k_t$ in an explicit form which enables us to investigate directly the behavior of $k_t$.

In Section 2 we consider the problem of random "death" and its effect on the optimal consumption function. We will develop the model
in the above set-up as follows:

\[ U(\cdot) \text{ satisfies (0.2) and} \]

\[
(0.8) \quad f_{r,T}(\cdot):= \begin{cases} 
    f_r(\cdot) & t \leq T \\
    0 & t > T 
\end{cases}
\]

where \( T \) is a random time and \( f(\cdot) \) satisfies (0.3).

A similar model was considered by Yaari [6]. We consider here the effect of "age" on the optimal consumption.

In Section 3 we consider aggregate consumption of individuals where the individuals operate in a model given by (0.6) and (0.7). We assume that different individuals differ in their initial capital \( k_0 \) and by their subjective discount factor \( \beta \). We compare the optimal consumption behavior to the consumption in an egalitarian society.

1. Limiting Behaviour of \( k_t \)

The limiting behavior of \( k_t \) is of interest both when we consider the optimal behavior of the individual and even more so when we consider aggregations of individuals. The random "death" model makes the capital \( k_T \) (\( T \) is the random death time) the amount that is left over to the new generation, so when we consider \( T \) large the limiting behavior of \( k_t \) plays an important role.

In the deterministic case, under some assumptions, there exists a steady state and \( k_t \rightarrow k^* \) as \( t \rightarrow \infty \) where \( k^* \) is the steady state. We are interested in investigating the problem in the stochastic case.

We will consider two special cases which are not covered by Brock and Mirman [1].
Case i: We repeat (0.6) of the introduction

\[ f_r(x) = rx, \quad 0 \leq r \]

\[ U(x) = \frac{x^{1-\alpha}}{1-\alpha}, \quad 0 < \alpha \quad 1 \neq \alpha \]

\[ r_i \text{ are i.i.d., } E[r^{1-\alpha}] < \frac{1}{\beta}, \quad P(r \geq 0) = 1. \]

L & S [3] considered the optimization problem here and got the following solution: the optimal consumption function is linear, i.e.,
\[ c(k) = \lambda k \] where:

\[ \lambda = 1 - \beta^\alpha E^\alpha \left( r^{1-\alpha} \right). \]

This solution determines \( k_t \) as follows:

\[ k_t = (1-\lambda)^t k_0 T \prod_{i=1}^t r_i. \]

In order to analyze \( k_t \) we will take logs on both sides and we get

\[ \log k_t = \log k_0 + \sum_{i=1}^{t} \log r_i + \log (1-\lambda). \]

By a theorem of Chung and Fuchs [2] the behavior of \( \log k_t \) depends only on \( E[\log r + \log (1-\lambda)] \), namely

\[ E[\log r] + \log (1-\lambda) < 0 \iff \log k_t \to -\infty \quad \text{a.s.} \]

\[ E[\log r] + \log (1-\lambda) > 0 \iff \log k_t \to \infty \quad \text{a.s.}. \]

Using (1.4) the left-hand side of (1.7) is equivalent to

\[ \alpha E[\log r] + \log E[r^{1-\alpha}] < -\log \beta. \]
On the other hand, the right-hand side of (1.7) is equivalent to

\[(1.10) \quad k_t \to 0 \ a.s. \]

Combining (1.7), (1.9), and (1.10) we get

\[(1.11) \quad k_t \to 0 \ a.s. \iff \alpha E[\log r] + \log E[r^{1-\alpha}] < -\log \beta \]

(1.3) and (1.11) are consistent if

\[(1.12) \quad E[\log r] < 0 \]

and hence

\[(1.13) \quad E[\log r] < 0 \implies k_t \to 0 \ a.s. \]

Since \( E[r] < 1 \Rightarrow E[\log r] < 0 \), hence

\[(1.14) \quad E[r] < 1 \implies k_t \to 0 \ a.s. \]

In an analogous way we get that the left-hand side of (1.8) is equivalent to

\[(1.15) \quad \alpha E[\log r] + \log E[r^{1-\alpha}] > -\log \beta \]

On the other hand, the right-hand side of (1.8) is equivalent to

\[(1.16) \quad k_t \to \infty \ a.s. \]

Combining (1.8), (1.15) and (1.16) we get

\[(1.17) \quad k_t \to \infty \ a.s. \iff \alpha E[\log r] + \log E[r^{1-\alpha}] > -\log \beta \]
(1.3) and (1.17) imply

(1.18) \[ k_t \to \infty \text{ a.s. } \Rightarrow E[lg r] \geq 0 \]

and hence

(1.19) \[ k_t \to \infty \text{ a.s. } \Rightarrow E[r] \geq 1 \]

A special case is the case of equality, i.e.,

(1.20) \[ \alpha E[lg r] + lg E[r^{1-\alpha}] = -lg \beta \]

In this case the almost sure behavior of $k_t$ is that of oscillation on the ray $(0, \infty)$. If we assume that

(1.21) \[ V[lg r] < \infty \]

we can use the Central Limit Theorem to conclude

(1.22) \[ \frac{lg k_t}{\sqrt{t V[lg r]}} \xrightarrow{D} N(0,1) \]

where $\xrightarrow{D}$ denotes convergence in distribution and $N(0,1)$ denotes the standard normal distribution.

(1.20) and (1.3) imply

(1.23) \[ E[lg r] > 0 \]

and hence (1.22) and (1.3) imply

(1.24) \[ E[r] > 1 \]
Case (ii): We repeat (0.7)

\[(1.25) \quad U(x) = \log x\]

\[(1.26) \quad f_{r,\alpha}(x) = rx^{\alpha}\]

\[(1.27) \quad \{r_i\} \text{ and } \{\alpha_i\} \text{ are a sequence of i.i.d. random variables, and we assume the sequence } \{r_t\} \text{ to be independent of the sequence } \{\alpha_t\}\]

\[(1.28) \quad P(0 < \alpha < 1) = 1, \quad P(\alpha \geq 0) = 1 .\]

We compute the optimal solution following the steps used by L & S to solve case (ii). The solution satisfies

\[(1.29) \quad U'(c(k)) = \beta E[U'(c(f_{r,\alpha}(k-c(k)))) - f_{r,\alpha}(k-c(k))]\]

which reduces under our assumptions to

\[(1.30) \quad \frac{1}{c(k)} = \beta E \left[ \frac{a(k-c(k))^{\alpha-1}}{c(r(k-c(k))^{\alpha})} \right] .\]

\[c(k) = \lambda \cdot k \text{ satisfies (1.30) where}\]

\[(1.31) \quad \lambda = 1 - \beta E[\alpha] .\]

Now we can write explicitly the expression for \(k_n\)

\[(1.32) \quad k_n = \prod_{i=1}^{k_n} (r_i^{\alpha_j}) \cdot (1-\lambda)^{\sum_{i=1}^{n} (\alpha_j)} \cdot \prod_{i=1}^{n-1} \alpha_j = 1 \quad \text{and hence}\]

\[\prod_{j=n+1}^{n} \alpha_j = 1 \quad \text{and hence}\]
(1.33) \[ \log k_n = \sum_{i=1}^{n} (\prod_{j=i+1}^{n} \alpha_j) \log r_i + \sum_{i=1}^{n} (\prod_{j=1}^{i} \alpha_j) \log (1-\lambda) \]
\[ + (\prod_{i=1}^{n} \alpha_i) \log k_0. \]

In order to determine the behavior of \( \log k_n \) we will first prove a lemma.

**Lemma 1.1:**

Let \( \{X_1: i=1,2,\ldots\} \) be i.i.d. and let \( \{Y_1: i=1,2,\ldots\} \) be i.i.d. where the sequence \( \{Y_i\} \) is independent of the sequence \( \{X_i\} \) define:

(1.34) \[ Z_n = \sum_{i=1}^{n} Y_i \prod_{j=i+1}^{n} X_j \quad (\prod_{j=n+1}^{n} X_j \equiv 1). \]

If \( P(0 < X_1 \leq 1) = 1 \) and \( E[X] < 1 \), then \( Z_n \) converges in distribution.

We will prove the lemma here under the additional assumption that \( E[|Y|] < \infty \).

**Proof:**

Let

(1.35) \[ \tilde{Z}_n = \sum_{i=1}^{n} Y_i \prod_{j=1}^{i-1} X_j. \]

Note that \( \tilde{Z}_n \) and \( Z_n \) have the same distribution. We will show that \( \tilde{Z}_n \to \tilde{Z} \) a.s. where \( \tilde{Z} \) is a random variable.

Consider

(1.36) \[ \tilde{Z}_{n+k} - \tilde{Z}_n = \sum_{i=n+1}^{n+k} Y_i \prod_{j=1}^{i-1} X_j = (\prod_{j=1}^{n+k} Y_j) \sum_{i=n+1}^{n+k} Y_i \prod_{j=n+1}^{i-1} X_j. \]

Let

(1.37) \[ W_{n,k} = \tilde{Z}_{n+k} - \tilde{Z}_n. \]
We will show that $W_{n,k}$ converges to zero a.s. uniformly in $k$

\[(1.38) \quad P\left(\frac{|W_{n,k}|}{\varepsilon} \geq \varepsilon\right) \leq \frac{E[|W_{n,k}|]}{\varepsilon} \leq \frac{E[|Y|]}{\varepsilon(1-E[X])}\]

and hence

\[(1.39) \quad \tilde{Z}_n \rightarrow \tilde{Z} \text{ a.s.} \]

We have to show that

\[(1.40) \quad \lim_{z \rightarrow \infty} P(\tilde{Z} \leq z) = 1 \quad \text{and} \quad \lim_{z \rightarrow -\infty} P(\tilde{Z} \leq z) = 0\]

\[(1.41) \quad P(\tilde{Z} \leq z) = P(\sum_{i=1}^{\infty} Y_i \prod_{j=1}^{i-1} X_j \leq z)\]

and hence for $z > 0$

\[(1.42) \quad P(\tilde{Z} > z) \leq P(\sum_{i=1}^{\infty} |Y_i| \prod_{j=1}^{i-1} X_j > z) \leq \frac{E[|Y|]}{z(1-E[X])}\]

and for $z < 0$

\[(1.43) \quad P(\tilde{Z} \leq z) = P(-\tilde{Z} \geq -z) \leq \frac{E[|Y|]}{-z(1-E[X])}\]

so that (1.42) and (1.43) imply (1.40). Since $\tilde{Z}_n$ and $Z_n$ have the same distribution, we can conclude that the sequence $\tilde{Z}_n$ has a limiting distribution which is equal to the distribution of $\tilde{Z}$.

Applying Lemma 1.1 to the first and second sums in the right hand of (1.33), and noticing that the third term converges to zero, we get
that \( \log k_n \) converges in distribution and hence so does \( k_n \).

Lemma 1.1 does not cover the case where \( r \) might take the value zero with a positive probability. In such a case it is easily seen from (1.32) that \( k_n \to 0 \) a.s.

Lemma 1.1 also does not cover the case where \( \alpha \) is equal to 1. Almost surely in this case we are back in our analysis of Case (i), and we get

\[
(1.44) \quad k_n \to 0 \text{ a.s. } \iff E[\log r] < -\log \beta.
\]

Hence

\[
(1.45) \quad E[r] < \frac{1}{\beta} \Rightarrow k_n \to 0 \text{ a.s.}
\]

\[
(1.46) \quad k_n \to \infty \text{ a.s. } \iff E[\log r] > -\log \beta
\]

hence

\[
(1.47) \quad k_n \to \infty \text{ a.s. } \Rightarrow E[r] > \frac{1}{\beta}.
\]

If \( E[\log r] = -\log \beta \) and \( V(\log r) < \infty \),

\[
(1.48) \quad \frac{\log k_n}{\sqrt{nV(\log r)}} \xrightarrow{D} N(0,1).
\]

2. **Random "Death" and Its Effect on the Consumption Function**

In this section we will develop a model of optimal consumption under uncertainty of lifetime. We will assume

\[
(2.1) \quad U(0) = 0, \ U'(\cdot) > 0, \ U''(\cdot) < 0, \ U'(0) = \infty
\]

\[
(2.2) \quad \frac{f_{r,T}(\cdot)}{f_r(T)} = \begin{cases} f_r(\cdot) & t < T \\ 0 & t > T \end{cases}
\]

10
where \( f \) satisfies

\[
(2.3) \quad f_{r_t}(0) = 0 \quad f_{r_t}'(\cdot) > 0 \quad f_{r_t}''(\cdot) \leq 0 \quad \text{for each} \quad r_t,
\]

\[\{r_t : t = 1, 2, \ldots\} \text{ are i.i.d., and} \ T, r_t \text{ are independent.}\]

It is useful to start the analysis by considering the optimal model in which the horizon is not a random variable and the discount factor depends on time, i.e., we have a sequence \((\beta_1, \beta_2, \ldots)\) where \( \beta_t \) is the discount factor in period \( t \). Later we will show that the random "death" model can be reduced to this model.

The problem is:

\[
(2.4) \quad V_{\beta}(k_0) = \max \mathbb{E} \sum_{t=0}^{\infty} \left( \pi \beta_t \right) U(c_t(k_t)), \quad \beta_0 = 1, \quad 0 \leq \beta_t \leq 1, \quad \beta = (\beta_1, \beta_2, \ldots)
\]

with the stochastic constraints \( 0 \leq c_t(k_t) \leq k_t, k_{t+1} = f_{r_{t+1}}(k_t - c_t(k_t)) \).

We first consider the finite horizon case.

Let \( V^N_{\beta} \) denote the maximized expected value of the sum of discounted utilities, and let \( C^N_{\beta}(k) \) denote the optimal initial consumption, when the length of the horizon is \( N \). From dynamic programming we have

\[
(2.5) \quad V^N_{\beta}(k_0) = U(C^N_{\beta}(k_0)) + \beta_1 \mathbb{E} V^{N-1}_{\beta}(f_{r_1}(k_0 - c^N_{\beta}(k_0)))
\]

where \( \beta = (\beta_2, \beta_3, \ldots, \beta_N) \).

Following L & S [3] we can characterize the optimum solution by
(2.6) \[ U'(c^N_{\beta}(k)) = \beta_1 E(\frac{N-1}{\beta}) [ f_r(k_{\beta} - c^N_{\beta}(k)) ] f'_r(k_{\beta} - c^N_{\beta}(k)) \]

(2.7) \[ \frac{d}{dk} V^N_{\beta}(k) = U'(c^N_{\beta}(k)) \]

where prime denotes derivative.

It is well known that \( V^N_{\beta}(k) \) is concave and differentiable.

We will prove a lemma for the finite horizon case.

**Lemma 2.1:** Let \( \beta = (\beta_1, \beta_2, \ldots, \beta_N) \) and \( \beta^* = (\beta_1^*, \beta_2^*, \ldots, \beta_N^*) \) be two sequences of discount factors, and assume \( \beta_t^* \geq \beta_t \) for \( t = 1, 2, \ldots, N \) then \( c^N_{\beta^*}(k) \leq c^N_{\beta}(k) \).

**Proof:** (by induction on \( N \)):

For \( N = 0 \), \( c^0_{\beta}(k) = c^0_{\beta^*}(k) = k \). Assume now that for \( N-1 \) \( c^{N-1}_{\beta}(k) \geq c^{N-1}_{\beta^*}(k) \) for all \( k \geq 0 \).

To show that the same inequality holds for \( N \), assume that there exists a \( k \) such that \( c^N_{\beta}(k) < c^N_{\beta^*}(k) \).

Consider now the two sides of equation (2.6) when \( \beta^* \) substitutes for \( \beta \). The right-hand side increases for the following reasons.

For all \( r \)

(2.8) \[ f'_r(k - c^N_{\beta^*}(k)) \geq f'_r(k - c^N_{\beta}(k)) \]

by hypothesis. Also, \( f_r(k - c^N_{\beta^*}(k)) < f_r(k - c^N_{\beta}(k)) \) for all \( r \). Hence

(2.9) \[ (V^N_{\beta})'(f_r(k - c^N_{\beta^*}(k))) \leq (V^N_{\beta})'(f_r(k - c^N_{\beta}(k))) \]
by the concavity of $v_{\beta}^{N-1}$. Denote

\[(2.10) \quad f_r(k-c_{\beta}(k)) = k_1,\]

then from (2.7) we have

\[(2.11) \quad \frac{d}{dk} v_{\beta}^{N-1}(k_1) = U'(c_{\beta}^{N-1}(k_1)) .\]

Using the induction hypothesis for $N-1$, the concavity of $U$ and (2.7) we get

\[(2.12) \quad U'(c_{\beta}^{N-1}(k_1)) \leq U'(c_{\beta^*}^{N-1}(k_1)) = \frac{d}{dk} v_{\beta^*}^{N-1}(k_1) .\]

Observing (2.9) - (2.12) we have

\[(2.13) \quad (v_{\beta}^{N-1})'(f_r(k-c_{\beta}(k))) \leq (v_{\beta^*}^{N-1})'(f_r(k-c_{\beta^*}(k))) \quad \text{for every} \quad r .\]

Since $\beta^*_1 \geq \beta_1$, (2.8) and (2.13) imply

\[(2.14) \quad \beta_1 E (v_{\beta}^{N-1})'(f_r(k-c_{\beta}(k)))f_r'(k-c_{\beta}(k)) \leq \beta^*_1 E (v_{\beta^*}^{N-1})'(f_r(k-c_{\beta^*}(k)))f_r'(k-c_{\beta^*}(k)) .\]

In other words, the right-hand side of (2.6) increases (or remains unchanged) upon substituting $\beta^*$ for $\beta$. On the other hand, the strict concavity of $U$ implies that the left-hand side of (2.6) decreases, thus equality (2.6) is violated.

Therefore we must have for all $k$
\[
\hat{c}_E^N(k) \geq c_{E*}^N(k) .
\]

This completes the proof of the lemma.

In the infinite horizon case we denote \( c_\beta(k) \) - the optimal initial consumption and we assume that \( V_\beta(k) \) in (2.4) is bounded. The counterpart of Lemma 2.1 in this case is the following lemma.

**Lemma 2.2:** Let \( \beta = (\beta_0, \beta_1, \ldots) \) and \( \beta^* = (\beta^*_0, \beta^*_1, \ldots) \) be two infinite sequences of discount factors and assume \( \beta^*_t \geq \beta_t \) for \( t = 1, 2, 3, \ldots \), then \( c_{\beta^*}(k) \leq c_{\beta}(k) \).

**Proof:** First observe that \( c_{\beta}^N(k) \) (the optimal initial consumption for the \( N \)-period problem) is monotonically decreasing in \( N \). To see this consider the sequences \( (\beta_1, \beta_2, \ldots, \beta_N, 0) \) and \( (\beta_1, \beta_2, \ldots, B_N, B_{N+1}) \). Since the second sequence is larger than or equal to the first one elementwise, it follows from Lemma 2.1 that

\[
c_{(\beta_1, \ldots, \beta_{N+1})}^N(k) \leq c_{(\beta_1, \beta_2, \ldots, \beta_N, 0)}^N(k) = c_{(\beta_1, \ldots, \beta_N)}^N(k) .
\]

This implies that \( \overline{c}_\beta(k) = \lim\limits_{N \to \infty} c_{\beta}^N(k) \) exists.

Since \( V_\beta(k) \) is monotonically increasing in \( N \) and is bounded by \( V_{\beta^*}(k) \) (the maximum of the infinite horizon problem) it has a limit \( \overline{V}_\beta(k) \).

\[(2.15) \quad \overline{V}_\beta(k) \leq V_\beta(k)\]

On the other hand, let \( \{c_t\}_{t=0}^\infty \) be the stochastic solution of the infinite horizon case. Thus \( \{c_t\}_{t=0}^N \) (the truncated sequence) is a
feasible stochastic solution for any $N$, which implies that

\begin{equation}
\mathcal{P}(k) \geq \mathcal{P}(k) \geq \sum_{t=0}^{N} \prod_{i=0}^{t} \beta_i U(c_t).
\end{equation}

Taking limits in both sides of (2.16) yields

\begin{equation}
\overline{V}(k) \geq \overline{V}(k).
\end{equation}

Therefore (2.15) and (2.17) imply

\begin{equation}
\mathcal{P}(k) = \overline{V}(k).
\end{equation}

To show that $\overline{c}(k)$ is an optimal initial consumption in the
infinite horizon case, we consider equation (2.5). It is legitimate
to take limits in both sides of equation (2.5), since $\mathcal{P}(k)$ and
$c(k)$ have limits, $U_i$ is continuous, and $\mathcal{P}(k-c(k))$ converges
monotonically (thus the limit of expected value is equal to expected
value of the limit).

Thereby using (2.18), we get

\begin{equation}
\mathcal{P}(k) = U(\overline{c}(k)) + \beta E \mathcal{P}(fr(k-c(k))).
\end{equation}

Equation (2.19) implies that $\overline{c}(k) = c(k)$ is an optimal consumption
in the infinite horizon problem.

From Lemma 2.1 we know that $c^N(k) \leq c_N(k)$ for all $N$, thus
$c_{\beta^*}(k) \leq c(k)$. This completes the proof.

We return now to the model of optimal consumption under uncertainty
of lifetime given by (2.1)-(2.3).
Since the sum of the discounted utilities is now a random variable, we will maximize the expectation of this variable over the distribution of $T$ and $r_t$. We have

\[
(2.9) \quad V^T(k) = \max_{\{c_t\}} \mathbb{E}_{T,r} \sum_{t=0}^{\infty} \beta^t U(c_t(k_t))
\]

subject to $0 \leq c_t(k_t) \leq k_t$ for all $t$, and $k_{t+1} = f(r_{t+1}, T)(k_t - c_t(k_t))$.

Let $Z(k_0)$ denote the expected value for the discounted sum of utilities for a feasible stochastic solution $\{c_t\}$, since $r_t$ and $T$ are independent.

\[
(2.10) \quad Z(k_0) = \sum_{N=0}^{\infty} P(T=N) \mathbb{E}_r \sum_{t=0}^{N} \beta^t U(c_t(k_t))
\]

Since $U(\cdot) \geq 0$, we can interchange the order of summation to get

\[
(2.11) \quad Z(k_0) = \mathbb{E}_r \left[ \sum_{t=0}^{\infty} \sum_{N=t}^{\infty} P(T=N) \beta^t U(c_t(k_t)) \right] = \mathbb{E}_r \sum_{t=0}^{\infty} P(T \geq t) \beta^t U(c_t(k_t))
\]

where $P(T \geq t) = \sum_{N=t}^{\infty} P(T=N)$ is the probability of surviving until $t$. Define now the probability of death at time $t$ conditional on surviving $t-1$ periods by $P_t$.

\[
P_t = P(T = t | T \geq t)
\]

therefore

\[
P(T \geq t) = (1-P_0)(1-P_1) \cdots (1-P_{t-1})
\]
We can see then that the model of random "death" is equivalent to the infinite horizon model with a sequence of discount factors \( \beta_0, \beta_1, \ldots \) which depends on time in the following way:

\[
\beta_t = \beta(1-P_{t-1}), \quad t \geq 1, \quad \beta_0 = 1.
\]

Substituting (2.12) into (2.11), we get

\[
Z(k_0) = \mathbb{E}_r \sum_{t=0}^{\infty} \left( \prod_{i=0}^{t} \beta_i \right) U(c_t).
\]

In the standard infinite horizon model (\( \beta_t = \beta \)), a stationary solution is obtained. In the random "death" case, the solution, in general, will not be stationary except in the special case in which the conditional probability of death time does not depend on age, i.e., \( P_t = P, \ t = 1, 2, \ldots \). Then (2.12) reduces to

\[
\beta_t = \beta(1-P)
\]

which leads to the standard infinite horizon model with a smaller discount factor.

We examine now the effect of "age" on optimal consumption. In order to investigate the pure effect of aging, we will analyze optimal consumption as a function of age holding constant the amount of capital.

Assume that the conditional probability of death increases with age, which means

\[
P_t \uparrow \text{ with } t.
\]
Then, \( \beta_t = \beta(1 - P_t) \) decreases with \( t \). Denote \( c_t(k) \) the optimal consumption which corresponds to the sequence of discount factors \( (\beta_t, \beta_{t+1}, \ldots) \). A straightforward application of Lemma 2.2 implies that the optimal consumption functions \( c_t(k) \) are increasing with \( t \).

We therefore conclude that when age increases the conditional probability of death, it will increase consumption (holding constant the amount of capital).

3. Aggregate Consumption

In Sections 1 and 2 we dealt with an individual (or a planner), whereas in this section we shall consider a "community".

We usually consider \( \beta \) to be subjective to an individual; thus for the community we have a distribution function over \( \beta \). We will simplify our analysis by assuming \( U(\cdot) \) the utility function to be the same over the individuals. Our purpose is to consider different community policies and compare the results.

We will assume first that we are in case i of Section 1 and that (1.1), (1.2) and (1.3) hold. Suppose that the community is using the services of a planner and this community is an egalitarian one; i.e., the constraints imposed on the planner require equal consumption. Therefore the objective function is given by

\[
(3.1) \quad \max_{t=0}^{\infty} E[\sum_{t=0}^{\infty} \beta_t U(c_t)]
\]

where the expectation is taken over the production function and the random variable \( \beta \). (In this case \( \beta \) is independent of \( r \)).
The constraints are given by

\[(3.2) \quad 0 \leq c_t(m_t) \leq m_t\]

where \(m_t\) is the average capital of the community and \(c_t(m_t)\) is the consumption of each individual at time \(t\).

For a stationary policy with \(c_t(\cdot) = c(\cdot)\) and initial average capital \(m\), we have

\[(3.3) \quad V(m) = U[c(m)] + \sum \{E[\beta_1 V(f_r(m - c(m)))]\}P(\beta_1)\]

and hence

\[(3.4) \quad V(m) = U[c(m)] + \bar{\beta} E[V(f_r(m - c(m)))]\]

where \(\bar{\beta}\) is the average \(\beta\) in the community.

This model is reduced to the previous one discussed in Section 1 and hence the solution is given by

\[(3.5) \quad c_t(m) = \lambda m_t\]

where

\[(3.6) \quad \lambda = 1 - \bar{\beta} \frac{1}{\alpha} \frac{1}{E[r^{1-\alpha}]}\]

We will compare \(c_t(m)\) given in (3.5) to the average consumption if the community did not impose the egalitarian consumption constraint. In the latter case each individual will behave optimally according to his \(\beta\) and also his \(k\) would differ. The average consumption of the community can be defined by

\[(3.7) \quad \tilde{c}_t(\tilde{k}) = \sum \lambda_0 \frac{k_i}{\beta_i} P(\beta_i, k_i)\]

where \(\tilde{k}\) denotes the amount of capital of the community, \(\tilde{c}\) denotes
the average consumption and \( P(\beta_i, k_i) \) is the weighting of the individual \( i \). Using (1.4) we get

\[
\tilde{c}_t(\tilde{r}) = \sum_{i=1}^{1/\alpha} \sum_{i=1}^{1/\alpha} \left( 1 - \beta_i \right) E \left[ r_i^{1-\alpha} \right] k_i P(\beta_i, k_i).
\]

Equation (3.8) is now compared with (3.5), and we have

\[
\tilde{c}_t(\tilde{r}) - c_t(m) = \sum_{i=1}^{1/\alpha} \left[ r_i^{1-\alpha} \right] (\beta_i - m) - \sum_{i=1}^{1/\alpha} k_i P(\beta_i, k_i).
\]

If \( \alpha < 1 \) and we assume that the \( k_i \) are the same over individuals, i.e., \( k_i = m \), we see that

\[
\tilde{c}_t(\tilde{r}) - c_t(m) < 0.
\]

If \( \alpha > 1 \) and we assume that \( k_i = m \), then

\[
\tilde{c}_t(\tilde{r}) - c_t(m) > 0.
\]

In case that \( \alpha < 1 \) (3.10) indicates that the egalitarian consumption is larger.

We note that individuals with high \( \beta \) consume less than individuals with small \( \beta \) having the same \( k \), so as a result \( k_i \) and \( \beta_i \) may be expected to be positively correlated (assuming that the "starting point" of all individuals was the same). In this case we can conclude that

\[
\sum_{i=1}^{1/\alpha} \beta_i^{1/\alpha} k_i P(\beta_i, k_i) > \left( \sum_{i=1}^{1/\alpha} \beta_i^{1/\alpha} P(\beta_i) \right) \sum_{i=1}^{1/\alpha} k_i P(k_i) = m \sum_{i=1}^{1/\alpha} \beta_i^{1/\alpha} P(\beta_i)
\]

and so (3.10) holds also without assuming \( k_i = m \). Unfortunately we cannot conclude that (3.11) holds universally.

Analyzing case (ii) of Section 1 in the same fashion as above, we get
\[ (3.15) \quad \tilde{c}_t(\tilde{k}) - c_t(m) = 0. \]

Otherwise, with a positive correlation between \( \beta_i \) and \( k_i \)

\[ (3.16) \quad \tilde{c}_t(\tilde{k}) - c_t(m) = E\alpha[\tilde{p}m - \sum_i \beta_i k_i P(\beta_i, k_i)] < 0. \]

In this case (3.16) indicates that egalitarian consumption is larger than the aggregate consumption in a community which does not impose the equal consumption constraint.
References


