NOTE ON RELATIVE EFFICIENCY OF TESTS

BY

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Colin R. Blyth

University of Illinois and Stanford University

1. SUMMARY. This note is concerned with possible definitions of relative efficiency for two sequences of tests of the same hypothesis. For two examples of one kind of definition, relative efficiencies of the Student test and sign test against normal alternatives are calculated for fixed sample size and asymptotically.

2. INTRODUCTION. Consider the following problem of relative efficiency of tests. Experiments $X_1, X_2, \ldots$ and two sequences $\{A_n(X_1, \ldots, X_n)\}$, $\{A^*_n(X_1, \ldots, X_n)\}$ of level $\alpha$ tests are available for testing the same hypothesis. We must decide whether to use an $A$ test or an $A^*$ test. Commonly one sequence, say the $A^*$'s, gives better power for given sample size, but for some reason such as wider validity we may prefer one of the "less efficient" $A$ tests.

The general decision formulation for this problem would use three loss functions (i) cost of experimentation (ii) loss from wrong decisions (iii) disadvantages of using $A^*$. The usual kinds of decision problems for three loss functions could then be discussed. In practice (iii) is hard to assess and there is no natural comparability between (i) and (ii). So what is usually done is to consider (i) and (ii) only, and having required a bound on one of them, to decide whether the decrease in the other is enough to compensate for the disadvantages of using $A^*$ instead of $A$. More specifically, the following two types of problems
(a). **Fixed power requirement problems.** For a given power requirement, shall we use \( A_n \) or \( A^*_n \)? Here \( n \) and \( n^* \) are the smallest sample sizes for which the respective kinds of tests satisfy the given power requirement. Some function \( K(n,n^*) \) such as \( C(n) - C(n^*) \) or \( 1 - C(n^*)/C(n) \) is chosen as measuring our loss (extra experimentation cost) from using \( A_n \) instead of \( A^*_n \). If \( K(n,n^*) \) is small enough we will prefer to use \( A_n \) because of the advantages (iii) of \( A \) tests. If the given power requirement is a function of an unknown parameter \( \theta \), the loss \( K(n,n^*) \) will also be a function of \( \theta \) and so cannot be used directly for deciding between \( A_n \) and \( A^*_n \). Some measure of loss not dependent on \( \theta \) is needed. One natural choice is the worst possible loss \( \sup_\theta K(n,n^*) \). (Weighted averages over \( \theta \) and limits over particular sequences of \( \theta \)'s have also been used.) Asymptotic behavior of \( K(n,n^*) \) and \( \sup_\theta K(n,n^*) \) can be investigated for sequences of power requirements \( \theta \) forcing \( n \to \infty \) and \( n^* \to \infty \). The particular choice \( K(n,n^*) = 1 - n^*/n \) (with \( n^*/n \) being called the efficiency of \( A \) relative to \( A^* \)) and its asymptotic properties has been of wide interest [1], [2], [3], [4].

(b). **Fixed sample size problems.** For a given sample size \( n \) shall we use \( A_n \) or \( A^*_n \)? Let \( \beta_n \) be the power of \( A_n \) and \( \beta^*_n \) the power of \( A^*_n \). Some function \( L(\beta_n,\beta^*_n) \) such as \( \beta^*_n - \beta_n \) or \( 1 - \beta_n/\beta^*_n \) is chosen as measuring our loss (extra wrong decisions) from using \( A_n \) instead of \( A^*_n \). If \( L(\beta_n,\beta^*_n) \) is small enough we will prefer to use \( A_n \) because of the advantages (iii) of \( A \) tests. If the powers \( \beta_n \) and \( \beta^*_n \) are functions
of an unknown parameter $\theta$, the loss $L(\beta_n, \beta^*_n)$ will also be a function of $\theta$ and so cannot be used directly for deciding between $A_n$ and $A^*_n$. Some measure of loss not dependent on $\theta$ is needed. One natural choice is the worst possible loss $\sup_{\theta} L(\beta_n, \beta^*_n)$. Asymptotic behavior of $L(\beta_n, \beta^*_n)$ and $\sup_{\theta} L(\beta_n, \beta^*_n)$ as $n \to \infty$ can be investigated.

Though interest has been mostly in type (a) problems, it would seem that type (b) problems should be about equal in interest and applicability. The purpose of the present note is to discuss, as an illustration of type (b) problems, the following simple example.

3. **SIGN TEST vs. STUDENT TEST.** Let $X_1, X_2, \ldots$ be independent, each with Normal $(\theta, \sigma^2)$ distribution. We are to test at level $\alpha$ the one-sided hypothesis $\{\theta \leq 0\}$ against the alternative $\{\theta > 0\}$. Let $\delta = \theta/\sigma$ and $p = p(s) = P(X_1 > 0) = F(s)$ where $F$ is the Normal $(0,1)$ cumulative. Then the number $R_n$ of positive observations among $X_1, \ldots, X_n$ has a Binomial $(n, p)$ distribution. And $T_n = \sqrt{n} \frac{X}{\sqrt{\sum (X_i - \bar{X})^2/(n-1)}}$ has a Student t distribution with $n-1$ degrees of freedom which is central when $\delta = 0$ and non-central with parameter $\sqrt{n} \delta$ in general.

The **sign test** $A_n$ of $\{\theta \leq 0\}$ is

$$\begin{cases} 
\text{Reject when } R_n - n/2 > k_n \\
\text{Reject with prob. } \gamma_n \text{ when } R_n - n/2 = k_n,
\end{cases}$$

where $k_n$, $\gamma_n$ are constants determined by
\[ P(R_n - n/2 > k_n | \delta = 0) + \gamma_n P(R_n - n/2 = k_n | \delta = 0) = \alpha. \]

The power function of this test is

\[ \beta_n(\delta) = P(R_n - n/2 > k_n) + \gamma_n P(R_n - n/2 = k_n). \]

Values of \( k_n, \gamma_n, \beta_n(\delta) \) can be obtained from tables such as [5] of

the binomial distribution. For large values of \( n \) the normal

approximation to binomial gives

\[ (1) \quad \beta_n(\delta) \approx F\left(\frac{\sqrt{n}\left(2p - 1\right) - c}{2/\sqrt{n}(1 - p)}\right) \quad \text{where} \quad F(c) = 1 - \alpha. \]

The Student test \( A_n^* \) of \( \{ \theta \leq 0 \} \) is

Reject when \( T_n > c_n \)

where \( c_n \) is a constant determined by

\[ P(T_n > c_n | \delta = 0) = \alpha. \]

The power function of this test is

\[ \beta_n^*(\delta) = P(T_n > c_n). \]

Values of \( c_n \) can be obtained from tables of the Student t distribution, and values of \( \beta_n^*(\delta) \) from tables such as [6] of the non-central

Student t distribution. For large values of \( n \) the normal approximation
to non-central Student $t$ gives

$$\beta^*_n(8) \approx F(\sqrt{n} \cdot 8 - c) \quad \text{where} \quad F(c) = 1 - \alpha. \quad (2)$$

Loss functions such as $L^n_1(8) = L_1(\beta_n, \beta^*_n) = \beta^*_n(8) - \beta_n(8)$ and $L^n_2(8) = L_2(\beta_n, \beta^*_n) = 1 - \beta_n(8)/\beta^*_n(8)$ can easily be plotted for particular values of $n$ and $\alpha$. The appearance of these two functions for $\alpha = .05$, $n = 11$ is shown in the following figure. As $\beta$ increases from 0 the function $L^n_1(8)$ increases from 0 to a maximum and then decreases toward 0.

If $\alpha$ is fixed and $n$ increased the appearance of these curves changes only by a horizontal compression. That is, $L^n_1(8)$ rises more quickly to its maximum and then falls more quickly toward 0. The position of the maximum tends to 0 at the rate $1/\sqrt{n}$ but the maximum value changes very little and has a limit. The following table gives values of $\sup_8 L^n_1(8)$ for $\alpha = .05$ and $n = 2, 3, \ldots, 13$.

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<th>$n$</th>
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The cases \( n=2,3,4 \) are special, because for them the power of the sign test does not tend to 1 as \( \delta \to \infty \). Once these values of \( n \) are passed, \( \sup_{\delta} L_1^n(\delta) \) stays fairly close to its asymptotic value \( \lim_{n \to \infty} \sup_{\delta} L_1^n(\delta) \). We will now conclude discussion of this example by finding these asymptotic values.

**Lemma.** \( \lim_{n \to \infty} \sup_{\delta} L_n(\delta) = \sup_{\Delta} \lim_{n \to \infty} L_n(\delta_n) \) if the former exists, where \( \Delta \) is the set of all sequences \( \{\delta_n\} \) for which \( \lim_{n \to \infty} L_n(\delta_n) \) exists. [If \( \lim \) be replaced throughout by \( \lim \) or by \( \lim \) the same result holds; existence provides unnecessary.]

**Proof:** By suitable choice of \( \delta_1, \delta_2, \ldots \) we can make

\[
L_n(\delta_n) = \sup_{\delta} L_n(\delta) \leq L_n(\delta_n) + 1/n.
\]

Taking \( \lim \) and \( \lim \) throughout, we see from the existence of \( \lim_{n \to \infty} L_n(\delta) \) that \( \lim_{n \to \infty} L_n(\delta_n) = \lim_{n \to \infty} \sup_{\delta} L_n(\delta) \) exists, so that \( \Delta \) is non-empty. Moreover, since

\[
\lim_{n \to \infty} \sup_{\delta} L_n(\delta) = \lim_{n \to \infty} L_n(\delta_n)
\]

for this particular choice of \( \{\delta_n\} \), we have

\[
\lim_{n \to \infty} \sup_{\delta} L_n(\delta) \leq \sup_{\Delta} \lim_{n \to \infty} L_n(\delta_n).
\]

But of course we also have
\[
\sup_{\delta} L_n(\delta) \geq L_n(\delta_n)
\]

for every \( \{\delta_n\} \in \Delta \), which implies

\[
\lim_{n \to \infty} \sup_{\delta} L_n(\delta) \geq \lim_{n \to \infty} L_n(\delta_n)
\]

for every \( \{\delta_n\} \in \Delta \), giving us

\[
\lim_{n \to \infty} \sup_{\delta} L_n(\delta) \geq \sup_{\Delta} \lim_{n \to \infty} L_n(\delta_n).
\]

This completes proof of the lemma.

We can now find \( R_1 = \lim_{n \to \infty} \sup_{\delta} L_{1n}(\delta) \) by using the lemma to find the corresponding \( \lim \) and \( \overline{\lim} \). Instead of carrying this out we will assume the limit exists and use the lemma to compute it. This argument contains the essentials of the correct proof and is much shorter. From (1) and (2) we have for large \( n \)

\[
\beta_n(\delta_n) \equiv F\left(\frac{\sqrt{n} (2p_n - 1) - c}{2 \sqrt{p_n (1 - p_n)}}\right),
\]

\[
\beta^*(\delta_n) \equiv F(\sqrt{n} a_n - c),
\]

where \( p_n = F(\delta_n) \) and \( F(c) = 1 - \alpha \). Writing \( \delta_n = a_n / \sqrt{n} \) these become
\[ \beta_n(\delta_n) \simeq F\left( \frac{a_n}{\sqrt{p_n(1-p_n)}} \cdot \frac{F(a_n \sqrt{n}) - 1/2}{a_n / \sqrt{n}} - \frac{c}{2 \sqrt{p_n(1-p_n)}} \right), \]

\[ \beta^*_n(\delta_n) \simeq F(a_n - c). \]

Now if \( a_n \to 0 \) we have

\[ \beta_n(\delta_n) \to \alpha, \quad \beta^*_n(\delta_n) \to \alpha. \]

And if \( a_n \to \infty \) we have

\[ \beta_n(\delta_n) \to 1, \quad \beta^*_n(\delta_n) \to 1. \]

And if \( a_n \to a \) we have

\[ \beta_n(\delta_n) \to F(a / \sqrt{\pi} - c), \quad \beta^*_n(\delta_n) \to F(a - c). \]

Thus the only sequences \( \{\delta_n\} \) for which

\[ L^n_1(\delta_n) = \beta^*_n(\delta_n) - \beta_n(\delta_n) \quad \text{and} \quad L^n_2(\delta_n) = 1 - \beta_n(\delta_n) / \beta^*_n(\delta_n) \]

have non-zero limits are those in which \( a_n \to a \) with \( 0 < a < \infty \). It only remains to find the values of \( a \) which maximize these limits.

If \( \{\delta_n\} = \{a_n / \sqrt{n}\} \) with \( a_n \to a \) we have

\[ \lim_{n \to \infty} L^n_1(\delta_n) = F(a - c) - F(a / \sqrt{2/\pi} - c). \]
Differentiating with respect to \( a \) and equating the result to zero gives

\[
\frac{1}{\sqrt{2\pi}} \exp\left\{ -\frac{1}{2}(a - c)^2 \right\} = \frac{1}{\sqrt{2\pi}} \exp\left\{ -\frac{1}{2} (a \sqrt{2/\pi} - c)^2 \right\} = 0,
\]

i.e., \((a - c)^2 = (a \sqrt{2/\pi} - c)^2 + \log(\pi/2)\). The root of this quadratic at which the maximum of (3) occurs is

\[
a' = \frac{c}{1 + \sqrt{2/\pi}} \left\{ 1 + \sqrt{1 + [\log(\pi/2)](1 + \sqrt{2/\pi})/(1 - \sqrt{2/\pi})c^2} \right\}.
\]

The maximum value \( R_1 = \lim \sup L_1^n(\delta) \) can now be found by substituting \( a' \) for \( a \) in (3). For example, if \( \alpha = .05 \) we have \( c = 1.6449 \) and \( a' = 2.3750 \), and \( R_1 = .1686 \) for the asymptotic maximum loss.

Again if \( \{ \delta_n \} = \{ a_n / \sqrt{n} \} \) with \( a_n \to a \) we have

\[
(4) \quad \lim_{n \to \infty} L_2^n(\delta_n) = F(a \sqrt{2/\pi} - c) / F(a - c).
\]

Differentiating with respect to \( a \) and equating the result to zero gives

\[
F(a - c) \sqrt{\frac{2}{\pi}} \cdot \frac{1}{\sqrt{2\pi}} \exp\left\{ -\frac{1}{2} (a \sqrt{2/\pi} - c)^2 \right\} = F(a \sqrt{2/\pi} - c) \cdot \frac{1}{\sqrt{2\pi}} \exp\left\{ -\frac{1}{2} (a - c)^2 \right\}.
\]

For given \( \alpha' \) the solution \( a'' \) of this equation can be found numerically.
and shown to maximize (4). The maximum value \( R_2 = \limsup_{n \to \infty} L_2^g(8) \) is then found by substituting \( a'' \) for \( a \) in (4). For example if \( \alpha = .05 \) we have \( c = 1.6449 \) and \( a'' = 1.5593 \), and \( R_2 = .2610 \) for the asymptotic maximum loss.

The following table gives, for several \( \alpha \) values, \( a', a'', R_1 \) (the asymptotic maximum loss in power) and \( R_2 \) (the asymptotic maximum amount by which the power ratio falls below 1).

<table>
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