ON CENTERING INFINITELY DIVISIBLE PROCESSES

BY
RONALD PYKE

TECHNICAL REPORT NO. 24
JANUARY 8, 1959

PREPARED UNDER CONTRACT Nonr-225(21)
(NR-042-993)
FOR
OFFICE OF NAVAL RESEARCH

APPLIED MATHEMATICS AND STATISTICS LABORATORY
STANFORD UNIVERSITY
STANFORD, CALIFORNIA
ON CENTERING INFINITELY DIVISIBLE PROCESSES

BY

RONALD PYKE

TECHNICAL REPORT NO. 24
JANUARY 8, 1959

PREPARED UNDER CONTRACT Nonr-22521
(NR-042-993)
FOR
OFFICE OF NAVAL RESEARCH

Reproduction in Whole or in Part is Permitted for any Purpose of the United States Government

APPLIED MATHEMATICS AND STATISTICS LABORATORY
STANFORD UNIVERSITY
STANFORD, CALIFORNIA
ON CENTERING INFINITELY DIVISIBLE PROCESSES

BY

Ronald Pyke

1. Summary:

The concept of centering stochastic processes having independent increments, introduced by Lévy, is applied to processes having both stationary and independent increments. The question of what centering functions preserve the stationarity of the increments is studied. In Section 3, certain characterizations of centered Infinitely Divisible Processes are given.

This paper is partially expository in nature, and represents some of the results obtained through discussions with some of the members of the 1957-58 Advanced Probability class at Stanford University. These members were A. Albert, M. Epling, J. Kullback, F. Proschan and W. Pruitt.

2. Stationarity Preserving Centering Functions for I. D. Processes:

In 1934, Lévy [1] proved that any stochastic process with independent increments may be transformed by subtraction of a sure function, called a centering function, into a process whose sample functions possess certain desirable smoothness properties. (cf. Lévy [2] and Doob [3]). It is clear that the transformed process, called the centered process, is also a process possessing independent increments. The purpose of this section is to show that a process having stationary and independent increments may be centered in such a way so as to preserve the stationarity as well as the independence of the increments.

To be more precise, consider the following definitions. (cf. Doob[3] p.407).
For a set $T \subset \mathbb{R}$, let $T^*$ denote the set of limit points of $T$ except that the supremum and infimum of $T$ are to be included in $T^*$ only if they belong to $T$.

**Definition 1:** A stochastic process $\{X_t : t \in T\}$ is said to be centered if and only if

(a) for every $\{t_n\} \subset T$ satisfying $t_n \searrow t, \forall t \in T^*$ ($t_n \nearrow t \in T^*$) there exists a random variable $X_t^-$ ($X_t^+$), independent of the particular sequence, such that

$$X_{t_n} \overset{a.s.}{\longrightarrow} X_t^- , \quad (X_{t_n} \overset{a.s.}{\longrightarrow} X_t^+)$$

(b) there exists a function $g$ defined and continuous on the closure of $T$ such that any difference $X_t - X_s, t, s \in T^*$, or any such difference with $t$ replaced by $t^+$ or $t^-$ and/or $s$ replaced by $s^+$ or $s^-$, is constant a.s. if and only if $X_t - X_s = g(t) - g(s)$ a.s.

(c) $X_{t^-} = X_t = X_{t^+}$ a.s. for all but at most a countable number of points of $T$.

This definition differs from that given by Doob only through condition (b). In Doob's definition, the function $g$ was restricted to be constant over $T^*$. The above modified definition has the advantage of making it unnecessary to distinguish between degenerate and non-degenerate processes in the theorems below, as well as of assuring the truth of the statement that if $\{X_t : t \in T\}$ is a centered process, then so is $\{X_t + h(t) : t \in T\}$ for
every continuous and bounded function \( h \) on \( T \). This statement would not be true without this less restrictive definition.

**Definition 2**: A function \( c:T \to \mathbb{R}_1 \) is said to be a centering function of a stochastic process \( \{X_t : t \in T_3\} \) if and only if the process \( \{X_t - c(t) : t \in T_3\} \) is centered.

It is clear that one may always find a centering function, such that the resulting centered process satisfies (b) of Definition 1 with \( g = 0 \).

**Definition 3**: A stochastic process, \( \{X_t : t \in T_3\} \), having stationary and independent increments and for which \( T = [0, +\infty) \) and \( X_0 = 0 \) a.s. is said to be an Infinitely Divisible (I.D.) process.

The terminology of the above definition is motivated by the fact that for each \( t \), \( X_t \) is an infinitely divisible random variable (r.v.). This may be seen from the representation

\[
X_t = X_{t/n} + (X_{2t/n} - X_{t/n}) + \ldots + (X_t - X_{t-t/n})
\]

which is possible for every \( n \geq 1 \). Moreover, as is evident from Lemma 1 below, a 1-1 correspondence may be defined in a natural way between the class of infinitely divisible r.v.'s and the class of I.D. processes. For properties of infinitely divisible r.v.'s used in this paper the reader is referred to [4] and [5].

In the case of a centering function for processes with independent increments, uniqueness is clearly impossible. One possible centering function
is that used by Doob ([3], p. 408), namely the solution to

\[ E \left\{ \arctan[X_t - c(t)] \right\} = 0 \]

It should be noted that this particular centering function would not preserve stationarity of increments in case the given process were a non-degenerate I.D. process.

Define for all \( \omega \in \mathbb{R}_1 \) and \( t \geq 0 \)

\[ f(\omega:t) = E \left\{ e^{i\omega X_t} \right\} \]

**Lemma 1:** A stochastic process \( \{ X_t : t \geq 0 \} \) having independent increments is an I.D. process if and only if there exist unique functions \( c : [0, \infty) \rightarrow \mathbb{R}_1 \) and \( \psi : \mathbb{R}_1 \rightarrow \text{complex plane satisfying} \)

(i) for all \( s, t \geq 0 \)

\[ c(s) + c(t) = c(s+t) \]

(ii) for all rational \( r \geq 0 \), \( c(r) = 0 \)

(iii) for all \( t \geq 0 \), \( \omega \in \mathbb{R}_1 \)

\[ \log f(\omega:t) = i\omega c(t) + t\psi(\omega) \]

**Proof:** The proof of the sufficiency is left to the reader. Our main interest is in the necessity of these conditions, two proofs of which are now given.

A: Since \( X_t \) is an infinitely divisible r.v. for all \( t \), it is known that
there exists a real number \( \mu(t) \) and a bounded non-decreasing right-continuous real function \( G(\cdot; t) \) satisfying \( G(-\infty; t) = 0 \) for which

\[
(2) \quad \log f(\omega; t) = i\omega \mu(t) + \int_{\mathbb{R}_1} \left( e^{i\omega x} - 1 - \frac{i\omega x}{1 + x^2} \right) \frac{1 + x^2}{x^2} \, dG(x; t) .
\]

It follows from the stationarity and independence of the increments of the process and from the uniqueness of the representation (2) that

\[
(3) \quad f(\omega; s) f(\omega; t) = f(\omega; s+t)
\]

and hence

\[
\mu(s) + \mu(t) = \mu(s+t)
\]

and

\[
(4) \quad G(x; s) + G(x; t) = G(x; s+t)
\]

for all \( s, t \geq 0 \) and \( x \in \mathbb{R}_1 \). Since \( G(x; \cdot) \geq 0 \) for each \( x \), it is well known that the solution of (4) must be

\[
G(x; t) = tG(x; 1) .
\]

Define

\[
c(t) = \mu(t) - t\mu(1)
\]

and

\[
(5) \quad \psi(\omega) = \log f(\omega; 1) .
\]
It is immediate that $c$ and $\psi$ are unique and satisfy the conditions of the theorem.

$\mathbb{P}$: The purpose of this second proof is to demonstrate that the powerful tool (2) is not essential for proving the necessity of the conditions of Lemma 1. This is important, it is felt, because the result stated as Lemma 1 should logically be proven very shortly after an I.D. process is defined, and because such a definition may well precede any discussion of infinitely divisible r.v.'s. In fact this author, in the advanced probability course mentioned in the Summary, used the concept of an I.D. process as a means of defining an infinitely divisible r.v.

It follows from (1) that for each $n$, $[f(\omega:t)]^{1/n}$ is a characteristic function. It is well known, therefore, and may easily be proved, that for all $p > 0$ $[f(\omega:t)]^p$, properly defined, is a characteristic function and that $f(\omega:t) \neq 0$ for all $\omega \in \mathbb{R}_+$ and $t \geq 0$. Consider the real function $g(\omega:t) = |f(\omega:t)|^2$ defined on $\mathbb{R}_+ \times [0, \infty)$. From (3), it follows that

$$g(\omega;s)g(\omega;t) = g(\omega;s+t).$$

Since $0 < g(\omega;s) \leq 1$, the solution of this functional equation is given by

(6) $$\log g(\omega:t) = t[\psi(\omega) + \psi(-\omega)]$$

for all $\omega,t$ where $\psi$ is defined by (5). Consequently, defining $q(\omega:t) = e^{t\psi(\omega)} [f(\omega:t)]^{-1}$ it follows from (6) that $|q(\omega:t)| = 1$. We now
show that \( q(\omega; t) \), a continuous function in \( \omega \), is a limit of characteristic functions and hence is a characteristic function itself. From (3) it follows that for all rational \( r \), \( f(\omega; rt) = [f(\omega; t)]^r \). Moreover, as \( n \to \infty \) through rationals,

\[
\left[ \frac{f(\omega; l)}{f(\omega; rt)} \right]^t = \frac{e^{t\psi(\omega)}}{[f(\omega; t)]^rt} \to q(\omega; t)
\]

However, by (3) \( f(\omega; l)/f(\omega; rt) = f(\omega; l-rt) \) is a characteristic function and hence so is \( q(\omega; t) \). The proof is then complete since for each \( t \geq 0 \), \( q(\omega; t) \) is of modulus one; which is to say that \( q(\omega; t) = e^{i\omega c(t)} \) for some real number \( c(t) \). It may be easily checked that the functions \( c \) and \( \psi \) thus defined satisfy the required conditions.

**Theorem 1:** For an I.D. process \( \xi X_t : t \geq 0 \), there exists a centering function \( c \) such that the resulting centered process \( \xi X_t - c(t); t \geq 0 \) is also an I.D. process.

**Proof:** Choose for the centering function the function \( c \) of Lemma 1. Set \( Z_t = X_t - c(t) \) and \( h(\omega; t) = E(e^{i\omega Z_t}) \). By Lemma 1, there exists a function \( \psi: \mathbb{R} \to \text{complex plane} \) such that

\[
(7) \quad \log h(\omega; t) = t \psi(\omega)
\]

For any sequence \( 0 \leq t_n \to t \), consider

\[
Z_{t_n} = \sum_{j=1}^{n} (Z_{t_j} - Z_{t_{j-1}}) + Z_{t_0}
\]
where the summands are independent. By (7) \( h(\omega; t_n) \rightarrow h(\omega; t) \). It therefore follows (cf. Doob [3] p. 115) that \( Z_{t_n} \xrightarrow{a.s.} Z_t \). Similarly, if \( t_n \downarrow t \geq 0 \), consider

\[
Z_{t_0} - Z_{t_n} = \sum_{j=0}^{n-1} (Z_{t_j} - Z_{t_{j-1}}).
\]

However \( h(\omega; t_0 - t_n) \rightarrow h(\omega; t_0 - t) \), so as before \( Z_{t_n} \xrightarrow{a.s.} Z_t \). Therefore (a) of Definition 1 is satisfied. (b) and (c) are vacuously true whenever the process is non-degenerate. If the process is degenerate, then (b) is true for \( g(t) = -i\psi(1) \), and (c) is again vacuously true. Since \( \xi Z_t : t \geq 0 \) is clearly an I.D. process, the proof is complete.

It is remarked that a stationarity preserving centering function for an I.D. process is unique up to the addition of straight lines through the origin.

It is evident that portions of Definition 1 are superfluous when applied to I.D. processes. In fact, from the above proof of Theorem 1 one obtains:

**Lemma 2:** An I.D. process is centered if and only if its characteristic function \( f(\omega; t) \) is continuous in \( t \).

**Corollary 1:** An I.D. process is centered if and only if for all sequences \( 0 \leq t_n \rightarrow t \), \( X_{t_n} \xrightarrow{a.s.} X_t \).

It is emphasized that the above results are neither difficult nor too surprising. The fact that a centered I.D. process has a characteristic function which satisfies
\[ (8) \quad \log f(\omega,t) = t\psi(\omega) \]

is well known (e.g., cf. Lévy [2] p. 186, Doob [3] p. 419, Ito [6]). The justification for the presentation of the above material is two-fold;

(i) Theorem 1 has not been located in the literature and

(ii) several recent papers in the literature indicate that Lemma 1 and Theorem 1 are not known.

Concerning (ii), several authors assume that (8) is true for all separable I.D. processes (cf. [7]) while in other papers the exact role played by centering in the case of I.D. processes seem to have been misunderstood (cf. [8]). Furthermore, as a consequence of Lemma 1, the assumption (retaining the notation of the papers referred to) that \( \phi(t;\lambda) \) be continuous in \( \lambda \) may be removed from Theorem 1 of [9] and from Theorem 1 of [10]. For example Theorem 1 (iii) of [9] could be strengthened to read:

\[ F(x;\lambda) \in C_\perp \text{ if and only if } \phi(t;\lambda) = [f(t)]^\lambda e^{ict(\lambda)} \]

where \( f(t) \) is a characteristic function and where \( c \) is a function satisfying the conditions of Lemma 1.

Some Characterizations of I.D. Processes:

In this section several equivalent definitions of a centered I.D. process are stated and proven under the assumption of separability. Although it is clear that a separable I.D. process need not be centered, it is possible to relate these two properties as well as the properties of measurability and of boundedness of sample functions.
Theorem 2: For a separable I.D. process \( X_t = \left\{ X_t : t \geq 0 \right\} \), the following conditions are equivalent:

(i) \( X \) is centered

(ii) \( X \) is measurable

(iii) there exists a separating sequence which is a subset of the rational numbers

(iv) there exists an open interval in \([0, \infty)\) over which almost all sample functions are bounded.

Proof: From Corollary 1, it follows that for a centered and separable process, 
\( X_s \overset{a.s.}{\to} X_t \) as \( s \to t \) for all \( t > 0 \) regardless of the manner in which \( s \to t \). Therefore, (i) implies (ii) (cf. [3] p. 60). Conversely, the characteristic function of a measurable process may be shown to be measurable (cf. [3] p. 62). Therefore, the function \( c \) of Lemma 1 must be measurable and hence is identically zero. (This last result depends heavily on the fact that \( c \) is a solution of the linear functional equation, since in general the measurability of \( f(\omega : \cdot) \) does not imply the measurability of every branch of \( \log f(\omega : \cdot) \)). That (i) implies (iv) may be easily deduced from a known theorem (cf. Doob [3], p. 411). If the function \( c \) of Lemma 1 is not identically zero, it is well known that it must be non-measurable and unbounded in every interval. Since by Theorem 1, and uncentered I.D. process differs from a centered one by such a function, it follows that (iii) and (iv) each imply (i). That (i) implies (iii) follows from Corollary 1 and [3] pp. 54-55.
In [6], Ito proves that a centered I.D. process may be decomposed into a sum of Wiener process and a countable number of compound Poisson processes, all of which are mutually independent. This same result has been stated recently by Benes [8]. As a simple consequence of Lemma 1, this characterization may be extended to uncentered I.D. processes, namely

**Lemma 2:** An I.D. process \( \{X_t : t \geq 0\} \) may be expressed as

\[
X_t = c(t) + W_t + \sum_{j=1}^{\infty} Y_t^{(j)} \quad (t \geq 0)
\]

where \( c \) is a sure function satisfying the conditions of Lemma 1, \( W_t \) is a Wiener process and \( Y_t^{(j)} \) is a compound Poisson process for each \( j \), all processes being mutually independent.

In conclusion, we remark that the restriction to \( T = [0, \infty) \) in Definition 3 may be removed, provided \( T \) remains a union of connected sets of positive Lebesque measure. In this case the theorems go through with only minor changes. If \( T \) contains isolated points more care is required. Moreover, for this latter case, it is clear that for some \( t \in T, X_t \) need not be an infinitely divisible r.v. .
BIBLIOGRAPHY


STANFORD UNIVERSITY

Technical Reports Distribution List
Contract Nonr-225(21)
(NR-042-993)

Armed Services Technical Information Agency
Arlington Hall Station
Arlington 12, Virginia

Chief, Bureau of Ships
Dept. of the Navy
Washington 25, D. C.
Attn: H. Weingarten
Code 223

Commanding Officer
U. S. Naval Powder Factory
Indianhead, Md.
Attn: F. Freshman
R. and D.

Director, Naval Research Lab.
Washington 25, D. C.
Attn: Technical Information Officer

Director
National Security Agency
Attn: REMP-1
Fort George Meade, Md.

Dept. of Mathematical Statistics
University of North Carolina
Chapel Hill, N. C.

Office of Naval Research
Dept. of the Navy
17th and Constitution Aves.
Washington 25, D. C.
Attn: Code 433

Office of the Asst. Naval Attache for Research
Naval Attache
American Embassy
Navy No. 100
Fleet Post Office
New York, N. Y.

Office of Technical Services
Dept. of Commerce
Washington 25, D. C.

U. S. Naval Avionics Facility
Indianapolis 18, Indiana
Attn: Library

Scientific Section
Office of Naval Research
Dept. of the Navy
1000 Geary St.
San Francisco 9, California

Statistical Engineering Lab.
National Bureau of Standards
Washington 25, D. C.

Dept. of Statistics
University of California
Berkeley 4, California

Statistical Laboratory
University of Washington
Seattle 5, Washington

Dr. Stephen G. Allen
Stanford Research Institute
Menlo Park, California

Professor Fred C. Andrews
Mathematics Dept.
University of Oregon
Eugene, Oregon

Professor T. W. Anderson
Center of Advanced Studies in Behavioral Sciences
Stanford, California

Professor David Blackwell
Dept. of Statistics
University of California
Berkeley 4, California
Dr. Julius R. Blum  
Dept. of Mathematics  
Indiana University  
Bloomington, Indiana  

Dr. Paul Blunk  
Box 532  
Fair Oaks, California  

Dr. Charles Boll  
General Electric Co.  
735 State Street  
Santa Barbara, California  

Professor W. G. Cochran  
Dept. of Statistics  
Harvard University  
2 Divinity Avenue, Room 311  
Cambridge 38, Massachusetts  

Professor Lee Cronbach  
Bureau of Education Research  
1007 S. Wright  
Champaign, Illinois  

Dr. Joseph Daly  
Bureau of the Census  
Washington 25, D. C.  

Dr. Francis Dresch  
Stanford Research Institute  
1915 University Ave.  
Palo Alto, California  

Professor Meyer Dwass  
Dept. of Mathematics  
Northwestern University  
Evanston, Ill.  

Professor D. A. S. Fraser  
Dept. of Mathematics  
University of Toronto  
Toronto 5, Canada  

Mr. Murray A. Geisler  
Logistics Section  
The RAND Corporation  
1700 Main Street  
Santa Monica, California  

Mr. Geoffrey Gregory  
4 Osborne Grove  
Gatley, Cheadle  
Cheshire, England  

Professor E. J. Gumbel  
Industrial Engr. Dept.  
409 Engineering Bldg.  
Columbia University  
New York 27, N. Y.  

Dr. Theodore E. Harris  
The RAND Corporation  
1700 Main Street  
Santa Monica, California  

Professor Leonid Hurwicz  
School of Business Administration  
University of Minnesota  
Minneapolis 14, Minn.  

Professor Stanley Isaacson  
4715 Pleasant Street  
Des Moines, Iowa  

Professor Leo Katz  
Statistics Dept.  
Michigan State University  
East Lansing, Michigan  

Professor Tosio Kitagawa  
Mathematical Institute  
Faculty of Science  
Kyusyu University  
Fukuoka, Japan  

Dr. Dennis V. Lindley  
Statistical Lab.  
University of Cambridge  
Cambridge, England  

Dr. Eugene Lukacs  
Dept. of Mathematics  
Catholic University  
Washington 17, D. C.  

Mr. Monroe Norden  
Research Division  
Engineering Statistics Group  
College of Engineering  
New York University  
401 W. 205 Street  
New York 54, N. Y.  
Professor Stanley Reiter
Dept. of Economics
Purdue University
Lafayette, Indiana

Professor George Resnikoff
Dept. of Industrial Engineering
Illinois Institute of Technology
Chicago 16, Illinois

Dr. A. R. Roy
Statistical Wing
Indian Council of Agricultural Research
Linlithgow Avenue
New Delhi, India

Professor Herman Rubin
Dept. of Mathematics
University of Oregon
Eugene, Oregon

Dr. Jagdish Rustagi
Dept. of Mathematics
College of Engineering and Science
Carnegie Institute of Technology
Pittsburgh 13, Pa.

Professor Seymour Sherman
Moore Sch. of Electrical Engineering
University of Pennsylvania
Philadelphia 4, Pa.

Dr. Milton Sobel
Bell Telephone Labs.
555 Union Blvd.
Allentown, Pa.

Professor Herbert Solomon
Teachers' College
Columbia University
New York 27, N. Y.

Professor Donald Truax
Mathematics Dept.
University of Kansas
Lawrence, Kansas

Mr. Cesareo Villegas
Institute de Mathematica y Estadistica
Av. J. Herrera y Reissig 565
Montevideo, Uruguay

Professor Irving Weiss
Bell Telephone Labs.
1600 Osgood Street
North Andover, Mass.

Dr. Oscar Wesler
Dept. of Mathematics
University of Michigan
Ann Arbor, Michigan

Dr. John D. Wilkes
Office of Naval Research
Code 200
Washington 25, D. C.

Professor S. S. Wilks
Room 110, Fine Hall
Box 708
Princeton, New Jersey

Professor J. Wolfowitz
Mathematics Dept.
Cornell University
Ithaca, N. Y.

Professor M. A. Woodbury
Dept. of Mathematics
New York University
New York 53, N. Y.

Additional copies for project leader and assistants and reserve for future requirements 50