THE RECURRENCE OF REAL MARKOV PROCESSES

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1. Introduction

The Markov processes to be considered in this paper have the non-negative reals as their state space, and will always have the property that the transitions made by the process are uniformly bounded. In other words, if \( \{ X_n \} \) are the random variables comprising the process, then for some constant \( M < \infty \)

\[
|X_{n+1} - X_n| \leq M \quad \text{a.s.,} \quad n = 0, 1, 2, \ldots
\]

The object of the paper is to find conditions under which \( X_n \to \infty \) a.s. \( \{ X_n \} \) is transient, and also conditions under which \( \{ X_n \} \) is recurrent; the precise meaning of "recurrent" will be discussed below. These conditions will be in terms of the behaviour of the function

\[
F(x, y) = \Pr(X_{n+1} \leq y \mid X_n = x)
\]

for large values of \( x \).

A sort of prototype for the processes we will study is the class of "random walks." This term, variously used in the literature, will here mean a Markov chain with states 0, 1, 2, ... such that from state \( j \), only transitions to state \( j+1 \) or \( j-1 \) are possible. These processes have been studied extensively \([2;3;4]\), and in particular conditions for the recurrence or transience of a random
walk have been known.

The general idea of the approach we shall take is quite simple, although it becomes somewhat involved when carefully carried out. Let \( \{\alpha_j\} \) be an increasing sequence of positive numbers, with \( \alpha_{j+1} - \alpha_j \to \infty \). We will define a random walk \( \{X_n^*\} \) associated with the original process \( \{X_n\} \) in such a way that \( \{X_n^*\} \) makes a transition from \( j \) to \( j+1 \) (\( j-1 \)) if the \( \{X_n\} \) process, having last been "at \( \alpha_j \)," reaches \( \alpha_{j+1} \) before \( \alpha_{j-1} \) (respectively \( \alpha_{j-1} \) before \( \alpha_{j+1} \)). Of course this cannot be taken quite literally, because \( \{X_n\} \) may "jump over" the values \( \{\alpha_j\} \). This difficulty can, however, be overcome, and it is then clear that \( \{X_n\} \) is recurrent or transient according as \( \{X_n^*\} \) is. We have next to estimate the transition probabilities of the random walk \( \{X_n^*\} \). This can be done by replacing \( \{X_n\} \) by a homogeneous process and using well-known results of A. Wald [5] on "gambler's ruin" probabilities. Knowing the transition probabilities, the recurrence or transience of \( \{X_n^*\} \) and so of \( \{X_n\} \) can be determined from the results on random walks mentioned above.

In order to actually carry out this program, some conditions must be imposed on the function \( F(x, y) \). Our main result (Theorem 1) thus has rather complicated hypotheses, which later (Theorem 4) are simplified at some cost in generality. Some interesting specializations are possible; for instance, to random walks with steps of arbitrary (unequal) length to the left or right. The elegant methods of Harris [2] and of Karlin and McGregor [4] are no longer applicable in this case, but a theorem on these "generalized random walks" and one on another special class of processes are derived from Theorem 1. (The work of Hodges and Rosenblatt [3] has
more contact with the present paper, especially with the techniques of Section 2 below.) All our results are stated in Section 5; Sections 2 to 4 contain preparations.

We shall always suppose that there is a sort of reflecting barrier at \( x = 0 \). This assumption, made precise below, is purely a convenience, because of the local character of the random fluctuations. Thus a process on the whole line with bounded transitions is recurrent if and only if the two one-sided processes which result when a reflecting barrier is erected at the origin are both recurrent. The presence of the barrier makes it easy to be precise about what is meant by recurrence: \( X_n = 0 \) infinitely often a.s. We shall treat the alternatives of recurrence or \( X_n \to \infty \) a.s. as though they were exhaustive. This, of course, is not true without some restriction on the transition law of the process \( \{X_n\} \). A sufficient condition is that for each \( N \) there is an \( \epsilon > 0 \) such that for all \( x \in [0, N] \), if \( X_n = x \) then

\[
\Pr(X_{n+1} \geq x + \epsilon) \geq \epsilon \quad \text{and} \quad \Pr(X_{n+1} \leq \max(x - \epsilon, 0)) \geq \epsilon.
\]

It is not hard to see that this implies the equivalence of the conditions "\( \{X_n\} \) is recurrent," and "\( \Pr(X_n \to \infty) < 1.\)" It will always be assumed hereafter that some condition sufficient to insure this equivalence is in force.

It is also natural to ask for criteria for the existence of first-passage time moments. It would indeed be possible to obtain some conditions ensuring the non-existence of moments, for if the random walk
\( \{X^*_n\} \) has a \( k \)'th passage-time moment which is infinite, so clearly does \( \{X_n\} \). However, it seems impossible to get a result in the opposite direction; there will be long time intervals between successive transitions of \( \{X^*_n\} \), so that finite moments for \( \{X^*_n\} \) do not imply the same for \( \{X_n\} \). That is why this topic is not pursued in the present paper.

2. Comparison Theorems

We say that a function \( F(x, y) \) is a "stochastic transition function" if for each \( x \), \( F(x, \cdot) \) is a distribution function, while \( F(\cdot, y) \) is measurable for each \( y \). It is known [1; Chapters 2,5] that given such a function \( F(x, y) \) and a number \( x_o \) there exists an essentially unique Markov process \( \{X_n\} \) in discrete time such that with probability one

\[
(1) \quad X_0 = x_0 \quad \text{and} \quad \Pr(X_{n+1} \leq y \mid X_n = x) = F(x, y).
\]

All the transition functions considered in this paper will have the additional properties that for all \( x \geq 0 \),

\[
(2) \quad F(x, y) = \begin{cases} 0 & \text{if } y \leq x - M \\ 1 & \text{if } y \geq x + M \end{cases} \quad \text{for some } M < \infty;
\]

\[
(3) \quad F(x, y) = 0 \quad \text{if } y < 0.
\]

These assumptions ensure that with probability one the process \( \{X_n\} \) has bounded steps and is non-negative. We shall sometimes impose the further assumption that for each \( t \),

\[
(4) \quad F(x, x+t) \text{ is non-decreasing in } x, x \geq M;
\]
such a transition function will be called "monotone."

Let \( \{X_n\} \) and \( \{Y_n\} \) be two real Markov processes with stochastic transition functions \( F(x, y) \) and \( G(x, y) \) respectively satisfying (2) and (3). Let \( \{\xi_n\} \) be a sequence of independent random variables, each uniformly distributed on \([0,1]\). We shall link together the two processes while providing each with the correct transition probabilities by letting

\[
X_{n+1} = F^{-1}(X_n, \xi_{n+1}) \quad \text{and} \quad Y_{n+1} = G^{-1}(Y_n, \xi_{n+1}),
\]

where \( F^{-1}(x, y) \) is for each \( x \) the function inverse to \( F(x, \cdot) \), and similarly for \( G^{-1} \). It is easily seen that (5) implies the second part of (1) for both processes.

**Lemma 1.** Let \( G(x, y) \) be monotone. If \( X_0 \geq Y_0 \) and \( G(x, y) \geq F(x, y) \) for all \( x, y \geq 0 \), then

\[
X_n \geq Y_n - 2M \quad \text{a.s.} \quad \text{for all} \quad n.
\]

Conversely, if \( X_0 \leq Y_0 \) and \( G(x, y) \leq F(x, y) \), then

\[
X_n \leq Y_n + 2M \quad \text{a.s.} \quad \text{for all} \quad n.
\]

**Proof.** This is easily seen by induction; suppose in the first case that \( X_k \geq Y_k - 2M \). If in fact \( X_k \geq Y_k \), then clearly \( X_{k+1} \geq Y_{k+1} - 2M \) since both processes have transitions of maximum length \( M \). If, alternatively, \( X_k < Y_k \), we have for all \( y \)

\[
F(X_k, y) \leq G(X_k, y) \leq G(Y_k, y + (Y_k - X_k))
\]
by (4). However, from (5) we obtain
\[ F(X_k, X_{k+1}) = \xi_{k+1} = G(Y_k, Y_{k+1}). \]
Putting \( y = X_{k+1} \) and using the fact that \( G \) (as a distribution) is monotone in its second argument yields
\[ Y_{k+1} \leq X_{k+1} + (Y_k - X_k). \]
It follows that in this case also \( X_{k+1} \geq Y_{k+1} - 2M \), and so the induction is complete. The second part of the lemma is proved in almost exactly the same way.

**Lemma 2.** (Comparison Theorem.) Let \( G(x, y) \) be monotone. If \( G(x, y) \geq F(x, y) \) for all sufficiently large \( x \) and \( y \) and \( Y_n \to \infty \) a.s., then \( X_n \to \infty \) a.s. On the other hand, if \( G(x, y) \leq F(x, y) \) for all large enough \( x \) and \( y \) and \( \{Y_n\} \) is recurrent, then \( \{X_n\} \) is also recurrent.

**Proof.** Let \( N \) be a number such that \( G(x, y) \geq F(x, y) \) if \( x, y \geq N - 2M \). Since \( Y_n \to \infty \), there is a value for \( Y_0 > N \) for which the probability that \( Y_n > N \) for all \( n \) and \( Y_n \to \infty \) is positive. Let \( X_0 \) be not less than \( Y_0 \), and let \( \{X_n\} \) be linked to \( \{Y_n\} \) as in (5). Then by the proof of Lemma 1, \( X_n \geq Y_n - 2M \) for all \( n \) for those sequences \( \{\xi_n\} \) such that \( Y_n > N \), since for these paths the inequality between the transition functions always holds. There is therefore a positive probability that \( X_n \geq Y_n - 2M \) for all \( n \) and \( Y_n \to \infty \); hence \( \Pr(X_n \to \infty | X_0) \) is positive for some \( X_0 \). It then follows from the general assumption of the
introduction that for any \( X_0, X_n \to \infty \) with probability one.

The converse is similar; for any \( Y_0 \) the probability is one that some \( Y_n \leq N - 2M \). (\( N \) plays the same role as before for the reversed inequality.) If \( X_0 = Y_0 \) and the two processes are linked, then from Lemma 1 \( X_n \leq Y_n + 2M \) as long as the inequality \( G(X_n, y) \leq F(X_n, y) \) is true for all \( y \) at each step. It follows that the \( \{ X_n \} \) process eventually makes a passage from \( X_0 \) to a position \( \leq N \) (with probability one) for any \( X_0 \). Therefore \( \lim \inf X_n < \infty \) a.s., and so \( \{ X_n \} \) is recurrent.

Suppose next that \( 0 < A < B < \infty \), and let \( R(S) \) denote the least integer such that \( X_R(Y_S) \) is not in the interval \( (A, B) \), where \( X_O(Y_O) \) does lie in \( (A, B) \). We now specialize \( \{ Y_n \} \) to a homogeneous process; that is, \( G(x, y) = G(x+t, y+t) \) for all \( t \). The next lemma provides a means of estimating the probabilities with which \( \{ X_n \} \) leaves an interval on the left or on the right.

**Lemma 3.** Let \( X_0 \geq Y_0 \), and \( G(x, y) \geq F(x, y) \) for all \( y \) and all \( x \in (A, B) \). Then

\[
\Pr(X_R \geq B) \geq \Pr(Y_S \geq B).
\]

Conversely if \( X_0 \leq Y_0 \) and \( G(x, y) \leq F(x, y) \), then

\[
\Pr(X_R \geq B) \leq \Pr(Y_S \geq B).
\]

**Proof.** Much the same argument applies as that used for Lemma 1. Let \( \{ X_n \} \) and \( \{ Y_n \} \) be linked by (5); then (in the first case) \( X_n \geq Y_n \) as long as both \( X_n \) and \( Y_n \) remain in \( (A, B) \). This may be proved by induction as
follows: if \( X_k \leq Y_k \) and both are in \((A, B)\), then
\[
F(X_k, y) \leq G(X_k, y) = G(Y_k, y + (Y_k - X_k)).
\]

But from (5) once again
\[
F(X_k, X_{k+1}) = \xi_{k+1} = G(Y_k, Y_{k+1}),
\]
and comparison shows that \( X_{k+1} + (Y_k - X_k) \geq Y_{k+1} \), implying
\( X_{k+1} \geq Y_{k+1} \). Therefore it is impossible for \( \{Y_n\} \) to leave \((A, B)\)
on the right and \( \{X_n\} \) not do so, which proves the lemma. The other
case is entirely similar. Note that the slightly sharper result here
than in Lemma 1 is due to the homogeneity of \( \{Y_n\} \).

One final result, also "intuitively obvious," will be needed in
the next section. Let \( \{X_n\} \) be as before, and let \( \eta_{n+1} \) be a measurable
function of \( X_0, X_1, \ldots, X_n \). Let \( \{X'_n\} \) be a modified process with the
transition law
\[
(6) \quad X'_0 = X_0, \quad \Pr(X'_{n+1} = \eta_{n+1}(x_0, x_1, \ldots, x_n) \mid X'_0 = x_0, \ldots, X'_n = x_n) = F(x_n, y).
\]

In other words, if \( X'_n = x \), \( \eta_{n+1} \) is chosen by the same law which would
govern \( X_{n+1} \) if \( X_n = x \), except that \( \eta_{n+1}(X'_0, \ldots, X'_n) \) is added to the
result. Of course, in general this modification destroys the Markov
property.
Lemma 4. Let $F(x, y)$ be monotone. If $\eta_n \leq 0$ a.s. for each $n$, and if $X'_n \to \infty$ a.s., then $X_n \to \infty$ a.s. Conversely if $\eta_n \geq 0$ a.s. for each $n$ and $\liminf X'_n < \infty$ a.s., then $\{X'_n\}$ is recurrent.

Proof. Let $\{X_n\}$ and $\{X'_n\}$ be referred after the manner of (5) to the same sequence $\{\xi_n\}$. For the $\{X'_n\}$ process this means that

$$X'_{n+1} = F^{-1}(X'_n, \xi_{n+1}) + \eta_n(X'_1, \ldots, X'_n).$$

An induction argument similar to that of Lemma 1 then shows that under the first set of hypotheses,

$$X_n \geq X'_n - 2M \text{ a.s.}, \quad n = 1, 2, 3, \ldots,$$

while under the second set

$$X_n \leq X'_n + 2M \text{ a.s.}, \quad n = 1, 2, 3, \ldots.$$

The lemma follows at once from these inequalities.

3. Auxiliary Processes

Let $\{\alpha_n\}$ be an increasing sequence of positive numbers such that $\alpha_0 = 0$ and $\alpha_{n+1} - \alpha_n > 2M$. Let $\{X_n\}$ be a Markov process with transition function $F(x, y)$ satisfying (1) to (4). We shall construct a modified process $\{X'_n\}$ as follows: let $X'_0 = X'_o = 0$. If $X'_k = \alpha_j$, the process $\{X'_n\}$ follows the transition law (1) as long as $X'_{k+1}, X'_{k+2}, \ldots$ lie in the interval $(\alpha_{j-1}, \alpha_{j+1} - M)$. The first time the law (1) leads to a value outside this interval, that value is replaced by $\alpha_{j-1}$ if it was less than or equal to $\alpha_{j-1}'$ but by $\alpha_{j+1}$ if it was greater than or equal to $\alpha_{j+1} - M$. The $\{X'_n\}$ process then again follows the transition law (1) of the $\{X_n\}$ process as long as it remains in the interval $(\alpha_{j-2}, \alpha_j - M)$ or $(\alpha_j, \alpha_{j+2} - M)$ (depending on whether the first or second of the above alternatives occurred),
and so on. The point is that the paths of \( \{X'_n\} \) cannot skip over the value \( \alpha_j \) in passing from \( \alpha_{j-1} \) to \( \alpha_{j+1} \) or the reverse. The Markov property of \( \{X_n\} \) is, in general, lost.

Note that the modifications imposed on \( \{X_n\} \) consist always of a positive addition (or none) to the state. Hence \( \{X'_n\} \) is of the form (6), and although it would be difficult to write out \( \eta_n \) as a function of the path, it is clear that \( \eta_n \geq 0 \) a.s. for all \( n \). Therefore Lemma 4 applies, and we have the result that if \( \lim \inf X'_n < \infty \) with probability one, then \( \{X_n\} \) is recurrent.

Next we define some random variables associated with the \( \{X'_n\} \) process: \( T(o) = 0 \), and \( T(i) \) are times defined recursively by requiring that if \( X'_T(i) = \alpha_j \) for some \( j \), then \( T(i+1) \) is the smallest integer exceeding \( T(i) \) such that \( X'_T(i+1) = \alpha_{j+1} \) or \( \alpha_{j-1} \). We then define

\[
(7) \quad X^*_i = j \quad \text{if} \quad X'_T(i) = \alpha_j.
\]

The \( \{X^n_\ast\} \) form a stochastic process with the non-negative integers as states. Since \( \{X'_n\} \) cannot skip over the \( \alpha_j \) (except possibly for the \( \alpha_j \) last visited), we have that if \( X^*_i = j \), \( X^*_i+1 \) can only be \( j+1 \) or \( j-1 \). Finally note that the times \( T(i) \) are "regeneration points" for the process \( \{X'_n\} \); that is, knowledge of the past beyond the present state (necessarily one of the \( \alpha_j \)'s) is of no predictive value. This implies that \( \{X^*_n\} \) is a Markov process, and in fact a random walk.

* There must be such an integer, since under our general assumption law (1) must eventually lead from \( \alpha_j \) to a value outside \( (\alpha_{j-1}, \alpha_{j+1} - M) \), whereupon \( X' \) will equal \( \alpha_{j-1} \) or \( \alpha_{j+1} \).
Lemma 5. If the random walk \( (X^*_n) \) is recurrent so is the original process \( (X_n) \).

Proof. If \( (X^*_n) \) is recurrent, some state, say \( J \), is visited infinitely often a.s. Therefore the probability is one that \( X'_n = \alpha_j \) infinitely often, so that \( \lim \inf X'_n < \infty \) a.s. As noted above, this implies that \( (X_n) \) is recurrent.

In a very similar way, we define another pair of processes. For \( (X''_n) \), the transition law is just the same as that of \( (X'_n) \) except that the interval \( (\alpha_{J-1} + M, \alpha_{J+1} - M) \) is used instead of \( (\alpha_{J-1}, \alpha_{J+1} - M) \). This has the effect of making \( \eta_n \) (the "additions" to the state) all \( \leq 0 \), and so, by the first part of Lemma 4 this time, if \( X'_n \to \infty \) a.s. so does \( X''_n \to \infty \) a.s. The random times \( T(i) \) are defined just as before, and again we put

\[
7' \quad X''_{i+1} = J \quad \text{if} \quad X''_{T(i)} = \alpha_j.
\]

For just the same reasons as before \( (X''_n) \) is a random walk process, and we have in the same way

Lemma 5'. If the random walk \( (X''_n) \) is transient (i.e., if \( X''_n \to \infty \) a.s.), then so is the original process \( (X_n) \).

4. Transition Probabilities

We define

\[
8 \quad p^*_j = \Pr(X^*_{n+1} = j+1 | X^*_n = j) \quad \text{and} \quad p''_j = \Pr(X''_{n+1} = j+1 | X''_n = j).
\]
The purpose of this section is to obtain estimates for $p_j^\star$ and $p_j^{\star\star}$. We actually shall carry out the calculation of an over-estimate for $p_j^\star$; the under-estimate of $p_j^{\star\star}$ is obtained in the same way, and the result turns out to be of the same form.

We are assuming that the transition function $F(x, y)$ of $\{X_n\}$ satisfies (1) to (4). Now observe that

(9) $p_j^\star = \Pr(\{X_n\} \text{ leaves } (\alpha_{j-1}, \alpha_{j+1} - M) \text{ on the right } | X_0 = \alpha_j)$.

Let $\{Y_n\}$ be a homogeneous process with the transition function

$G(x, y) = F(\alpha_{j-1}, \alpha_{j-1} + y - x)$, and let $Y_0 = \alpha_j$. Then for all $x \geq \alpha_{j-1}$ (and so all $x$ in the interval under consideration) we have from (4)

$G(x, y) = F(\alpha_{j-1}, \alpha_{j-1} + y - x) \leq F(x, y)$.

By Lemma 3, therefore, the probability that, starting at $\alpha_j$, $\{Y_n\}$ leaves the interval $(\alpha_{j-1}, \alpha_{j+1} - M)$ on the right exceeds the analogous probability for $\{X_n\}$, which by (9) is $p_j^\star$.

We turn therefore to the study of the homogeneous process $\{Y_n\}$. This process may be considered as successive sums of independent random variables $Z_i$, each with the distribution

$\Pr(Z_1 \leq y) = G(x, x+y)$ (for all $x$).

In [5], A. Wald has given a method of calculating the "absorption" probability for such processes; let $Y_0 = \alpha_j$, $Y_n = Z_1 + Z_2 + \ldots + Z_n + Y_0$, and let $S$ be the first time at which $Y_S \notin (\alpha_{j-1}, \alpha_{j+1} - M)$. Then

(10) $p_j^\star \leq \Pr(Y_S \geq \alpha_{j+1} - M) = \frac{E_0 - 1}{E_0 - E_1}$,
where \( E_0 = E[\exp \lambda_0 (Y_\alpha - \alpha_j) | Y_\alpha \leq \alpha_{j-1}] \) and \( E_1 = E[\exp \lambda_0 (Y_\alpha - \alpha_j) | Y_\alpha \geq \alpha_{j+1} - M] \). Here \( \lambda_0 \) is the unique non-zero solution of the equation

\[
E[\exp \lambda Z_1] = \int_0^\infty e^{\lambda y} \, dG(x, x+y) = 1,
\]

which exists provided that \( E[Z_1] \neq 0 \). We shall use this method below when \( E(Z_1) > 0 \), and so \( \lambda_0 \) will always be negative. Since \( |Z_1| \leq M \) a.s. we then have estimates for \( E_0 \) and \( E_1 \):

(11) \[ 1 < \exp[-\lambda_0 (\alpha_j - \alpha_{j-1})] \leq E_0 \leq \exp[-\lambda_0 (\alpha_j - \alpha_{j-1} + M)]; \]

(12) \[ \exp[\lambda_0 (\alpha_{j+1} - \alpha_j)] \leq E_1 \leq \exp[\lambda_0 (\alpha_{j+1} - \alpha_j - M)] < 1. \]

It remains to calculate \( \lambda_0 \), and then (10) together with (11) and (12) permit an estimate of \( p_j^* \).

Let us put

(13) \[ \mu(x) = \int_{-\infty}^{\infty} F(x, y) \, dy; \quad s(x) = \int_{-\infty}^{\infty} (y-x)^2 \, dy \quad F(x, y). \]

From properties (2) and (4), \( F(x, x+y) \) form, as \( x \to \infty \), a monotone increasing family of distributions (in \( y \)) on \((-M, M)\). It follows that \( \mu(x) \) decreases, and that \( \mu(x) \) and \( s(x) \) tend to limits as also does

\[ \Phi_\alpha(x) = \int_0^\infty e^{\lambda (y-x)} \, dF(x, y) = \int_{-M}^{M} e^{\lambda y} \, dF(x, x+y). \]

We can also deduce that

(14) \[ |\lim_{x \to \infty} s(x) - s(x)| \leq 2M |\lim_{x \to \infty} \mu(x) - \mu(x)|. \]

**Lemma 6.** Suppose that \( \mu(x) \) is positive for all \( x \) but approaches zero, and let \( \sigma^2 = \lim_{x \to \infty} s(x) \neq 0 \). Let \( \lambda_0(x) \) be the unique negative root of the equation
\[ \varphi_x(\lambda) = 1. \]

Then as \( x \) tends to \( \infty \) we have

\[ \lambda_0(x) = -\frac{2\mu(x)}{\sigma^2} + O(\mu^2(x)). \]  

**Proof.** As \( x \to \infty \) and \( F(x, x+u) \) approaches a distribution \( F(y) \) with mean 0, \( \varphi_x(\lambda) \) approaches the Laplace transform of \( F(x) \) which has both its real \( \lambda \)-values at \( \lambda = 0 \). Therefore \( \lambda_0(x) \to 0 \) as \( x \to \infty \). Hence we can write

\[
\varphi_x(\lambda_0(x)) = \int_{-M}^{M} \left[ 1 + u\lambda_0(x) + \frac{[u\lambda_0(x)]^2}{2} + \lambda_0^3(x) u^3 K(x) \right] \delta p(x, x+u) \frac{d}{u} \\
= 1 + \lambda_0(x) \mu(x) + \frac{\lambda_0^2(x)}{2} s(x) + \lambda_0^3(x) K^*(x),
\]

where \( K(x) \) and \( K^*(x) \) are bounded for large \( x \). The equation \( \varphi_x(\lambda_0) = 1 \) becomes then

\[
1 + \frac{\lambda_0(x)}{2\mu(x)} s(x) + \frac{\lambda_0^2(x)}{\mu(x)} K^*(x) = 0.
\]

Since the last term is of smaller order of magnitude than the second, we have that \( -\lambda_0(x) s(x)/2\mu(x) \to 1 \). Put now \( \lambda_0(x) = -2\mu(x)/s(x) + \epsilon \); the result is

\[
\frac{\epsilon}{2} s(x) = -K^*(x) \left[ \frac{\mu^2(x)}{s(x)} - \frac{\mu(x)\epsilon}{s(x)} + \epsilon^2 \right].
\]

But from what we have shown \( \epsilon(x) = o(\mu(x)) \), so that the first term on the right-hand side dominates; in other words, \( \epsilon = 0(\mu^2) \). By (14) we have

\[
\frac{s(x)}{\sigma^2} = [1 + o(\mu(x))],
\]
and combining all these things gives (15).

We shall now specify that \( \{\alpha_j\} = \{j^3\} \times \). This is arbitrary in that any power greater than 2 would do, but the cube is convenient. (Actually formula (17) below is different if \( \alpha_j = j^p \) with \( p \neq 3 \), but the final result does not depend on \( p \).)

Lemma 7. Suppose that \( F(x, y) \) satisfies (1) to (4), that \( s(x) \) approaches \( \sigma^2 > 0 \), and that for some constant \( \beta \),

\[
\mu(x) = \frac{\beta}{x} + O\left(\frac{1}{x^2}\right) \quad \text{as} \quad x \to \infty.
\]

Then

\[
\phi_j^* \leq \frac{1}{2} \left(1 + \left(\frac{3\beta}{\sigma^2} - 1\right) \frac{1}{j} + O\left(\frac{1}{j^2}\right)\right).
\]

Proof. We have only to calculate using the tools already set out: with \( \alpha_j = j^3 \), substitute (11) and (12) into (10) in such a way as to increase the right-hand side. \( \lambda_0(x) \) is given by (15) with \( x = \alpha_{j-1} = (j-1)^3 \) and \( \mu(x) \) as in the hypothesis. When the exponentials in (10) are expanded as far as terms in \( j^{-2} \) and the resulting expression simplified, (17) is the result.

This entire procedure may be imitated with only trivial changes in order to estimate the \( \phi_j^{**} \). The result is the following

Lemma 7'. Under the hypotheses of Lemma 7,

\[
\phi_j^{**} \geq \frac{1}{2} \left(1 + \left(\frac{3\beta}{\sigma^2} - 1\right) \frac{1}{j} + O\left(\frac{1}{j^2}\right)\right).
\]

\[\star\] Strictly speaking, we must have \( \alpha_{j+1} - \alpha_j > 2M \), so some terms at the start of the sequence \( \{j^3\} \) may have to be omitted.
5. Results

We say that the stochastic transition function $F(x, y)$ "eventually dominates" $G(x, y)$ provided

$$F(x, y) \geq G(x, y)$$

for all $y$ and large $x$. Our main result is

Theorem 1. Let $\{X_n\}$ be a real Markov process whose transition function $F(x, y)$ satisfies (1) to (3). Suppose that there exists another transition function $G(x, y)$ satisfying (2), (3), (4) and (16), and such that $F(x, y)$ eventually dominates $G(x, y)$. Then if for the function $G$ we have $\beta \leq \frac{\sigma^2}{2}$, the process $\{X_n\}$ is recurrent. Alternatively, if there exists $G(x, y)$ which eventually dominates $F(x, y)$, and satisfies (2), (3), (4) and (16) with $\beta > \frac{\sigma^2}{2}$, then $X_n \to \infty$ with probability one.

Proof. We need the following result, which is to be found (in slightly different forms) in [2] and [3]: let $p_j$ be the probability of a step from $j$ to $j+1$ for a random walk on the non-negative integers; suppose that $0 < p_j < 1$ and $p_j = \frac{1}{2} [1 + \frac{\gamma}{j} + O\left(\frac{1}{j^2}\right)]$. Then the walk is recurrent if $\gamma \leq \frac{1}{2}$, and transient if $\gamma > \frac{1}{2}$.

In the first part of Theorem 1, consider a process $\{U_n\}$ with $G(x, y)$ for its stochastic transition function, and form the random walk $\{U_n^*\}$ as outlined in Section 3. By Lemma 7 and the hypothesis $\beta \leq \frac{\sigma^2}{2}$, together with the theorem quoted above, the right transition probabilities $p_j^*$ of $\{U_n^*\}$ are less than or equal to the corresponding probabilities for a recurrent random walk. By Theorem 2 of [3], $\{U_n^*\}$ is then recurrent, and
by Lemma 5, it follows that the process \( \{U_n\} \) is recurrent. Finally, Lemma 2 and the hypothesis of eventual dominance allow the conclusion that \( \{X_n\} \) is also recurrent. The proof of the other part of the theorem is very much the same, except that Lemmas 7' and 5' are used.

We shall now consider two applications of this theorem. The first is to "generalized random walks"; that is, to processes obeying a transition law of the form

\[
X_{n+1} = \begin{cases} 
X_n + a & \text{with probability } \varphi(X_n) \\
\max(X_n - b, 0) & \text{with probability } 1 - \varphi(X_n).
\end{cases}
\]

(18)

Here \( a \) and \( b \) are positive constants, and \( \varphi \) is a measurable function such that \( 0 \leq \varphi(x) \leq 1 \).

Theorem 2. Let \( \{X_n\} \) be a generalized random walk satisfying (18).

If

\[
\varphi(x) \leq \frac{b}{a+b} + \frac{1}{2x} \frac{ab}{a+b} + O\left(\frac{1}{x^2}\right),
\]

(19)

then \( X_n \) is recurrent. On the other hand, if for \( x \) large

\[
\varphi(x) \geq \frac{b}{a+b} + \frac{\beta}{x} \text{ for some } \beta > \frac{ab}{2(a+b)},
\]

(20)

then \( X_n \to \infty \) a.s.

Proof. For a process of this form, (4) is equivalent to \( \varphi(x) \) being non-increasing. We can construct another stochastic transition function \( G(x, y) \) by using again transition law (18) with the same \( a \) and \( b \), but another function \( \psi(x) \) in place of \( \varphi(x) \); the condition \( F(x, y) \) eventually dominates \( G(x, y) \) is then equivalent to the inequality \( \psi(x) \geq \varphi(x) \) for all large enough \( x \).
To prove the first part of Theorem 2, let \( \psi(x) \) be chosen to satisfy the three conditions

\[
\psi(x) = \frac{b}{a+b} + \frac{1}{2x} \frac{ab}{a+b} + O\left(\frac{1}{x^2}\right); \quad \psi(x) \downarrow \text{ for } x \geq 0;
\]

\[\psi(x) \geq \phi(x) \text{ for all large enough } x.\]

(Such a choice is possible if (19) holds.) For the transition law \( g(x, y) \) we then have

\[\mu(x) = \frac{ab}{2x} + O\left(\frac{1}{x^2}\right), \quad \sigma^2 = ab,\]

and it follows from Theorem 1 that \( \{X_n\} \) is recurrent. The proof of the second assertion is very similar.

Remarks. In particular, if \( \phi(x) \leq b/(a+b) \) for all large \( x \), \( \{X_n\} \) is recurrent; if \( \liminf \phi(x) > b/(a+b) \), then \( \{X_n\} \) is transient. These much weaker results follow directly from Lemma 2, using for comparison a process consisting of sums of random variables with the values \( a \) and \( -b \). Another observation is that Theorem 2 still holds if with positive probability \( X_{n+1} \) can equal \( X_n \); it is only necessary to reinterpret \( \phi(X_n) \) to be

\[\Pr(X_{n+1} = X_n + a \mid X_{n+1} \neq X_n).\]

Another interesting special class of Markov processes are those of the form

\[X_{n+1} = \max \left(0, X_n + \mu(X_n) + \xi_{n+1}\right),\]

where \( \{\xi_n\} \) is a sequence of independent, bounded, identically distributed random variables with (at no loss of generality) mean \( 0 \) and variance \( \sigma^2 \), and \( \mu(x) \) is a bounded measurable function.
Theorem 3. Let \((X_n)\) be a real Markov process with a transition law of the form (21). If

\[
\mu(x) \leq \frac{\sigma^2}{2x} + 0 \left( \frac{1}{x^2} \right),
\]

then \((X_n)\) is recurrent. If for all large \(x\),

\[
\mu(x) \geq \frac{\beta}{x} \text{ for some } \beta > \frac{\sigma^2}{2},
\]

then \(X_n \to \infty\) a.s.

Proof. For a process of this type, the function \(\mu(x)\) in (21) is also the \(\mu(x)\) defined in (13), and the same is true of \(\sigma^2\). The monotonicity condition (4) is equivalent to \(\mu(x)\) being non-increasing, and for two transition laws of the form (21) with the same distribution of \(\xi_n\), the "eventual dominance" condition reduces to an inequality between the corresponding functions \(\mu(x)\) and \(\nu(x)\).

We shall prove in detail only the second part of the theorem. Let \(\nu(x)\) be a bounded, non-increasing function such that for all large \(x\)

\[
\nu(x) \leq \mu(x), \quad \text{and} \quad \nu(x) = \frac{\gamma}{x} + 0 \left( \frac{1}{x^2} \right), \quad \gamma > \frac{\sigma^2}{2};
\]

it is clear from the hypothesis of the theorem that such a function exists. Let \(G(x,y)\) be the stochastic transition function for a process obeying law (21) with \(\nu(x)\) in place of \(\mu(x)\). This function then satisfies the requirements of Theorem 1, and the conclusion may be drawn that \(X_n \to \infty\) a.s.

Finally, we consider the task of putting Theorem 1 into a form more easily applied. This entails a loss of generality, so the theorem as previously stated remains the main result of the paper. The idea of the
modified version is that when $F(x, y)$ is of sufficiently regular behavior for large $x$, it is no longer necessary to postulate the existence of the comparison function $G(x, y)$.

Let us suppose that we can write

$$F(x, x+y) = F(y) + \frac{\theta(y)}{x^\gamma} + \epsilon(x, y),$$

where $F(y)$ is a distribution function on $[-M, M]$ with mean 0 and variance $\sigma^2$, $\theta(y) \leq 0$, $\theta(y) = 0$ for $y \notin [-M, M]$; the function $F(y) + \theta(y)/x^\gamma$ is monotone in $y$ for some $x > 0$, and $\epsilon(x, y) = o(x^{-\gamma} \theta(y))$ uniformly in $y$; put

$$\beta = \int_{-M}^{M} y \, d\theta(y) \geq 0.$$

**Theorem 4.** Let the stochastic transition function $F(x, y)$ of $\{X_n\}$ satisfy (1) to (3) and (22). Then $\{X_n\}$ is recurrent when $\gamma > 1$, or when $\gamma = 1$ and $\beta < \sigma^2/2$; if $\gamma < 1$, or if $\gamma = 1$ and $\beta > \sigma^2/2$, then $X_n \to \infty$ a.s.

**Proof.** We shall only examine the first case with $\gamma = 1$; all the rest is much the same. Since $\sigma^2/2\beta > 1$, there is an $N$ such that for $x \geq N$ the function

$$G(x, x+y) = F(y) + \frac{\sigma^2 \theta(y)}{2\beta x}$$

satisfies $G(x, x+y) \leq F(x, y)$. In addition, take $N$ large enough to ensure that for $x \geq N$, $G(x, x+y)$ is monotone in $y$. Define a transition function $G(x, y)$ by (23) for $x \geq N$, and by
\begin{equation*}
G(x, x+y) = G(N, N+y)
\end{equation*}

for \( x \leq N \) (except that a modification is necessary near \( x = 0 \)). Then \( G(x, y) \) satisfies (4) and eventually \( F(x, y) \) dominates \( G(x, y) \). But for \( G(x, y) \) we have

\begin{equation*}
\mu(x) = \int_{-M}^{M} y d[F(y) + \frac{\sigma^2}{2\beta} \frac{\theta(y)}{x}] = \frac{\sigma^2}{2x}.
\end{equation*}

Thus all the requirements of Theorem 1 have been met, and the conclusion that \( \{X_n\} \) is recurrent follows.
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