ON TESTS WITH LIKELIHOOD RATIO CRITERIA
IN SOME PROBLEMS OF MULTIVARIATE ANALYSIS

BY
NARAYAN CHANDRA GIRI

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Introduction.

The likelihood ratio principle has been found quite successful in leading to satisfactory test procedures in many testing problems of univariate and multivariate analysis, though it is not based on any clearly defined optimum consideration. It has been shown (see for example Wald [7] and Le Cam [3]) that these procedures are nearly optimum when the sample size is sufficiently large. Furthermore since it is based on the ratio of maximum likelihoods it is invariant under all transformations which leave the problem invariant, and it often leads to a test criterion which is uniformly most powerful in the class of all invariant tests. On the other hand there exist examples (see for example Lehmann [5], p. 252) where this procedure is worse than useless, where in fact it is so bad that one can do better without taking any observations, and also cases where it does not lead to the uniformly most powerful test. In this investigation we will examine the behavior of this principle in two different testing problems concerning the mean vector and the covariance matrix of a non-singular multivariate normal population. Let us assume that the random vector

\[
\mathbf{X} = \begin{pmatrix}
\mathbf{x}_1 \\
\mathbf{x}_2 \\
\vdots \\
\mathbf{x}_p
\end{pmatrix}
\]
has a multivariate normal distribution with mean vector

$$\xi = E(X) = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_p \end{pmatrix}$$

and the non-singular covariance matrix $\Sigma = E(X-\xi)(X-\xi)'$. Its probability density with respect to Lebesgue measure in p-space is;

$$\frac{1}{(2\pi)^{\frac{p}{2}}|\Sigma|^{\frac{1}{2}}} e^{-\frac{1}{2}(X-\xi)' \Sigma^{-1}(X-\xi)}.$$ 

We will consider here the following two problems: (1) to test the hypothesis that the mean vector $\xi$ belongs to $\mathcal{Z}$ against the alternative that it belongs to $\mathcal{Y}$, where $\mathcal{Z} \subset \mathcal{Y}$ are sub-spaces of the space $\Pi_\Omega$ of $\xi$'s such that $\mathcal{Z}$ is of strictly lower dimension than $\mathcal{Y}$ and $\Sigma$ is unknown; (ii) to test the hypothesis that $\Sigma^{-1} \xi$ belongs to $\mathcal{Z}'$ against the alternative that $\Sigma^{-1} \xi$ belongs to $\mathcal{Y}'$, where $\mathcal{Z}' \subset \mathcal{Y}'$ are linear sub-spaces of the adjoint space $\mathcal{L}'$ on the space of $X$'s such that $\mathcal{Z}'$ is of strictly lower dimension than $\mathcal{Y}'$ and $\Sigma, \xi$ are both unknown.

It will be shown later that the likelihood ratio test for problem (ii) is the uniformly most powerful invariant similar test, whereas if $N-p, p-q$, and $q$ ($N = \text{Sample size}, p-q = \text{dim. of } \mathcal{Y}$) are of same order of magnitude and
large then the likelihood ratio test for problem (i) is nearly uniformly most powerful. For problem (ii) a test based on the discriminant function has been suggested by Fisher [Ref. Rao [6], Secs. 7b.4, 7b.5] which is identical to the one we will obtain in section 2 from the likelihood ratio principle. But as far as I am aware nothing is known about its optimum property.

Section 1

1.1. Formulation of the problem:

Let the random vector

\[
X = \begin{pmatrix}
X_1 \\
X_2 \\
\vdots \\
X_p
\end{pmatrix}
\]

have a multivariate normal distribution with mean \( \xi \) and non-singular covariance matrix \( \Sigma \). Its density with respect to ordinary Lebesgue measure in the \( p \)-space is

\[
n(X|\xi,\Sigma) = \frac{1}{(2\pi)^{p/2}|\Sigma|^{1/2}} e^{-\frac{1}{2} \text{trace } \Sigma^{-1}(X-\xi)(X-\xi)'}.
\]

Let the distribution law of \( X \) be denoted by \( N(\xi,\Sigma) \). In this section we are interested in testing the hypothesis \( H_{10} \) that the mean vector \( \xi \) lies in a \( (p-p') \) dimensional subspace \( \mathcal{Z} \) against
the alternative \( H_1 \) that it lies in a \((p-q)\) dimensional sub-space \( \gamma' \) of the \( p \)-dimensional space \( \prod_{\Omega} \gamma' \) of \( \xi \)'s where \( q < p' < p \). In general this testing problem can be reduced to a more convenient form, viz., to test the hypothesis

\[
(1.1) \quad H_{10} : \xi_1 = \xi_2 = \cdots = \xi_{p'} = 0
\]

against \( H_1 : \xi_1 = \xi_2 = \cdots = \xi_q = 0 \) with \( q < p' < p \), when \( \Sigma \) is unknown.

1.2. **Likelihood ratio test of** \( H_{10} \) **against** \( H_1 \):

Let \( \{ \chi^\alpha \} \quad \alpha = 1, 2 \ldots N \quad (N > p) \) be a random sample of size \( N \) from \( N(\xi, \Sigma) \). The likelihood function of \( \{ \chi^\alpha \} \) is

\[
(1.2) \quad L(\xi, \Sigma) = \prod_{\alpha=1}^{N} n(\chi^\alpha | \xi, \Sigma)
\]

\[
= \frac{1}{(2\pi)^{(Np/2)} |\Sigma|^{(N/2)}} e^{-\frac{1}{2} \text{trace} \sum_{\alpha=1}^{N} \Sigma^{-1}(\chi^\alpha - \xi)(\chi^\alpha - \xi)'}
\]

The observations are given and \( L \) is a function of the variables \( \xi \) and \( \Sigma \). The likelihood criterion is:

\[
(1.3) \quad \lambda = \frac{\max_{H_{10}} L(\xi, \Sigma)}{\max_{H_1} L(\xi, \Sigma)}
\]
Lemma 1.1

$$\max_{H_{10}} L(\xi, \Sigma) = \frac{|S_{22}|^{(N/2)} e^{-(N\beta/2)}}{(2\pi)^{(N\beta/2)}|S_{22} + \bar{N}\bar{x}_{[2]} \bar{x}_{[2]}'|^{(N/2)}|S|^{(N/2)}}$$

where

$$S_{22} = \sum_{\alpha=1}^{N} (x_{[2]}^{\alpha} - \bar{x}_{[2]}) (x_{[2]}^{\alpha} - \bar{x}_{[2]})'$$

$$S = \sum_{\alpha=1}^{N} (x^{\alpha} - \bar{x}) (x^{\alpha} - \bar{x})'$$

$$\bar{N} \bar{x}_{[2]} = \sum_{\alpha=1}^{N} x_{[2]}^{\alpha}$$

$$\bar{N} \bar{x} = \sum_{\alpha=1}^{N} x^{\alpha}$$

$$x_{[2]} = \begin{pmatrix} x_{1} \\ x_{2} \\ \vdots \\ x_{p} \end{pmatrix}$$

and

$$x = \begin{pmatrix} x_{[2]} \\ x_{[3]} \end{pmatrix}$$
Proof:

\[ L(\xi, \Sigma) = \frac{1}{(2\pi)^{(Np/2)}|\Sigma|^{(N/2)}} \cdot e^{-\frac{1}{2} \sum_{\alpha=1}^{N} (x_\alpha - \xi)^\top \Sigma^{-1} (x_\alpha - \xi)} \]

\[ = \frac{1}{(2\pi)^{(Np'/2)}|\Sigma_{22}|^{(N/2)}} \cdot e^{-\frac{1}{2} \sum_{\alpha=1}^{N} (x_{2\alpha} - \xi_{2\alpha})^\top \Sigma_{22}^{-1} (x_{2\alpha} - \xi_{2\alpha})} \]

\[ \cdot \frac{1}{(2\pi)^{(N(p'-p)/2)}|\Sigma_{33\cdot2}|^{(N/2)}} \cdot e^{-\frac{1}{2} \sum_{\alpha=1}^{N} [(x_{3\alpha} - \xi_{3\alpha})^\top \Sigma_{32} \Sigma_{22}^{-1} (x_{3\alpha} - \xi_{3\alpha})]} \]

\[ \cdot \Sigma_{33\cdot2}^{-1} [(x_{3\alpha} - \xi_{3\alpha})^\top \Sigma_{32} \Sigma_{22}^{-1} (x_{3\alpha} - \xi_{3\alpha})] \cdot \]

where

\[ E(x_{2\alpha}) = \xi_{2\alpha} \] \[ E(x_{3\alpha}) = \xi_{3\alpha} \]

\[ E(x_{2\alpha} - \xi_{2\alpha}) (x_{2\alpha} - \xi_{2\alpha})^\top = \Sigma_{22} \]

\[ E(x_{3\alpha} - \xi_{3\alpha}) (x_{3\alpha} - \xi_{3\alpha})^\top = \Sigma_{33} \]

\[ E(x_{3\alpha} - \xi_{3\alpha})^\top (x_{2\alpha} - \xi_{2\alpha}) = \Sigma_{32} \]

\[ E(x_{2\alpha} - \xi_{2\alpha}) (x_{3\alpha} - \xi_{3\alpha})^\top = \Sigma_{23} \]

and \[ \Sigma_{33\cdot2} = (\Sigma_{33} - \Sigma_{32} \Sigma_{22}^{-1} \Sigma_{23}) \]
By Anderson\(^1\), p. 46,

\[
\max_{H_{10}} L(\xi, \Sigma) = \frac{e^{-(N\beta'/2)}}{(2\pi)^{(N\beta'/2)} |S_{22} + N \bar{X}_{[2]} |^{(N/2)}} \cdot \frac{e^{-(N(p-p')/2)}}{(2\pi)^{(N(p-p')/2)} |S_{33} - S_{32} s_{22}^{-1} s_{23} |^{(N/2)}}
\]

(1.5)

\[
= \frac{e^{-(N\beta/2)}}{(2\pi)^{(N\beta/2)} |S_{22} + N \bar{X}_{[2]} |^{(N/2)} |S_{33} - S_{32} s_{22}^{-1} s_{23} |^{(N/2)}}
\]

\[
= |s_{22}|^{(N/2)} e^{-(N\beta/2)}
\]

\[
= \frac{e^{-(N\beta/2)}}{(2\pi)^{(N\beta/2)} |S_{22} + N \bar{X}_{[2]} |^{(N/2)} |S|^{(N/2)}}
\]

where

\[
S = \begin{pmatrix} s_{22} & s_{23} \\ s_{32} & s_{33} \end{pmatrix}
\]

Q.E.D.

**Lemma 1.2**

\[
\max_{H_1} L(\xi, \Sigma) = \frac{e^{-(N\beta/2)} |s_{11}|^{(N/2)}}{(2\pi)^{(N\beta/2)} |S_{11} + N \bar{X}_{[1]} |^{(N/2)} |S|^{(N/2)}}
\]

(1.6)
where

\[ S_{11} = \sum_{\alpha=1}^{N} (X_{[1]}^\alpha - \bar{X}_{[1]}) (X_{[1]}^\alpha - \bar{X}_{[1]})' \]

\[ N \bar{X}_{[1]} = \sum_{\alpha=1}^{N} X_{[1]}^\alpha \]

and

\[ X_{[1]} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_q \end{pmatrix} \]

Proof:

Exactly similar to Lemma 1.1 \quad Q.E.D.

Before proceeding further, to evaluate the likelihood ratio we need lemmas 1.3 and 1.4. The proofs of these lemmas are available in any standard text book of matrix theory, so we will only state them, omitting the proofs.

**Lemma 1.3**

If \( |B| \neq 0 \) then,

\[ \begin{vmatrix} B & C \\ D & E \end{vmatrix} = |B| \left| E - DB^{-1}C \right| \]

\[ (1.7) \]

**Lemma 1.4**

If \( S \) is symmetric then

\[ N \bar{X}' (S + N \bar{X} \bar{X}')^{-1} \bar{X} = \frac{N \bar{X}' S^{-1} \bar{X}}{1 + N \bar{X}' S^{-1} \bar{X}} \]

\[ (1.8) \]
From (1.5) and (1.6) we get

\[
\lambda = \max_{H_0} \frac{L(\xi, \Sigma)}{\max_{H_1} L(\xi, \Sigma)} = \frac{|S_{11}^{\frac{1}{2}} X_{[1]}^{t} \bar{X}_{[1]}^{t} S_{11}^{\frac{1}{2}}|^{(N/2)}}{|S_{22}^{\frac{1}{2}} X_{[2]}^{t} \bar{X}_{[2]}^{t} S_{22}^{\frac{1}{2}}|^{(N/2)}}
\]

Applying Lemma 1.3, we get from (1.9)

\[
\lambda^{(2/N)} = \frac{1 + N \bar{X}_{[1]}^{t} S_{11}^{-1} \bar{X}_{[1]}}{1 + N \bar{X}_{[2]}^{t} S_{22}^{-1} \bar{X}_{[2]}}
\]

From (1.10) it is evident that the components \(X_{p'+1}, X_{p'+2}, \ldots, X_{p}\) of \(\bar{X}\) do not enter into our picture. Hence for notational convenience we will take from now on \(p' = p\) i.e., \(X_{[2]} = \bar{X}\) without disturbing the original set-up of the problem. Thus, on the basis of observations \(X_{[1]}, X_{[2]}^{t}, \ldots, X_{[N]}^{t}\) from \(N(\xi, \Sigma)\) with \(N > p > q\), the likelihood ratio criterion for testing,

\[
H_{10} : \xi_1 = \xi_2 = \cdots = \xi_p = 0
\]

against \(H_1 : \xi_1 = \xi_2 = \cdots = \xi_q = 0\), when \(\Sigma\) is unknown, is

\[
\lambda^{(2/N)} = \frac{1 + N \bar{X}_{[1]}^{t} S_{11}^{-1} \bar{X}_{[1]}}{1 + N \bar{X}_{[1]}^{t} S_{11}^{-1} \bar{X}_{[1]}}
\]

which we will call the \(Z\) statistic. Hence we get the following theorem:

**Theorem 1.1**

The likelihood ratio test of the hypothesis \(H_{10} : \xi = 0\) for the distribution \(N(\xi, \Sigma)\) is given by \(Z \leq Z_0\), where \(Z\) is defined above.
and \( Z_0 \) is chosen in such a way that the probability that \( Z \leq Z_0 \) under \( H_{10} \) is equal to the chosen level of significance.

**Corollary to theorem 1.1:**

The rejection region of the likelihood ratio test of the hypothesis \( H_{10} : \xi = 0 \) for the distribution \( N(\xi, \Sigma) \) is given by

\[
(1,12) \quad R_2 + (1-Z_0)R_1 \geq 1 - Z_0
\]

where

\[
R_1 = N \overline{X}^T_{[1]} (S_{11} + N \overline{X}_{[1]} \overline{X}_{[1]}^T)^{-1} \overline{X}_{[1]}
\]

\[
R_1 + R_2 = N \overline{X}^T (S + N \overline{X} \overline{X}^T)^{-1} \overline{X}
\]

and \( Z_0 \) is determined by the fact that \( P(Z \leq Z_0) \) is equal to the chosen level of significance.

**Proof:**

By Lemma 1.4

\[
R_1 = \frac{N \overline{X}^T_{[1]} S^{-1}_{11} \overline{X}_{[1]}}{1 + N \overline{X}^T_{[1]} S^{-1}_{11} \overline{X}_{[1]}}
\]

or

\[
N \overline{X}^T_{[1]} S^{-1}_{11} \overline{X}_{[1]} = \frac{R_1}{1-R_1}
\]

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Similarly

\[ N^{-1} S^{-1} \frac{\bar{X}}{X} = \frac{R_1 + R_2}{1 - R_1 - R_2} \]

Hence from the above theorem the rejection region of the likelihood ratio test is \[ \frac{1 - R_1 - R_2}{1 - R_1} \leq Z_0 \], which can be expressed equivalently as

\[ R_2 + (1 - Z_0) R_1 \geq 1 - Z_0 \quad \text{Q.E.D.} \]

We will show later that if \( N-p, p-q \) are large then \( Z_0 \) is given by

\[ Z_0 = \frac{N-p}{N-q} + Z_\alpha \left( \frac{2(N-p)(p-q)}{(N-q)^2} \right)^{1/2} \]

where \( Z_\alpha \) is \( \alpha \) percentile point of a unit normal distribution.

1.3 Distribution of \( Z \) Under \( H_{10} : \xi = 0 \)

The probability density of \( X \) under \( H_{10} \) is,

\[
(1.13) \quad n(X|0, \Sigma) = \frac{1}{(2\pi)^{p/2} |\Sigma|^{1/2}} \ e^{-(1/2) \text{trace} \Sigma^{-1} X X'}
\]

Since \( \Sigma \) is positive definite, it is well-known that there exists a \( p \times p \) non-singular matrix \( g = \begin{pmatrix} g_{11} & 0 \\ g_{21} & g_{22} \end{pmatrix} \) such that \( g \Sigma g' = I \)

where \( g' \) is the transpose of \( g \), \( I \) is the identity matrix, and \( g_{11} (q \times q) \), \( g_{22} [(p-q) \times (p-q)] \) are sub-matrices of \( g \). Let us assume that \( X \) has been transformed to the random variable \( Y \) by
means of the transformation $g$. The density of $Y$ under $H_{10}$ is then

\begin{equation}
(1.14) \quad n(Y|0,1) = \frac{1}{(2\pi)^{(p/2)}} e^{-(1/2) \text{ trace } Y Y'}.
\end{equation}

It can be verified that $Z$ is invariant under $g$. Hence by Rao [6], p. 71, $Z$ is a Beta distributed random variable with density function,

\begin{equation}
B_{(N-P, \frac{P-q}{2}, Z)} = \frac{\Gamma(\frac{N-q}{2})}{\Gamma(\frac{N-P}{2}) \Gamma(\frac{P-q}{2})} Z^{(N-P)/2}(1-Z)^{(p-q)/2} \text{ if } 0 < Z < 1
\end{equation}

\begin{equation}
= 0 \quad \text{otherwise.}
\end{equation}

**Asymptotic case:**

In deriving all asymptotic results in this section we will assume tacitly that $N-p$, $p-q$, $q$ are large and are of same order of magnitude as $N$. $B_{(N-P, \frac{P-q}{2}, Z)}$ is maximum at $Z = \frac{N-P}{N-q}$ if $N-p$, $p-q$ are both large. Expanding $\log_e B$ by Taylor's series at the point $Z = \frac{N-P}{N-q}$, we get

\begin{equation}
\log_e B_{(\frac{N-P}{2}, \frac{P-q}{2}, Z)} = \log_e \frac{1}{\sqrt{2\pi} \sigma_o^2} - \frac{1}{2} \left( \frac{Z - \frac{N-P}{N-q}}{\sigma_o} \right)^2 + O(\frac{1}{N})
\end{equation}

where

\begin{equation}
\sigma_o^2 = \frac{2(N-p)(p-q)}{(N-q)^2}
\end{equation}

Thus with an error of $O(\frac{1}{N})$ in this asymptotic case, the distribution of $Z$ is normal with mean $\frac{N-P}{N-q}$ and variance $= \sigma_o^2$. Hence when $N-p$, $p-q$ are both large the level $\alpha$ likelihood test of $H_{10}$ is given by [with an error of $O(\frac{1}{N})$]
\[ R_1 + bR_2 \geq 1 \quad \text{where} \]

\[ b = \left( \frac{p q}{N - q} - Z_\alpha \right)^{-1}, \quad \text{and} \quad Z_\alpha \quad \text{is the} \alpha \quad \text{percentile point} \]

of a unit normal distribution.

1.4 Invariance of the problem and maximal invariants:

The notion of invariance of a statistical decision problem under transformation is essentially the same as the notion of invariance in any branch of mathematics. It is generally true that if a problem with a unique solution is invariant under certain transformation, then that solution is also invariant under that transformation. Perhaps the principal reason for the strong intuitive appeal of invariant decision rules is the feeling that there should be a unique best way to analyze a collection of statistical data. I will not try to formulate the notion of invariance of a statistical decision problem as it is available in complete details in modern statistical literature (see for example Lehmann [5], Chapter 6, or Blackwell and Girshick [2], Sections 8.5 - 8.8). However it asserts that if a problem remains invariant under a group of transformations, then we should restrict ourselves to invariant tests and attempt to find out the best among them.

We have already pointed out that the likelihood ratio test is invariant under all transformations which keep the problem invariant. In our case it is easy to check that this problem is invariant under the groups \( G_1 \) and \( G_2 \).
where $G_1$ : groups of translations of the components

\[ X_{p' + 1}, X_{p' + 2} \ldots X_p \] of $X$;

$G_2$ : groups of $(p' \times p')$ non-singular matrices

\[(1.17) \quad g = \begin{pmatrix} g_{11} & 0 \\ g_{21} & g_{22} \end{pmatrix} \] where $g_{11}$ is $q \times q$,

which transforms the components $X_1, X_2 \ldots X_{p'}$ of $X$. Since all invariant tests are functions of maximal invariants, we need to find out the maximal invariants for this problem and their joint distribution. Once we do that we will assume all throughout that we have observed the values of the maximal invariants instead of $(X^\alpha) \alpha = 1, 2, \ldots N$ and find the most powerful invariant test of $H_{10}$ against $H_1$ based on maximal invariants. Finally we will show that this test and the likelihood ratio test are nearly the same if $N-p, p-q$, and $q$ are sufficiently large.

Let us now recall the definition of maximal invariant $T(X)$, $X \in \mathcal{X}$ under the group of transformations $G$ on $\mathcal{X}$. I will prefer to use the following definition instead of the abstract one (in terms of invariant partitions)

**Definition:**

$T(X)$ is maximal invariant under $G$ on $\mathcal{X}$ if (i) $T(X) = T(gX)$ for all $X \in \mathcal{X}$ and $g \in G$, (ii) $T(X) = T(Y)$ for $X, Y \in \mathcal{X}$ iff there exist a $g \in G$ such that $Y = gX$.
Lemma 1.5:

For testing $H_{10}$ against $H_{1}$ by means of observations

$(x^2)_{\alpha = 1, 2, \ldots, N}$ from $N(\xi, \Sigma)$, the maximal invariants under $G_{1}$ and $G_{2}$ are

$$T(x^{[1]}) = R_{1} = N \bar{x}^{'[1]} (S_{11} + N \bar{x}^{[1]} \bar{x}^{'[1]})^{-1} \bar{x}^{[1]}$$

and

$$T(x) = R_{1} + R_{2} = N \bar{x}^{'} (S + N \bar{x} \bar{x}^{'})^{-1} \bar{x}$$

It may be remarked that here we can also consider $p' = p$ i.e., $X^{[2]} = X$ without any loss of freedom; for it is easy to check that the components $x_{p' + 1}^{p}', x_{p' + 2}^{p}', \ldots, x_{p}^{p}$ of $X$ which are of no use to us can always be eliminated by means of $G_{1}$. Thus the only group we will consider is $G_{2}$ on the space of $X$'s ($=X^{[2]}$). To prove lemma 1.5 we need another well-known result which I will state in the form of a lemma.

Lemma 1.6:

If $XX' = YY'$ then there exists an orthogonal $(p \times p)$ matrix $O$ such that $OX = Y$, and conversely.

Proof of lemma 1.5:

Let $g : X \rightarrow gX$, $g \in G_{2}$ and $X \in \mathcal{X}$ then it can be shown that $(gX)^{[1]} = g_{11} X^{[1]}$, where $(gX)^{[1]}$ is the partition of $gX$ similar to $X$.
\[ T((gX)[1]) = T(g_{11}X[1]) \]
\[ = N \bar{X}'[1] g_{11} g_{11}^{-1} (S_{11} + N \bar{X}'[1])^{-1} g_{11}^{-1} g_{11} \bar{X}[1] \]
\[ = N \bar{X}'[1] (S_{11} + N \bar{X}'[1])^{-1} \bar{X}[1]. \]

Similarly \( T(gX) = T(X) \). Hence \( R_1, R_1 + R_2 \) are invariants under \( G_2 \). To show they are maximal invariants, let

\[ T(Y) = N \bar{Y}'(\bar{S} + N \bar{Y} \bar{Y}')^{-1} \bar{Y} \]

and

\[ T(Y[1]) = N \bar{Y}'[1] (\bar{S}_{11} + N \bar{Y}[1] \bar{Y}'[1])^{-1} \bar{Y}[1], \]

where \( Y \in \mathcal{X} \) and \( \bar{S}, \bar{S}_{11} \bar{Y}[1] \) and \( \bar{Y} \) are defined in terms of \( Y \) in the same way as \( S, S_{11} \bar{X}[1] \) and \( \bar{X} \) are defined.

Furthermore let

\[ T(X) = T(Y), \]

(1.18)
\[ T(X[1]) = T(Y[1]), \]

and let \( T \) be invariant under \( G_2 \). Since \((S + N \bar{X} \bar{X}')\) and \((\bar{S} + N \bar{Y} \bar{Y}')\) are positive definite there exist \( g^1, g^2 \in G_2 \) such that

\[ g^1(S + N \bar{X} \bar{X}') g^1' = I \]

and

\[ g^2(\bar{S} + N \bar{Y} \bar{Y}') g^2' = I. \]
From (1.16) we get

\[(g^1X)' (g^1X) = (g^2Y)' (g^2Y)\]

and

\[(g^3X)'[1] (g^3X)[1] = (g^2Y)'[1] (g^2Y)[1]\]

Thus by lemma 1.6 there exist \( O_1 (q \times q) \), \( O (p \times p) \) orthogonal matrices such that,

\[
Og^1X = g^2Y \quad \text{and} \\
O_1g^3X[1] = g^2Y[1]
\]

(1.19)

Now it is easy to see that the required form of \( O \) is

\[
O = \begin{pmatrix}
O_1(q \times q) & 0 \\
0 & O_2[(p-q) \times (p-q)]
\end{pmatrix}
\]

where \( O_2 \) is also orthogonal. From (1.19) we get

\[
Y = g^1X
\]

where

\[
g^1 = (g^2)^{-1} \quad g^1 \in G_2
\]

Q.E.D.

From lemma 1.5 it follows trivially that \( R_1 \) and \( R_2 \) as defined above are also maximal invariants under \( G_2 \). Let us now define
\[ \xi_1 = N \xi [1] \Sigma_{11}^{-1} \xi [1] \]

\[ \xi_1 + \xi_2 = N \xi' \Sigma^{-1} \xi \]

where

\[ \xi = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_p \end{pmatrix} \quad \text{and} \quad \xi [1] = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_q \end{pmatrix} . \]

clearly \( \xi_1 \geq 0 \) and \( \xi_2 \geq 0 \) and they are maximal invariants in the parametric space under \( G_1 \) and \( G_2 \). Our problem in terms of \( \xi_1 \) and \( \xi_2 \) reduces to that of testing the hypothesis

\[ H_{10} : \xi_1 = 0, \xi_2 = 0 \]

against the alternative \( H_1 : \xi_1 = 0, \xi_2 > 0 \).

1.5 **The most powerful invariant test of** \( H_{10} : \)

We have already remarked that any invariant test depends on the observations only through maximal invariants. Thus in order to obtain the most powerful test of \( H_{10} : \xi_1 = 0, \xi_2 = 0 \) against the alternative: \( \xi_1 = 0, \xi_2 = \xi_2 (> 0) \) we have to compute the probability ratio

\[
\frac{\text{dP}^* (R_1, R_2)}{\text{dP}^* (R_1, R_2)}
\]

where \( P^* \) and \( P^* \) are the distributions of \( R_1, R_2 \) under the alternative and \( H_{10} \) respectively. Before attempting to evaluate (1.21) we
need to state some familiar results and develop certain ideas concerning invariant measure and distribution of maximal invariants. Let $G$ be a group operating (not necessarily transitively) on the topological space $\mathcal{Y}$ and $\lambda$ be a left invariant measure (under $G$) in $\mathcal{Y}$. Assume there are given two probability densities $p_1$ and $p_2$ with respect to $\lambda$; i.e.,

$$p_1(S) = \int_S p_1(Z) \, d\lambda(Z),$$

$$p_2(S) = \int_S p_2(Z) \, d\lambda(Z), \quad S \subseteq \mathcal{Y},$$

and $p_1$ and $p_2$ vanish simultaneously. Let $f(Z)$ be a maximal invariant under $G$. Further let $P^*_i$ be the distribution of $f(Z)$ when $Z$ has distribution $P_i$ ($i=1,2$). Then under certain conditions (which are obviously true in this problem),

$$(1.22) \quad \frac{dP^*_2(f)}{dP^*_1(f)} = \frac{\int_G p_2(gZ) \, d\mu(g)}{\int_G p_1(gZ) \, d\mu(g)}$$

where $\mu$ is left invariant Haar measure in $G$. It may be remarked that in our problem $f(Z)$ is the pair $(R_1, R_2)$ and $Z$ is the pair $(\sqrt{N \, \overline{X}}, S)$.

**Lemma 1.7:**

$$(1.23) \quad \omega(\sqrt{N \, \overline{X}}, S) = \frac{d(\sqrt{N \, \overline{X}})}{|S| \sqrt{p+2/2}}$$
is a left invariant measure under $G_2$ in the space of $(\sqrt{N} \bar{x}, S)$

**Proof:**

The transformation $g : x \rightarrow g x$, $g \in G_2$ transforms $\sqrt{N} \bar{x}$ to $\sqrt{N} g \bar{x}$, and $S$ to $g S g'$. The Jacobian of the transformation from $\sqrt{N} \bar{x}$ to $\sqrt{N} g \bar{x}$ is $|g|$ and that of the transformation from $S$ to $g S g'$ is $|g|^{p+1}$.

Hence

$$
d\lambda (\sqrt{N} g \bar{x}, g S g') = \frac{d(\sqrt{N} g \bar{x}) \cdot d(g S g')}{|g S g'|^{(p+2)/2}}
$$

$$
= \frac{d(\sqrt{N} \bar{x}) \cdot dS |g|^{p+2}}{|S|^{(p+2)/2} |g|^{p+2}}
$$

$$
= d\lambda (\sqrt{N} \bar{x}, S) \quad . \quad Q.E.D.
$$

**Lemma 1.8:**

A left invariant Haar measure in $G_2$ is

$$(1.24) \quad d\mu(g) = \frac{dg}{|g_{11}|^q |g_{22}|^p}$$

where

$$dg = \prod_{i,j} dg_{i,j}$$
Proof:

Let

\[ g = \begin{pmatrix} g_{11} & 0 \\ g_{21} & g_{22} \end{pmatrix} \]

\[ h = \begin{pmatrix} h_{11} & 0 \\ h_{12} & h_{22} \end{pmatrix} \quad \text{and} \quad h, g \ \text{belong to} \ G_2. \]

\[ hg = \begin{pmatrix} h_{11} g_{11} & 0 \\ h_{21} g_{11} + h_{22} g_{21} & h_{22} g_{22} \end{pmatrix}. \]

It can be verified that

\[ \frac{\partial h g}{\partial g} = |h_{11}|^q |h_{22}|^p. \]

Hence

\[ d\mu(h g) = d(h g) = \frac{\partial(h g)}{|h_{11} g_{11}|^q |h_{22} g_{22}|^p} \]

\[ = \frac{d g |h_{11}|^q |h_{22}|^p}{|h_{11} g_{11}|^q |h_{22} g_{22}|^p} \]

\[ = d\mu(g). \quad \text{Q.E.D.} \]
The joint probability density of \( \sqrt{N} \bar{x} \) and \( S \) is

\[
(1.25) \quad P_{\xi, \Sigma}(\sqrt{N} \bar{x}, S) = K |S|^{(N-p-1)/2} e^{-(1/2) \text{trace } \Sigma^{-1}[S+N(\bar{x}-\xi)(\bar{x}-\xi)']} \\
= P_{\xi, \Sigma}(\sqrt{N} \bar{x}, S) \cdot \frac{1}{|S|^{(p+1)/2}},
\]

where

\[
(1.26) \quad P_{\xi, \Sigma}(\sqrt{N} \bar{x}, S) = K |S|^{(N/2)} e^{-(1/2) \text{trace } \Sigma^{-1}[S+N(\bar{x}-\xi)(\bar{x}-\xi)']} ,
\]

and \( K \) is an absolute constant depending on \( \Sigma \). Thus to obtain

\[
\frac{d\Omega_2^* (R_1, R_2)}{d\Omega_1^* (R_1, R_2)}
\]

we need to evaluate the integral:

\[
I = \int_{G_2} |g g'|^{(N/2)} e^{-(1/2) \text{trace } \Sigma^{-1}[g S g' + N(g \bar{x}-\xi)(g \bar{x}-\xi)']} \frac{dg}{|g_{11}|^q |g_{22}|^p}
\]

\[
= \int_{G_2} |g g'|^{(N/2)} |S|^{(N/2)} e^{-(1/2) \text{trace } \Sigma^{-1}[g(S+N \bar{x} \bar{x}')(g g' - 2 \xi \bar{x}' g' + \xi \xi')]} \times \frac{dg}{|g_{11}|^q |g_{22}|^p} .
\]

(1.27)
Since $S + N \overline{X} \overline{X}'$ is positive definite we can always find a non-singular $(p \times p$ lower triangular) matrix.

$$C : X \to Y$$

Such that

$$C(S + N \overline{X} \overline{X}') C' = I \quad \text{i.e.,} \quad N(\overline{C}X)(\overline{C}X)' = R_1 + R_2$$

and $N(\overline{C}X)_{[1]}(\overline{C}X)'_{[1]} = R_1$.

Let $CSC' = S^{**}$. Then from (1.27) we get

$$I = \frac{|S^{**}|(N/2)_K}{|S + N \overline{X} \overline{X}'|^{(1/2)}} \int_{G_2} gg' \left( \frac{N/2}{e^{-(1/2) \text{trace} \Sigma^{-1}(gg' - 2N\overline{X}'g' + \overline{X}' \overline{X}')} \times \frac{dg}{|g_{11}|^q |g_{22}|^p} \right)$$

$$= \frac{|S^{**}|(N/2)_K}{|S + N \overline{X} \overline{X}'|^{(1/2)}} \int_{G_2} e^{-(1/2)(\xi'_1 + \xi'_2)} |gg'| \left( \frac{N/2}{e^{-(1/2) \text{trace} \Sigma^{-1}(gg' - 2N\overline{X}'g' + \overline{X}' \overline{X}')} \times \frac{dg}{|g_{11}|^q |g_{22}|^p} \right)$$

Since we are interested in the ratio $\frac{dP^*}{dP^*_1}$ we can assume $\Sigma = I$, (which we will do throughout the remaining section) and try to evaluate the integral $I'$ instead of $I$. 

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Where

\[(1.28)\]

\[I' = \int_{G_2} e^{-\frac{1}{2}((\zeta_1 + \zeta_2) + (N/2) - (1/2) \text{trace}(gg' - 2N\hat{\gamma} g'))} \left| g_{11} \right|^p |g_{22}|^p \]

\[-\frac{(1/2)(\zeta_1 + \zeta_2)}{\left| g_{22} \right|} \left| g_{11} \right| - \frac{(N-q/2)}{2} \left( \sum_{j=1}^{2} g_{ij} g_{ij}' - 2N \sum_{j=1}^{2} \hat{\gamma}_{ij} g_{ij}' \right) \cdot \left| g_{11} \right|^p |g_{22}|^p \]

\[-\frac{(1/2)(\zeta_1 + \zeta_2)}{\left| g_{22} \right|} - \frac{(1/2)(\zeta_1 + \zeta_2 + \zeta_1 R_1 + \zeta_2 R_2)}{\left| g_{11} \right|} \cdot \left| g_{11} \right|^p |g_{22}|^p \]

\[= e^{-\frac{1}{2}((\zeta_1 + \zeta_2) + (\zeta_1 R_1 + \zeta_2 R_2))} \left| g_{11} \right|^p |g_{22}|^p \cdot I_1 \cdot I_2 \cdot I_3 \]

where

\[(1.29)\]

\[I_1 = \int_{G_2} |g_{22} g_{22}'|^p e^{-\frac{1}{2}(\zeta_1 R_1 + \zeta_2 R_2) + (N-p/2) - (1/2)(g_{22} - N\hat{\gamma}[2])(g_{22} - N\hat{\gamma}[2])'} d\mu_{g_{22}} \]

\[= \int_{G_2} \left| (g_{22}^\ast + N\hat{\gamma}[2] \bar{\gamma}[2]) (g_{22}^\ast + N\hat{\gamma}[2] \bar{\gamma}[2])' \right|^p e^{-\frac{1}{2}(g_{22}^\ast - 2N - N\hat{\gamma}[2])'} d\mu_{g_{22}^\ast} \]

\[= (2\pi)^{-\frac{p-q}{2}} E[\chi^2_{p-q}(R_2 \zeta_2)]^{-\frac{N-p}{2}} \left( \frac{p-q-1}{1} \right) E(\chi^2_{1})^{-\frac{N-p}{2}} \]

where \(\chi^2_{p-q}(R_2 \zeta_2)\) is a non-central chi-square with \(p-q\) d.f. and non-centrality parameter \(R_2 \zeta_2\) and \(\chi^2_{1}\) is a central chi-square with 1 d.f.
\((1.30)\)

\[
I_2 = \int_{\mathcal{D}_2} \left| g_{11} g_{11}^t \right|^{((N-q)/2)} e^{-(1/2) \text{ trace } (g_{11} - N_\varepsilon[1]) (g_{11} - N_\varepsilon[1])^t} \, dg_{11}
\]

\[
= \int_{\mathcal{D}_2} \left| g_{11}^* + N_\varepsilon[1] \right| (g_{11}^* + N_\varepsilon[1])^t \, e^{-(1/2) \text{ trace } g_{11}^* g_{11}^t} \, dg_{11}^*
\]

\[
= \left(2\pi\right)^{q^2/2} E[x_q^2(R_1^t)^t] \left((N-q)/2\right) \prod_{j=1}^{q-1} E(x_j^2) \left((N-q)/2\right)
\]

and

\[(1.31)\]

\[
I_3 = \int_{\mathcal{D}_2} e^{-(1/2) \text{ trace } (g_{21} - N_\varepsilon[2]) (g_{21} - N_\varepsilon[2])^t} \, dg_{21}
\]

\[
= \left(2\pi\right)^{q(p-q)/2}
\]

Hence

\[
\frac{dP_2(R_1, R_2)}{dP_1(R_1, R_2)} = e^{-(1/2) \zeta_2(1-R_1-R_2)} \frac{E[x_{p-q}^2(R_2)^t] \left((N-p)/2\right)}{E(x_{p-q}^2)^t \left((N-p)/2\right)}
\]

\[(1.32)\]

\[
e^{-(1/2) \zeta_2(1-R_1)} \sum_{r=0}^{\infty} \frac{\left(R_2 \frac{\zeta_2}{2}\right)^r}{r!} \frac{\Gamma \left(\frac{N-q}{2} + r\right) \Gamma \left(\frac{p-q}{2}\right)}{\Gamma \left(\frac{N-q}{2}\right) \Gamma \left(\frac{p-q}{2} + r\right)}
\]

By Neyman and Pearson's Fundamental Lemma, the rejection region of the most powerful level \(\alpha\) test of \(H_0 : \zeta_1 = 0; \zeta_2 = 0\) against the alternative \(\zeta_1 = 0, \zeta_2 = \zeta_2\) is
\[ \Phi(R_1, R_2) = \begin{cases} 1 & \text{if } e^{-(1/2)\zeta_2^2(1-R_1)} \sum_{r=0}^{\infty} \frac{\left(\frac{R_2 \zeta_2^2}{2}\right)^r}{\frac{r!}{\Gamma\left(\frac{N-q}{2} + r\right) \Gamma\left(\frac{p-q}{2} + r\right)}} \geq C \\ 0 & \text{otherwise} \end{cases} \]

where \( C \) is determined in such a way that \( \sum_{R_{10}} \Phi = \alpha \). The series

\[ \sum_{r=0}^{\infty} \frac{\left(\frac{R_2 \zeta_2^2}{2}\right)^r}{\frac{r!}{\Gamma\left(\frac{N-q}{2} + r\right) \Gamma\left(\frac{p-q}{2} + r\right)}} \]

is the well-known confluent hypergeometric series and is tabulated by Slater [8]. Though we will be mainly concerned with large sample cases, it is interesting to consider certain typical small sample cases as examples.

Example 1.1

\[ N = 6, \quad p = 4, \quad q = 2 \]

The test \( \Phi \) defined above reduces to:

\[ \Phi(R_1, R_2) = \begin{cases} 1 & \text{if } e^{-(\zeta_2/2)(1-R_1-R_2)} \left(1 + \frac{2R_2 \zeta_2^2}{2}\right) \geq C \\ 0 & \text{otherwise} \end{cases} \]

It will follow from the results we are going to derive later, that under \( H_{10} \), the joint distribution of \( R_1, R_2 \) in this particular case is uniform with probability density function.
\[ f_{R_1, R_2} (R_1, R_2) = 2 \quad \text{if } 0 \leq R_1 + R_2 \leq 1 \]
\[ = 0 \quad \text{otherwise.} \]

Case 1:
\[ \alpha = 0.05 \quad \zeta_2 = 1 \]
\[ \Phi(R_1, R_2) = 1 \quad \text{if } R_1 + R_2 + 2 \log_e (1 + \frac{R_2}{2}) \geq 1.55 \]
\[ = 0 \quad \text{otherwise.} \]

Case 2:
\[ \alpha = 0.05 \quad \zeta_2 = 2 \]
\[ \Phi(R_1, R_2) = 1 \quad \text{if } R_1 + R_2 + \log_e (1 + R_2) \geq 1.50 \]
\[ = 0 \quad \text{otherwise.} \]

The rejection region of likelihood ratio test in this case is
\[ R_1 + 1.05 R_2 \geq 1 \]

Example 1.2
\[ N = 10, \quad p = 0.5, \quad q = 2 \]

In this case the density of \( R_1, R_2 \) under \( H_{10} \) is
\[ f_{R_1, R_2} (r_1, r_2) = \frac{r_1(5)}{\Gamma(5)} (1-r_1-r_2)^2 \quad \text{if } 0 \leq r_1 + r_2 \leq 1 \]
\[ = 0 \quad \text{otherwise.} \]
For \( \alpha = 0.05 \), \( \xi_2 = 1 \). The test \( \Phi \) reduces to

\[
\Phi(R_1, R_2) = 1 \quad \text{if} \quad R_1 + R_2 + 2 \log_e (1 + 1.5 R_2 + 3.74 R_2^2 + 0.21 R_2^3) \geq 1.70
\]

\[
= 0 \quad \text{otherwise}.
\]

The rejection region for the likelihood ratio test is

\[
R_1 + 1.56 R_2 \geq 1.
\]

From Slater [8], p. 56, (1.32) can be written as

\[
e^{-\frac{(1/2)\xi_2^2(1-R_1-R_2)}{2}} \left[1 + \frac{N-p}{p-q} \frac{R_2 \xi_2}{2} + \frac{(N-p)(N-p-2)}{(p-q)(p-q+2)} \frac{R_2^2 \xi_2^2}{2} \right] + \ldots \ldots \ldots (1.33)
\]

Asymptotic case:

The ratio of the probability densities of \( R_1, R_2 \) under the alternative and under the null hypothesis has been obtained in (1.33). Here our aim is to approximate this ratio with a relative error of \( O\left(\frac{1}{N}\right) \) when \( N-p, p-q, \) and \( q \) are large and of same order of magnitude.

To achieve this we need to approximate with a relative error of \( O\left(\frac{1}{N}\right) \)

\[
\Phi_{N,p,q}(r) = \sum_{j=0}^{\infty} \frac{r^j}{j!} \frac{(N-p)(N-p-2) \cdots (N-p-2j+2)}{(p-q)(p-q+2) \cdots (p-q+2j-2)} ;
\]

where \( r = \frac{R_2 \xi_2}{2} = O(\sqrt{N}) \), and \( N-p, p-q \) and \( q \) are sufficiently large.
and of same order of magnitude. We first observe that the terms for which \( j \neq O(\sqrt{N}) \) are negligible. Then we approximate (using Stirling's formula \( n! = \sqrt{2\pi n} n^n e^{-n} + O(\frac{1}{n}) \))

\[
\frac{(\frac{N-p}{2})!}{(\frac{N-p}{2} - j)!} = \frac{(\frac{N-p}{2})^j e^{-j + O(\frac{1}{N})}}{\left(\frac{N-p}{2} - j\right)^{\frac{N-p}{2} - j} \sqrt{\frac{N-p}{N-p-2j}}}
\]

\[
eq e^{-j + O(\frac{1}{N})} \left[ 1 + \frac{j}{N-p} \right] \left(\frac{N-p}{2}\right)^j \left(\frac{N-p}{N-p-2j}\right)^{\frac{N-p}{2} - j}
\]

\[
= (\frac{N-p}{2})^j \left(1 + \frac{j}{N-p}\right) e^{-j + O(\frac{1}{N}) - (\frac{N-p}{2} - j) \left[ - \frac{2j}{N-p} - \frac{2j^2}{(N-p)^2} \right.}
- \frac{8}{3} \frac{j^3}{(N-p)^3} \left. \right]}
\]

\[
= (\frac{N-p}{2})^j \left(1 + \frac{j}{N-p}\right) e^{-\frac{j^2}{N-p} - \frac{2j}{3} \frac{j^3}{(N-p)^2} + O(\frac{1}{N})}
\]

Also

\[
\frac{(\frac{p-q-2}{2})!}{[\frac{p-q+2(j-1)}{2}]!} = \frac{1}{[\frac{p-q+2(j-1)}{2}]^j \left(1 + \frac{j}{p-q+2(j-1)}\right)}
\]

\[
eq e^{-\frac{j^2}{p-q+2(j-1)} - \frac{2j}{3} \left[\frac{p-q+2(j-1)}{2}\right]^2 + O(\frac{1}{N})}
\]

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\[
\left(\frac{N-P}{2}\right) \cdot (\frac{P-q-2}{2})! \cdot \frac{1}{(N-P)^2 - j!} = \frac{N-P}{j-2p-q} - j \frac{1}{N-p} + j \frac{1}{p-q} - \frac{2}{3} j^3 \left[ \frac{1}{(N-p)^2} - \frac{1}{(p-q)^2} \right] \cdot e^{\frac{j}{N}}.
\]

(1.35)

\[
\frac{N-P}{j(2p-q)} - \frac{1}{N-p} \cdot e^{\frac{j}{N}}.
\]

Thus

\[
\mathcal{F}_{N,p,q}(r) = e^{\frac{j}{N}} \sum_{j=0}^{\infty} \frac{(N-P) \cdot (2p-q) \cdot r^j}{j!} \left( - \frac{N-p}{(N-p)\cdot(p-q)} \cdot j^2 + \frac{N-q}{(N-p)\cdot(p-q)} \cdot j^3 \right) - \frac{2}{3} \left[ \frac{1}{(N-p)^2} - \frac{1}{(p-q)^2} \right] \cdot j^3 \cdot e^{\frac{j}{N}}.
\]

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Let us define $j_0$ by

\[(1.37) \quad j_0 = \frac{N-p}{p-q} r\]

The maximum term in (1.34) is that $j = j'$ for which

\[\frac{r}{j'} \left( \frac{N-p-2j+2}{p-q+2j-2} \right) = 1\]

i.e., $j' = j_0 + O(1)$

From (1.36)

\[
\frac{\phi_{N,p,q}(r)}{N} = e^{O(\frac{1}{N})} \sum_{j=0}^{\infty} \left[ \frac{N-p}{p-q} r \right]^j \left/ \frac{N-q}{(N-p)(p-q)} \right. 
- \frac{1}{2} \prod_{j=0}^{j_0} \left[ \frac{N-q}{(N-p)(p-q)} \right] j_0^2 - \frac{1}{3} \prod_{j=0}^{j_0} \left[ \frac{N-q}{(N-p)(p-q)} \right] j_0^3 + j_0^2 (j-j_0)\]

\[= e^{O(\frac{1}{N}) + \frac{(N-q)}{(N-p)(p-q)} (j_0-j_0^2) - \frac{2}{3} \prod_{j=0}^{j_0} \left[ \frac{N-q}{(N-p)(p-q)} \right] j_0^3} \]

\[(1.38) \quad \sum_{j=0}^{\infty} \left[ \frac{N-p}{p-q} r \right]^j \left/ j! \right. \]
\[ \cdot \left(1 - \frac{2}{(N-p)(p-q)} j_o(j-j_o) + \frac{1}{2} \left[ \frac{N-q}{(N-p)(p-q)} j_o(j-j_o) \right]^2 \right) \]

\[ - \frac{1}{6} \left[ 2 \frac{N-q}{(N-p)(p-q)} j_o^2(j-j_o) \right] \]

\[ \cdot \left(1 + \frac{N-q}{(N-p)(p-q)} (j-j_o) \right) \]

\[ \cdot \left(1 - \frac{2}{3} \left[ \frac{1}{(N-p)^2} - \frac{1}{(p-q)^2} \right] 3 j_o^2(j-j_o) \right) \]

But

\[ \sum_{j=0}^{\infty} \frac{(N-p) r}{p-q} j^j / j! = e^{p-q} r, \]

\[ \sum_{j=0}^{\infty} (j-j_o) \frac{(N-p) r}{p-q} j^j / j! = 0, \]

\[ \sum_{j=0}^{\infty} (j-j_o)^2 \frac{(N-p) r}{p-q} j^j / j! = j_o e^{p-q} r, \]

and

\[ \sum_{j=0}^{\infty} (j-j_o)^3 \frac{(N-p) r}{p-q} j^j / j! = e^{p-q} r O(\sqrt{N}). \]

Thus

\[ \bar{\phi}_{N,p,q}(r) = e^{O(1/N)} + \frac{N-q}{(N-p)(p-q)} (j_o - j_o^2) - \frac{2}{3} \left[ \frac{1}{(N-p)^2} - \frac{1}{(p-q)^2} \right] j_o^3 \]

\[ \cdot j_o^2 + \frac{N-p}{p-q} r \left(1 + 2 \frac{(N-q)^2 j_o^3}{(N-p)^2 (p-q)^2} \right) \]
Hence from (1.39) we obtain for the probability ratio, putting
\[ r = \frac{R_2^\xi_2}{2} = o(\sqrt{N}) \]

\[
\frac{dP_x}{dP_1}(R_1, R_2) = e \left[ -\frac{1}{2} \xi_2 \left[ 1 - R_1 - \frac{N-q}{p-q} R_2 + o\left(\frac{1}{N}\right) \right] \right.
\]
\[
- \frac{(N-p)(N-q)}{(p-q)^3} \frac{R_2^\xi_2}{2}
\]
\[
+ 2 \left[ (N-q)^2 - \frac{1}{3} (p-q)^2 + \frac{1}{3} (N-p)^2 \right] \frac{N-p}{(p-q)^5} \frac{R_2^\xi_2}{2} \]
\[
+ o\left(\frac{1}{N}\right) \right] .
\]

But
\[
\frac{(N-p)(N-q)}{(p-q)^3} \frac{R_2^\xi_2}{2} = \frac{(N-p)(N-q)}{N^2 (p-q)} \frac{\xi_2^2}{4} - \frac{(N-p)(N-q)}{(p-q)^2 N} \frac{\xi_2^2}{2} R_2 + o\left(\frac{1}{N}\right),
\]

and
\[
\left[ (N-q)^2 - \frac{1}{3} (p-q)^2 + \frac{1}{3} (N-p)^2 \right] \frac{N-p}{(p-q)^5} \frac{R_2^\xi_2}{2} \]
\[
= \text{constant} + o\left(\frac{1}{N}\right) .
\]

Thus in this asymptotic case the most powerful level \( \alpha \) invariant test of \( H_{10} \) against the alternative \( \xi_1 = 0, \xi_2 = \xi_2 \) with an error of \( o\left(\frac{1}{N}\right) \) (when \( \xi_2 = o(\sqrt{N}) \)) is given by

\[
\Phi(R_1, R_2) = \begin{cases} 
1 & \text{if } R_1 + \left( \frac{N-q}{p-q} + \frac{(N-p)(N-q)}{(p-q)^2 N} \xi_2 \right) R_2 \geq K^1 \\
0 & \text{otherwise}
\end{cases}
\]

(1.41)
where \( K_1 \) is such that \( \mathbb{E}_{H_{10}} \Phi(R_1, R_2) = \alpha \). We will show later that

\[
R_1 + \left( \frac{N-q}{p-q} + \frac{(N-p)(N-q)}{(p-q)^2N} \right) \zeta_2 R_2 \quad \text{under } H_{10}
\]

is normally distributed with mean \( 1 + \frac{(N-p)(N-q)}{(p-q)^2N} \zeta_2 \) and variance \( \sigma^2 = 2 \frac{(N-p)(N-q)}{N^2(p-q)} \). Hence (1.41) reduces to

\[
\Phi(R_1, R_2) = 1 \quad \text{if} \quad R_1 + \left( \frac{N-q}{p-q} + \frac{(N-p)(N-q)}{(p-q)^2N} \right) \zeta_2 R_2 \geq 1 + \frac{(N-p)(N-q)}{(p-q)^2N} \zeta_2 + \sigma Z_{1-\alpha}
\]

\[
= 0 \quad \text{otherwise}
\]

where \( Z_{1-\alpha} \) is the \((1-\alpha)\) percentile point of a unit normal distribution.

Now to justify our claim made in the introduction that in this asymptotic case the likelihood ratio test is nearly uniformly most powerful invariant, we have to compute the difference of the powers of \( \Phi \) and the likelihood ratio test and to show that it is of \( O\left(\frac{1}{N}\right) \). This we will do after we have derived the joint distribution of \( R_1, R_2 \) under \( H_{10} \).

1.6 The joint distribution of \( R_1, R_2 \) under \( H_{10} \):

Let

\[
\tau_1^2 = N \bar{X}'[1] S^{-1} \bar{X}[1]
\]

and

\[
\tau_1^2 + \tau_2^2 = N \bar{X}' S^{-1} \bar{X}.
\]

Since \( S \) is positive definite, we can express \( S \) as \( S = KK' \), where \( K \) is a lower triangular (non-singular) \( p \times p \) matrix. Let \( K \) be
partitioned similar to $S$, as

\[(1.43) \quad K = \begin{pmatrix} K_{11} & 0 \\ K_{21} & K_{22} \end{pmatrix} \]

Hence

\[K^{-1} = \begin{pmatrix} K_{11}^{-1} & 0 \\ -K_{22}^{-1} K_{21}^{-1} & K_{22}^{-1} \end{pmatrix} \]

It is easy to check that

\[K_{11} K_{11}' = S_{11} \]

\[(1.44) \quad K_{22} K_{22}' = S_{22} \]

and

\[K_{21} K_{11}^{-1} = S_{21} S_{11}^{-1} \]

Thus

\[N \bar{X}' S^{-1} \bar{X} = N \begin{pmatrix} \bar{X}[1] \\ X[2] \end{pmatrix}' (KK')^{-1} \begin{pmatrix} \bar{X}[1] \\ X[2] \end{pmatrix} \]

\[= N \bar{X}[1]' S^{-1} \bar{X}[1] + N(\bar{X}[2] - S_{21} S_{11}^{-1} \bar{X}[1])' (S_{22} - S_{21} S_{11}^{-1} S_{12})^{-1} (\bar{X}[2] - S_{21} S_{11}^{-1} \bar{X}[1]) \]

Hence

\[(1.45) \quad \bar{\tau}^2 = N(\bar{X}[2] - S_{21} S_{11}^{-1} \bar{X}[1])' (S_{22} - S_{21} S_{11}^{-1} S_{12})^{-1} (\bar{X}[2] - S_{21} S_{11}^{-1} \bar{X}[1]) \]
It is a well-known result in multivariate analysis (see for example Anderson [1]) that under $H_{10}$, $T^2_1$ is distributed as $\frac{\chi^2}{\chi^2_{N-q}}$ and the conditional distribution of $\sqrt{N} (\bar{\mathbf{x}}_{[2]} - S_{21} S^{-1}_{11} \bar{\mathbf{x}}_{[1]})$, given $\bar{\mathbf{x}}_{[1]}$ and $S_{11}$, i.e., $T^2_1$, is multivariate normal with mean vector $= 0$ and covariance $= \text{E}[N (\bar{\mathbf{x}}_{[2]} - S_{21} S^{-1}_{11} \bar{\mathbf{x}}_{[1]})(\bar{\mathbf{x}}_{[2]} - S_{21} S^{-1}_{11} \bar{\mathbf{x}}_{[1]}) | S_{11}, \bar{\mathbf{x}}_{[1]}] = (1 + T^2_1) I$. Furthermore $(S_{22} - S_{21} S^{-1}_{11} S_{12})$ which is independent of $S_{11}$ and $\bar{\mathbf{x}}_{[1]}$ is distributed as $\sum_{\alpha=1}^{N-q-1} U_{\alpha}^2 U_{\alpha}^2$ independently of $(\bar{\mathbf{x}}_{[2]} - S_{21} S^{-1}_{11} \bar{\mathbf{x}}_{[1]}),$ where $U_{\alpha}$'s (of dim $(p-q)$) are independently and identically distributed with probability law $N(0, I)$. Thus it is well-known (see for example Anderson [1], p. 106) that under $H_{10}$ the conditional distribution of $\frac{T^2_2}{1+T^2_1}$, given $T^2_1$, is $\frac{\chi^2_{p-q}}{\chi^2_{N-p}}$. By Lemma 1.4

$$T^2_1 = \frac{R_1}{1-R_1} \quad \text{and} \quad \frac{T^2_2}{1+T^2_1} = \frac{R_2}{1-R_1-R_2} \ .$$

Hence one can easily verify that the joint density of $R_1, R_2$ under $H_{10}$ is

$$(1.46) \ f(R_1, R_2) = \frac{\Gamma(N/2)}{\Gamma(N-p/2) \Gamma(p-q/2) \Gamma(q/2)} R_1^{(q/2)-1} R_2^{((p-q)/2)-1} (1-R_1-R_2)^{(N-p)/2}-1$$

if $0 \leq R_1 + R_2 \leq 1$ .

1.7 Asymptotic expression of $f(R_1, R_2)$ (when $q, p-q, N-p$ are all large).

It is easy to check that $f(R_1, R_2)$ is maximum at the point

$$(R_1 = R_1^* = \frac{q}{N}, \ R_2 = R_2^* = \frac{p-q}{N}) \ .$$

Expanding $\log_e f$ by Taylor's series...
at the point \((R_1^*, R_2^*)\) we get (if \(q, p-q, N-p\) are all large)

\[
\log f(R_1, R_2) = \log \left( \frac{1}{2\pi \sigma_1 \sigma_2 (1-\xi^2)^{1/2}} \right)
\]

\[
1.47 \quad - \frac{1}{2(1-\xi^2)} \left[ \left( \frac{R_1 - r_1}{\sigma_1} \right)^2 + \left( \frac{R_2 - r_2}{\sigma_2} \right)^2 - 2\xi \left( \frac{R_1 - r_1}{\sigma_1} \right) \left( \frac{R_2 - r_2}{\sigma_2} \right) \right]
\]

where

\[
\begin{align*}
  r_1 &= \frac{q}{N}, & r_2 &= \frac{p-q}{N} \\
  \sigma_1^2 &= \frac{2q(N-q)}{N^3}, & \sigma_2^2 &= \frac{2(p-q)(N-p+q)}{N^3} \\
  \xi &= -\sqrt{\frac{q(p-q)}{(N-q)(N-p+q)}}
\end{align*}
\]

and

Since \(\sigma_1, \sigma_2\) both are of \(O\left(\frac{1}{N}\right)\), it follows from (1.38) that when \(q, p-q, N-p\) are all large, \(R_1, R_2\) (with an error of \(O\left(\frac{1}{N}\right)\)) have a bivariate normal distribution under \(H_{10}\) with probability density function,

\[
1.48 \quad f(R_1, R_2) = \quad \frac{1}{2\pi \sigma_1 \sigma_2 (1-\xi^2)^{1/2}} e^{- \frac{1}{2(1-\xi^2)} \left[ \left( \frac{R_1 - r_1}{\sigma_1} \right)^2 + \left( \frac{R_2 - r_2}{\sigma_2} \right)^2 - 2\xi \left( \frac{R_1 - r_1}{\sigma_1} \right) \left( \frac{R_2 - r_2}{\sigma_2} \right) \right]}
\]

where \(r_1, r_2, \sigma_1, \sigma_2\) and \(\xi\) are defined in (1.38). Thus under \(H_{10}\),
\[ R_1 + \left( \frac{N-p}{p-q} \right) \frac{(N-p)(N-q)}{(p-q)^2 N} \zeta_2 \] is normally distributed

with mean \( 1 + \frac{(N-p)(N-q)}{(p-q) N^2} \zeta_2 \) and variance \( \frac{2(N-p)(N-q)}{N^2 (p-q)} \).

1.8 Analytical evaluation of the difference of the powers of the most powerful invariant test and the likelihood ratio test of \( H_{10} \) with a relative error of \( O\left( \frac{1}{N} \right) \) (when \( N-p, p-q, \) and \( q \) are large and are of same order of magnitude).

We have already obtained the likelihood ratio test and the most powerful invariant test of \( H_{10} \). Now we are going to show that the difference of the powers of the most powerful invariant test and the likelihood ratio test is of \( O\left( \frac{1}{N} \right) \), which will justify the claim that the likelihood ratio test is nearly uniformly most powerful invariant in this asymptotic case.

The likelihood ratio test (level \( \alpha \)) for testing \( H_{10} \) against \( H_1 \) with an error of \( O\left( \frac{1}{N} \right) \) is given by reject \( H_{10} \) if \( R_1 + (1-Z_0)^{-1} R_2 \geq 1 \), or equivalently,

\[
\text{reject } H_{10} \quad \text{if} \quad V = \frac{R_1 + (1-Z_0)^{-1} R_2 - E_{H_{10}}\left( R_1 + (1-Z_0)^{-1} R_2 \right)}{\sqrt{\text{Var}_{H_{10}}\left( R_1 + (1-Z_0)^{-1} R_2 \right)}}
\]

(1.49)

\[
1 - E_{H_{10}}\left[ R_1 + (1-Z_0)^{-1} R_2 \right] \geq \frac{1}{\sqrt{\text{Var}_{H_{10}}\left( R_1 + (1-Z_0)^{-1} R_2 \right)}} = v_0 \quad \text{(say)},
\]

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and accept

\[ H_{10} \] if \( V < v_0 \),

where

\[
(1.50) \quad (1-Z_0)^{-1} = \frac{N-q}{P-q} + Z_\alpha \sqrt{\frac{2(N-p)}{(p-q)(N-q)}}.
\]

From (1.40) the most powerful level \( \alpha \) invariant test \( \phi \) of \( H_{10} \) with an error of \( O\left(\frac{1}{N}\right) \) is given by,

\[
\phi(R_1, R_2) = 1 \quad \text{if} \quad f(R_1, R_2) = e^{\frac{1}{2} \xi_2 (1-R_1) - \frac{(N-q)(N-q)}{(p-q)^2 N} \xi_2 R_2}
\]

\[ + \text{constant} + O\left(\frac{1}{N}\right) \]

\[
(1.51) \quad \geq e^c
\]

\[ = 0 \quad \text{otherwise} \]

where \( e^c \) is chosen in such a way that \( E_{H_{10}} \phi(R_1, R_2) = \alpha \).
Let

$$(1.52) \quad f(R_1, R_2) = f(U, V) = e^{\lambda V - \frac{1}{2} \lambda^2} + g^*(U, V)$$

where $\lambda = E_{H_1}(V)$, $U$ is some function of $R_1, R_2$ which is uncorrelated with $V$, and $g^*(U, V)$ is some function of $U, V$.

Furthermore let us define a set function $\psi(U, V)$ as follows

$$\psi(U, V) = 1 \quad \text{if} \quad V < v_o \quad \text{and} \quad f(U, V) \geq e^c$$

$$(1.53) \quad = -1 \quad \text{if} \quad V \geq v_o \quad \text{and} \quad f(U, V) < e^c$$

$$= 0 \quad \text{otherwise}.$$ 

Since both the tests have the same level of significance we have

$$(1.54) \quad E_{H_{10}}(\psi(U, V)) = 0.$$
Also $\Psi(U,V) = 0$ unless $V$ is close to $V_0$ (with high probability).

Now the difference of the powers of $\Phi$ and the likelihood ratio test is

$$\mathbb{E}_{H_1} \Psi(U,V) = \mathbb{E}_{H_0} \left[ \Psi(U,V) f(U,V) \right]$$

$$= \mathbb{E}_{H_0} \left[ \Psi(U,V)(f(U,V) - e^c) \right]$$

(Since $\mathbb{E}_{H_0} \Psi(U,V) = 0$)

$$\approx \mathbb{E}_{H_0} \Psi(U,V) \left[ \frac{1}{2} (\lambda v_0 - \frac{1}{2} \lambda^2 + g^*(U,v_0) - e^c) \right]$$

$$\approx \mathbb{E}_{H_0} \Psi(U,V) \left[ \frac{1}{2} (e^{\lambda v_0} - \frac{1}{2} \lambda^2 + g^*(U,v_0) - e^c) \right]$$

$$\approx \mathbb{E}_{H_0} \Psi(U,V) \left[ \frac{1}{2} (e^{(v_0-v_0^*)} - 1) e^c \right] ,$$

where $v_0^* = v_0^*(U)$ is the solution of the equation

$$(1.56) \quad \frac{\lambda v_0^*}{2} \lambda^2 + g^*(U,v_0^*) = e^c .$$
From (1.55) and (1.56) we get

\[ E_{H_1} \psi(U,V) \approx E_{H_{10}} \psi(U,V) \frac{1}{2} e^{c} \left[ g^*(U,v_o) + \text{constant} \right] \]

(Since \( v_o \) and \( v^* \) are nearly the same)

\[ \approx \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} v_o^2} \frac{e^{c}}{2} E_{H_{10}} \left( \frac{1}{\lambda} [g^*(U,v_o) + \text{constant}] \right)^2 \]

\[ \approx \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} v_o^2} \frac{e^{c}}{2\lambda} \text{Var}_{H_{10}} [g^*(U,v_o)] \]

as under \( H_{10} \), \( V \) is nearly normally distributed with mean 0 and variance 1, and

\[ 0 = E_{H_{10}} \psi(U,V) \]

\[ \approx \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} v_o^2} \frac{1}{\lambda} E_{H_{10}} \left( g^*(U,v_o) + \text{constant} \right) \]

From (1.57) we get

\[ (1.58) \quad E_{H_1} \psi(U,V) \approx \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} v_o^2} \frac{e^{c}}{2\lambda} \text{Var}_{H_{10}} (g^*(U,V)|V) \]

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From (1.46) it can be checked that

\[
\text{Var}_{H_{10}}(R_1) = \frac{2q(N-q)}{N^2(N+2)},
\]

\[
\text{Var}_{H_{10}}(R_2) = \frac{2(p-q)(N-p+q)}{N^2(N+2)},
\]

and

\[
\text{Cov}_{H_{10}}(R_1, R_2) = -\frac{2q(p-q)}{N^2(N+2)}.
\]

Hence

\[
\text{Var}_{H_{10}}(R_1 + (1-Z_0)^{-1}R_2) \simeq \frac{2(N-q)(N-p)}{N(N+2)(p-q)},
\]

and

\[
\nu_0 \simeq Z_{1-\alpha}.
\]

Furthermore from (1.40)

\[
\lambda = E_{H_1}(V)
\]

\[
\simeq \frac{(N-p)(N-q)}{N^2(p-q)} \frac{t^2}{4}
\]

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From (1.52)

\[ g^*(U, V) = \frac{\left( \frac{(N-p)(N-q)}{(p-q)^2 N} \right)^{\frac{1}{2}} z_2 - z_2 \sqrt{\frac{2}{N} \left( \frac{N-p}{N(N+2)(p-q)} \right)}}{\sqrt{\frac{2(N-p)(N-q)}{N(N+2)(p-q)}}} \cdot R_2 + \text{constant} \]

\[ = K R_2 + \text{constant} \]

where

\[ K = \frac{\left( \frac{(N-p)(N-q)}{(p-q)^2 N} \right)^{\frac{1}{2}} z_2 - z_2 \sqrt{\frac{2}{N} \left( \frac{N-p}{N(N+2)(p-q)} \right)}}{\sqrt{\frac{2(N-p)(N-q)}{N(N+2)(p-q)}}} \]

So

\[ \text{Var}_{H_{10}} (g^*(U, V) | V) \]

\[ = \text{Var}_{H_{10}} (KR_2 | V) \]

\[ = K^2 \text{Var}_{H_{10}} (R_2 | V) \]

\[ = K^2 \text{Var}_{H_{10}} (R_2) (1 - \tau^2) \]

where

\[ \tau^2 = \frac{[\text{Cov}_{H_{10}} (R_2, R_1) + (1-Z_0)^{-1} \text{Var}_{H_{10}} (R_2)]^2}{\text{Var}_{H_{10}} (R_2) \cdot \text{Var}_{H_{10}} (R_1 + (1-Z_0)^{-1} R_2)} \]

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Hence
\[
E_{H_1} \psi(U, V) \approx \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} \mu^2 \lambda} e^{(\frac{\lambda}{2}) (R_2)(1-\tau^2)}
\]
\[
\approx \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} \mu^2 \lambda + \mu^2 \lambda} e^{(\frac{\lambda}{2}) \frac{(p-q)^2}{N^2(N+2)(N-q)}}
\]
(1.59)

since
\[
\tau^2 \approx \frac{N(N-p)}{(N-q)(N-p+q)}
\]
and
\[
e^{\lambda} \approx e^{\frac{\lambda}{2} \mu^2 - \frac{1}{2} \lambda^2}
\]

Substituting for \( \lambda \) the above value we get from (1.59)
\[
E_{H_1} \psi(U, V) \approx O\left(\frac{1}{N}\right).
\]

Thus we have succeeded in showing that in this asymptotic case
the difference of the powers of the most powerful invariant test and the
likelihood ratio test is \( O\left(\frac{1}{N}\right) \). Hence the likelihood ratio test with an
error of \( O\left(\frac{1}{N}\right) \) is nearly uniformly most powerful invariant if \( N-p, p-q, \)
and \( q \) are large and of same order of magnitude.
1.9. Conclusion

Let $Y$ be a $p$-dimensional column vector distributed normally with mean,

$$
\eta = \begin{pmatrix}
\eta_1 \\
\eta_2 \\
\vdots \\
\eta_p
\end{pmatrix}
$$

and unknown covariance matrix $\Sigma$ (non-singular). On the basis of $N$ observations from $N(\eta, \Sigma)$ [$N > p$], we have obtained the likelihood ratio test of the hypothesis

$$H_{10} : \eta_1 = \eta_2 = \cdots = \eta_p = 0$$

against the alternative

$$H_{1} : \eta_1 = \eta_2 = \cdots = \eta_{q} = 0$$

where $p > p' > q$ and $\Sigma$ is unknown. We have also shown that if $q, p'-q$ and $N-p'$ are large, then the likelihood ratio test is nearly uniformly most powerful invariant.

The original problem we had in mind was to test by means of observations $X_1, X_2, \ldots, X_N$ ($p$-dimensional column vectors) from $N(\xi, \Sigma)$ the hypothesis

$$H_{10} : \xi \in \mathcal{Z}$$
against the alternative

\[ H_1 : \xi \in \mathcal{Y} \]

where \( \Sigma \) is unknown and \( \mathcal{Z} \subset \mathcal{Y} \) are sub-spaces of the space of \( \xi \)'s, \( \prod_0 \), of dimensions \( p-p' \) and \( p-q \) respectively. In general \( \mathcal{Z} \) and \( \mathcal{Y} \) can be defined in many different ways. We are not going to cover all of them. Instead we will consider only a case which generally arises in practice. Let \( \alpha, \beta \) be the one to one inclusion mappings (arbitrary matrices) defined by,

\[ x \leftarrow \eta \leftrightarrow z \]

where \( x \) is the space of \( X \)'s. We shall consider the case where under \( H_1 \), \( \xi \) is explicitly expressible as \( \xi = \alpha \eta \) (\( \xi \), \( \eta \) are unknown) and \( H_{10} \) may further specify \( \eta \) as \( \eta = \beta \xi \) (\( \xi \) is unknown). In this case two meaningful (natural: - no choice of coordinate system is necessary to define them) maximal invariants are,

\[
Q_1 = \text{squared length of } X - \text{square of the orthogonal projection of } X \text{ on } \mathcal{Z}; \text{ and}
\]

\[
Q_2 = \text{square of the orthogonal projection of } X \text{ on } \alpha \mathcal{Y} - \text{square of the orthogonal projection of } X \text{ on } \mathcal{Z}.
\]

In terms of the sample covariance matrix \( S \) and the sample mean \( \bar{X} \) they can be defined by

\[
(1.60) \quad Q_1 = \bar{X}' S^{-1} \bar{X} - \bar{X}' S^{-1} \beta' \alpha' (S^{-1} \alpha \beta)^{-1} \alpha \beta S^{-1} \bar{X}
\]
and

\[ q_2 = \bar{N} \alpha' s^{-1} (\alpha' s^{-1} \alpha)^{-1} \alpha s^{-1} \bar{x} - \bar{N} s^{-1} \beta' \alpha' (s^{-1} \alpha \beta)^{-1} \alpha \beta s^{-1} \bar{x} \]

where \( \alpha', \beta' \) are adjoints of \( \alpha \) and \( \beta \) respectively. If we assign \( \alpha, \beta \) the values

\[
\alpha = \begin{pmatrix}
0 & 0 \\
I_1 & 0 \\
0 & I_2
\end{pmatrix}
\]

and

\[
\beta = \begin{pmatrix}
0 \\
0 \\
I_2
\end{pmatrix}
\]

where \( I_1, I_2 \) are identity square matrices of orders \( p'-q \) and \( p-p' \) respectively, then this problem is reduced to that of testing \( H_{10}^1 \) against \( H_1^1 \). Since \( Q_1, Q_2 \) remain invariant for all arbitrary \( \alpha, \beta \), we can in particular assign \( \alpha, \beta \) the above values to define \( Q_1, Q_2 \). Hence to obtain the likelihood ratio test statistic for the original problem we need to find the relations between \( Q_1, Q_2 \) (with the above values of \( \alpha, \beta \)) and \( R_1, R_2 \) and replace \( R_1, R_2 \) in terms of \( Q_1, Q_2 \) in (1.12). We have observed that in testing \( H_{10}^1 \) against \( H_1^1 \) we can take \( p' = p \) (for notational convenience) without disturbing the original set-up of the problem. Hence we can take, in particular

\[
\alpha = \begin{pmatrix}
0 \\
I_1
\end{pmatrix}
\]

and

\[
\beta = 0
\]

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which we shall do in finding the relations between $Q_1$, $Q_2$ and $R_1$, $R_2$. Assigning $\alpha$, $\beta$ the above values, we get

$$Q_1 = \frac{R_1 + R_2}{1 - R_1 - R_2}$$

and

$$Q_2 = \frac{R_1 + R_2}{1 - R_1 - R_2} - \frac{R_1}{1 - R_1}$$

where $R_1 = N\overline{X}_{[1]} (S_{11} + N\overline{X}_{[1]} \overline{X}_{[1]}')^{-1} \overline{X}_{[1]}$ and $R_1 + R_2 = N\overline{X}'(S + N \overline{X} \overline{X})^{-1}\overline{X}$, $[X_{[1]} = X]$. In other words the likelihood ratio test of $H_{10}$ against $H_1$ is given by (1.16) where

$$R_1 = \frac{Q_1 - Q_2}{1 + Q_1 - Q_2},$$

$$R_2 = \frac{Q_1}{1 + Q_1} - \frac{Q_1 - Q_2}{1 + Q_1 - Q_2},$$

and $Q_1$, $Q_2$ are as defined in (1.60).
2.1 Formulation of the problem

Let \( X^1, X^2, \ldots, X^N \) with \( N > p \) be independently normally distributed p-dimensional column vectors with the common mean \( \xi = E(X) \), and non-singular covariance matrix \( \Sigma = E(X-\xi)(X-\xi)' \). We are interested in this section in testing the hypothesis,

\[
H_{20} : \Sigma^{-1} \xi \in \mathcal{Z}'
\]

against the alternative,

\[
H_2 : \Sigma^{-1} \xi \in \mathcal{Y}'
\]

when \( \Sigma, \xi \) are both unknown and \( \mathcal{Z}' \subset \mathcal{Y}' \) are linear sub-spaces of \( \mathbb{R}^p \) of dimensions \( q \) and \( p' \) respectively, with \( q < p' < p \). In general, the hypothesis can be given a more convenient form as follows:

Let \( \Sigma^{-1} \xi = \Gamma = \begin{pmatrix} \Gamma_1 \\ \Gamma_2 \\ \vdots \\ \Gamma_p \end{pmatrix} \),

then

\[
H_{20} : \Gamma_{q+1} = \Gamma_{q+2} = \cdots = \Gamma_p = 0
\]

and

\[
H_2 : \Gamma_{p'+1} = \Gamma_{p'+2} = \cdots = \Gamma_p = 0
\]

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2.2 Origin of the problem:

In recent years the method of discriminant function analysis introduced by Fisher [4] has been found extremely useful in deriving test statistics suitable for multivariate problems. This technique aims at reducing a multivariate problem to a univariate problem by using a linear combination of the variates, where the compounding coefficients are chosen in such a way that the linear combination can afford maximum discrimination between any two classes such as taxonomic species, the two sexes, and so on, and then constructing a test statistic suitable for the reduced univariate case. As a representative case let us consider the problem of testing equality of mean vectors of two p-variate normal populations with a common unknown non-singular covariance matrix $\Sigma$. Let their probability density functions be given by $n(X|\xi, \Sigma)$ and $n(Y|\eta, \Sigma)$ where,

$$
X = \begin{pmatrix}
X_1 \\
X_2 \\
\vdots \\
X_p
\end{pmatrix}
$$

and

$$
Y = \begin{pmatrix}
Y_1 \\
Y_2 \\
\vdots \\
Y_p
\end{pmatrix}
$$

According to this technique we have to replace $X$, $Y$ by the linear combination (according to Fisher, these linear combinations are called discriminant functions)

$$
Z_1 = \ell X \quad \text{and} \quad Z_2 = \ell^T Y
$$

where

$$
\ell = \begin{pmatrix}
\ell_1 \\
\ell_2 \\
\vdots \\
\ell_p
\end{pmatrix}
$$
is a p-vector and is determined in such a way that \( Z_1 \) and \( Z_2 \) afford maximum discrimination between the populations. The linear functions \( \ell' X \) and \( \ell' Y \) have the same variance \( \ell' \Sigma \ell \), and the square of the difference in the mean values of \( Z \)'s is \((\ell' d)^2\) where

\[
d = \begin{pmatrix}
\xi_1 - \eta_1 \\
\xi_2 - \eta_2 \\
\vdots \\
\xi_p - \eta_p
\end{pmatrix}.
\]

Now to choose \( \ell \) we have to maximize \((\ell' d)^2\) subject to the condition that \((\ell' \Sigma \ell)\) is a constant, which is equivalent to maximizing unconditionally the ratio \(\frac{\ell d)^2}{\ell' \Sigma \ell} \). It can be shown that the required value of \( \ell \) is given by \( \ell = \hat{\Sigma}^{-1} \hat{d} \). As the parameters \( \Sigma, \xi, \eta \) occurring in the probability distributions are unknown, the usual practice in setting up the discriminant function is to replace these by their best possible estimates. The test statistic based on these discriminant functions is then

\[
\left( \frac{N_1}{N_1 + N_2} \right)^{\frac{1}{2}} \left( \frac{d^T}{\hat{\Sigma}} \right) = \frac{N_1}{N_1 + N_2} \hat{d}^T \hat{\Sigma}^{-1} \hat{d},
\]

where \( d, \Sigma \) are the best possible estimates of \( d \) and \( \Sigma \) based on \( N_1 \) observations from \( n(X|\xi, \Sigma) \) and \( N_2 \) observations from \( n(Y|\eta, \Sigma) \).

In this approach the decision maker is generally confronted with the problem to test whether elimination of some variables can be done without losing any precision of the test procedure. In other words he is confronted with a problem of the above type.
2.3 The Likelihood Ratio Test of $H_{20}$ against $H_{2}$

The likelihood of the observations $x^1, x^2, \ldots, x^N$ is

$$
\frac{1}{\sqrt{(2\pi)^N \Sigma}} \, e^{-\frac{1}{2} \sum_{i=1}^{N} (x^i - \bar{x})' \Sigma^{-1} (x^i - \bar{x})}
$$

$$
= \frac{1}{\sqrt{(2\pi)^N \Sigma}} \, e^{-\frac{1}{2} \text{trace} \Sigma^{-1} [S + N \bar{x} \bar{x}']}
$$

(2.2)

$$
= \frac{1}{\sqrt{(2\pi)^N \Sigma}} \, e^{-\frac{1}{2} \text{trace} \Sigma^{-1} [S^* - 2N \bar{x}' \bar{x} + N \bar{x} \bar{x}']}
$$

$$
= \frac{1}{\sqrt{(2\pi)^N \Sigma}} \, e^{-\frac{1}{2} \text{trace} \left[ \Sigma^{-1} S^* - 2N \bar{x}' \bar{x} \right] - \frac{N}{2} \bar{x}' \Sigma \bar{x}}
$$

where

(2.3) $S^* = (S + N \bar{x} \bar{x}')$

Given the observations $x^1, x^2, \ldots, x^N$ the likelihood is a function of the variables $\Gamma$ and $\Sigma$, and we will denote it by $L(\Gamma, \Sigma)$. The likelihood criterion for testing $H_{20}$ against $H_{2}$ is

$$
\lambda = \frac{\max_{H_{20}} L(\Gamma, \Sigma)}{\max_{H_2} L(\Gamma, \Sigma)}
$$

(2.4)
Lemma 2.1

\[ \max_{H_{20}} L(\Gamma, \Sigma) = \frac{e^{-\frac{p}{2}}}{(2\pi)^{2} \frac{Np}{2} \frac{N}{2} \frac{1}{|S^*|^2} |1-N \overline{x}_{[1]}|^{2} S_{11}^{-1} \overline{x}_{[1]}|^{2}} \]

where

\[ x = \begin{pmatrix} x_{[1]} \\ x_{[2]} \end{pmatrix}, \quad x_{[1]} = \begin{pmatrix} x_{1} \\ x_{2} \\ \vdots \\ x_{q} \end{pmatrix}, \quad x_{[2]} = \begin{pmatrix} x_{q+1} \\ x_{q+2} \\ \vdots \\ x_{p} \end{pmatrix}, \]

\[ N \overline{x}_{[1]} = \sum_{\alpha=1}^{N} x_{\alpha}^{[1]}, \]

\[ N \overline{x} = \sum_{\alpha=1}^{N} x_{\alpha}^{[1]}, \]

\[ S_{11}^* = S_{11} + N \overline{x}_{[1]} \overline{x}_{[1]}^{*} \]

and

\[ S_{11} = \sum_{\alpha=1}^{N} (x_{[1]}^{\alpha} - \overline{x}_{[1]}^{*})(x_{[1]}^{\alpha} - \overline{x}_{[1]}^{*})^{*} \]
Proof:

From (2.2)

\[
\max_{H_{20}} L(\Gamma, \Sigma) = \max_{H_{20}} \frac{1}{N_p} \frac{1}{2} \left[ \frac{N}{2} \right] e^{-\frac{1}{2} \operatorname{trace} \left[ \Sigma^{-1} \Sigma^* \right] - 2N \bar{X}' + N \Sigma \Gamma \Gamma'}
\]

\[
= \max_{\Gamma[1], \Sigma} \frac{1}{N} \frac{1}{2} \left[ \frac{N}{2} \right] e^{-\frac{1}{2} \left[ \operatorname{trace} \Sigma^{-1} \Sigma^* \right] + N \operatorname{trace} \Gamma[1] \bar{X}' + N \operatorname{trace} \Sigma \Gamma \Gamma'}
\]

\[
= \max_{\Gamma[1], \Sigma} \left[ L(\Gamma[1], \Sigma) \right] ,
\]

where

\[
\Gamma = \left( \begin{array}{c}
\Gamma[1] \\
\Gamma[2]
\end{array} \right)
\]

is partitioned similar to \(X\) into subvectors \(\Gamma[i](i=1,2)\) and \(\Sigma\) partitioned into submatrices \(\Sigma_{ij}(i,j=1,2)\) such that

\[
\Sigma = \left( \begin{array}{cc}
\Sigma_{11} & \Sigma_{12} \\
\Sigma_{21} & \Sigma_{22}
\end{array} \right)
\]

and

\[
\Sigma_{ij} = \mathbb{E}(X_{[i]} - \mathbb{E}(X_{[i]}))(X_{[j]} - \mathbb{E}(X_{[j]}))'
\]

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From (2.6) it is easy to check that

\[
\max_{\Gamma[1], \Sigma} L(\Gamma[1], \Sigma) = \max_{\Sigma} \frac{1}{N} \frac{1}{2} \text{trace} \Sigma^{-1} S^* + \frac{N}{2} \bar{X}_{[1]} \Sigma^{-1} \bar{X}_{[1]} - \frac{N}{2} \bar{X}_{[1]} \Sigma^{-1} \bar{X}_{[1]}
\]

\[
= \max_{\Sigma} \frac{1}{2} \text{trace} \Sigma^{-1} S^* + \frac{N}{2} \bar{X}_{[1]} \Sigma^{-1} \bar{X}_{[1]}
\]

Since \(\Sigma\) and \(S^*\) are positive definite, there exist non-singular \(p \times p\) upper triangular matrices \(K\) and \(T\) such that

\[
\Sigma = K K' \quad \text{and} \quad S^* = T T'
\]

Let us partition \(K\) and \(T\) similar to \(\Sigma\) as

\[
K = \begin{pmatrix} K_{11} & K_{12} \\ 0 & K_{22} \end{pmatrix}
\]

and

\[
T = \begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix}
\]
It is easy to verify that
\[
K^{-1} = \begin{pmatrix}
K_{11}^{-1} - (K_{11}^{-1} K_{12} K_{22}^{-1}) \\
0 & K_{22}^{-1}
\end{pmatrix}
\]
(2.11)
\[
T^{-1} = \begin{pmatrix}
T_{11}^{-1} - (T_{11}^{-1} T_{12} T_{22}^{-1}) \\
0 & T_{22}^{-1}
\end{pmatrix}
\]

\[
K_{11} K_{11}' = \Sigma_{11} \quad \text{and} \quad T_{11} T_{11}' = S_{11}^* .
\]

From (2.8) - (2.11) we get
(2.12)
\[
\max_{\Gamma_{[1]}, \Sigma} L(\Gamma_{[1]}, \Sigma) = \max_{K} \frac{1}{N p} \frac{1}{\sqrt{(2\pi)^2 |KK'|^2}} e^{-\frac{1}{2} \text{trace} \left( (KK')^{-1} TT' \right) + \frac{N}{2} \bar{X}_{[1]} (K_{11} K_{11}')^{-1} \bar{X}_{[1]}'}
\]
\[
= \max_{K} \frac{1}{N p} \frac{1}{\sqrt{(2\pi)^2 |KK'|^2}} e^{-\frac{1}{2} \text{trace} \left( (T'K' K^{-1} K^{-1} T) \right) + \frac{N}{2} \bar{X}_{[1]} (K_{11} K_{11}')^{-1} \bar{X}_{[1]}'}
\]

as
\[
\text{trace} \left( (KK')^{-1} (TT') \right) = \text{trace} \left( (TT')(KK')^{-1} \right)
\]
\[
= \text{trace} \left( (TT'K' K^{-1} K^{-1}) \right)
\]
\[
= \text{trace} \left( (T'K' K^{-1} K^{-1} T) \right)
\]
Let us define \( L \) by

\[
(2.13) \quad L = T^{-1}K
\]

then

\[
(LL')^{-1} = (T^{-1}KK'T^{-1})^{-1} = (T'K'K^{-1}T^{-1}) \quad ,
\]

and

\[
|KK'| = |T LL' T'| = |TT'| |LL'| \quad = |S^*| |LL'| \quad .
\]

Let \( L \) be partitioned into submatrices \( L_{ij} \) \((i,j=1,2)\) similar to \( \Sigma \) in (2.7). Then it is easy to check that

\[
(2.14) \quad T^{-1}_{11}K_{11} = L_{11} \quad \text{or} \quad K_{11} = T_{11}L_{11}
\]

From (2.10) - (2.12) one can easily get,

\[
\max_{\Gamma[1], \Sigma} L(\Gamma[1], \Sigma) = \max_{L} \frac{1}{\frac{NP}{2} \left| S^* \right|^2 \left| LL' \right|^2}
\]

\[
\times \exp \left( \frac{1}{2} \text{trace} \left( LL' \right)^{-1} + \frac{N}{2} (T^{-1}_{11}X_1[1])' (L_{11}L_{11})^{-1} (T^{-1}_{11}X_1[1]) \right)
\]

\[
= \max_{\Sigma^*} \frac{1}{\frac{NP}{2} \left| S^* \right|^2 \left| LL' \right|^2} \exp \left( \frac{1}{2} \text{trace} \Sigma^*^{-1} + \frac{N}{2} \frac{Z_1[1]}{\Sigma_{11}} \Sigma_{11}^{-1} \frac{Z_1}{\Sigma_{11}} \right)
\]

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where

\begin{equation}
(2.14) \quad LL' = \Sigma^* = \begin{pmatrix} \Sigma^*_{11} & \Sigma^*_{12} \\ \Sigma^*_{21} & \Sigma^*_{22} \end{pmatrix}
\end{equation}

and \( \Sigma^*_{ij} \) are sub-matrices of \( \Sigma^* \) similar to \( \Sigma_{ij} \) \((i,j=1,2)\) of \( \Sigma \),
and \( \overline{Z}'[1] = T_{11}^{-1} \overline{X}[1] \).

Further let \( (\Sigma^*)^{-1} = \Lambda \), and \( \Lambda \) is similarly partitioned into

\begin{equation}
(2.15) \quad \Lambda = \begin{pmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{pmatrix}
\end{equation}

Then \( |\Sigma^*| = \frac{1}{|\Lambda|} = \frac{1}{|\Lambda_{22}|} \frac{1}{|\Lambda_{11} - \Lambda_{12} \Lambda_{22}^{-1} \Lambda_{21}|} \).

\begin{equation}
(2.16) \quad (\Sigma^*_{11})^{-1} = \left( \Lambda_{11} - \Lambda_{12} \Lambda_{22}^{-1} \Lambda_{21} \right)
\end{equation}

and trace \( \Lambda = \text{trace} \Lambda_{11} + \text{trace} \Lambda_{22} \).

From (2.14) - (2.16),

\[
\max_{\Gamma[1], \Sigma} L(\Gamma[1], \Sigma) = \max_{\Lambda} \frac{|\Lambda_{22}| |\Lambda_{11} - \Lambda_{12} \Lambda_{22}^{-1} \Lambda_{21}|}{\Lambda} \frac{N_p}{2} \frac{N}{2} \left( \frac{2\pi}{N} \right)^{-\frac{N}{2}} |\Sigma^*|^{-\frac{N}{2}}
\]

\[
\cdot e^{-\frac{1}{2} \text{trace} (\Lambda_{11}) - \frac{1}{2} \text{trace} \Lambda_{22}}
\]

\[
\cdot e^{-\frac{1}{2} \text{trace} (\Lambda_{11} - \Lambda_{12} \Lambda_{22}^{-1} \Lambda_{21}) (\sqrt{N} \overline{Z}[1])(\sqrt{N} \overline{Z}[1])'}
\]

From (2.17) it can be shown that the maximum likelihood estimates
of \( \Lambda_{22} \) is the \((p-q) \times (p-q)\) identity matrix \( I_{11} \) and that of \( \Lambda_{12} \)
and \( \Lambda_{11} \) are 0 and \((I_2 - N \overline{Z}[1] \overline{Z}[1])\) respectively, where \( I_2 \) is
the $q \times q$ identity matrix. Hence

$$\max_{\Gamma[1], \Sigma} \mathcal{L}(\Gamma[1], \Sigma) = \frac{e^{-\frac{D}{2}}}{N p \pi \frac{N}{2} |S|^2 |I_2 - N Z[1] Z[1]^T|^2}.$$  

Finally by the lemma 1.3 we get

$$\max_{\Gamma[1], \Sigma} \mathcal{L}(\Gamma[1], \Sigma) = \frac{e^{-\frac{D}{2}}}{N p \pi \frac{N}{2} |S|^2 |1 - N Z[1] Z[1]^T|^2}.$$  

$$\quad = \frac{e^{-\frac{D}{2}}}{N p \pi \frac{N}{2} |S|^2 |1 - N Z[1] S^{-1} Z[1]^T|^2}. \quad \text{Q.E.D}$$

**Lemma 2.2**

$$\max_{H_2} \mathcal{L}(\Gamma, \Sigma) = \frac{e^{-\frac{D}{2}}}{N p \pi \frac{N}{2} |S|^2 |1 - N X[3] S^{-1} X[3]^T|^2}.$$  

where $X$ is partitioned into subvectors $X[3]$ and $X[4]$ such that

$$X = \begin{pmatrix} X[3] \\ X[4] \end{pmatrix} \quad \text{with} \quad X[3] = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{pmatrix}, \quad X[4] = \begin{pmatrix} x_{p+1} \\ \vdots \\ x_{p'} \end{pmatrix}.$$  

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$$S_{33}^* = (S_{33} + N \bar{X}_{[3]} \bar{X}'_{[3]})$$,

$$N \bar{X}_{[3]} = \sum_{\alpha=1}^{N} X_{[3]}^\alpha$$

and

$$S_{33}^* = \sum_{\alpha=1}^{N} (x_{[3]}^\alpha - \bar{X}_{[3]}) (x_{[3]}^\alpha - \bar{X}_{[3]})'$$

The proof is exactly similar to that of Lemma 2.1. From (2.5) and (2.19) we get

$$\lambda = \frac{\max_{H_2} L(\Gamma, \Sigma)}{\max_{H_2} L(\Gamma, \Sigma)} = \left| \frac{1 - N \bar{X}_{[3]} S_{33}^{-1} \bar{X}_{[3]}}{1 - N \bar{X}_{[1]} S_{11}^{-1} \bar{X}_{[1]}} \right|^2.$$  

Hence on the basis of the observations $X^1 X^2 \cdots X^N$ with $N > p$ from $N(\xi, \Sigma)$ the likelihood ratio criterion for testing $H_2$ against $H_2$ is given by

$$Z = \frac{1 - N \bar{X}_{[3]} (S_{33} + N \bar{X}_{[3]} \bar{X}_{[3]})^{-1} \bar{X}_{[3]}}{1 - N \bar{X}_{[1]} (S_{11} + N \bar{X}_{[1]} \bar{X}_{[1]})^{-1} \bar{X}_{[1]}}.$$  

Thus we have the following theorem:

**Theorem 2.1**

The likelihood ratio test of the hypothesis,

$$H_2 : \Gamma_{q+1} = \Gamma_{q+2} = \cdots = \Gamma_p = 0$$

against

$$H_2 : \Gamma_{p'+1} = \Gamma_{p'+2} = \cdots = \Gamma_p = 0; \text{ when } \Gamma_1, \Gamma_2 \cdots \Gamma_q$$

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are unknown, is given by \( Z \leq Z_0 \), where \( Z \) is defined in (2.21) and \( Z_0 \) is determined in such a way that the probability that \( Z \leq Z_0 \) under \( H_{20} \) is equal to the chosen level of significance.

2.3 The distribution of \( Z \) under \( H_{20} \):

The probability density of \( X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{pmatrix} \) is,

\[
\pi(X|\mathbf{G},\Sigma) = \frac{1}{(2\pi)^{p/2} |\Sigma|^{1/2}} e^{-\frac{1}{2} [\text{trace } \Sigma^{-1} XX' - 2 \text{trace } \Sigma^{-1} \mathbf{G} X' + \text{trace } \Sigma^{-1} \mathbf{G} \mathbf{G}' ]},
\]

(2.22)

\[
= \frac{1}{(2\pi)^{p/2} |\Sigma|^{1/2}} e^{-\frac{1}{2} [\text{trace } \Sigma^{-1} XX' - 2 \text{trace } \Gamma X' ] - \frac{1}{2} \Gamma^T \Sigma \Gamma}.
\]

Transform \( X \) to \( Y \) by means of the transformation

(2.23) \hspace{1cm} g : X \rightarrow Y ,

where \( g \) is a non-singular \( p \times p \) lower triangular matrix such that \( g \Sigma g' = I \). Let \( g \) be partitioned into sub-matrices \( g_{ij}(i,j=1,2) \) similar to \( \Sigma_{ij} \) of \( \Sigma \). By this transformation \( \Gamma = \Sigma_{ij}^{-1} \mathbf{G} \) is transformed to

\[
\Gamma^* = (g')^{-1} \Gamma = \begin{pmatrix} g_{11}^{-1} - (g_{11}^{-1} g_{21} g_{22}^{-1}) \\ 0 \end{pmatrix} \begin{pmatrix} \Gamma[1] \\ \Gamma[2] \end{pmatrix}
\]

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The probability density of \( Y = gX \) is

\[
\frac{1}{\pi^p} \frac{e^{-\frac{1}{2} \text{trace} \left[ YY' \right] - 2 \text{trace} \left[ \Gamma \Gamma' \right]} + \Gamma' \Gamma}{(2\pi)^2}
\]

(2.24)

\[
= \frac{1}{\pi^p} \frac{e^{-\frac{1}{2} \text{trace} \left[ \left( Y_{[1]} - \Gamma_1 \right) \left( Y_{[1]} - \Gamma_1 \right)' \right]}}{(2\pi)^2}
\]

\[
\times e^{-\frac{1}{2} \text{trace} \left[ \left( Y_{[2]} - \Gamma_2 \right) \left( Y_{[2]} - \Gamma_2 \right)' \right]}
\]

where \( Y_{[1]}, Y_{[2]} \) are subvectors of \( Y \) similar to \( X_{[1]}, X_{[2]} \) of \( X \), and \( \Gamma \) is partitioned similarly into subvectors \( \Gamma_1, \Gamma_2 \). Further transform \( Y \) to \( X \) by an orthogonal transformation \( O \) where

(2.25)

\[
O(p \times p) = \begin{pmatrix}
O^1(q \times q) & 0 \\
0 & O^2(p-q \times p-q)
\end{pmatrix}
\]

and \( O^1(q \times q), O^2(p-q \times p-q) \) are \( q \times q \), \( (p-q) \times (p-q) \) orthogonal matrices such that,

(2.26)

\[
\begin{pmatrix}
O^1(q \times q) \Gamma_1 \end{pmatrix} = \begin{pmatrix}
\sqrt{\Gamma_{[1]}' \Gamma_{[1]}} \\
0 \\
0 \\
\vdots \\
0
\end{pmatrix}
\]

and

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\[
O^2((p-q) \times (p-q))_{\Gamma^*[2]} = \begin{pmatrix}
\sqrt{\Gamma^*_{[2]} \Gamma^*_{[2]}} \\
0 \\
0 \\
\vdots \\
0
\end{pmatrix}.
\]

The probability density of \( X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{pmatrix} \) is then

\[
\frac{1}{(2\pi)^{p/2}} e^{-\frac{1}{2} \left[ (x_1 - \delta_1)^2 + p \sum_{j=q+2}^{p} x_j^2 \right]}
\]

(2.27)

where \( \delta_1 = \sqrt{\Gamma^*_{[1]} \Gamma^*_{[1]}} \)

and \( \delta_2 = \sqrt{\Gamma^*_{[2]} \Gamma^*_{[2]}} \)

Hence under \( H_{20} (\delta_2 = 0) \) the probability density of \( X \) is

\[
\frac{1}{(2\pi)^{p/2}} e^{-\frac{1}{2} \left[ (x_1 - \delta_1)^2 + p \sum_{j=q+2}^{p} x_j^2 \right]}
\]

(2.28)

It is easy to check that \( Z \) and \( R \quad [R \quad \text{is defined in Section 1}] \)

are invariant under the transformation \( G = Q_p \times p \) of \( X \). It may

be remarked that we are assuming in this subsection \( p' = p \), i.e.,

\( X[3] = X \). It will be shown later that this can be done without dis-

turbing the original set up of the problem. Now by pages 71-72 of Rao [6],
Z has Beta-distribution with parameters \( \frac{N-p}{2}, \frac{p-q}{2} \). Furthermore Z is independent of \( R_1 \).

2.4 The invariance of the problem and maximal invariants:

We have obtained the likelihood ratio test of \( H_{20} \) against \( H_2 \). Now our aim is to show that it is the uniformly most powerful similar invariant test of \( H_{20} \) against \( H_2 \). We have mentioned earlier that the likelihood ratio test is invariant under all transformations which keep the problem invariant. Furthermore since it is an invariant test it must depend on the observations only through maximal invariants. Thus to achieve our aim, we need to find out the maximal invariants for this problem. It may be verified that the problem remains invariant under the groups \( G_2 \) and \( G_4 \) where, \( G_2 \) is the group of non-singular \( p' \times p' \) matrices defined in (1.17) which transform the coordinates \( X_1, X_2, \ldots, X_{p'} \) of \( X \), and \( G_4 \) is the group of linear transformations of the coordinates \( X_{p'+1}, X_{p'+2}, \ldots, X_p \) of \( X \). Let us define, as in Section 1,

\[ R_1 = N \overline{X}_{[1]} \left( S_{11} + N \overline{X}_{[1]} \overline{X}_{[1]} \right)^{-1} \overline{X}_{[1]} \]

and

\[ R_1 + R_2 = N \overline{X}_{[3]} \left( S_{33} + N \overline{X}_{[3]} \overline{X}_{[3]} \right)^{-1} \overline{X}_{[3]} \]

Following the method developed in Section 1 it is easy to check that \( R_1, R_2 \) defined above are maximal invariants under \( G_2 \) and \( G_4 \). From
the above it is clear that the components $X_{p+1}$, \ldots, $X_p$ have no relation to this problem. Hence as in Section 1 we will consider from now on $p' = p$, i.e., $X_{[3]} - X$ without disturbing the original setup of the problem. With this change in notation the likelihood ratio test statistic of $H_{20} : \Gamma[2] = 0$ against $\Gamma[2] \neq 0$ where

$$
\Gamma[1] = \begin{pmatrix}
\Gamma_1 \\
\Gamma_2 \\
\vdots \\
\Gamma_q
\end{pmatrix}, \quad \Gamma[2] = \begin{pmatrix}
\Gamma_{q+1} \\
\Gamma_{q+2} \\
\vdots \\
\Gamma_p
\end{pmatrix}
$$

and $\Gamma[1]$ is unknown can be rewritten as

$$(2.29) \quad Z = \frac{1 - N\bar{X}'(S + N\bar{X}\bar{X}')^{-1}\bar{X}}{1 - N\bar{X}'_{[1]}(S_{11} + N\bar{X}_{[1]}\bar{X}'_{[1]})^{-1}\bar{X}_{[1]}}$$

$$= \frac{1 - R_1 - R_2}{1 - R_1}$$

2.5 The uniformly most powerful similar invariant test of $H_{20}$ against $H_2$:

The notion of uniformly most powerful similar test of the composite hypothesis against composite alternatives is well-known in statistical literature, (see for example Lehmann [5] pp. 130-131). It has been shown that we can restrict our attention to invariant tests having "Neyman structure" and find the uniformly most powerful one among all similar tests provided every similar test has "Neyman structure". We have already obtained the likelihood ratio test of $H_{20}$ against $H_2$. Here our aim is to establish the fact that it is uniformly most powerful similar among the group of invariant tests of $H_{20}$ against $H_2$. 

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Since invariant tests depend on the observations only through maximal invariants, we may assume that we have observed the values of the maximal invariants, viz., $R_1$ and $R_2$ instead of the observations $X^1, X^2, \ldots, X^N$. From Rao [6], p. 71, it can be shown that the joint distribution of $R_1$ and $R_2$ under $H_{20}$ is

\[
(2.30) \quad e^{-\frac{1}{2} \bar{r}_1^2} \sum_{r=0}^{\infty} \frac{\bar{r}_1^r}{r!} \frac{\binom{N-p}{2} R_1^{\frac{q+r-1}{2}}}{R_2^{\frac{p-q-1}{2}}} \frac{(1 - R_1 - R_2)^{\frac{N-p}{2}}}{B\left(\frac{N-p}{2}, \frac{q}{2} + r\right) B\left(\frac{N-p}{2}, \frac{p-q}{2}\right)}
\]

where $\bar{r}_1 = N \Gamma[1] \sum \Gamma [1]$. From (2.30) it is easy to see that $R_1$ is sufficient for $\bar{r}_1$. Furthermore the probability density of $R_1$ under $H_{20}$ is

\[
dP_{\bar{r}_1} (R_1) = e^{-\frac{1}{2} \bar{r}_1^2} \sum_{r=0}^{\infty} \frac{\bar{r}_1^r}{r!} \frac{B\left(\frac{p-q}{2}, \frac{N-p}{2} + 1\right) R_1^{\frac{q+r-1}{2}}}{B\left(\frac{N-p}{2}, \frac{q}{2} + r\right) B\left(\frac{N-p}{2}, \frac{p-q}{2}\right)} (1 - R_1)^{\frac{N-p}{2}}
\]

if $0 < R_1 < 1$

= 0 otherwise.

Let $\phi(R_1, R_2)$ be any invariant level $\alpha$ similar test of $H_{20}$ against $H_2$. According to the remark made earlier, in order to find the uniformly most powerful one among all $\phi$'s, it is necessary to check whether the family of distributions $\{P_{\bar{r}_1} (R_1), \bar{r}_1 \geq 0\}$ is boundedly complete or not. This we will do after we have defined the notion of bounded completeness of a family of distributions.
Definition:

A family of distributions \( \{ P_{\overline{R}_1}(R_1) \}; \overline{R}_1 \in \Omega \) is boundedly complete if

\[
E_{\overline{R}_1} [h(R_1)] = \int h(r_1) \, d P_{\overline{R}_1}(r_1) = 0
\]

for all \( \overline{R}_1 \in \Omega \) (some arbitrary space of \( \overline{R}_1 \)) and for any real valued measurable bounded function \( h(r_1) \), implies that \( h(r_1) = 0 \) almost everywhere with respect to each of the measure \( P_{\overline{R}_1}(R_1) \).

Lemma 2.3:

The family of distributions \( \{ P_{\overline{R}_1}(R_1) \}; \overline{R}_1 > 0 \) is boundedly complete.

Proof:

Let \( \Phi(R_1) \) be any real valued measurable bounded function of \( R_1 \). Then

\[
E_{\overline{R}_1} \Phi(R_1) = e^{-\frac{1}{2} \overline{R}_1} \int_{0}^{\infty} \frac{\overline{R}_1}{r_1} \left( \frac{r_1}{2} \right)^r a_r \int_{0}^{1} \Phi(r_1) \frac{q}{2} + r - 1 \frac{N-q}{(1-r_1)^2} \, dr_1
\]

\[
= e^{-\frac{1}{2} \overline{R}_1} \int_{0}^{\infty} \frac{\overline{R}_1}{r_1} \left( \frac{r_1}{2} \right)^r a_r \int_{0}^{1} \Phi(r_1) r_1^r \, dr_1
\]

(2.32)

where \( a_r = \frac{B\left(\frac{p-q}{2}, \frac{N-p}{2} + 1\right)}{B\left(\frac{N-q}{2}, \frac{q}{2} + r\right) \cdot B\left(\frac{N-p}{2}, \frac{p-q}{2}\right)} \)

and \( \Phi^*(r_1) = \Phi(r_1) r_1^\frac{q}{2} - 1 \frac{N-q}{(1-r_1)^2} \)

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Hence \( B^r_{\gamma_1} \phi(R_1) = 0 \) implies that

\[
(2.33) \quad \sum_{r=0}^{\infty} \left( \frac{\gamma_1}{r} \right)^r a_r \int_0^1 \phi^+(r_1) r_1^r dr_1 = 0.
\]

But the left hand side of (2.33) is a polynomial in \( \frac{\gamma_1}{r} \), all the coefficients of which must be zero, viz.,

\[
(2.34) \quad \int_0^1 \phi^+(r_1) r_1^r dr_1 = 0 \quad \text{for} \quad r = 0, 1, 2, \ldots.
\]

Let \( \phi^*(R_1) = \phi^+(R_1) - \phi^-(R_1) \), where \( \phi^+ \) and \( \phi^- \) denote the positive and negative parts of \( \phi^* \) respectively. Then from (2.34) we get

\[
(2.35) \quad \int_0^1 \phi^+(r_1) r_1^r dr_1 - \int_0^1 \phi^-(r_1) r_1^r dr_1 = 0 \quad \text{for} \quad r = 0, 1, 2, \ldots.
\]

But (2.35) implies that \( \phi^+(r_1) = \phi^-(r_1) \) for all \( r_1 \), except possibly on a set of measure zero. Hence \( \phi^*(r_1) = 0 \ a.e. \{ B^r_{\gamma_1} (R_1), \gamma_1 \geq 0 \} \) i.e., \( \phi(R_1) = 0 \ a.e. \{ B^r_{\gamma_1} (R_1), \gamma_1 \geq 0 \} \). Q.E.D.

Since \( R_1 \) is complete it is well-known that \( \phi \) has got Neyman structure with respect to \( R_1 \) (see for example Lehmann [5], p. 134, Theorem 2), i.e.,

\[
(2.36) \quad E_{[2]}^{0} [\phi(R_1, R_2) | R_1] = \alpha.
\]

Thus to find the uniformly most powerful test among all similar invariant tests we have to find the probability ratio.
(2.37) \[
\frac{d P^*_2 (R_2 | R_1)}{d P^*_1 (R_2 | R_1)} = \frac{d P^*_2 (R_2, R_1)}{d P^*_1 (R_2, R_1)} \frac{d P^*_1 (R_1)}{d P^*_2 (R_1)}
\]

where \( P^*_2 \) and \( P^*_1 \) are distributions of \( R_1, R_2 \) under \( H_2 \) and \( H_{20} \) respectively. From (1.22)

(2.38) \[
\frac{d P^*_2 (R_2, R_1)}{d P^*_1 (R_2, R_1)} = \frac{\int q_2 p_2 (gZ) \, d\mu (g)}{\int q_2 p_1 (gZ) \, d\mu (g)}
\]

where

\[
p_2 (Z) = p_2 (\sqrt{N} \bar{X}, S)
\]

\[
= Ke^{-\frac{1}{2} \text{trace} \, S^* + N \text{trace} \, \Gamma \bar{X}^t - \frac{1}{2} \left( \bar{\zeta}_1 + \bar{\zeta}_2 \right) \frac{N}{|S|^2}}
\]

\[
p_1 (Z) = p_1 (\sqrt{N} \bar{X}, S)
\]

\[
= Ke^{-\frac{1}{2} \text{trace} \, S^* + N \text{trace} \, \Gamma [1] [1] - \frac{1}{2} \bar{\zeta}_1 \frac{N}{|S|^2}}
\]

\[
S^* = S + N \bar{X} \bar{X}^t
\]

\[
\bar{\zeta}_1 = N \Gamma [1] \Sigma \Gamma [1] \quad ,
\]

\[
\bar{\zeta}_1 + \bar{\zeta}_2 = N \Gamma ^t \Sigma \Gamma , \quad (\bar{\zeta}_1 \geq 0 \text{ and } \bar{\zeta}_2 \geq 0)
\]

and \( K \) is a constant depending on \( \Gamma, \Sigma \).
Proceeding in the same way as in Section 1 we can have from (2.37) and (2.38),

\[
\frac{dp_{2}}{dp_{2}}(R_{2}|R_{1}) = \frac{dp_{1}}{dp_{1}}(R_{1}) = e^{-\frac{1}{2}r_{2}(1-R_{1})} \sum_{r=0}^{\infty} \frac{(R_{2} \bar{r}_{2})^{r}}{r!} \frac{\Gamma \left( \frac{N-q}{2} + r \right)}{\Gamma \left( \frac{P-q}{2} + r \right)} \frac{\Gamma \left( \frac{N-q}{2} \right)}{\Gamma \left( \frac{P-q}{2} \right)}
\]

In terms of \( \bar{r}_{1} \) and \( \bar{r}_{2} \), our problem is reduced to that of testing the hypothesis

\[
H_{0} : \bar{r}_{2} = 0 \quad \text{against} \quad H_{1} : \bar{r}_{2} \neq 0
\]

when \( \bar{r}_{1} \) is unknown. Furthermore since the distribution of \( R_{2} \) on each surface \( R_{1} = r_{1} \) is independent of \( \bar{r}_{1} \), the condition (2.36) reduces the above problem to the problem of testing a simple hypothesis: \( \bar{r}_{2} = 0 \) against the alternative: \( \bar{r}_{2} \neq 0 \) for each value of \( r_{1} \). In this conditional situation by Neyman and Pearson's fundamental lemma, the most powerful level \( \alpha \) invariant test \( \phi(R_{2}|R_{1}=r_{1}) \) of the hypothesis \( \bar{r}_{2} = 0 \) against \( \bar{r}_{2} \neq 0 \) is [from (2.39)] given by

\[
\phi(R_{2}|R_{1}=r_{1}) = 1 \quad \text{if} \quad \sum_{r=0}^{\infty} \frac{(R_{2} \bar{r}_{2})^{r}}{r!} \frac{\Gamma \left( \frac{N-q}{2} + r \right)}{\Gamma \left( \frac{P-q}{2} + r \right)} \frac{\Gamma \left( \frac{N-q}{2} \right)}{\Gamma \left( \frac{P-q}{2} \right)} \geq C(r_{1})
\]

\[
(2.41)
\]

\[= 0 \quad \text{otherwise}
\]

where \( C(r_{1}) \) is chosen in such a way that \( E_{\bar{r}_{2}=0} \phi(R_{2}|R_{1}=r_{1}) = \alpha \).

Since \( R_{2} = (1-R_{1})(1-Z) \), (2.41) reduces to

\[
\phi(Z|R_{1}=r_{1}) = 1 \quad \text{if} \quad Z \leq C'
\]

\[= 0 \quad \text{otherwise}
\]

(2.42)
where $C'$ is chosen in such a way that,

$$E_{\xi_2=0} \Phi(Z|\Gamma_1 = r_1) = \alpha .$$

We have remarked earlier that $Z$ is independent of $r_1$. Hence $C'$ is independent of $r_1$. Furthermore the above test $\Phi$ is independent of $\xi_2$. Hence we get the following theorem:

**Theorem 2.2:**

Given the observations $X^1, X^2 \cdots X^N (N > p)$ from $N(\xi, \Sigma)$, the likelihood ratio test of $\mathbb{H}_{20} : \Gamma_{[2]} = 0$ against $\mathbb{H}_2 : \Gamma_{[2]} \neq 0$ when $\Gamma_{[1]}$ is unknown is the uniformly most powerful similar invariant test.

### 2.6 Conclusion.

Let $Y$ be a $p$-dimensional random column vector with distribution law $N(\eta, \Sigma)$. Let

$$\Sigma^{-1} \eta = \underline{\Gamma} = \begin{pmatrix} \Gamma_1 \\ \Gamma_2 \\ \vdots \\ \Gamma_p \end{pmatrix} .$$

On the basis of the observations $Y^1, Y^2 \cdots Y^N (N > p)$ from $N(\eta, \Sigma)$ we have derived the likelihood ratio test of the hypothesis

$$\mathbb{H}_{20}' : \Gamma_{q+1} = \Gamma_{q+2} = \cdots = \Gamma_p = 0$$

against the alternative

$$\mathbb{H}_2' : \Gamma_{p'+1} = \Gamma_{p'+2} = \cdots = \Gamma_p = 0$$

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where $\Sigma, \eta$ are unknown, and $p > p' > q$. We have also established that this test is uniformly most powerful invariant similar.

The original problem was, to test by means of observations $X^1, X^2, \cdots X^N$ ($N > p$) from a p-variate non-singular normal population with distribution law $N(\xi; \Sigma)$, the hypothesis

$$H_{20} : \Gamma = \Sigma^{-1} \xi \in \mathcal{Z}$$

against the alternative

$$H_{2} : \Gamma \in \mathcal{Y}$$

where $\mathcal{Z} \subseteq \mathcal{Y}$ are subs-spaces of the adjoint space $\mathcal{Z}'$ of $\mathcal{Z}$ (the space of $X$'s) of dimensions $q$ and $p'$ respectively, and $\Sigma, \xi$ are unknown. In general $\mathcal{Z}'$ and $\mathcal{Y}'$ can be defined in many different ways. We will not consider all of them here. Instead, we will consider a case where under $H_{2}'$, $\mathcal{Z}$ can be expressed explicitly as $\Gamma = i_2 \overline{\Gamma}$ ($\Gamma$, $\overline{\Gamma}$ are unknown) and $H_{20}$ may further specify $\overline{\Gamma}$ as $\overline{\Gamma} = i_2 \xi$ ($\xi$ is unknown) where $i_2$, $i_1$ are the one to one inclusion mappings (arbitrary matrices) defined by

$$\mathcal{Z}' \leftarrow \mathcal{Y}' \leftarrow \mathcal{Z}$$

We recall that there is a natural isomorphism from $\mathcal{Z}$ to $\mathcal{Z}'$ where $\mathcal{Z}'$ is the adjoint space of $\mathcal{Z}'$. Since the isomorphism is natural we can identify $\mathcal{Z}'$ with $\mathcal{Z}$. Let $i_{2}', i_{1}'$ be the adjoints of $i_{2}$ and $i_{1}$.
\( \mathbf{x} \xrightarrow{i_1'} \mathbf{y} \xrightarrow{i_2'} \mathbf{z} \)

where \( \mathbf{y}, \mathbf{z} \) are adjoint spaces of \( \mathbf{y'} \) and \( \mathbf{z'} \) respectively. Now with the above specialization two meaningful maximal invariants for the problem are

\[
\overline{Q}_1 = \text{the squared length of } i_1' \mathbf{x} \\
= N \overline{x}' i_1 (i_1' s i_1)^{-1} i_1' \overline{x} \quad ;
\]

(2.43)

and \( \overline{Q}_2 = \text{the squared length of } i_2' i_1' \mathbf{x} \\
= N \overline{x}' i_1 i_2 (i_2' i_1' s i_1 i_2)^{-1} i_2' i_1' \overline{x} \quad ;
\]

where \( s \) and \( \overline{x} \) are sample covariance matrix and sample mean based on \( N \) observations. It is clear that if we take

\[
i_1 = \begin{pmatrix} I_1 & 0 \\ 0 & I_2 \end{pmatrix}
\]

and

\[
i_2 = \begin{pmatrix} I_1 \\ 0 \\ 0 \end{pmatrix}
\]

where \( I_1, I_2 \) are identity square matrices of order \( q \) and \( p'-q \) respectively, then this problem is reduced to that of testing \( H_0' \) against \( H_2' \). Since \( i_1 \) and \( i_2 \) are arbitrary we can in particular assign \( i_1, i_2 \) the above values to define \( \overline{Q}_1 \) and \( \overline{Q}_2 \). Thus to obtain
the likelihood ratio test statistic for the original problem we need only to find the relationship between \( \bar{Q}_1, \bar{Q}_2 \) (for the above values of \( i_1, i_2 \)) and \( R_1, R_2 \) (defined in Section 2). Assigning \( i_1, i_2 \) the above values we get

\[
\bar{Q}_1 = \frac{R_1 + R_2}{1 - R_1 - R_2}
\]

and

\[
\bar{Q}_2 = \frac{R_1}{1 - R_1}
\]

Hence the likelihood ratio test of \( H_{20} \) against \( H_2 \) can be obtained from that of \( H_{10}' \) against \( H_2' \), by replacing

\[
R_1, R_2 \text{ by } \frac{\bar{Q}_2}{1 + \bar{Q}_2}, \quad \frac{\bar{Q}_1}{1 + \bar{Q}_1} - \frac{\bar{Q}_2}{1 + \bar{Q}_2},
\]

where \( \bar{Q}_1, \bar{Q}_2 \) are as defined in (2.43).
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