ON SEVERAL "DISTINCT" HYPOTHESES

by

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ON SEVERAL "DISTINCT" HYPOTHESES

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1. Introduction and Summary

A quite general multiple-decision problem is the following. A numerical or vector random variable \( X \) has an elementary probability density \( f(x, \theta) \). The unknown parameter \( \theta = (\theta_1, \ldots, \theta_m) \), which specifies the distribution of \( X \) lies in a Borel subset \( \Omega \) of the \( m \)-dimensional Euclidean space \( \mathbb{E}_m \) (possibly \( \Omega = \mathbb{E}_m \)).

Consider a partition of \( \Omega \) into \( k \) subsets \( \omega_1, \ldots, \omega_k \), i.e.,

\[
\Omega = \omega_1 \cup \ldots \cup \omega_k ,
\]

\[
\omega_i \cap \omega_j = \emptyset , \quad i \neq j, \quad i, j = 1, \ldots, k .
\]

Let \( H_i \) designate the composite hypothesis that \( \theta \in \omega_i \), \( i = 1, \ldots, k \). Let \( \varphi = (\varphi_1, \ldots, \varphi_k) \) denote a (possibly randomized) decision rule (d.r.) accepting \( H_i \) with probability \( \varphi_i(x) \) when \( x \) is the observed value of \( X \). The question of interest is: What can we say about the minimum probability of correct decision? For every d.r. \( \varphi \) let us define

\[
(1) \quad \beta_i(\varphi) = \inf_{\theta \in \omega_i} \mathbb{E}_\theta \varphi_i(X) ,
\]

and
\begin{equation}
\beta(\phi) = \min_{1 \leq i \leq k} \beta_i.
\end{equation}

Since we allow randomized d.r.'s, we notice that $\beta$ can be made at least equal to $\frac{1}{k}$, for we can use the "guess" d.r. $\phi^0$, say, which, no matter what $x$ we observe, accepts each $H_1$ with probability $\frac{1}{k}$. Thus, we will look for conditions on $f_\theta(x)$ and $\omega_1, ..., \omega_k$ such that we can find a $\phi^*$ with
\[ \beta(\phi^*) > \frac{1}{k}. \]

Wald and Berger [1] consider the case $k = 2$; under the assumption that $\omega_1$ and $\omega_2$ are disjoint, compact and connected subsets of the parameter space $\Omega = \omega_1 \cup \omega_2$. They prove that a necessary and sufficient condition for the existence of a $\phi^* = (\phi^*_1, \phi^*_2)$ such that $\beta(\phi^*) > \frac{1}{2}$ is that $H_1$ and $H_2$ will be distinct (cf. Definition 2.2 below).

The purpose of this paper is to show how the results of [1] extend for $k > 2$. In particular, the results of [1] can be strengthened by a weakening of the assumption that $\omega_1$ and $\omega_2$ are compact.

2. Definitions

2.1 Two densities $f_1(x)$ and $f_2(x)$ with respect to a $\sigma$-finite measure $\mu$ will be called different and we shall write $f_1 \neq f_2$ if there exists a measurable subset $A$ of $x = \{x\}$ such that
\begin{enumerate}
\item $\mu(A) > 0$
\item $\int_A f_1(x) d\mu(x) \neq \int_A f_2(x) d\mu(x)$.
\end{enumerate}

Otherwise $f_1$ and $f_2$ will be called equal and we shall write $f_1 = f_2$.
2.2 Two composite hypotheses \( H_1 = \{f_1(x)\}, \ H_2 = \{f_2(x)\} \) are called distinct (cf. [1]) if
\[
f_1 \neq f_2 \quad \text{for all } f_1 \in H_1, \ f_2 \in H_2
\]

For the case of more than two composite hypotheses we introduce the following definition of "distinctness":

2.3 The composite hypotheses \( H_i = \{f_i(x)\}, \ i = 1, \ldots, k \) will be called "overlapping" if there exist \( f_i \in H_i, \ i = 1, \ldots, k \) such that
\[
f_1 = f_2 = \ldots = f_k.
\]

Otherwise, the \( H_i \) will be called "distinct". Below we are going to compare the performance of a d.r. \( \varphi \) to that of the completely randomized \( \varphi^0 \), which is formally defined by
\[
\varphi_1(x) = \ldots = \varphi_k(x) = \frac{1}{k}
\]
identically in \( x \). Thus, we introduce the definition

2.4 A d.r. \( \varphi \) will be called superior to the "guess" rule \( \varphi^0 \) in (2) (henceforth "superior") if
\[
\beta(\varphi) > \beta(\varphi^0) = \frac{1}{k}.
\]
3. Preliminary Results

Before we return to the discussion about the composite hypotheses of Section 1, we prove some auxiliary results which are of interest in themselves.

Theorem 3.1. If the composite hypotheses \( H_i = \{f_i(x)\}, i = 1, \ldots, k \) are overlapping there is no superior d.r.

Proof: By hypothesis (cf. definition 2.3) there exist densities \( f_1^0 \in H_i, i = 1, \ldots, k, \) such that

\[
(3) \quad f_1^0 = \ldots = f_k^0 = f^0, \quad \text{say}.
\]

If \( \varphi \) is any d.r. then by (1)

\[
\beta_i(\varphi) \leq E_{f_1} \varphi_1(x), \quad f_1 \in H_i, \quad i = 1, \ldots, k,
\]

and by (3)

\[
E_{f_1^0} \varphi_1(x) = E_{f^0} \varphi_1(x), \quad i = 1, \ldots, k.
\]

Since for every \( x \)

\[
(4) \quad \sum_{i=1}^{k} \varphi_i(x) = 1,
\]

we get

\[
4
\]
\[
\sum_{i=1}^{k} \beta_i(\varphi) \leq \sum_{i=1}^{k} E_{f_i} \varphi(X) = \sum_{i=1}^{k} E_{f_i} \varphi(X) = E_{\varphi^0} \sum_{i=1}^{k} \varphi(X) = 1.
\]

Hence for every \( \varphi \)
\[
\beta(\varphi) = \min_{1 \leq i \leq k} \beta_i(\varphi) \leq \frac{1}{k},
\]

and our assertion is established.

This result is intuitively justified as follows: the \( b_i \) are the minimum probabilities of "discriminating" among \( H_1, \ldots, H_k \) on the basis of an observation \( x \); however, when the state of nature is described by \( (3) \), \( x \) is equally likely to have come from any \( H_i \), and therefore it has no informative value for discriminating among \( H_1, \ldots, H_k \). Thus no d.r. can improve the \( \beta \) of the "guess" rule \( \varphi^0 \).

As a matter of fact from Theorem 3.1 we conclude

**Corollary 3.1.** If the \( k \) composite hypotheses \( H_1, \ldots, H_k \) are overlapping, then a necessary and sufficient condition that a d.r. \( \varphi \) be minimax is that \( \beta(\varphi) \geq \frac{1}{k} \), i.e., for each \( f_i \in H_i, i = 1, \ldots, k \),
\[
E_{f_i} \varphi(X) \geq \frac{1}{k}.
\]

**Lemma 3.1.** Let \( h_i \) denote the simple hypothesis that \( X \) has density \( f_i(x), i = 1, 2 \). If \( f_1 \neq f_2 \) then for each \( \gamma, 0 < \gamma < 1 \) there exists a test \( \varphi^{(\gamma)} = (\varphi_1^{(\gamma)}, \varphi_2^{(\gamma)}) \) such that
\[
E_{h_1} \varphi_1^{(\gamma)}(x) > \gamma, \quad E_{h_2} \varphi_2^{(\gamma)}(x) > 1 - \gamma.
\]
Proof: Let $S$ be the set of all points $(\alpha, \beta)$ for which there exists a test $\varphi$ such that

$$
\alpha = E_{n_1} \varphi_2(x), \quad \beta = E_{n_2} \varphi_2(x).
$$

It is known (see, e.g., Lehmann [2], p. 67) that $S$ is a convex set, it contains the points $(0,0)$ and $(1,1)$, and is symmetric with respect to the point $(\frac{1}{2}, \frac{1}{2})$ in the sense that if $(\alpha, \beta) \in S$ then $(1 - \alpha, 1 - \beta) \in S$. Moreover, $S$ is closed. Thus, it suffices to show that there exists a point $(\alpha_\gamma, \beta_\gamma) \in S$ such that $\alpha_\gamma < 1 - \gamma$, $\beta_\gamma > 1 - \gamma$.

Since $f_1 \neq f_2$, $S$ is different from the segment of the diagonal between $O = (0,0)$ and $(1,1)$. Hence, there is a point $B = (1 - \gamma, \beta^*)$ with $\beta^* > 1 - \gamma$; by the convexity of $S$ every point on the line segment $OB$ belongs to $S$; it follows that $OB$ lies above the diagonal (see Fig. 1). Therefore, there exists an interior point $\Gamma = (\alpha_\gamma, \beta_\gamma)$ of $OB$ which lies above the line $\beta = 1 - \gamma$; clearly, $\alpha_\gamma < 1 - \gamma$ and $\beta > 1 - \gamma$. Hence our assertion follows.

![Figure 1](image-url)
Remark: This result for $\gamma = \frac{1}{2}$ was proven in [1] (Lemma 1) by exhibiting a critical region $W$ such that $P[W|h_1] < \frac{1}{2}$, $P[W|h_2] > \frac{1}{2}$.

4. The case of parametric families of hypotheses

Let us now return to the parametric hypotheses $H_i$ described in Section 1. As an immediate corollary of Theorem 3.1 and the definition 2.3 we have

Corollary 4.1. A necessary condition for the existence of a superior d.r. for discriminating among $H_1: \theta \in \omega_1$, ..., $H_k: \theta \in \omega_k$ is that the $\omega_i$ have an empty intersection, i.e.,

$$\omega_1 \cap \omega_2 \cap \ldots \cap \omega_k = \emptyset .$$

That the above is not a sufficient condition can be very easily verified. As an example, consider the case $k = 2$ and suppose that $X$ is $N(\mu,1)$. Then the hypotheses

$$H_1: \mu < 0, \quad H_2: \mu > 0$$

are clearly distinct but there is no $\varphi = (\varphi_1, \varphi_2)$ such that $\beta(\varphi) > \frac{1}{2}(\sup_\varphi \beta(\varphi) = \frac{1}{2})$.

In order to describe another necessary condition for the existence of superior d.r.'s, we introduce the following.
For each \( i = 1, \ldots, k \), let

\[
\mathcal{Z}_i = \{ \xi_i : \int_{\omega_1} \xi_i (\theta) \, d\xi_i (\theta) = 1 \}
\]

be the set of distribution functions on \( \omega_1 \), and let

\[
\Omega_1 = \{ h_1 : h_1 (x) = \int_{\omega_1} f(x, \theta) \, d\xi_i (\theta), \, x \in \mathcal{X}, \, \xi_i \in \mathcal{Z}_i \}.
\]

Then we have

Theorem 4.1. A necessary condition for the existence of a superior d.r. for discriminating among \( H_1 : \theta \in \omega_1, \ldots, H_k : \theta \in \omega_k \) is that the \( \Omega_1 \) in (6) satisfy

\[
\Omega_1 \cap \Omega_2 \cap \ldots \cap \Omega_k = \emptyset
\]

Proof: Suppose that (7) is not true. Then there exist \( \xi_i^0 \in \mathcal{Z}_i \), \( i = 1, \ldots, k \) such that for all \( x \), except possibly in a set of measure zero,

\[
h_i^0 (x) = \int_{\omega_1} f(x, \theta) d\xi_i^0 (\theta) = h_i^0 (x), \quad i = 1, \ldots, k.
\]

Since, however, for each \( i = 1, \ldots, k \), and every d.r. \( \varphi \)

\[
\beta_i (\varphi) = \inf_{\theta \in \omega_1} E_{\theta} \varphi_1 (X) \leq \int_{\omega_1} \int_{\mathcal{X}} \varphi_1 (x) f(x, \theta) \, dx \, d\xi_i^0 (\theta)
\]
we obtain by Fubini's theorem and (4) also

\[ \sum_{i=1}^{k} \beta_i(\varphi) \leq \sum_{i=1}^{k} \int_{\omega_i} \varphi_i(x) \int f(x, \theta) d\lambda_1^o(\theta) dx \int O_i^o(\theta) dx = \sum_{i=1}^{k} \int_{\omega_i} \varphi_i(x) h^o(x) dx \]

\[ = \int h^o(x) dx = 1. \]

Hence for every \( \varphi \)

\[ \beta(\varphi) = \min_{1 \leq i \leq k} \beta_i(\varphi) \leq \frac{1}{k}, \]

and (7) follows.

For the discussion of sufficient conditions for the existence of superior d.r.'s we require the following. Let

\[ \omega_i^c = \Omega - \omega_i, \quad i = 1, \ldots, k, \]

and consider the family

\[ \Lambda_i = \{ \lambda_i: \int_{\omega_i} d\lambda_i(\theta) = 1 \}, \quad i = 1, \ldots, k \]

of all probability distributions over \( \omega_i^c \). Define the set \( \Omega^{*}_i \) of elementary probability densities \( g_i(x) \) by

\[ \Omega^{*}_i = \{ g_i: g_i(x) = \int_{\omega_i} f(x, \theta) d\lambda_i(\theta), \quad x \in \mathcal{X}, \lambda_i \in \Lambda \} \]

for \( i = 1, \ldots, k \).
Definition 4.1. The pair \((\xi_1^0, \lambda_1^0)\) with \(\xi_1^0 \in \Xi_1, \lambda_1^0 \in \Lambda_1\) is called least favorable (l.f.) (cf. [2], p. 325) for the problem \(P_1\) of testing

\[ H_{11} : \theta \in \omega_1 \] versus \[ H_{21} : \theta \in \omega_1^c \]

if

\[
\sup_{\omega_1} E_{\theta} \varphi_{2, \xi_1^0, \lambda_1^0} (X) \leq \alpha,
\]

and

\[
\inf_{\omega_1^c} E_{\theta} \varphi_{2, \xi_1^0, \lambda_1^0} (X) = \beta_{\xi_1^0, \lambda_1^0},
\]

where for every \(\alpha, \ 0 < \alpha < 1, \ \beta_{\xi_1^0, \lambda_1^0}\) is the power of the most powerful \(\xi_1^0, \lambda_1^0\) test \(\varphi_{\xi_1^0, \lambda_1^0}\) for testing

\[
h_1(x) = \int_{\omega_1} f(x, \theta) d\xi_1(\theta)
\]

at level \(\alpha\) against

\[
g_1(x) = \int_{\omega_1^c} f(x, \theta) d\lambda_1(\theta).
\]

Now we can state a sufficient condition for the existence of a superior rule.
Theorem 4.2. Suppose that $(\xi_{i_1^*}^0, \lambda_{i_1^*}^0)$ for some $i^* \in (1, \ldots, k)$ is a least favorable pair of distributions for the problem $P_{i^*}$. If furthermore

$$\Omega_{i^*} \cap \Omega_{i^*}^* = \emptyset,$$

then there exists a superior d.r. $\varphi^* = (\varphi_1^*, \ldots, \varphi_k^*)$ for discriminating among $H_1, \ldots, H_k$.

Proof: Let

$$h_{i^*}^0(x) = \int_{\omega_{i^*}} f(x, \theta) d\xi_{i^*}^0(\theta) \in \Omega_{i^*},$$

$$g_{i^*}^0(x) = \int_{\omega_{i^*}} f(x, \theta) d\lambda_{i^*}^0(\theta) \in \Omega_{i^*}^*.$$

Since $\Omega_{i^*}$ and $\Omega_{i^*}^*$ are, by hypothesis disjoint, $h_{i^*}^0(x)$ and $g_{i^*}^0(x)$ are different densities, and by Lemma 3.1 there exists a test

$$(\varphi_{\frac{1}{k}}^k, \varphi_{\frac{1}{k}}^k, \varphi_{\frac{1}{k}}^k)$$

such that for the problem $P_{i^*}$

$$E_{h_{i^*}^0} \varphi_{\frac{1}{k}}^k(X) > \frac{1}{k},$$

$$E_{g_{i^*}^0} \varphi_{\frac{1}{k}}^k(X) > \frac{k-1}{k}.$$

Let $\varphi' = (\varphi_1', \varphi_2')$ be the most powerful test for testing the simple hypothesis that $X$ has density $h_{i^*}^0$ against the alternative that $X$
has density $g_{i*}$ at level

$$\alpha_k = E_{\phi_2} \frac{(\frac{1}{k})}{h_{i*}} = E_{\phi_2} [1 - \frac{(\frac{1}{k})}{h_{i*}}] \leq 1 - \frac{1}{k} = \frac{k-1}{k}.$$ 

Then

$$E_{\phi_1} \frac{(\frac{1}{k})}{h_{i*}} > \frac{1}{k}$$

and since $\phi'$ is the most powerful test

$$E_{\phi_2} \frac{(\frac{1}{k})}{\theta} \phi_2(X) > \frac{k-1}{k}.$$ 

By (10) we have

$$\inf_{\theta \in \omega_{i*}} E_{\phi_1} \phi_2(X) = E_{\phi_2} \frac{(\frac{1}{k})}{\phi_2(X)} > \frac{k-1}{k}$$

and by (9)

$$\inf_{\theta \in \omega_{i*}} E_{\phi} \phi_1(X) \geq 1 - \alpha_k > \frac{1}{k}.$$ 

Define a d.r. $\phi^* = (\phi_1^*, \ldots, \phi_k^*)$ by

$$\phi_{i*}^*(x) = \phi_1^*(x),$$

$$\phi_{j}^*(x) = \frac{1}{k-j} \phi_2^*(x), \quad j \neq i^*, \quad j = 1, \ldots, k.$$
Noting that

\[ \omega_{1*}^c = \Omega - \omega_{1*} = \bigcup_{j \neq i*} \omega_1, \]

we conclude by (11) and (12) also that \( \phi* \) is a superior d.r. as required by the theorem.

Remark: The problem of existence of a least favorable pair of distributions for testing

\[ H_1^*: \theta \in \omega_1 \text{ versus } H_2^*: \theta \in \omega_2 \]

is beyond the scope of the present investigation. However, at this point we should like to remark that the theorems of [1] can be strengthened by relaxing the condition of compactness of \( \omega_1 \) and \( \omega_2 \), which is required by Wald for the existence of a least favorable a priori distribution on \( \Omega = \omega_1 \cup \omega_2 \) (cf. [1]). From Lehmann [3] (Section 5), we conclude that if \( f(x,\theta) \) is continuous in \( \theta \) for almost all \( x \), and measurable in \( x \) and \( \theta \) then a sufficient condition for the existence of a least favorable a priori distribution on \( \Omega \) is that the following are satisfied:

1. Given any \( \epsilon > 0 \) and any compact subset \( \omega_i^j \) of \( \omega_i \), \( i = 1,2 \) there exists a Borel set \( A \) in the sample \( X \) such that

\[ \int_{X-A} f(x,\theta)dx \leq \epsilon, \quad \text{for all } \theta \in \omega_i^j, \quad i = 1,2, \]
and

\[
\int_A f(x, \theta) dx \to 0
\]

uniformly as \( \theta^2 = \theta_1^2 + \ldots + \theta_m^2 \to \infty \) (cf. Assumption A of [3]). In order to obtain necessary and sufficient conditions for the existence of superior d.r.'s further conditions have to be imposed on \( \Theta \). Results in this direction will appear in a subsequent paper.

REFERENCES


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