CONDITIONAL RANK-ORDER TESTS FOR EXPERIMENTAL DESIGNS

BY

K. L. MEHRA

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1. Introduction.

Consider an experimental design with K treatments, n blocks
and \( m_{ij} \) observations \( X_{ij\ell} \), \( \ell = 1, 2, \ldots, m_{ij} \), corresponding to the
i-th block and the j-th treatment. For testing the equality of the
treatment effects Hodges and Lehmann had proposed in [5] certain con-
dditional rank tests based on interblock rank comparison of the observa-
tions after "alignment" (defined below). Unlike the previously proposed
rank-procedures (Friedman [3], Bernard and van Elteren [1]), based on
independent rankings of observations in each block, a distinct feature
of the conditional rank tests is that the test-functions are based on a
joint-ranking of the totality of aligned observations. In Mehra and
Sarangi [8] the asymptotic efficiency \( e_{L_n, F} \) of these conditional
tests relative to the normal theory \( F \)-test was shown to be never
less than \( 3/\pi = .955 \) under normality. Under very mild restrictions,
whatever be K, the convergence in distribution of the conditional
test statistics, as \( n \to \infty \) is uniform in the configuration \( \xi \),
(where the "configuration" \( \xi \) stands for the observed distribution of
ranks over the blocks).

The purpose of this paper is to extend the above results by con-
sidering a general class of conditional test procedures of this nature,
namely, the conditional rank-order test procedures, and investigate their asymptotic efficiency. It is shown, under very general conditions, that the convergence is again uniform in the "configuration". For the normal-score versions of the conditional rank-order tests, the asymptotic efficiency \( e_{L_n} \) obtains the value one, under normality. It is also shown (using the results of [4] and [2]) that for the Wilcoxon and the normal score versions, \( e_{L_n} \) is bounded below by .864 and 1, respectively.

2. Assumptions, Notations and the Test Functions.

Let the \( m \) random observations \( X_{ij} \), \( i = 1, 2, \ldots, m \), corresponding to the \((i,j)\)-th cell, \( j = 1, 2, \ldots, K \), \( i = 1, 2, \ldots, n \), be distributed according to a common continuous distribution function (d.f.),

\[
F_{ij}(x) = F_j(x + \xi_i),
\]

where \( \xi_i \) may represent the unknown block effects. Then the hypothesis of equality of treatment-effects can be expressed as \( H_0: F_1 = F_2 = \cdots = F_K \).

Our test functions are based on a joint-ranking of the totality of observations after "alignment". By "alignment" is essentially meant rendering the observations comparable through some transformation. In the above context, it means removing the block effects \( \xi_i \) through a transformation such that, under \( H_0 \),

(C1) the aligned observations in each block have a symmetric joint distribution;

(C2) marginally or otherwise, the joint distribution of the aligned observations do not vary from block to block.
The symmetry condition (C1) is satisfied if we align the observations by subtracting from each observation in a block, say the i-th, a function \( \mu = \mu(X_i) = \mu(X_{i1l}, \ldots, X_{iKm}, \ldots, X_{iKm_1}) \) of observations in the block, which is symmetric in the arguments and satisfies the invariance condition (2.2) of [8]. The condition (C2) above, then, would also be satisfied if the number \( N_i = \sum m_{ij} \) of observation is the same for all blocks and the same function \( \mu \) is used for alignment in all blocks.

We observe that the conditions (C1) and (C2) above are desirable—although not necessary—for the applicability of the conditional tests. However, for the study of the asymptotic null distribution below we will assume (C1) and for studying the asymptotic efficiency both (C1) and (C2).

Let the aligned observations be denoted by \( Z_{ijl}, l=1,2,\ldots,m_{ij} \), \( 1 \leq j \leq K, 1 \leq i \leq n \), and \( r_{ijl} \) be the rank of \( Z_{ijl} \) in a combined ranking of all the \( N = \sum_{i=1}^{n} m_{ij} \) aligned observations. Consider now the conditional situation given the set of ranks for each block.

Each conditional situation (an event in the original sample space) will be referred to as a configuration. More specifically, if \( r_i^{(1)} < r_i^{(2)} < \ldots < r_i^{(N_i)} \) represent the ordered ranks for the i-th block and \( r_i = (r_i^{(1)}, \ldots, r_i^{(N_i)}) \), then a configuration is simply an event \( \xi = (r_1, r_2, \ldots, r_n) \) in our sample space. Observe that given a configuration, the only randomness that remains is due to independent assignment of ranks to the treatments. Consider now a double sequence of numbers \( \{\xi, k, k=1,2,\ldots,N\} \), where \( \xi, 1 \leq \xi, 2 \leq \cdots \leq \xi, \) with at least one strict inequality, and define a step function \( \xi, (u) \) on (0,1) with
(2.2) \[ \xi_N(u) = \xi_N\left(\frac{k}{N+1}\right) \text{ for } \frac{k-1}{N} < u \leq \frac{k}{N} , \]

for \( k=1,2,\ldots,N \). Assume further the existence of a non-constant non-decreasing function \( \xi(u) \) on \((0,1)\) such that

(2.3a) \[ \int_0^1 \xi^2(u)du < \infty \quad \text{and} \]

(2.3b) \[ \lim_{N \to \infty} \int_0^1 [\xi_N(u) - \xi(u)]^2 du = 0 . \]

Now let

(2.4) \[ T_{Nj} = \sum \sum \xi_{N,r_{ij}} = \sum \sum \xi_N\left(\frac{r_{ij}}{N+1}\right) \]

be the sum of the rank scores for the \( j \)-th treatment,

and let \( \widetilde{E}(T_{Nj}), \sigma_j^2, \sigma_{jj'}, \) and \( \sigma_{jj'} \) stand for the conditional expectation, variance and covariance of \( T_{Nj} \) and \( T_{Nj'} \), under \( H_0 \) and the condition (Cl), given a configuration \( \xi_N \). Then

(2.5) \[ \widetilde{E}(T_{Nj}) = \sum m_{ij} \xi_N^{(1)} \]

\[ \sigma_{jj'} = \sum_{i=1}^{n} (N_i \xi_{j,j'} - m_{ij} \xi_{j,j'} [\tau_{ii}]) , \]

where \( \tau_{ii}^2 = \sum \sum \xi_{N,r_{ij}} - \xi_N^{(1)} \frac{2}{N_i} \), \( \xi_N^{(1)} = (\sum \sum \xi_{N,r_{ij}}) / N_i \) and \( \xi_{j,j'} = 1 \) or zero according as \( j = j' \) or \( j \neq j' \).

In this paper, we consider mainly the complete case when \( m_{ij} = m_j \) for all \( i \) and each \( j \) (which also covers the equal observations...
per cell case). Under this condition, the $N_i$'s are all equal to \( N' = \sum_j m_j \), \( E(T_{N_j}) = m_j n \bar{\xi}_N \) with \( \bar{\xi}_N = \sum_k \xi_{N,k} / N \), so that the covariance matrix (2.5) and consequently) the proposed (conditional) test function takes the simple form

\[
L_n = \left( \frac{(N'-1)/N'}{\sum_{i=1}^{n} z_i^2} \right) \sum_{j=1}^{K} \frac{1}{m_j} \left( T_{N_j} - m_j n \bar{\xi}_N \right)^2,
\]

with the test statistic consisting in rejecting \( H_0 \) if \( L_n \) is too large. In the general case when the \( m_{ij} \) do not satisfy the above condition, whether the design is complete or not, one can base the test on a statistic of the type (2.13) of [8], with ranks replaced by the rank-scores \( \{\xi_{N,k} : 1 \leq k \leq N\} \), viz.,

\[
Y' \Delta^{-1} Y
\]

where \( Y = (V_{N1}, \ldots, V_{N,K-1}) \), with \( V_{Nj} = [T_{Nj} - \tilde{E}(T_{Nj})] \) and \( \Delta \) is the exact covariance matrix of \( Y \) given by (2.5). The only condition required is the non-singularity of \( \Delta \), a sufficient condition for which according to Theorem II of [1] is that each pair of treatments is compared in at least one block. For a balanced incomplete blocks design, in particular, this condition is always satisfied.

For application of these tests for small sample sizes, one can compute the exact (conditional) null distribution of \( L_n \) for any given set of scores by using the fact that conditionally, given a configuration, all permutations of ranks over the treatments in a block have the same probability. Computational techniques for the evaluation of such exact distributions are discussed in [5] for the Wilcoxon case and \( K = 2 \).
Ideally, exact distribution tables should be made available for various values of $K$, $n$ and $m_{ij}$.

3. **Uniform Convergence in the "Configuration".**

To prove that the convergence in distribution, under $H_0$, of the test functions (2.6) is uniform in the configuration, we make the additional assumptions:

(3.1a) $\xi(u)$ is strictly increasing in $u$;

and that, as $N \to \infty$,

(3.1b) $\frac{\xi_N(u)}{N} \to \xi(u)$, at each continuity point of $\xi(u)$.

The following theorem extends theorem (2.1) of [8].

**Theorem 3.1.** Suppose that for the functions $\xi_N(u)$ and $\xi(u)$, the conditions (2.3a), (2.3b), (3.1a) and (3.1b) are satisfied. Then if $H_0$ is true and the method of alignment is such that the assumptions (A) and (B) of [8] hold, (i) the conditional statistic $L_n$ defined by (2.6), given a configuration, converges in distribution, as $n \to \infty$, to a $\chi^2$-variable with $(K-1)$ degrees of freedom, and (ii) the convergence is uniform in the configuration.

**Proof.** Consider an arbitrary linear combination $U_i = \sum_{j=1}^{K-1} c_j T_j^{(1)}$ of $T_j^{(1)}$, $j=1,2,\ldots,K-1$, where

$$T_j^{(1)} = \sum_{k=1}^{m_j} \xi_{N,r_{ijk}} = \sum_{k=1}^{m_j} \xi_N \left( \frac{r_{ijk}}{N+1} \right).$$

Let $b_{i}^2$, $\beta_i$ denote the conditional, given a configuration $\xi$, 

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variance and third absolute moment of $U_i$ and set $S_n^2 = \sum_{i=1}^n b_i^2$.

First note that by setting $\xi_N(r_{i,j}/(N+1)) = \eta_{i,j}$,

$$\beta_i = \sum_{j=1}^{K-1} c_j (T_j^{(1)} - \bar{T}_j^{(1)}) |^3$$

$$\leq (K-1)^2 \sum_{j=1}^{K-1} |c_j|^3 \sum_{j=1}^{K-1} |T_j^{(1)} - \bar{T}_j^{(1)}|^3$$

$$\leq (K-1)^2 (N')^{2} c^{*} \sum_{j=1}^{m_j} \sum_{j=1}^{m_j} |\eta_{i,j}|^3$$

$$\leq (K-1)^2 4(N')^3 c^{*} \sum_{j=1}^{K} \sum_{j=1}^{m_j} |\eta_{i,j}|^3,$$

with $c^{*} = \max_j |c_j|^3$, so that

$$\sum_{i=1}^{n} \beta_i \leq (K-1)^2 (N')^3 c^{*} \sum_{k=1}^{N} |\xi_{N,k}|^3$$

$$\leq a^{*}_{1} \max_{1 \leq k \leq N} |\xi_{N,k}| (\sum_{k=1}^{N} \xi_{N,k}^2),$$

where $a^{*}_{1} > 0$ is a constant independent of the configuration. Further, letting

$$r'_1 = \min_{(j,\ell)} \eta_{i,j,\ell} \quad \text{and} \quad r''_1 = \max_{(j,\ell)} \eta_{i,j,\ell}$$

and following arguments similar to (2.8a)-(2.11) of [8], we obtain that

$$S_n^2 \geq a^{*}_{2} \sum_{i=1}^{n} (a''_i - a'_i)^2$$

where $a'_1 < a'_2 < \cdots < a'_n$ and $< a''_1 < a''_2 < \cdots < a''_n$ are, respectively,
the ordered \( r_1', r_2', \ldots, r_n' \) and \( < r_1'', r_2'', \ldots, r_n'' \), and \( a_2^* > 0 \) is a constant independent of the configuration. (On account of (3.1a) and (3.1b) we may assume that \( \xi_{n,1} < \xi_{n,2} < \cdots < \xi_{n,N} \).) Also since \( a_1' < a_2' < \cdots < a_n' < a_1'' < \cdots < a_n'' \), (3.3) gives

\[
(3.4) \quad s_n^2 > a_2^* n \min_{1 \leq k \leq N-n} [\xi_N^\left(\frac{k + n}{N + 1}\right) - \xi_N^\left(\frac{k}{N + 1}\right)]^2.
\]

We now show that there exists a constant \( c > 0 \) such that

\[
(3.5) \quad \liminf_{n} \min_{1 \leq k \leq N-n} |\xi_N^\left(\frac{k + n}{N + 1}\right) - \xi_N^\left(\frac{k}{N + 1}\right)| \geq c > 0.
\]

On account of (3.1a), it would suffice to prove (3.5) for continuous \( \xi(u) \). First note that there exists a \( c > 0 \) such that

\[
(3.6) \quad \inf_{0 < u < 1 - \lambda} [\xi(u + \lambda) - \xi(u)] \geq c > 0,
\]

for any \( 0 < \lambda < 1 \). To see this, suppose to the contrary that no such \( c \) exists, then there exists a sequence \( \{u_k\} \) such that

\[ [\xi(u_k + \lambda) - \xi(u_k)] \to 0, \text{ as } k \to \infty. \]

Let \( u_0 \) be an accumulation point of \( \{u_k\} \) (\( u_0 \) may be zero) and \( \{u'_k\} \) be a subsequence which converges to \( u_0 \), say, from above. Then on account of the continuity of \( \xi(u) \), there exists a \( k'_o = k'_o(u_0) \) such that for \( k' \geq k'_o \)

\[
(3.7) \quad \xi(u'_k) < \xi(u_0 + \frac{\lambda}{4});
\]

also, since \( \xi(u) \) is strictly increasing, we have

\[
(3.8) \quad \xi(u_k + \lambda) \geq \xi(u_0 + \lambda) > \xi(u_0 + \frac{\lambda}{4}).
\]
But (3.7) and (3.8) contradict the assertion that \(\xi(u_k + \lambda) - \xi(u_k)\) → 0, as \(k \to \infty\), so that (3.6) is proved. Now set \(u_k = k/(N+1), \lambda_N = n/(N+1)\) and \(\lambda_0 = 1/(N'+1)\), then \((\lambda_0) < \lambda_N < (1 - \lambda_0)\), so that

\[
(3.9) \quad \min_{1 \leq k \leq N-n} \left[ \frac{\xi_N(k+n)}{N+n+1} - \frac{\xi_N(k+1)}{N+n+1} \right] \geq \min_{0 < u < 1 - \lambda_N} \left[ \frac{\xi_N(u + \lambda_0)}{N+n+1} - \frac{\xi_N(u)}{N+n+1} \right]
\]

\[
\geq \inf_{0 < u < 1 - \lambda_0} \left[ \frac{\xi_N(u + \lambda_0)}{N+n+1} - \frac{\xi_N(u)}{N+n+1} \right].
\]

Since (3.1b) holds for all \(u\), R.H.S. of (3.9) is approximated for large \(N\) by \(\inf_{0 < u < 1 - \lambda_0} [\xi(u + \lambda_0) - \xi(u)]\), so that taking \(\lim \inf\) on both sides of (3.9), (3.5) follows forthwith on account of (3.6).

Now let \(G(n)(x)\) denote the conditional d.f. of \(\sum_{i=1}^{n} [U_i - \overline{E}(U_i)]/\overline{S}_n\) given a configuration \(\xi = \theta\), and \(\phi(x)\) the \(N(0,1)\) d.f., then by the well-known Berry-Esseen theorem, (3.2) and (3.4) follows the existence of a constant \(a_3^* > 0\), independent of the configuration, such that

\[
(3.10) \quad |G^{(n)}_\theta(x) - \phi(x)| \leq a_3^* \max_{1 \leq k \leq N} \left| \frac{\xi_N,k}{N} \right| (\sum_{k=1}^{N} \xi_N,k)^2/(N \min_{1 \leq k \leq N-n} \left[ \frac{\xi_N(k+n)}{N+n+1} - \frac{\xi_N(k+1)}{N+n+1} \right])^{3/2}
\]

Now observe that the condition (2.6) implies uniform absolute continuity of the set functions \(v_N(A) = \int_A \xi_N^2(u)du\), where \(A\) is a Borel subset of \((0,1)\), so that

\[
(3.11) \quad \frac{1}{N} \sum_{k=1}^{N} \xi_N,k^2 \rightarrow \int_0^1 \xi_N^2(u)du < \infty,
\]

and

\[
(3.12) \quad \frac{1}{N} \max_{1 \leq k \leq N} \left| \xi_N,k \right|^2 = \max_{1 \leq k \leq N} \int_{\left\{ \frac{k-1}{N}, \frac{k}{N} \right\}} \xi_N^2(u)du \rightarrow 0,
\]
as \( N \to \infty \). From (3.10) and (3.5), (3.11), (3.12), it follows that,
as \( n \to \infty \), \( G_{\theta}^{(n)}(x) \to \phi(x) \) uniformly in \( \theta \).

From the last result it follows, as in the proof of Theorem 2.1 of
[8], that the conditional distribution of \( Y^* = (V_{N_j}/[m_{j}]^{1/2}; 1 \leq j \leq K-1) \),
where \( V_{N_j} = T_{N_j} \widetilde{Z}(T_{N_j}) \) and \( d = [N'/(N'-1)](\sum_{i=1}^{n} r_{i}^{2}) \), converges as \( n \to \infty \)
to the multivariate \( N(Q,z) \) with \( z = \| \tilde{y}_{jj} \|^{1/2}(m_{j}/N')^{1/2} \) and
part (i) is proved. Further, since the limiting distributions of
\( (V_{N_j}/[d]^{1/2}; 1 \leq j \leq K-1) \) and the arbitrary linear combination
\[
\sum_{j=1}^{K-1} c_j V_{N_j}/[d]^{1/2} = \sum_{i=1}^{n} [U_i - \widetilde{Z}(U_i)]/[d]^{1/2}
\]
do not depend on the configuration, the equicontinuity conditions of
Lemmas 3.1 and 3.2 below are satisfied and their application proves part
(ii). The proof is complete.

In the statements of the following two lemmas as in [8] an essential
condition had been incorrectly omitted. For our purposes the following
versions would suffice: Let \( F_{\theta}(x) \) be a continuous d.f. and each \( \theta \in \Omega \).

Definition. The family \( \{ F_{\theta}: \theta \in \Omega \} \) is then said to be equicontinuous
in \( \theta \) (i) at each finite \( x \) if, \( \lim_{h \to 0} F_{\theta}(x+h) = F_{\theta}(x) \) uniformly in
\( \theta \in \Omega \) and (ii) at \( \pm \infty \) if \( \lim_{x \to \pm \infty} F_{\theta}(x) = F_{\theta}(\pm \infty) \) uniformly in \( \theta \in \Omega \), or
simply equicontinuous in \( \theta \) if both (i) and (ii) are satisfied.

Lemma 3.1. Let: \( V^{(n)} = (V_{l}^{(n)}, \ldots, V_{c}^{(n)}) \), \( n=1,2,\ldots \), be a sequence of
random vectors with d.f. \( F_{\theta}^{(n)}(v) \) depending on a parameter \( \theta \in \Omega \) and let
\( G_{\theta}^{(n)}(u) \) denote the d.f. of an arbitrary linear function
\( u^{(n)} = f(v^{(n)}) = \sum_{j=1}^{c} c_{j} V_{j}^{(n)} \) of the components, such that, as \( n \to \infty \),

(i) \( F_{\theta}^{(n)}(v) \) converges to \( F_{\theta}(v) \) for each \( \theta \) and \( v \), where \( F_{\theta} \)
is the d.f. of a r.v. \( V \) and the family \( \{ F_{\theta}: \theta \in \Omega \} \) is equicontinuous in \( \theta \).
(ii) for every linear $f$, $G^{(n)}_{\theta}(u)$ converges to $G_{\theta}(u)$ uniformly in $\theta$, where $G_{\theta}(u)$ is the d.f. of $f(V)$ and the family $\{G_{\theta}: \theta \in \Theta\}$ is equicontinuous in $\theta$.

Then $F^{(n)}_{\theta}(v)$ also converges to $F_{\theta}(v)$ uniformly in $\theta$ (and $v$).

Proof. Let $t = (t_1, t_2, \ldots, t_c)$ and $\varphi_{n, \theta}(t) (v_{\theta}(t))$ be the characteristic function of $V^{(n)}(v)$. Then

\begin{equation}
\varphi_{n, \theta}(t) = \varphi_{n, \theta}^*(\lambda) \quad \text{and} \quad \varphi_{\theta}(t) = \varphi_{\theta}^*(\lambda) \quad (\lambda \neq 0),
\end{equation}

where $\varphi_{n, \theta}^*(\lambda) (\varphi_{\theta}^*(\lambda))$ is the characteristic function of

\[ f(V^{(n)}) = \sum_{j=1}^{c} \left( t_j / \lambda \right)^{V_j^{(n)}} f(V). \]

But on account of (ii), it follows by Theorem 7C of Parzen [10] that for each $\lambda$, $\varphi_{n, \theta}^*(\lambda) \to \varphi_{\theta}^*(\lambda)$ uniformly in $\theta$, as $n \to \infty$. By (3.13), therefore, we have for each $t$,

$\varphi_{n, \theta}(t) \to \varphi_{\theta}(t)$ uniformly in $\theta$, as $n \to \infty$. The conclusion now follows by applying the same Theorem 7C of Parzen.

For the statement of Lemma 3.2, we use the following stronger equicontinuous condition: Let $\mu_{\theta}$ denote the probability measure on $(E^{(c)}, \mathcal{B}^{(c)})$ corresponding to the d.f. $F_{\theta}$, where $E^{(c)} = c$-dimensional Euclidean space and $\mathcal{B}^{(c)}$ the Borel field in $E^{(c)}$. Assume that for every non-increasing sequence of sets $\{A_m\} \subset \mathcal{B}^{(c)}$ converging to the empty set,

\begin{equation}
\mu_{\theta}(A_m) \to 0, \text{ uniformly in } \theta, \text{ as } m \to \infty.
\end{equation}

Lemma 3.2. Let the sequences $\{V^{(n)}\}$ and $\{F_{\theta}^{(n)}(v)\}$ be as defined in Lemma 3.1, and suppose that (3.14) as well as the conclusion of
Lemma 3.1 hold. Let $f: E(\mathbb{C}) \to E(\mathbb{I})$, be a continuous function such that the d.f. $G_\theta(u)$ of $f(v)$ is continuous in $u$ for every $\theta \in \Omega$.

Then (i) the d.f. $G_\theta^{(n)}(u)$ of $f(v^{(n)})$ converges to the d.f. $G_\theta(u)$ for each $\theta$ and $u$, and (ii) the convergence is uniform in $\theta$ (and $u$).

**Proof.** Since $\{F_\theta\}$ is an equicontinuous family and $F_\theta^{(n)}(v) \to F_\theta(v)$ uniformly in $\theta$, it follows by Theorem 4B of Parzen [10] that for every continuous bounded function $g(v)$,

$$
(3.15) \quad \int g(v) dF_\theta^{(n)}(v) \to \int g(v) dF_\theta(v), \text{ uniformly in } \theta,
$$

as $n \to \infty$. Suppose now that $A_\theta^{(c)}$ is a $\mu_\theta$-continuity set for each $\theta$, and let $\overline{A}$ be the closure of $A$, $A^c$ the complement of $A$ and $S(\overline{A}, \epsilon)$ the open $\epsilon$-neighborhood of $\overline{A}$. Then for each $m = 1, 2, \ldots$, there exists a continuous function $f_m$, $0 \leq f_m \leq 1$, such that $f_m(v) = 1$ or zero according as $v \in A$ or $v \in (S(\overline{A}, 1/m))^c$. Consequently, by $(3.15)$ for every $\epsilon > 0$ there exists an $n_0(\epsilon, m)$ such that for $n \geq n_0(\epsilon, m)$

$$
(3.16) \quad \mu_\theta^{(n)}(\overline{A}) \leq \int f_m dF_\theta^{(n)} \leq \int f_m dF_\theta + \epsilon \leq \mu_\theta[S(\overline{A}, 1/m)] + \epsilon
$$

Further by $(3.14)$, there exists an $m_0(\epsilon, A)$ such that for $m \geq m_0(\epsilon, A)$,

$$
(3.17) \quad \mu_\theta[S(\overline{A}, \frac{1}{m})] \leq \mu_\theta[\overline{A}] + \epsilon.
$$

From $(3.16)$ and $(3.17)$ it follows that there exists an $n_0 = n_0(\epsilon, A)$, not depending on $\theta$, such that for $n \geq n_0$

$$
(3.18) \quad \mu_\theta^{(n)}(\overline{A}) \leq \mu_\theta(\overline{A}) + 2\epsilon.
$$

Similarly, there exists an $n^* = n^*(\epsilon, A)$, not depending on $\theta$, such
that for \( n \geq n^*_\theta \), \( (A^{(o)}) = \text{interior of } A \)

\[
\mu_{\theta}^{(n)}(A^{(o)}) \geq \mu_{\theta}(A^{(o)}) - 2\varepsilon.
\]

From (3.18) and (3.19), and \( \mu_{\theta}(\overline{A}) = \mu(A^{(o)}) \) for each \( \theta \), it follows that

\[
\mu_{\theta}^{(n)}(A) \rightarrow \mu_{\theta}(A), \text{ uniformly in } \theta,
\]

as \( n \rightarrow \infty \).

Since \( G_{\theta}(u) \) is continuous in \( x \) for each \( \theta \), it follows that \( f^{-1}((\infty, u]) \) is a \( \mu_{\theta} \)-continuity set for each \( \theta \), so that from (3.20) we have for each \( u \),

\[
G_{\theta}^{(n)}(u) = P[V_{\theta}^{(n)} \in f^{-1}(\infty, u)]
\]

\[
\rightarrow P[V_{\theta} \in f^{-1}(\infty, u)] = G_{\theta}(u)
\]

uniformly in \( \theta \), as \( n \rightarrow \infty \). The proof is complete.

**Remarks:**

(3.21) In the general case of the statistic (2.7), part (i) of Theorem 3.1 remains true, provided in addition to \( N_i < t \) for all \( i \), the conditions (2.92) and (2.93) of [1] are satisfied. In particular, these conditions are always satisfied for any balanced incomplete design. Under these conditions part (ii) on uniform convergence also continues to hold: The same proofs extend easily except for the proof of the equicontinuity conditions for the limiting distribution of \( Y \). These equicontinuity conditions then follow by showing the existence of constants \( a_i^+ \) and \( a_i^- \), independent of the configuration, such that

\[
\frac{\tilde{\sigma}_j^2}{a_j} \leq n \cdot a_i^+ \text{ and the}
\]
\[ \det(\Delta) \geq n^{K-1} a_x^2, \] and using Theorems 5 and 8A of Parzen [10].

For reasons of space, we shall not discuss the details.

(3.22) If in the model (2.1), we also assume additivity for the treatment effects viz., \( F_{ij}(x) = F(x + \xi_i + \theta_j) \), one can use similar conditional tests for testing the equality of block effects. More generally, if one assumes the linear model \( X_{ij} = \mu + \xi_i + \theta_j + \gamma_{ij} + \epsilon_{ij} \) where the errors \( \epsilon_{ij} \) are independent and identically distributed and \( \bar{\xi} = \bar{\theta} = \bar{\gamma}_i = \bar{\gamma}_j = 0 \), one can also test conditionally the hypothesis of no-interactions, (i.e., \( \gamma_{ij} = 0 \)) using observations aligned suitably through transformations that eliminate both \( \xi_i \)'s and \( \theta_j \)'s.

(3.23) It follows from Theorem 4.1 of the next section that the statistic (2.6) is distributed in the limit, under \( H_0 \), (C1) and (C2) (and without the condition (4.2)), as a \( \chi^2_{K-1} \)-variable. This result also follows under (A) and (B) from Theorem 3.1: To see this note that given \( \epsilon > 0 \), there exists an \( n_0 = n_0(\epsilon) \), not depending on the configuration, such that for \( n \geq n_0 \)

\[ \chi^2_{K-1}(x) - \epsilon \leq H_0^{(n)}(x) \leq \chi^2_{K-1} + \epsilon, \]

where \( H_0^{(n)} \) is the conditional d.f. of \( L_n \), given \( \theta \), and \( \chi^2_{K-1}(x) \) is the d.f. of a \( \chi^2_{K-1} \)-variable. On the other hand, if \( H^{(n)} \) is the unconditional d.f. of \( L_n \), it follows from the above inequalities and \( H^{(n)}(x) = E[H_0^{(n)}(x)] \) that for \( n \geq n_0(\epsilon) \)

\[ \chi^2_{K-1}(x) - \epsilon \leq H^{(n)}(x) \leq \chi^2_{K-1}(x) + \epsilon; \]

and the assertion is proved.
In view of the above result, one may regard the test based on \( L_n \) as an unconditional test for large samples. The distinct usefulness of the conditional tests based on \( L_n \) is that they provide (unlike the otherwise remarkable nonparametric procedures of Lehmann [6], [7]) exact significance levels and consequently are distribution free rank-order tests for all sample-sizes.

4. **Asymptotic Distribution Under Alternatives.**

Suppose now that the treatment effects are also additive and consider, for each \( n \), the alternative hypothesis

\[
K_n: F_{i,j}(x) = F(x + \bar{x} + \theta_j n^{-1/2}) \quad \text{(not all } \theta_j \text{ equal).}
\]

Let \( Z_{i,j} = X_{i,j} - \bar{X}_i \), where \( \bar{X}_i = \frac{\sum \sum X_{i,j}/N'}{} \) and \( G(x) = G(x_1, \ldots, x_t) \), with \( t = N' - 1 \), denote under \( H_0 \) the joint d.f. of all \( Z_{i,j} \) except the \( Z_{1,K_m} \). Suppose now that \( G \) possesses a density \( g(x_1, x_2, \ldots, x_t) \), such that

\[
(4.1) \quad \begin{cases} 
\text{\( g \) is absolutely continuous in each argument } x_{\alpha}, \text{ for almost all (w.r. to the Lebesgue measure } \mu(t-1) \text{ on } E(t-1) \text{)} \\
\{ x^{(\alpha)} \in E^{(t-1)} \text{, } (y^{(\alpha)}) = (y_1, \ldots, y_{\alpha-1}, y_{\alpha+1}, \ldots, y_t) \text{, } -\infty < y_i < \infty \} 
\end{cases}
\]

and that for each \( \alpha, \ 1 \leq \alpha \leq t \), whatever be \( N' \),

\[
(4.2) \quad \int_{E^{(t)}} \left[ (g^{(\alpha)}(x))^2 / g(x) \right] dx < \infty
\]

where \( g^{(\alpha)}(x) = \partial g / \partial x_{\alpha} \) and \( E^{(t)} \) = \( t \)-dimensional Cartesian space.
(We shall follow the convention that in (4.2) and similar expressions, the integrand is zero at points where \( g(x) = 0 \)).

For the proof of Theorem 4.1 below we need

Lemma 4.1. Suppose that the method of alignment is such that the conditions (C1) and (C2) are satisfied. Then, under the conditions (2.3a) and (2.3b), \( \Pr \lim_{n \to \infty} \left( \frac{1}{n} \sum_{i=1}^{n} \tau_i^2 / n \right) = \frac{1}{(N'-1)/N'} A^2 \), where

\[
A^2 = \int_0^1 \xi^2(u)du - \int_0^1 \int_0^1 \xi(u)\xi(v)dH(u,v)
\]

with \( H(u,v) \) as the joint d.f. of \( (G_1(Z_{1j}), G_1(Z_{1j}'), j) \) and \( G_1 \) as the marginal distribution of any aligned observation \( Z_{1j} \) (under \( H_0 \)).

Proof. Let for convenience the aligned observations in block \( i \) be denoted by \( Z_{11}, \ldots, Z_{1N} \) and \( r_{i\alpha} = \text{Rank of } Z_{i\alpha} \). Then it easily follows that

\[
\frac{1}{n} \sum_{i=1}^{n} \tau_i^2 = \frac{1}{n(N')^2} \sum_{i=1}^{n} \sum_{1 \leq \alpha < \alpha' \leq N'} (\eta_{i\alpha} - \eta_{i\alpha'})^2
\]

\[
= \frac{(N'-1)^2}{N'} \frac{1}{N} \sum_{k=1}^{N} \xi_{N,k}^2 - \frac{1}{n(N')^2} \sum_{i=1}^{n} \sum_{\alpha \neq \alpha'} \xi_N \left( \frac{r_{i\alpha}}{N+1} \right) \xi_N \left( \frac{r_{i\alpha'}}{N+1} \right)
\]

\[
\to \frac{(N'-1)}{N'} \int_0^1 \xi^2(u)du - \frac{1}{(N')^2} \Pr \lim_{n \to \infty} B_n
\]

where

\[
B_n = \frac{1}{n} \sum_{i=1}^{n} \sum_{\alpha \neq \alpha'} \xi_N \left( \frac{r_{i\alpha}}{N+1} \right) \xi_N \left( \frac{r_{i\alpha'}}{N+1} \right)
\]

If we let \( U_{i\alpha} = G_1(Z_{i\alpha}) \) and

\[
B_n^* = \frac{1}{n} \sum_{i=1}^{n} \sum_{\alpha \neq \alpha'} \xi(U_{i\alpha}) \xi(U_{i\alpha'})
\]

we obtain
\begin{align*}
(4.5) \quad \mathbb{E}|B_n-B^*_n| &\leq \frac{1}{n} \sum_{i=1}^{\infty} \sum_{\alpha \neq \alpha'} \mathbb{E}[|\xi_N^{r_1}(N+1)_{N+1}^{r_1} \xi(U_{1\alpha})| \cdot |\xi(U_{1\alpha'})|] \\
&+ \frac{1}{n} \sum_{i=1}^{\infty} \sum_{\alpha \neq \alpha'} \mathbb{E}[|\xi_N^{r_1}(N+1)_{N+1}^{r_1} \xi(U_{1\alpha})| \cdot |\xi(N+1)|] \\
&\leq N'(N'-1)[(\int_0^1 \xi^2(u)du)^{1/2} + (\frac{1}{N} \sum_{i=1}^{N} \xi^2_{i,1})^{1/2}] \cdot \mathbb{E}[\xi(U_{1\alpha}) - \xi(N+1)|^{2}]^{1/2}.
\end{align*}

The last inequality follows by Schwartz inequality and the conditions (C1) and (C2). The first term on the right in (4.5) remains finite and the second term tends to zero, as \( n \to \infty \), on account of Lemma 3.1 of [9] and (3.11) and (3.12). Consequently,

\begin{align*}
(4.6) \quad \lim_{n \to \infty} \text{Pr} (B_n - B^*_n) = 0.
\end{align*}

But by the strong law of large numbers,

\begin{align*}
(4.7) \quad B^*_n &\xrightarrow{a.s.} N'(N'-1) \int_0^1 \int_0^1 \xi(u)\xi(v)d\mathbb{H}(u,v),
\end{align*}

so that from (4.4), (4.6) and (4.7), the result follows.

**Theorem 4.1.** Suppose that for each \( n \), \( K_n \) holds. Then, under the conditions (C1), (C2), (4.1) and (4.2), the statistic \( L_n \), defined by (2.6), is distributed in the limit, as \( n \to \infty \), as a non-central \( \chi^2 \)-variable with \( (K-1) \) degrees of freedom and the noncentrality parameter

\begin{align*}
(4.8) \quad \delta^2 = [(\int_0^1 \xi(u)\eta(u)du)^2/A^2] \sum_{j=1}^{K} m_j (\theta_j - \bar{\theta})^2,
\end{align*}

where \( \eta(u) = -[g_1^{-1}(G_1^{-1}(u))/g_1(G_1^{-1}(u))] \), \( 0 < u < 1 \), \( g_1' = dg_1/dx \) a.e. and \( G_1^{-1}(u) = \text{Inf}\{x: G_1(x) = u\} \).
Proof. Let \( Q_N \) and \( P_N \) denote the joint probability distributions of the totality of aligned observations \( Z_{ij}^N \) under \( K_n \) and \( H_0 \), respectively. Under the conditions of the theorem, it was proved in [9] that \( \{Q_N\} \) are "contiguous" to \( \{P_N\} \). Accordingly, from Lemma 4.1, we obtain that under \( K_n \) also,

\[
\text{Pr} \lim_{n \to \infty} \left( \frac{1}{n} \sum_{i=1}^{n} r_i^2 \right) = \left[ (N'-1)/N' \right] A^2.
\]

We now apply the arguments, with \( r_i = t \), of Theorem 5.2 of [9]. For this, first note that \( Z_{iN'} = \sum_{i=1}^{N'-1} Z_{i\alpha} \) for each \( i \), so that the condition (5.1) (i) concerning the existence of second moments may be omitted. Secondly, on account of Lemma 4.1 of [9], the conditions (2.1) (ii) and (iii), with \( r_i = t \), can be replaced by the weaker conditions (4.1) and (4.2) of this paper. It follows, therefore, under the assumed conditions that the vector \( \mathbf{Y} = (V_{N1}, V_{N2}, \ldots, V_{N, K-1}) \) is distributed in the limit according to the multivariate \( N(\mu^*, \Sigma^*) \) distribution, where \( \mu^* = (\mu_1^*, \mu_2^*, \ldots, \mu_{K-1}^*) \) and \( \Sigma^* \) are given by

\[
\mu_j = m_j (\bar{\theta}_j - \bar{\theta}) \int_0^1 \xi(u) \eta(u) du, \quad 1 \leq j \leq K-1 \tag{4.10}
\]

\[
\Sigma^* = \left\| \xi_{jj}' \right\| - \left( m_j / N' \right)^{1/2} (m_j / N')^{1/2} \| A \|^{1/2}.
\]

From (4.9)(4.10), the result follows through standard arguments (see the proof of Theorem 3.1 of [8]); the proof is complete.

For a specified univariate distribution \( F(x) \) for the unaligned observations \( X_{ij} \), with density \( f(x) \), the conditions (4.1) and (4.2) would, in general, be hard to check unless the density \( g(x_1, x_2, \ldots, x_t) \)
can be explicitly derived, which is often not the case. We will, accordingly, relate (4.1) and (4.2) to similar conditions on the distribution \( F(x) \): Let

\[
\left\{
\begin{array}{l}
\text{f(x) be absolutely continuous and} \\
\int_{-\infty}^{\infty} [(f'(x))^2/f(x)] \, dx < \infty.
\end{array}
\right.
\] (4.11)

We now state a more general theorem from which such a relation will follow: Let \( Z = (Z_1, Z_2, \ldots, Z_t) \) be a random vector with density \( g(x) = g(x_1, x_2, \ldots, x_t) \) and \( Y = (Y_1, Y_2, \ldots, Y_m) \), with \( m \leq t \), be a random vector obtained from \( Z \) by

\[
(4.12) \quad Y = BZ,
\]

where \( B = \|b_{ij}\| \) is an \((m \times t)\) matrix of rank \( m \). Then the density \( h(y) \) of \( Y \) exists and is given by

\[
(4.13) \quad h(y) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} g(B_1 y_1^*, \ldots, B_m y_m^*, y_{m+1}, \ldots, y_t) \left| J \right| dy_{m+1} \cdots dy_t,
\]

where \( y = (y_1, y_2, \ldots, y_m) \), \( y^* = (y_1^*, y_2^*, \ldots, y_t) \), \( B_i \) is the \( i \)-th row of the inverse \((B^*)^{-1} = \|b_{ij}\|\) of the \((t \times t)\) matrix

\[
(4.14) \quad B^* = \begin{bmatrix} B^{(1)} & B^{(2)} \\ \| & 0 & I \end{bmatrix}
\]

In (4.14), \( B^{(1)} \) and \( B^{(2)} \) are \((m \times m)\) and \( m \times (t-m) \) matrices given by \( B = \|B^{(1)}, B^{(2)}\| \), \( 0 \) consists of zeros and \( I \) is the \((t-m) \times (t-m) \) identity matrix. (Since the rank of \( B \) is \( m \) we can so arrange that \( B^* \) is non-singular.) In (4.13) \( \left| J \right| \) is the Jacobian of the inverse transformation \((B^*)^{-1}\).
Theorem 4.2. If the density \( g(x) \) satisfies (4.1) and (4.2), then (i) the density \( h(y) \) also satisfies these conditions with \( h \) in place of \( g \), and (ii) the marginal density of any \( m < K \) components of \( Z = (Z_1, \ldots, Z_t) \) also satisfies these conditions.

Proof. Let for \( 1 \leq j \leq m \),

\[
(4.15) \quad h^{(j)}(y) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} (\sum_{i=1}^{m} \frac{\partial g^{(i)}}{\partial x_i} \cdot B_{i,j}) |J| dy_{m+1}, \ldots, dy_t
\]

where \( g^{(i)} = (\partial g/\partial x_i) \) at the point \( (B_1 y^*, \ldots, B_m y^*, y_{m+1}^*, \ldots, y_t^*) \). Now from (4.2) and Schwarz inequality it follows that

\[
(4.16) \quad (\int |g^{(i)}(x)|^2 dx)^2 \leq \int [(g^{(i)}(x))^2/g(x)] dx < \infty ,
\]

for each \( 1 \leq i \leq m \), so that from (4.15)

\[
(4.17) \quad \int_{E(m)} h^{(j)}(y) dy \leq \sum_{i=1}^{m} |B_{i,j}| |J| \int_{E(t)} |g^{(i)}(x)| dx < \infty .
\]

From (4.17), it follows that for each \( 1 \leq j \leq m \) and for almost all \( y^{(j)} \in E^{(m-1)}_{y^{(j)}} \),

\[
(4.18) \quad \int_{-\infty}^{\infty} |h^{(j)}(y)| dy < \infty \quad \text{and} \quad h(y) = \int_{-\infty}^{y_j} h^{(j)}(y) dy_j.
\]

From (4.18), we obtain that \( h(y) \) is absolutely continuous in each argument \( y_j \) for almost all \( y^{(j)} \in E^{(m-1)}_{y^{(j)}} \) and that \( h^{(j)}(y) = (\partial h/\partial y_j) \) a.e. Finally for \( 1 \leq j \leq m \),
\[
\begin{align*}
\int_E \frac{(h^{(1)}(y))^2}{h(y)} dy & \leq m \sum_{i=1}^{m} (b_{ij})^2 \int_E \frac{1}{h(y)} \left( \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} (g^{(1)}(y_1 \cdots y_t))^2 dy_1 \cdots dy_t \right)^2 dy \\
& = m \sum_{i=1}^{m} (b_{ij})^2 \int_E \frac{1}{h(y)} \left( \int_{-\infty}^{\infty} (g^{(1)}(y_1 \cdots y_t))^2 dy_1 \cdots dy_t \right)^2 dy \\
& \leq m \sum_{i=1}^{m} (b_{ij})^2 |J| \int_{E(t)} \frac{(g^{(1)}(x))^2}{g(x)} dx < \infty,
\end{align*}
\]

using (4.2). The last inequality follows by Schwarz inequality and (4.13). Thus both (4.1) and (4.2) are satisfied for the density \( h(y) \).

This completes the proof of part (i). Part (ii) is a simple application of part (i).

Now it is easily seen that if \( X_1, X_2, \ldots, X_N \) are independent random variables whose distributions satisfy the condition (4.11), then their joint distribution satisfies the conditions (4.1) and (4.2). Thus if the common underlying distribution \( F \) in \( K \) satisfies (4.11), it follows by part (i) of Theorem 4.2 that the density \( g(x_1, x_2, \ldots, x_t) \), under \( H_0 \), of any \( t = N' - 1 \) aligned (on the mean) observations satisfies (4.1) and (4.2). Consequently, in the statement of Theorem 4.1, one can replace the conditions (4.1) and (4.2) with the condition (4.11).
5. **Asymptotic Efficiency.**

From (3.18) of [8] and (4.8), it follows that the asymptotic efficiency \( e_{L, \mathcal{H}} \) of \( I_n \) relative to the \( \mathcal{H} \)-test is

\[
e_{L, \mathcal{H}} = \frac{\sigma^2 \left( \int_0^1 \xi(u)\eta(u)du \right)^2}{\int_0^1 \xi^2(u)du - \int_0^1 \int_0^1 \xi(u)\xi(v)dH(u,v)} ,
\]

where \( \sigma^2 \) is the variance of the distribution \( P \). Letting \( \xi(u) = u \) and \( \xi = \Phi^{-1}(u) \) in (5.1), one obtains \( e_{W, \mathcal{H}} \) and \( e_{L^*, \mathcal{H}} \), respectively, the asymptotic efficiencies of the Wilcoxon and the normal-score versions of the conditional tests based on \( I_n \). We shall now put these efficiency expressions in a more familiar form: From Theorem 4.2, it follows that

\[
\int_{-\infty}^{\infty} |g_1'(x)|dx < \int_{-\infty}^{\infty} \left[ (g_1'(x))^2/g_1(x) \right]dx < \infty \quad \text{and that } g_1(x) \text{ is absolutely continuous, so that } g_1(x) \to 0, \text{ as } x \to \pm \infty .
\]

Since \( \xi(u) \) is nondecreasing in \( u \), we obtain therefore that \( \lim[\xi(G_1(x))g_1(x)] \) exists as \( x \to \infty \) or \( -\infty \). From \( \int_{-\infty}^{\infty} |\xi(G_1(x))g_1(x)|dx < \infty \), it follows that these limits must be zero, so that by integration by parts,

\[
\int_0^1 \xi(u)\eta(u)du = \int_{-\infty}^{\infty} \xi'(G_1(x))g_1^2(x)dx .
\]

Consequently,

\[
e_{W, \mathcal{H}} = 3\sigma^2 \left( \int_{-\infty}^{\infty} g_1^2(x)dx \right)^2 / \left[ 1 - 3 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g_1(x)G_1(y)dG_2(x,y) \right]
\]

and
\[
(5.3) \quad e_{L*}, \mathcal{L} = \frac{\sigma^2 \left( \int_{-\infty}^{\infty} [\varphi^{-1}(G_1(x))] dx \right)^2}{\left[ 1 - \int_{-\infty}^{\infty} \varphi^{-1}(G_1(x)) \varphi^{-1}(G_1(y)) dG_2(x,y) \right]}
\]

where \( G_2(x,y) \) is the d.f. of any \( (Z_{ij}', Z_{ij}, \xi_i) \), under \( H_0 \) and \( \varphi \) and \( \Phi \) are, respectively, the \( N(0,1) \) density and d.f. The expression (5.2) has been discussed at length in [8]. For the efficiency \( e_{L*}, \mathcal{L} \) of the normal-score statistics, if one assumes \( F \) to be \( N(a, \sigma^2) \) (and so \( G_1(x) = \Phi(x(N')^{1/2}/\sigma(N'-1)^{1/2}) \)), it follows that

\[
e_{L*}, \mathcal{L} (\text{Normal}) = 1.
\]

More generally, if we let \( \xi(u) = G^{-1}_1(u) \) in (5.1), we obtain \( e_{L}, \mathcal{L} = 1 \).

Lower bounds. Now observe that if \( Z_1, Z_2, \ldots, Z_N \) denote the aligned observations in any block, and \( U_\alpha = G_1(Z_\alpha) \), then, under \( H_0 \),

\[
\text{Var}(\sum_{\alpha=1}^{N'} \xi(U_\alpha)) = \sum_{\alpha=1}^{N'} \text{Var}(\xi(U_\alpha)) + \sum_{\alpha \neq \alpha'} \text{Cov}(\xi(U_\alpha), \xi(U_{\alpha'}))
\]

\[
= N' \left[ \left( \int_0^1 \xi^2(u) du \right) - \left( \int_0^1 \xi(u) du \right)^2 \right] + N(N'-1) \left[ \int_0^1 \int_0^1 \xi(u)\xi(v) dH(u,v) \right]
\]

so that

\[
(5.4) \quad \int_0^1 \int_0^1 \xi(u)\xi(v) dH(u,v) \leq \left( \int_0^1 \xi^2(u) du / (N'-1) \right) - \left( \int_0^1 \xi(u) du \right)^2.
\]

From (5.1) to (5.4), it follows that

23
\begin{align*}
(5.5) \quad e_{w, \mathcal{H}} \geq 12\left(\frac{N' - 1}{N}\right) \sigma^2 \int_{-\infty}^{\infty} g_1(x)dx^2 \\
\quad \geq 0.864
\end{align*}

and

\begin{align*}
(5.6) \quad e_{L^*, \mathcal{H}} \geq \left(\frac{N' - 1}{N'}\right) \sigma^2 \int_{-\infty}^{\infty} \frac{g_1(x)}{\Phi^{-1}[G_1(x)]} dx^2 \\
\quad \geq 1,
\end{align*}

the last inequalities in (5.5) and (5.6) follow, respectively, from Theorem 1 of Hodges and Lehmann \cite{14} and Theorem 3 of Chernoff and Savage \cite{2}. (The inequality (5.5) we had missed observing in \cite{8}).

Thus the conditional tests $W$ and $L^*$ possess the same (asymptotic) robustness properties relative to the $\mathcal{H}$-test as the Wilcoxon and normal-score one and two-sample tests possess relative to the $t$-tests.

We simply comment (without details of proof) that (5.5) and (5.6) also hold for the balanced incomplete blocks case.
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13. ABSTRACT
The present work investigates the asymptotic efficiency of certain rank-order tests for testing the absence of main-effects in linear models under additivity. A distinct feature of these tests, unlike some previously available rank-tests of Friedman, Durbin, etc., is that they are based on a joint-ranking of the totality of observations after "alignment", and yet can be applied conditionally (given $\mathcal{E}$ = the ordered set of ranks for each block) at exact significance level for all sample sizes.

It is shown in this paper that the Wilcoxon version $W$ and the normal-score version $L^k$ of these conditional tests, besides possessing high degrees of asymptotic efficiency relative to the classical $\chi^2$-test under normality, namely, $e_{W,\mathcal{E}}$ (normal) $\geq 3/\pi$, $e_{L^k,\mathcal{E}}$ (normal) = 1, are also highly robust relative to the $\mathcal{F}$-test: $e_{W,\mathcal{F}}$ $\geq .864$ and $e_{L^k,\mathcal{F}}$ $\geq 1$.

It is also shown under fairly mild conditions that, for large number of blocks, the conditional null distribution of these test-statistics can be approximated by a $\chi^2$-distribution uniformly in $\mathcal{E}$. The last result is an extension of Theorem 3.1 of Mehra and Sarangi (Ann. Math. Statist. [1967], pp. 90-107).
### Rank-order Tests

**Distribution-free Tests for Linear Models**

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