AN ITERATED LOGARITHM RESULT FOR AUTOCORRELATIONS OF A STATIONARY LINEAR PROCESS

BY

C. C. HEYDE

TECHNICAL REPORT NO. 74
NOVEMBER 28, 1972

PREPARED UNDER CONTRACT
NO0014-67-A-0112-0051 (NR-042-993)
FOR THE OFFICE OF NAVAL RESEARCH

DEPARTMENT OF STATISTICS
STANFORD UNIVERSITY
STANFORD, CALIFORNIA
AN ITERATED LOGARITHM RESULT FOR AUTOCORRELATIONS
OF A STATIONARY LINEAR PROCESS

by

C. C. HEYDE

TECHNICAL REPORT NO. 74
November 28, 1972

PREPARED UNDER CONTRACT
NO0014-67-A-0112-0051 (NR-042-993)
FOR THE OFFICE OF NAVAL RESEARCH

Reproduction in Whole or in Part is Permitted for
any purpose of the United States Government

DEPARTMENT OF STATISTICS
STANFORD UNIVERSITY
STANFORD, CALIFORNIA
AN ITERATED LOGARITHM RESULT FOR AUTOCORRELATIONS
OF A STATIONARY LINEAR PROCESS

by

C. C. Heyde

Stanford University and Australian National University

1. Introduction.

In this paper we shall be concerned with a stationary linear stochastic process \( \{x(n)\} \) which may be represented in the form

\[
x(n) - \mu = \sum_{j=-\infty}^{\infty} \alpha(j) \varepsilon(n-j), \quad \sum_{j=-\infty}^{\infty} \alpha^2(j) < \infty,
\]

where the \( \varepsilon(n) \) are independent and identically distributed with mean zero and variance \( \sigma^2 \). Most of the standard inferential results for this process (e.g. Hannan [2]) are based on the autocorrelations

\[
r(j) = \frac{\sum_{n=1}^{N-j} \{x(n) - \bar{x}\} \{x(n+j) - \bar{x}\}}{\sum_{n=1}^{N} \{x(n) - \bar{x}\}^2}, \quad j \geq 0,
\]

\( x(1), x(2), \ldots, x(N) \) being a sample of \( N \) consecutive observations on the process \( \{x(n)\} \) and \( \bar{x} \) denoting the sample mean. The prime result is that \( r(j) \) converges almost surely (a.s.) to \( \rho(j) = \gamma(j)/\gamma(0) \) where \( \gamma(j) \) is given by

\[
\gamma(j) = \sigma^2 \sum_{n=-\infty}^{\infty} \alpha(n) \alpha(n+j).
\]
The most common application of this theory would be in the estimation of parameters in an autoregressive or moving average model or a mixture of both models.

It is our object here to give the following iterated logarithm result which provides information on the rate of a.s. convergence of \( r(j) \) to \( \rho(j) \).

**Theorem.** Suppose that \( E E^4(n) < \infty \) and \( \sum_{n=1}^{\infty} \left[ \sum_{|r| \geq n} \alpha^2(r) \right]^{\frac{1}{2}} < \infty \). Then, for any fixed \( j \geq 1 \),

\[
r(j) = \rho(j) + A(j) \zeta_j(N) N^{\frac{1}{2}} (2 \log \log N)^{\frac{1}{2}}
\]

where \( \zeta_j(N) \) has a.s. as its set of limit points the interval \([-1, 1]\), in particular \( \limsup_{N \to \infty} \zeta_j(N) = 1 \) a.s. and \( \liminf_{N \to \infty} \zeta_j(N) = -1 \) a.s., while

\[
A(j) = \left\{ 1 + 2 \rho^2(j) \right\} \sum_{s=\infty}^{\infty} \rho(s) + \sum_{s=\infty}^{\infty} \rho(s+j) (\rho(s-j) - 4 \rho(j) \rho(s)) \right\}^{\frac{1}{2}}.
\]

This result provides a detailed supplement to those obtained in [4] for the linear combination of autocorrelations which arises in estimation of parameters in an autoregressive model. Here we have an iterated logarithm result for each individual component.

2. **Proof of Theorem.**

This rests on use of an invariance principle for the law of the
iterated logarithm for processes with stationary increments which was established in [5].

Our setting is that of a probability space $(\Omega, \mathcal{F}, P)$ with ergodic measure preserving transformation $T$ such that $\varepsilon(n)\{\omega\} = \varepsilon(T^n\omega)$, $\omega \in \Omega$ (since the $\varepsilon(n)$ are independent and identically distributed and hence stationary and ergodic). Let $\mathcal{F}_n$ denote the $\sigma$-field generated by $\varepsilon(m)$, $m \leq n$. Then, $\mathcal{F}_k = T^{-k}(\mathcal{F}_0)$ and the framework of Theorem 2 of [5] is applicable.

We shall first establish the following Lemma which is of independent interest.

**Lemma.** Suppose that $\sum_{n=1}^{\infty} \left[ \sum_{|r| \geq n} \sigma^2(r) \right]^{1/2} < \infty$. Then,

$$
\bar{x} = \mu + \sum_{j=-\infty}^{\infty} \gamma(j) N^{-1/2} (2 \log \log N)^{1/2}
$$

where $\varepsilon(N)$ has a.s. as its set of limit points the interval $[-1,1]$.

**Proof.** Put $y(n) = x(n) - \mu$. We have

$$
N^{-1} \text{var} \bar{x} = \sum_{i=-(N-1)}^{N-1} (1-|i|N^{-1}) \text{cov}(x(0), x(i))
$$

$$
\longrightarrow \text{var} x(0) \sum_{i=-\infty}^{\infty} \rho(i) = \sum_{i=-\infty}^{\infty} \gamma(i)
$$

as $N \to \infty$. Also,
\[ E\{y(0)\mid \mathcal{F}_m\} = \sum_{r=-\infty}^{\infty} \alpha(r)E\{e(-r)\mid \mathcal{F}_m\} = \sum_{r=m}^{\infty} \alpha(r)e(-r) \]

and

\[ \sum_{m=1}^{\infty} \left[ E\{E(y(0)\mid \mathcal{F}_m)\} \right]^2 \leq \sum_{m=1}^{\infty} \left[ \sum_{r=-\infty}^{\infty} \alpha^2(r) \right] / 2 < \infty. \]

Similarly,

\[ \sum_{m=1}^{\infty} \left[ E\{y(0) - E(y(0)\mid \mathcal{F}_m)\} \right]^2 \leq \sum_{m=1}^{\infty} \left[ \sum_{r=-\infty}^{m-1} \alpha^2(r) \right] / 2 < \infty. \]

The result of the lemma then follows immediately from Theorem 2 of [5].

We now resume the proof of the theorem. Put \( \chi_n = y(n)y(n+j) - \rho(j)y^2(n) \) and note that \( \{\chi_n\} \) is a stationary sequence with zero mean and finite variance under the conditions of the theorem. Further, writing

\[ c^*(j) = N^{-1} \sum_{n=1}^{N-j} y(n)y(n+j), \]

it is well-known that (e.g. Hannan [1], p. 40)

\[ N \text{ cov}(c^*(k), c^*(k+j)) \]

\[ \longrightarrow \sum_{s=-\infty}^{\infty} \{ \gamma(s)\gamma(s+j) + \gamma(s+j+k)\gamma(s-k) \} + k_4 \gamma(k)\gamma(k+j) \]

as \( N \to \infty \) where \( k_4 \) is the fourth cumulant of \( e(n) \). Thus,
\[ N^{-1} \text{var} \left( \sum_{n=1}^{N} X_n \right) = N^{-1} \mathbb{E} \left[ \sum_{n=N-j+1}^{N} (c^*(n) - \mathbb{E}c^*(n) - \rho(n)(c^*(0) - \mathbb{E}c^*(0))) \right] + \sum_{n=N-j+1}^{N} \{(y(n)y(n+j) - \mathbb{E}y(n)y(n+j))\}^2 \]

\[ \longrightarrow \lim_{N \to \infty} \mathbb{E} \left[ (c^*(n) - \mathbb{E}c^*(n)) - \rho(n)(c^*(0) - \mathbb{E}c^*(0)) \right]^2 \]

\[ = \lim_{N \to \infty} N \text{ var } c^*(n) + \rho^2(n) \lim_{N \to \infty} N \text{ var } c^*(0) \]

\[ - 2\rho(n) \lim_{N \to \infty} N \text{ cov}(c^*(n), c^*(0)) \]

\[ = (1+2\rho^2(n)) \sum_{s=-\infty}^{\infty} \gamma^2(s) + \sum_{s=-\infty}^{\infty} \gamma(s+j)\gamma(s-j-4\rho(n))\gamma(s) \]

\[ = \gamma^2(0)A^2(n), \]

using the fact that \( \text{var} \left[ \sum_{n=N-j+1}^{N} y(n)y(n+j) \right] \) is independent of \( N \).

In order to apply Theorem 2 of [5] we need to show that

\[ \sum_{m=1}^{\infty} \left[ \mathbb{E} \{ E(X_0 | \mathcal{F}_m) \}^2 \right]^{1/2} < \infty \quad \text{and} \quad \sum_{m=1}^{\infty} \left[ \mathbb{E} \{ X_0 - E(X_0 | \mathcal{F}_m) \}^2 \right]^{1/2} < \infty. \]

We have

\[ E(X_0 | \mathcal{F}_m) = \sum_{r=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} \alpha(r)\alpha(s)\{E(\epsilon(-r)\epsilon(j-s) | \mathcal{F}_m) - \rho(n)E(\epsilon(-r)\epsilon(-s) | \mathcal{F}_m)\} \]

\[ = \sum_{r=m}^{\infty} \sum_{s=m+j}^{\infty} \alpha(r)\alpha(s)\epsilon(-r)\epsilon(j-s) + \sum_{r=m}^{\infty} \alpha(r)\alpha(r+j)\{\epsilon^2(-r) - \sigma^2\} \]

\[ - \rho(n) \sum_{r=m}^{\infty} \sum_{s=m}^{\infty} \alpha(r)\alpha(s)\epsilon(-r)\epsilon(-s) - \rho(n) \sum_{r=m}^{\infty} \alpha^2(r)\{\epsilon^2(-r) - \sigma^2\}, \]

\[ s \neq j+r \]

\[ s \neq r \]
and after some further algebra

\[
E(E(X_0 | \mathcal{F}_m))^2 = \{ Ee^4(0)-3a^4 \} \sum_{r=m}^\infty a^2(r)[a(r+j)-\rho(j)a(r)]^2 \\
+ \sigma^4 \sum_{r=m}^\infty a^2(r) \sum_{s=m}^\infty [a(s+j)-\rho(j)a(s)]^2 \\
+ \sigma^4 \left[ \sum_{r=m}^\infty a(r)[a(r+j)-\rho(j)a(r)] \right]^2.
\]

Thus, \( \sum_{m=1}^\infty \left[ E(E(X_0 | \mathcal{F}_m))^2 \right]^{\frac{1}{2}} < \infty \) if the following three conditions hold:

(i) \( \sum_{m=1}^\infty \left[ \sum_{r=m}^\infty a^2(r)[a(r+j)-\rho(j)a(r)]^2 \right]^{\frac{1}{2}} < \infty \)

(ii) \( \sum_{m=1}^\infty \left[ \sum_{r=m}^\infty a^2(r) \sum_{s=m}^\infty [a(s+j)-\rho(j)a(s)]^2 \right]^{\frac{1}{2}} < \infty \)

(iii) \( \sum_{m=1}^\infty \left| \sum_{r=m}^\infty a(r)[a(r+j)-\rho(j)a(r)] \right| < \infty \).

To show that these are satisfied we first observe that (i) and (iii) hold if (ii) holds since, using Schwarz's inequality and then Minkowski's inequality,

\[
\sum_{r=m}^\infty a^2(r)[a(r+j)-\rho(j)a(r)]^2 \leq \left[ \sum_{r=m}^\infty a^4(r) \sum_{s=m}^\infty [a(s+j)-\rho(j)a(s)]^4 \right]^{\frac{1}{2}} \\
\leq \sum_{r=m}^\infty a^2(r) \sum_{s=m}^\infty [a(s+j)-\rho(j)a(s)]^2,
\]

and, again using Schwarz's inequality,
\[ \sum_{r=m}^{\infty} |\alpha(r)\alpha(r+j) - \rho(j)\alpha(r)| \leq \left[ \sum_{r=m}^{\infty} \alpha^2(r) \sum_{s=m}^{\infty} [\alpha(s+j) - \rho(j)\alpha(s)] \right]^{1/2}. \]

To deal with (ii) we note that

\[ \sum_{s=m}^{\infty} [\alpha(s+j) - \rho(j)\alpha(s)]^2 \leq 2\left\{ \sum_{s=m}^{\infty} \alpha^2(s+j) + \rho^2(j) \sum_{s=m}^{\infty} \alpha^2(s) \right\} \]

\[ \leq 2(1 + \rho^2(j)) \sum_{s=m}^{\infty} \alpha^2(s) \]

since \( j \geq 1 \) and

\[ \sum_{m=1}^{\infty} \sum_{r=m}^{\infty} \alpha^2(r) = \sum_{r=1}^{\infty} r \alpha^2(r) < \infty. \]

Next,

\[ E[X_0 - E(X_0|\mathcal{F}_m)]^2 = EX_0^2 - E[E(X_0|\mathcal{F}_m)]^2 \]

and a routine calculation gives

\[ EX_0^2 = \{E(e^4(0) - 3\sigma^4) \sum_{r=-\infty}^{\infty} \alpha^2(r)[\alpha(r+j) - \rho(j)\alpha(r)]^2 + \gamma^2(0) - \gamma^2(j)\}, \]

so that using (1) (with \(-m\) replaced by \(m\)) and after some algebra,

\[ E[X_0 - E(X_0|\mathcal{F}_m)]^2 = \{E(e^4(0) - 3\sigma^4) \sum_{r=-\infty}^{-m} \alpha^2(r)[\alpha(r+j) - \rho(j)\alpha(r)]^2 \]

\[ - \sigma^4 \left[ \sum_{r=-\infty}^{-m-1} \alpha(r)[\alpha(r+j) - \rho(j)\alpha(r)] \right]^2 \]

\[ - \sigma^4 \left[ \sum_{r=-\infty}^{-m-1} \alpha(r)[\alpha(r+j) - \rho(j)\alpha(r)] \right]^2 \]
\[
+ \sigma^2 \gamma(0) \sum_{r=\infty}^{-m-1} [a(r+j) - \rho(j) a(r)]^2
+ \sigma^2 \sum_{r=\infty}^{-m-1} a^2(r) \left\{ \gamma(0)(1-\rho^2(j)) - \sigma^2 \sum_{s=\infty}^{-m-1} [a(s+j) - \rho(j) a(s)]^2 \right\}.
\]

Thus, \( \sum_{m=1}^{\infty} [E(X_0^2 - E(X_0 \mid \mathcal{F}_m))^2]^{\frac{1}{2}} < \infty \) if the following four conditions hold:

(i) \( \sum_{m=1}^{\infty} \left[ \sum_{r=\infty}^{-m-1} a^2(r) (a(r+j) - \rho(j) a(r))^2 \right]^{\frac{1}{2}} < \infty \),

(ii) \( \sum_{m=1}^{\infty} \left[ \sum_{r=\infty}^{-m-1} a(r) (a(r+j) - \rho(j) a(r)) \right] < \infty \),

(iii) \( \sum_{m=1}^{\infty} \left[ \sum_{r=\infty}^{-m-1} (a(r+j) - \rho(j) a(r))^2 \right]^{\frac{1}{2}} < \infty \),

(iv) \( \sum_{m=1}^{\infty} \left[ \sum_{r=\infty}^{-m-1} a^2(r) \right]^{\frac{1}{2}} < \infty \).

Furthermore, as in the case of \( \sum_{m=1}^{\infty} [E(X_0^2 \mid \mathcal{F}_m)]^{\frac{1}{2}} \), we readily find that (i) and (ii) are satisfied for \( \sum_{m=1}^{\infty} \sum_{r=\infty}^{-m-1} a^2(r) < \infty \). On the other hand, (iii) and (iv) are satisfied when \( \sum_{m=1}^{\infty} \left[ \sum_{r=\infty}^{-m-1} a^2(r) \right]^{\frac{1}{2}} < \infty \).

Consequently, the conditions of Theorem 2 of [5] are satisfied and

\[
(2N \log \log N)^{-\frac{1}{2}} \sum_{n=1}^{N} \{y(n)y(n+j) - \rho(j)y^2(n)\}
\]

has a.s. as its set of limit points the interval \( [-\gamma(0)A(j), \gamma(0)A(j)] \).
But, the sequence \((y(n)y(n+j))\) is stationary with finite variance so a simple application of the Borel-Cantelli lemma gives

\[
\lim_{N \to \infty} (N \log \log N)^{-\frac{1}{2}} \sum_{n=N-j+1}^{N} y(n)y(n+j) = 0 \quad \text{a.s.}
\]

and thus

\[
N^{\frac{1}{2}}(2 \log \log N)^{-\frac{3}{2}} c^*(j) - p(j)c^*(0)
\]

has a.s. as its set of limit points the interval \([-\gamma(0)A(j), \gamma(0)A(j)]\).

Now, if

\[
c(j) = N^{-1} \sum_{n=1}^{N-J} \{x(n)-\overline{x}\} \{x(n+j)-\overline{x}\}, \quad j \geq 0,
\]

we have

\[
c^*(j)-c(j) = (\overline{x}-\mu)^2 N^{-1} \left\{ \sum_{n=N-j+1}^{N} x(n) + \sum_{j=1}^{j-1} x(n-j(\overline{x}+\mu)) \right\} (\overline{x}-\mu),
\]

and with the aid of the lemma it is easily seen that for any fixed \(k \geq 0,\)

\[
\lim_{N \to \infty} N^{\frac{3}{2}}(\log \log N)^{-\frac{3}{2}} \{c^*(k)-c(k)\} = 0 \quad \text{a.s.}
\]

The result of the theorem then follows upon noting that, from the ergodic
3. **Remarks.**

1. Since the result of the theorem does not explicitly involve $E_{\varepsilon^4}(n)$ it seems possible that it will continue to hold in the absence of $E_{\varepsilon^4}(n) < \infty$. This is certainly the case with the corresponding central limit result.

2. It is easy to see that similar methods can be applied to investigate the rate of convergence of $c(j)$ to $\gamma(j)$. In this case Theorem 2 of [5] would be applied to the stationary ergodic sequence $\{Y_n\}$ where

$$Y_n = y(n)y(n+j) - \gamma(j).$$

The result that emerges is, again under $E_{\varepsilon^4}(n) < \infty$ and

$$\sum_{n=1}^{\infty} \left( \sum_{r > n} a^2(r) \right)^{1/2} < \infty,$$

$$c(j) = \gamma(j) + B(j) n_j^2 (N) N^{-1/2} (2 \log \log N)^{1/2}.$$

Here $n_j(N)$ has a.s. as its set of limit points the interval $[-1,1]$, while $B(j)$ is given by

$$B(j) = \left[ \sum_{s=-\infty}^{\infty} \gamma^2(s) + \gamma(s+j)\gamma(s-j) + k_4 \gamma^2(j) \right]^{1/2},$$

$k_4$ being the fourth cumulant of $\varepsilon(n)$. In this case we explicitly need the finite fourth moment condition.
3. If a restriction is made to the case where the process \( \{x(n)\} \) is purely non-deterministic \( (\alpha(n) = 0, \ n < 0) \), the assumption that the \( \epsilon(n) \) are independent and identically distributed can be replaced by a more general martingale condition along the lines of [3] and [4]. However, it is then necessary to make certain unattractive assumptions on third and fourth moments and also to impose slightly stronger series conditions on the \( \alpha \)'s than that used here.

4. From an examination of the proof of the theorem it seems likely that, at least in the purely non-deterministic case, the result will continue to hold under only

\[
\sum_{n=1}^{\infty} \sum_{|r| \geq n} \alpha^2(r) = \sum_{r=-\infty}^{\infty} |r| \alpha^2(r) < \infty.
\]

A convergence rate result for \( \bar{x} - \mu \) considerably weaker than that provided by the lemma is all that is required. For example, \( N^{1/2}(\bar{x} - \mu) \xrightarrow{a.s.} 0 \) would suffice.
REFERENCES


Let \( r(j) \) denote the \( j^{\text{th}} \) autocorrelation based on a sample of \( N \) consecutive observations on a stationary linear stochastic process. Under mild regularity conditions on the process, an iterated logarithm result is given for the convergence of \( r(j) \) as \( N \to \infty \) to the corresponding process autocorrelation \( \rho(j) \).
Iterated logarithm law;
stationary linear processes;
estimation of autocorrelations;
time series estimation.

1. ORIGINATING ACTIVITY: Enter the name and address of the contractor, subcontractor, grantee, Department of Defense activity or other organization (corporate author) issuing the report.

2a. REPORT SECURITY CLASSIFICATION: Enter the overall security classification of the report. Indicate whether "Restricted Data" is included. Marking is to be in accordance with appropriate security regulations.

2b. GROUP: Automatic downgrading is specified in DoD Directive 5200.10 and Armed Forces Industrial Manual. Enter the group number. Also, when applicable, show that optional markings have been used for Group 3 and Group 4 as authorized.

3. REPORT TITLE: Enter the complete report title in all capital letters. Titles in all cases should be unclassified. If a meaningful title cannot be selected without classification, show title classification in all capitals in parenthesis immediately following the title.

4. DESCRIPTIVE NOTES: If appropriate, enter the type of report, e.g., interim, progress, summary, annual, or final. Give the inclusive dates when a specific reporting period is covered.

5. AUTHOR(S): Enter the name(s) of author(s) as shown on or in the report. Enter last name, first name, middle initial. If necessary, show rank and branch of service. The name of the principal author is an absolute minimum requirement.

6. REPORT DATE: Enter the date of the report as day, month, year, or month, year. If more than one date appears on the report, use date of publication.

7a. TOTAL NUMBER OF PAGES: The total page count should follow normal pagination procedures, i.e., enter the number of pages containing information.

7b. NUMBER OF REFERENCES: Enter the total number of references cited in the report.

8a. CONTRACT OR GRANT NUMBER: If appropriate, enter the applicable number of the contract or grant under which the report was written.

8b. & 8d. PROJECT NUMBER: Enter the appropriate military department identification, such as project number, subproject number, system numbers, task number, etc.

9a. ORIGINATOR’S REPORT NUMBER(S): Enter the official report number by which the document will be identified and controlled by the originating activity. This number must be unique to this report.

9b. OTHER REPORT NUMBER(S): If the report has been assigned any other report numbers (either by the originator or by the sponsor), also enter this number(s).

10. AVAILABILITY/LIMITATION NOTICES: Enter any limitations on further dissemination of the report, other than those imposed by security classification, using standard statements such as:

(1) “Qualified requesters may obtain copies of this report from DDC.”

(2) “Foreign announcement and dissemination of this report by DDC is not authorized.”

(3) “U. S. Government agencies may obtain copies of this report directly from DDC. Other qualified DDC users shall request through ______.”

(4) “U. S. military agencies may obtain copies of this report directly from DDC. Other qualified users shall request through ______.”

(5) “All distribution of this report is controlled. Qualified DDC users shall request through ______.”

If the report has been furnished to the Office of Technical Services, Department of Commerce, for sale to the public, indicate this fact and enter the price, if known.

11. SUPPLEMENTARY NOTES: Use for additional explanatory notes.

12. SPONSORING MILITARY ACTIVITY: Enter the name of the departmental project office or laboratory sponsoring (paying for) the research and development. Include address.

13. ABSTRACT: Enter an abstract giving a brief and factual summary of the document indicative of the report, even though it may also appear elsewhere in the body of the technical report. If additional space is required, a continuation sheet shall be attached.

It is highly desirable that the abstract of classified reports be unclassified. Each paragraph of the abstract shall end with an indication of the military security classification of the information in the paragraph, represented as (TS), (S), (C), or (U).

There is no limitation on the length of the abstract. However, the suggested length is from 150 to 225 words.

14. KEY WORDS: Key words are technically meaningful terms or short phrases that characterize a report and may be used as index entries for cataloging the report. Key words must be selected so that no security classification is required. Identifiers, such as equipment model designation, trade name, military project code name, geographic location, may be used as key words but will be followed by an indication of technical context. The assignment of links, rules, and weights is optional.