ON THE CENTRAL LIMIT THEOREM AND ITERATED LOGARITHM LAW FOR STATIONARY PROCESSES

BY

C. C. HEYDE

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ABSTRACT

Invariance principles are given for both the central limit theorem and iterated logarithm law for a wide class of stationary processes. The results are derived from corresponding results for martingales with stationary ergodic increments. This is accomplished via a representation for the stationary process in terms of stationary martingale differences plus other terms whose sum telescopes and disappears under suitable norming. An application is given to show how previously known results for stationary uniformly mixing processes can be improved.
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1. Introduction and Principal Results. In this paper our setting is that of a probability space \((\Omega, \mathcal{F}, P)\) with ergodic measure preserving transformation \(T\). Let \(L_2(P)\) be the Hilbert space of random variables with finite second moment. Define \(U\) on \(L_2(P)\) by \(UX(\omega) = X(T\omega)\) for \(X \in L_2(P)\), \(\omega \in \Omega\) and write \(X_k = U^kX_0\) for some particular \(X_0 \in L_2(P)\) with \(EX_0 = 0\). Set also, \(S_0 = 0, S_n = \sum_{k=1}^{n}X_k\) for \(n \geq 1\) and

\[\sigma_n^2 = ES_n^2.\]

We shall be concerned with giving invariance principles for both the central limit theorem and iterated logarithm law for appropriate random functions in \(C\) or \(D\) defined from the partial sums \(S_k\). Here \(C = C[0,1]\) is the space of continuous functions on \([0,1]\) and \(D = D[0,1]\) the space of functions on \([0,1]\) which are right continuous with left hand limits. In both cases we employ the supremum metric which we denote by \(\rho\).

Let \(\{\theta_n(\cdot)\}\) be a sequence of random functions on \([0,1]\) defined by

\[\theta_n(t) = S_j/\sigma_n, \quad j/n \leq t < (j+1)/n, \quad j = 0,1,\ldots,n-1,\]

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and

$$\theta_n(1) = S_n / \sigma_n.$$ 

Also, let \( \{\eta_n(\cdot)\} \) be a sequence of random functions on \([0,1]\) defined by

$$\eta_n(t) = (2\sigma_n^2 \log \log \sigma_n)\frac{1}{2}(S_k + (nt-k)X_k),$$

$$k \leq nt \leq k+1; \quad k = 0,1,\ldots,n-1.$$

Let \( K \) be the set of absolutely continuous \( x \in C \) such that

$$x(0) = 0$$

and

$$\int_0^1 \left[ x'(t) \right]^2 dt \leq 1$$

where \( x' \) denotes the derivative of \( x \) determined almost everywhere with respect to Lebesgue measure. Also define

$$g = \sup \{ n : \sigma_n^2 \leq \varepsilon \}.$$

Let \( \mathcal{M}_0 \) be a \( \sigma \)-field such that \( \mathcal{M}_0 \subset \mathcal{B} \) and \( \mathcal{M}_0 \subset \mathcal{T}^{-1}(\mathcal{M}_0) \) and
write \( \mathcal{M}_k = T^{-k}(\mathcal{M}_0) \), \( \mathcal{M}_\infty = \bigcap_{k=\infty}^{\infty} \mathcal{M}_k \) and \( \mathcal{M}_\infty = \sigma \)-field generated by \( \bigcup_{k=\infty}^{\infty} \mathcal{M}_k \).

Our object is the following theorem.

**Theorem**  If

\[
(1) \quad \sum_{n=1}^{\infty} \left( \sum_{r=n}^{\infty} x_r \right)^2 \in \mathcal{B}_{\text{\tiny \mathbb{M}}} (\sum_{r=n}^{\infty} x_r)^2 < \infty
\]

where

\[
x_r = \mathbb{E}(X_r|\mathcal{M}_0) - \mathbb{E}(X_r|\mathcal{M}_{-1})
\]

and \( \mathbb{E}(X_0|\mathcal{M}_\infty) = X_0 \) a.s., \( \mathbb{E}(X_0|\mathcal{M}_\infty) = 0 \) a.s., then \( \lim_{n \to \infty} \sigma_n / \sqrt{n} = \sigma \)

exists for \( 0 < \sigma < \infty \). If \( \sigma > 0 \) then \( \sigma_n \to \mathcal{W} \) in the sense \( (D, \rho) \)

where \( \mathcal{W} \) is a standard Wiener process. Also, \( g < \infty \), \{\( \eta_n ; n > g \)\} is relatively compact and the set of its limit points coincides with \( \mathcal{K} \).

This result extends those of Theorem 3 of Scott \( [10] \) (central limit case) and Theorem 2 of Hayde and Scott \( [4] \) (iterated logarithm case) which were both given with the condition

\[
(2) \quad \sum_{n=1}^{\infty} \left\{ \left( \mathbb{E}(X_0|\mathcal{M}_{-m}) \right)^2 + \left( \mathbb{E}(X_0 - \mathbb{E}(X_0|\mathcal{M}_{-m})) \right)^2 \right\} < \infty
\]

replacing those above. The idea in each case involves a representation for the \( X \)'s of the form

\[
(3) \quad X_0 = Y_0 + UZ_0 - Z_0
\]
where the $Y_0, Z_0 \in L_2(P)$ and $\{U_k Y_0\}$ forms a stationary ergodic sequence of martingale differences. This idea is due to Gordin [2] and the conditions of the theorem appear to be the most general under which (3) will hold as above. Of course the limit behaviour of $S_n$ is then easy to study via the corresponding behaviour of \[ \sum_{k=1}^n U_k Y_0 \] since

\[ S_n = \sum_{k=1}^n U_k Y_0 = \sum_{k=1}^n U_k Y_0 + U^{n+1} Z_0 - U Z_0 \]

and the effect of $U^{n+1} Z_0 - U Z_0$ disappears under suitable norming.

That the representation (3) holds under the asserted conditions of the theorem can easily be extracted from the proof of Theorem 3 of [10]. Condition (1) gives \[ \lim_{n \to \infty} \lim_{m \to \infty} \mathbb{E}(\sum_{r=n}^m x_r)^2 = \lim_{n \to \infty} \lim_{m \to \infty} \mathbb{E}(\sum_{r=n}^m x_r)^2 = 0 \] so that \[ \sum_{r=0}^\infty x_r \]
and \[ \sum_{r=0}^\infty x_{-r} \] converge in $L_2(P)$. That they also converge almost surely follows simply from the Borel-Cantelli lemma and (1). The $L_2(P)$ convergence result gives

\[ \lim_{m \to \infty} \mathbb{E}(\sum_{r=n}^m x_r)^2 + \lim_{m \to \infty} \mathbb{E}(\sum_{r=n}^m x_{-r})^2 = \mathbb{E}(\sum_{r=n}^\infty x_r)^2 + \mathbb{E}(\sum_{r=n}^\infty x_{-r})^2 \]

and our condition (1) is just the condition (46) of [10]. Once the representation (3) is obtained the remainder of the proof is exactly that of Theorem 3 of [10] in the central limit case and Theorem 2 of [4] in the case of the iterated logarithm law.

We remark that (1) arises via the restriction $Z_0 \in L_2(P)$. In fact

\[ EZ_0^2 = \sum_{n=1}^{\infty} \mathbb{E}(\sum_{r=n}^{\infty} x_r)^2 + (\sum_{r=n}^{\infty} x_{-r})^2. \]
Note also that under the conditions of the theorem we have that (3) holds and \( \lim_{n \to \infty} \frac{\sigma_n}{\sqrt{n}} = \sigma \) exists for \( 0 \leq \sigma < \infty \). Now \( z_0 \in L_2(\mathbb{P}) \) so \( \sigma^2 = EY_0^2 \) and we can only obtain \( \sigma = 0 \) in the case where \( X_n = u_{n+1}z_0 - u_nz_0 \), some \( z_0 \in L_2(\mathbb{P}) \).

We shall show that a useful sufficient condition for (1) is

\[
(4) \quad \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \left| E(X_k^* E(X_0|\mathcal{M}_{-n})) + E(X_k^* X_0) - E(X_k^* E(X_0|\mathcal{M}_{n-1})) \right| < \infty.
\]

To obtain this we first note that, under the condition (1),

\[
E\{ (\sum_{r=n}^{\infty} x_r)^2 + (\sum_{r=1}^{\infty} x_r)^2 \}
\]

\[
(5) \quad = \lim_{m \to \infty} \left\{ \sum_{r=n}^{m} E(x_r^2 x_{-r}^2) + 2 \sum_{k=1}^{m-n} \sum_{r=n}^{m-k} E(x_r^* x_{r+k}^* x_{-r}^* x_{-r-k}^*) \right\}.
\]

Now for any \( k, r, \)

\[
E(x_r x_{r+k}) = E(E(x_r | M_0^r)E(x_{r-k} | M_0^k)) - E(E(x_r | \mathcal{M}_{-r-k})E(x_{r-k} | \mathcal{M}_{-r-k}^k))
\]

\[
= E(x_{r-k}^* E(x_r | \mathcal{M}_{-r-k}^k)) - E(x_{r-k}^* E(x_{r-k} | \mathcal{M}_{-r-k}^{k+1})).
\]

so that for \( k \geq 0, \)

\[
\sum_{r=n}^{m-k} E(x_r x_{r+k}) = E(x_{-k}^* E(X_0 | \mathcal{M}_{n-k}^m)) - E(x_{-k}^* E(X_0 | \mathcal{M}_{n-1}^{m-k})),
\]

\[
\sum_{r=n}^{m-k} E(x_{-r} x_{-r-k}) = E(x_k^* E(X_0 | \mathcal{M}_{-n}) - E(x_k^* E(X_0 | M_{n-m+k-1})).
\]
Standard martingale results give \( \mathbb{E}(X_0^{M_{-n}}) \sim \mathbb{E}(X_0^{M_{-\infty}}) = 0 \) a.s. and
\( \mathbb{E}(X_0^{M_n}) \sim \mathbb{E}(X_0^{M_{-\infty}}) = x_0 \) a.s. so that \( \mathbb{E}(\mathbb{E}(X_0^{M_{-n}})^2) \rightarrow 0 \) and
\( \mathbb{E}(X_0 - \mathbb{E}(X_0^{M_n}))^2 \rightarrow 0 \) as \( n \rightarrow \infty \) and hence

(6) \[ \lim_{m \rightarrow \infty} \sum_{r=n}^{m} \mathbb{E}(x_r^2 + x_r^{-2}) = \mathbb{E}x_0^2 - \mathbb{E}(X_0 \mathbb{E}(X_0^{M_{n-1}})) + \mathbb{E}(X_0 \mathbb{E}(X_0^{M_{-n}})). \]

Also,

\[ \sum_{k=1}^{m-n} \sum_{r=n}^{k-m} \mathbb{E}(x_r x_r + x_{-r} x_{-r-k}) \]

\[ = \sum_{k=1}^{m-n} \{ \mathbb{E}(x_k \mathbb{E}(X_0^{M_{-n}})) + \mathbb{E}(X_{-k} x_0) - \mathbb{E}(X_{-k} \mathbb{E}(X_0^{M_{n-1}})) \}
\]

\[ - \sum_{k=1}^{m-n} \{ \mathbb{E}(x_k \mathbb{E}(X_0^{M_{-m+k-1}})) + \mathbb{E}(X_{-k} x_0) - \mathbb{E}(X_{-k} \mathbb{E}(X_0^{M_{n-k}})) \}\]

and under the condition (4),

(7) \[ \lim_{m \rightarrow \infty} \sum_{k=1}^{m-n} \sum_{r=n}^{m-k} \mathbb{E}(x_r x_r + x_{-r} x_{-r-k}) \]

\[ = \sum_{k=1}^{\infty} \{ \mathbb{E}(x_k \mathbb{E}(X_0^{M_{-n}})) + \mathbb{E}(X_{-k} x_0) - \mathbb{E}(X_{-k} \mathbb{E}(X_0^{M_{n-1}})) \}. \]

It is then clear from (5), (6) and (7) that (1) holds under the condition (4).

Conditions (1) and (4) do not appear to simplify in any really
convenient way in general. However, we remark that we are free to choose a convenient $\mathcal{M}_0$. For example, if $\mathcal{M}_0$ is the $\sigma$-field generated by $X_k$, $k < 0$, then the condition (4) becomes just

$$(5) \quad \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} |E(X_k E(X_0 | \mathcal{M}_n))| < \infty.$$ 

In some cases it may be useful to translate (1) (or (4)) into a condition on the Fourier coefficients of the expansion of $X_0$ in terms of a suitable complete orthonormal set. Certainly (1) represents a significant improvement over (2) and this justifies its use. An example is given in section 3 to illustrate this point. It should also be remarked that conditions (1) (or (4)) and (2) provide a convenient vehicle for the study of central limit and iterated logarithm results for stationary processes satisfying uniform mixing or strong mixing conditions. This will be the subject of a forthcoming paper but we illustrate here by considering the case of uniformly mixing processes.

2. Uniformly Mixing Processes. Here we suppose that $\{X_j\}$ is a (strictly) stationary process defined on $(\Omega, \mathcal{F}, \mathbb{P})$ with $E X_0 = 0$, $E X_0^2 < \infty$ and satisfying the uniform mixing condition:

$$\sup_{A \in \mathcal{F}_k^{\infty}, \ B \in \mathcal{F}_k^{\infty+n}} \frac{1}{\mathbb{P}(A)} |\mathbb{P}(A \cap B) - \mathbb{P}(A) \mathbb{P}(B)| = \phi(n) \downarrow 0 \quad \text{as} \quad n \to \infty$$

where $\mathcal{F}_a^{b}$ denotes the $\sigma$-field generated by $\{X_j, \ a \leq j \leq b\}$. We shall
obtain the following result.

**Corollary** Let \( \{X_j\} \) satisfy the uniform mixing condition and suppose that

\[
(\mathfrak{q}) \quad \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \left[ \phi(n) \phi(k) \phi(n+k) \right]^{\frac{1}{2}} < \infty
\]

then (1) holds and the results of the theorem apply.

This corollary extends previous results in this area. Ibragimov [5] proved a central limit result under the condition \( \sum_{n=1}^{\infty} \left[ \phi(n) \right]^{\frac{1}{2}} < \infty \) and Billingsley [1] later obtained an invariance principle for the central limit theorem under the same conditions, as did Scott [10]. Iterated logarithm results have been obtained by various authors under the condition \( \sum_{n=1}^{\infty} \left[ \phi(n) \right]^{\frac{1}{2}} < \infty \) together with a variety of additional assumptions. The classical Hartman-Wintner form was obtained by Iosifescu [6] and by Reznik [9] under the additional condition \( E|X_0|^{2+\delta} < \infty \), some \( \delta > 0 \). Iosifescu [7] later obtained an invariance principle form under the same conditions. An earlier invariance principle form was obtained by Oodaira and Yoshihara [8] under the weaker additional condition that

\[
\int_{|x| > N} x^2 d\mathbb{P}(X_0 \leq x) = O((\log N)^{-5}) \quad \text{as} \quad N \to \infty.
\]

Heyde and Scott [4] obtained the invariance principle form under \( \sum_{n=1}^{\infty} \left[ \phi(n) \right]^{\frac{1}{2}} < \infty \) without additional assumptions.

**Proof of Corollary.** We choose \( F_{-\infty}^0 \) as the \( \mathcal{M}_0 \) in the theorem and the framework there applies. Clearly \( E(X_0 | \mathcal{M}_{-n}) \xrightarrow{a.s.} 0 \) as \( n \to \infty \) via the
martingale convergence theorem and it suffices to show that the condition (4) ensures that (9) holds.

We have

\[ \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} |E(X_k E(X_0 | M_{-n})| \]

\[ = \sum_{n=1}^{\infty} \sum_{r=0}^{\infty} |E(X_{2r} E(X_0 | M_{-n})| + \sum_{n=1}^{\infty} \sum_{r=0}^{\infty} |E(X_{2r+1} E(X_0 | M_{-n})| \]

\[
\leq 4 \sum_{n=1}^{\infty} \sum_{r=0}^{\infty} (\phi(r+n)) E[|E(X_0 | M_{-r})|^2] E[|E(X_0 | M_{-n})|^2]^{1/2},
\]

using stationarity, monotonicity of the \( \phi \)'s and a well-known Hölder type inequality for uniformly mixing processes (e.g. [1], Lemma 1, p. 170). Also, for \( s \geq 0 \),

\[ E[|E(X_0 | M_{-s})|^2] = E[X_0 E(X_0 | M_{-s})]\]

\[ \leq 2(\phi(s)) E[X_0 E(X_0 | M_{-s})] E[X_0^2]^{1/2} \]

so that

\[ (E[|E(X_0 | M_{-s})|^2])^{1/2} \leq 2(\phi(s)) E[X_0^2]^{1/2} \]

(16)
and using (8) in (10) we obtain the desired result.

3. **An Example.** To illustrate the improvement of (1) over (2) we introduce the stationary linear process \( \{x(n)\} \) given by

\[
x(n) - \mu = \sum_{j=-\infty}^{\infty} \alpha(j) e(n-j), \quad \sum_{j=-\infty}^{\infty} \alpha^2(j) < \infty,
\]

where the \( \{e(n)\} \) are independent and identically distributed with zero mean and variance \( \sigma^2 \). Suppose that \( x(1), x(2), \ldots, x(N) \) is a sample of \( N \) consecutive observations on the process \( \{x(n)\} \) and \( \bar{x} \) denotes the sample mean. It follows from the ergodic theorem that \( \bar{x} \xrightarrow{\text{s.s.}} \mu \) as \( N \to \infty \) and it is of interest to obtain central limit and iterated logarithm results which give information on the rate of this convergence. Here we have \( X_k = x(k) - \mu \) and we can take \( \mathcal{M}_k \) as the \( \sigma \)-field generated by \( e(m), m \leq k \). Then,

\[
x_r = E(X_{-r} | \mathcal{M}_0) - E(X_{-r} | \mathcal{M}_{-1}) = \alpha(-r) e(0)
\]

and the theorem of this paper applies if

\[
(\Omega) \quad \sum_{n=1}^{\infty} \left\{ \left( \sum_{r=n}^{\infty} \alpha(r) \right)^2 + \left( \sum_{r=n}^{\infty} \alpha(-r) \right)^2 \right\} < \infty.
\]

On the other hand, the corresponding results based on the use of condition (2) hold if
\[ (13) \quad \sum_{n=1}^{\infty} \left( \sum_{|r| > n} a^2(r) \right)^{\frac{1}{2}} < \infty, \]

as in the lemma of Heyde [3]. (12) represents a significant improvement over (13) in the case where the \( a's \) continually vary in sign. If the \( a's \) are ultimately all positive, an example where (12) holds but (13) does not is provided by \( a(|r|) \sim C r^{-3/2} (\log r)^{-1} \) as \( r \to \infty \) for some \( C > 0 \).
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