COMPETITIVE OPTIMALITY OF LOGARITHMIC INVESTMENT

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TECHNICAL REPORT NO. 34
DECEMBER 1978

PREPARED UNDER THE AUSPICES
OF
NATIONAL SCIENCE FOUNDATION
GRANT ENG 76-03684

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Consider the two-person zero sum game in which two investors are each allowed to invest in a market with stocks \((X_1, X_2, \ldots, X_n) \sim F\), where \(X_i \geq 0\). Each investor has one unit of capital. The goal is to achieve more money than one's opponent. Allowable portfolio strategies are random investment policies \(B \in \mathbb{R}^n\), \(B \geq 0\), \(E \sum_{i=1}^{n} B_i = 1\). The payoff to player 1 for policy \(B_1\) vs. \(B_2\) is \(P\{B_1^t X \geq B_2^t X\}\). The optimal policy is shown to be \(B^* = U b^*\), where \(U\) is a random variable uniformly distributed on \([0,2]\), and \(b^*\) maximizes \(E u b^t X\) over \(b \geq 0\), \(\sum b_i = 1\).

Curiously, this competitively optimal investment policy \(b^*\) is the same policy that achieves the maximum possible growth rate of capital in repeated independent investments (Breiman (1961) and Kelly (1956)). Thus the immediate goal of outperforming another investor is perfectly compatible with maximizing the asymptotic rate of return.

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\(\ddagger\dagger\) Stanford University. This work was partially supported by National Science Foundation Grant ENG 76-03684.
1. **Introduction**

An investor is faced with a collection of stocks \((X_1, X_2, \ldots, X_n)\) drawn according to some known joint distribution function \(F\). We shall assume that stock values \(X_i\) are nonnegative. A **portfolio** is a vector \(\mathbf{b} = (b_1, \ldots, b_n)^t\), \(b_i \geq 0\), \(\sum b_i = 1\), with the interpretation that \(b_i\) is the proportion of capital allocated to stock \(i\). The capital return \(S\) from investment portfolio \(\mathbf{b}\) is

\[
S = \mathbf{b}^t \mathbf{X} = \sum_{i=1}^{n} b_i X_i .
\]

(1)

How should \(\mathbf{b}\) be chosen? A currently accepted procedure is the efficient portfolio selection approach of Markowitz (1952, 1959). A portfolio \(\mathbf{b}\) is said to be **efficient** if \((E \mathbf{b}^t \mathbf{X}, \text{Var} \mathbf{b}^t \mathbf{X})\) is undominated. Criticisms of this approach are many. Only the first two moments are used in the analysis; there is no optimality of this procedure with respect to other obvious investment goals; and no choice procedure among the efficient portfolios is provided. (See Thorp (1971) and Samuelson (1969) for further comments.)

Another criterion for selecting \(\mathbf{b}\), that of maximizing \(E \ln S\), has been put forth by Kelly (1956) and Breiman (1961), and persuasively advocated by Thorp (1969, 1971, 1973). (Also see Latané (1959) and Williams (1936).) The resulting portfolio investment policy \(\mathbf{b}^*\) has been demonstrated by Breiman to have the following properties:

P1. In repeated independent sequential investments, \(\mathbf{b}^*\) maximizes \(\lim (1/N) \ln S_N\). Thus the asymptotic "interest rate" is maximized.

P2. The time required to achieve a certain capital \(W\) is minimized by \(\mathbf{b}^*\) (in a sense that can be made precise), in the limit as \(W \to \infty\).
Yet \( b^* \) is not accepted in current economic practice. Perhaps one reason is that maximizing \( E \ln S \) suggests that the investor has a logarithmic utility for money. However, the criticism of the choice of utility functions ignores the fact that maximizing \( E \ln S \) is a consequence of the goals represented by properties P1 and P2, and has nothing to do with utility theory. (See Thorp (1971)).

Another objection, held by Samuelson (1967,69) and others, is that not all investors are interested in long term goals. Samuelson (1969, p. 246), writes, "Our analysis enables us to dispel a fallacy that has been borrowed into portfolio theory from information theory of the Shannon type. Associated with independent discoveries by J.B. Williams (1936), John Kelly (1956), and H.A. Latané (1959) is the notion that if one is investing for many periods, the proper behavior is to maximize the geometric mean of return rather than the arithmetic mean. I believe this to be incorrect (except in the Bernoulli logarithmic case where it happens to be correct for reasons quite distinct from the Williams-Kelly-Latané reasoning)... It is a mistake to think that, just because a \( w^{**} \) decision ends up with almost-certain probability to be better than a \( w^* \) decision, this implies that \( w^{**} \) must yield a better expected value of utility."

Another possible objection is that \( b^* \) may be too conservative, since it optimizes a concave (risk averse) function of the return \( S \). One interpretation of "too conservative" could be that \( b^* \) will be outperformed (i.e., \( b^t \bar{x} > b^{*t} \bar{x} \)) with high probability by a more ambitious policy \( b \). Alternatively, too conservative might mean that with substantial probability \( b^* \) will be outperformed by a large factor (i.e., \( b^t \bar{x} > c b^{*t} \bar{x} \), for some constant \( c > 1 \)) by a more risky policy \( b \). Thus a reasonable goal for
an individual investor or a mutual fund would be good short term competitive performance.

With the above objections in mind, we are led to the analysis of one-stage investments. Consider the game in which two investors seek portfolio policies that are competitively best in the sense that at least half the time one achieves more capital than one's opponent. Surprisingly, the game theoretic optimal strategy will be shown to be \( U \, b^* \), where \( U \) is an independent uniform \([0,2]\) random variable, and \( b^* \) is the same log optimal policy as before. Furthermore, among non-randomized strategies, \( b^* \) is shown to be competitively best in the sense that it will not be beaten by very much very often. Thus the alleged conservatism of \( b^* \) must be established on other grounds; and the short term value of \( U \, b^* \) is established competitively.

In the next section we shall argue for the naturalness of the random variable \( U \) in the competitive investment game. Section 3 will prove Theorem 1, establishing \( U \, b^* \) as the solution of the game. Section 4 shows that \( b^* \) is stochastically admissible.

We shall then reinvestigate the St. Petersburg Paradox in Section 5 to show how much that investment is worth as a function of the entry fee.
2. A Game-Theoretic Digression

Before proceeding, we establish the necessity for randomization in the competitive investment game.

Suppose 2 players each have 1 unit of capital. Their competitive positions are equal. However, let us now suppose that player 2 has available to him any fair gamble \( E X = 1, X \geq 0 \). By selecting the distribution of the gamble \( X \) judiciously, he can beat player 1 with probability \( 1 - \varepsilon \).

Simply let \( P(X = 1/(1 - \varepsilon)) = 1 - \varepsilon, P(X = 0) = \varepsilon \). Then \( P(X > 1) = 1 - \varepsilon \).

Therefore, player 1 must protect himself by randomizing his capital. This is a purely game theoretic maneuver and has nothing to do with maximizing investment return.

We now solve the following game. Let players 1 and 2 choose d.f.'s \( F \) and \( G \) respectively, \( \int x dF = \int y dG = 1, F(0) = G(0) = 0 \). Then \( X \sim F \) and \( Y \sim G \) are independently drawn. The freedom of choice we allow in the choice of \( F \) and \( G \) makes physical sense, since any capital distribution \( F(x), \int x dF(x) = 1, F(0) = 0 \), is achievable from initial capital 1 by a sequential gambling scheme on fair coin tosses (Cover (1974)). The payoff to player 1 is

\[
P(X > Y) = \int G dF.
\]

(2)

**Lemma:** The value of this game is \( 1/2 \), and the unique optimal strategies are

\[
F^*(t) = G^*(t) = \begin{cases} 
  t/2, & 0 \leq t \leq 2 \\
  1, & t \geq 2.
\end{cases}
\]

(3)

**Proof:** For \( F^* \) and for any \( G \),
\[ P(Y \geq X) = \int F^* dG = \int_0^\infty \min\{t/2,1\} dG(t) \leq 1/2 \int_0^\infty t dG(t) = 1/2. \quad (4) \]

Thus \( F^* \) achieves 1/2 against any \( G \).

Uniqueness of the optimal distribution \( F^*(x) \) is proved by assuming \( P(Y \geq X) \leq 1/2 \) for (i) \( Y \) uniform \([0,2]\), (ii) \( Y \) a two point distribution at 0 and a point \( c \in [1,2] \), and (iii) \( Y \) a two point distribution at \( c \in [0,1] \) and 2. Then (i) \( \Rightarrow F^*(2) = 1 \); (ii) \( \Rightarrow F^*(c) \leq c/2 \), \( 1 \leq c \leq 2 \); and (iii) \( \Rightarrow F^*(c) \leq c/2 \), \( 0 \leq c \leq 1 \). Since \( \int t dF^*(t) = 1 \), we see \( F^*(t) = t/2 \), \( 0 \leq t \leq 2 \). The proof of the uniqueness of \( G^* \) follows by symmetry.

We see that a gambler must exchange his unit capital for a r.v. \( U \) uniformly distributed on \([0,2]\) in order to protect himself.
3. **The Competitive Investment Game**

Let $B$ be the set of all r.v.'s $B = (B_1, B_2, \ldots, B_n)^t$, $B \geq 0$, a.e., $\sum_{i=1}^{n} B_i = 1$. Let the investment vector $X = (X_1, X_2, \ldots, X_n)^t$ be a r.v. with known distribution function $F(x)$. We assume that $X \geq 0$, a.e. To eliminate degeneracy, we also assume that

$$-\infty < \sup_{b} \mathbb{E} \ln b^{t} X < \infty.$$ 

Consider the game in which players 1 and 2 choose $B^{(1)} \in B$, $B^{(2)} \in B$, and player 1 receives payoff

$$p \{ B^{(1)^{t}} X > B^{(2)^{t}} X \}. \quad (5)$$

It is assumed that $B^{(1)}$, $B^{(2)}$, and $X$ are jointly independent.

**Theorem 1:** The solution for the competitive investment game is $B^* = U b^*$, where $U$ is unif. on $[0, 2]$, independent of $X$, and $b^*$ maximizes $\mathbb{E} \ln b^{t} X$. The value of the game is $1/2$.

**Proof:** The Kuhn Tucker Theorem (1951) implies that the $b^*$ maximizing $\mathbb{E} \ln \sum_{i=1}^{n} b_i^* X_i$ subject to the constraint $\sum_{i=1}^{n} b_i = 1$, $b_i \geq 0$, satisfies

$$\begin{align*}
\frac{X_i}{\sum_{i=1}^{n} b_i^* X_i} &= \lambda, \ b_i^* > 0 \\
\leq \lambda, \ b_i^* = 0, \ i = 1, 2, \ldots, n
\end{align*} \quad (6)$$

where $\lambda$ is chosen so that $\sum_{i=1}^{n} b_i^* = 1$. But we see $\lambda = 1$, since

$$\lambda = \sum_{i} b_i^* \lambda = \sum_{i} b_i^* \mathbb{E} X_i / (\sum_{i} b_i^* X_i)$$

$$= \mathbb{E} (\sum_{i} b_i^* X_i) / (\sum_{i} b_i^* X_i) = 1. \quad (7)$$
We now investigate the payoff of $B^* = UB^*$ against any other investment policy $B \in B$:

$$P\{B^tX \geq B^*tX\} = P\{B^tX \geq UB^*tX\}$$

$$= P\{U \leq (B^tX)/(b^*tX)\}$$

$$\leq 1/2 \ E\{(B^tX)/(b^*tX)\}$$

$$= 1/2 \ \sum_{i=1}^{n} E B_i \ E(X_i/\sum b_j^* X_j)$$

$$\leq 1/2 \ \sum E B_i \lambda$$

$$= \lambda/2 \ E \sum B_i = \lambda/2 = 1/2.$$ (8)

Thus $B^* = UB^*$ achieves the value of the game against any $B$, and the proof is complete.

The above strategy $B^*$ can be implemented by first exchanging the 1 unit initial capital for the fair gamble $U$, uniformly distributed over [0,2], then distributing $U$ on the investments according to the solution $b^*$ maximizing $E \ln b^tX$.

This result can be generalized to show that $B^* = UB^*$ will not be beaten by very much very often:

**Corollary 1:** $P\{B^tX \geq c UB^*tX\} \leq 1/2c$, for all $B \in B$, $c > 0$.

**Proof:** $P\{cU \leq B^tX/ b^*tX\} \leq (1/2c)E(B^tX/b^*tX)$, and the proof proceeds as in Theorem 1.

Dropping the randomization $U$ increases this probability by at most a factor of 2:
Corollary 2: \( P(B^t X \geq c b^t X) \leq 1/c \), for all \( B \in \mathcal{B} \), \( c > 0 \).

Proof: By Markov's inequality and (8),

\[ P(B^t X \geq c b^t X) \leq (1/c)E(B^t X / b^t X) \leq 1/c. \]

Remark 1: This is the best that can be attained by any non-randomized strategy, as can be seen from the discussion at the beginning of Section 2.

Remark 2: Corollary 2 bears a strong resemblance to Markov's lemma; i.e., \( Y \geq 0 \), \( E Y = \mu \) \( \Rightarrow P(Y \geq c\mu) \leq 1/c \), \( \forall c > 0 \). This suggests that \( b^t X \) acts like the fixed amount of capital \( \mu \) in Markov's lemma and that \( b^t X \) can be improved only by fair randomization. This is true despite the fact that \( E B^t X \) may be greater than \( E b^t X \).
4. Admissibility of $b^*$

We see from the definition of a Markowitz efficient portfolio that undominated portfolios are selected in the $(ES, \text{Var}(S))$ plane. A primary notion of dominance is stochastic dominance, i.e., the r.v. $S$ is said to stochastically dominate the r.v. $S'$ iff $P(S \geq t) \geq P(S' \geq t)$, for all $t$. Following Hakansson (1971), we shall say that a portfolio $b^*$ is admissible if it is not stochastically dominated by any other portfolio $B \in B$. We then have, in agreement with Thorp's assertion (1971), the following theorem.

**Theorem 2**: The log optimal policy $b^*$ is admissible.

**Proof**: Let $F$ be the d.f. of $b^* X$. Let $G$ be the d.f. of $B^t X$, where $B \in B$ is any other given allowable portfolio strategy. We wish to show that $G(t)$ is not $\leq F(t)$ for all $t \in [0, \infty)$. Let $\tilde{F}, \tilde{G}$ be the d.f.'s of $\ln b^* X$ and $\ln B^t X$ respectively. Clearly, by the monotonicity of $\ln$ on $[0, \infty)$ we need only show that $\tilde{G}(t)$ is not less than or equal to $\tilde{F}(t)$ for all $t$.

Suppose, on the contrary, that $\tilde{G}(t) \leq \tilde{F}(t)$, $\forall t$. Then

$$E \ln b^* X = \int_{-\infty}^{\infty} t \tilde{F}(t) dt$$

$$= -\int_{-\infty}^{0} \tilde{F}(t) dt + \int_{0}^{\infty} (1-\tilde{F}(t)) dt$$

$$\leq -\int_{-\infty}^{0} \tilde{G}(t) dt + \int_{0}^{\infty} (1-\tilde{G}(t)) dt$$

$$= E \ln B^t X,$$

contradicting the assumed optimality of $b^*$. 

**Remark**: Obviously any $b$ maximizing $E v(b^t X)$ for any monotonic increasing utility function $v$ is undominated. For example, $\max E b^t X$
yields an undominated \( b \). However, the Markowitz class of efficient portfolios contains stochastically inadmissible portfolios and does not contain all admissible portfolios.

**Example:** Let \( X_1 = 1 \) and let \( X_2 \) be uniform on the interval \([1,2]\). Then any \( b \geq 0 \) such that \( b_1 + b_2 = 1 \) is efficient in Markowitz's sense while only \( b_1 = 0 \), \( b_2 = 1 \), is admissible. For further examples, see Hakansson (1971, p.529) and Thorp (1971, p. 20).
5. The St. Petersburg Paradox

In the St. Petersburg paradox, a gambler pays an entry fee $c$. He receives in return a random amount of capital $X$, where $P(X=2^i) = 2^{-i}$, $i = 1, 2, \ldots, \infty$. Since $EX = \infty$, it is argued that any entry fee $c$ is therefore justifiable. This argument is unsatisfying; and various attempts to deal with $c$ in a more realistic way appear in the literature (e.g., Feller (1950, Vol. 1; pp. 235-237). It is commonly stated that the expected log return strategy does not remove the paradoxical conclusions. We do not agree, for reasons that will be developed in this section. This investment situation will be investigated both as a one-shot investment and as an investment imbedded in a sequence of i.i.d. such investments.

To fix ideas, suppose that a gambler has total initial capital $S_0$. He is allowed to receive 1 unit of St. Petersburg investment for each $c$ units that he pays as an entry fee. Let him invest the amount $bS_0$, $0 \leq b \leq 1$, and retain $(1-b)S_0$ in cash. Thus his return $S$ is given by $S = S_0 ((1-b) + (b/c)X)$.

Let $b^* \in [0, 1]$ maximize $E \ln S$. We calculate

$$\frac{dE \ln S}{db} = \mathbb{E} \frac{-1 + X/c}{(1-b) + (b/c)X}$$

$$= \sum_{i=1}^{\infty} \frac{(1/2)^i (2^i/c - 1)}{2^i b/c + (1-b)} = 0 \quad (10)$$

Letting $b = 1$, we see that $dE \ln S/db = 1 - (c/3)$, which is $\geq 0$ for $c \leq 3$. For $c > 3$, the solution $b^*$ to Eq. (10) tends monotonically to zero as the entry fee $c \to \infty$. Finally it can be seen that $b^*$ and $\max E \ln S$ are always strictly positive.
Investing a proportion of capital \( b^* \) guarantees that

1. The investor is acting in accordance with an investment policy maximizing \( \lim(1/n)\ln S_n \), regardless of whether or not the other investment opportunities are of the St. Petersburg form, and

2. The investor investing \( Ub^* \) is competitively optimal in the St. Petersburg game.

Moreover, we see that all entry fees \( c \) are "fair". However, the proportion \( b^* \) of total capital invested varies as a function of \( c \). Also, \( b^* \) is independent of the total initial capital \( S_0 \).

Finally, if the investment fee is low enough, i.e., \( 0 \leq c \leq 3 \), then \( b^* = 1 \) and all of the capital is invested. This results in

\[
E \log_2(S/S_0) = \sum_{i=1}^{\infty} \left(\frac{1}{2}\right)^i \log_2(2^i/c) \\
= \sum_{i=1}^{\infty} i 2^{-i} - \log_2 c \quad (11)
\]

\[
= 2 - \log_2 c \quad , \text{for} \quad 0 \leq c \leq 3 .
\]

Thus \( S_n \sim S_0(4/c)^n \) in the sense that \( (1/n)\log_2 S_n \to 2 - \log_2 c \), for \( 0 \leq c \leq 3 \).
6. Conclusions

It should now be clear that the investment policy $b^*$ achieving
$\max \mathbb{E} \ln b^T X$ has good short run as well as good long run properties. In
addition, $b^*$ is admissible in the sense that no other policy $b$ stochas-
tically dominates $b^*$.

We wish to comment on the use of $Ub^*$ (as opposed to $b^*$ alone) in
practice. We have seen in Section 2 that the use of $U$ is a purely game
theoretic protection against competition and has nothing to do with increas-
ing a player's capital. Thus we feel that $b^*$ alone is sufficient to
achieve all of the reasonable competitive investment goals, and we do not
choose to advocate the additional randomization $U$.

Finally, it is tantalizing that $b^*$ arises as the solution to such
dissimilar problems as maximizing $\lim \inf (1/n) \ln S_n$ and maximizing
$P\{B_1^T X > B_2^T X\}$. The underlying reason for this coincidence is yet to be
revealed.
References


