ASYMMETRIES IN RELATIVISTIC INFORMATION FLOW

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KEITH JARETT and THOMAS M. COVER

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Keith Jarett†
Thomas M. Cover‡

Abstract

In the so-called twin paradox of relativity theory, one spaceship leaves another at velocity $v = \beta c$ and returns to find that the other ship has aged more by a factor $\gamma = (1 - \beta^2)^{-1/2}$. We first isolate the asymmetry to resolve the paradox. Then we use the principle of relativity to derive the relative aging factor $\gamma$ and the relativistic doppler shift.

Using the doppler factor, we investigate asymmetries in information transmission between moving spaceships. We consider an additive white gaussian noise channel with Shannon capacity $C = W \log(1 + P/NW)$. After accounting for the effect of the relativistic doppler shift on signal power and bandwidth, we find that for a given transmission rate and bandwidth, the traveler needs $\gamma$ times the energy of the stationary spaceship to transmit $1/\gamma$ times as much information. The asymmetry in efficiency is thus $\gamma^2$. We give a simple proof that the round-trip asymmetry in efficiency for constant rate transmission is always the square of the relative aging factor for all trajectories regardless of accelerations of either spaceship and the presence of gravitational fields.

Two alternative formulations to that of constant transmission rate are

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also considered. We exhibit a water-filling solution for the optimal time
distribution of power under a round trip energy constraint. Both of these
results exhibit trajectory-dependent asymmetries in addition to the aging
factor. For constant power transmission, as well as for constant rate
transmission, we conclude that the traveler is less efficient than the
stationary transmitter.

1. Introduction

The twin paradox provides a suitable foundation for a discussion of
relativistic information flow. We present a new resolution [1] of the twin
paradox in Section 2, introduce the necessary concepts of relativity, and
derive the pertinent relativistic doppler shift factors. We shall consider
communication over an additive white Gaussian noise channel with receiver
noise spectral density N, transmitter power P, and transmitter band-
width W. The Shannon capacity for this continuous-time channel is C =
W \log(1+P/NW) bits per second. We let \( \alpha \) denote the relativistic doppler
shift factor and let primes denote quantities measured at the receiver. Thus
the reception capacity is \( C' = W' \log(1+P'/NW') \), since the receiver can
deceive the message at this rate if the received signal-to-noise ratio \( P'/NW' \)
satisfies this constraint. Reasoning that \( W' = \alpha W \) and \( P' = \alpha^2 P \) yields
\( C' = \alpha W \log(1+\alpha P/NW) \). This gives a transmission capacity \( C = C'/\alpha = 
W \log(1+\alpha P/NW) \) bits/second.

We apply these results in Section 4 to the circular trajectory problem,
in which B circles A at a constant radius. We show the close relation-
ship between the doppler shifts and the aging ratio for this problem, and
we exhibit the transmission capacities for A and B in terms of the
relative aging factor.
Then, in Section 5, we extend this capacity result to time-varying problems, such as the twin problem in which the doppler shift $\alpha$ is time-dependent. We show that the instantaneous transmission capacity is $C(t) = W \log(1 + \alpha(t)P(t)/NW)$, where $\alpha(t)$ is the doppler shift appropriate for signals transmitted at time $t$. Thus $\alpha(t)$ incorporates knowledge of the current trajectory of the transmitter and the future trajectory of the receiver. Also in Section 5, we prove for a round trip, where terminals $A$ and $B$ start and finish together, that

$$\int_0^{T_A} \frac{1}{\alpha_A(t)} dt = T_B \quad \text{and} \quad \int_0^{T_B} \frac{1}{\alpha_B(\tau)} d\tau = T_A.$$ 

This result is used in Section 6 to develop the fundamental rate equations. From these equations, it can be shown that if $A$ and $B$ transmit at the same constant rate and bandwidth, and $\gamma_0$ is the relative aging factor $T_A/T_B$, then $B$ will need $\gamma_0$ times as much energy as $A$ to transmit $1/\gamma_0$ times as many bits during the round trip. Regardless of acceleration or gravitational fields, the trajectory dependence of the asymmetry in efficiency reduces to the square of the aging factor.

Next, we demand that $P(t)$ be constant and show that if $A$ remains fixed, the traveler $B$ needs more energy per bit sent. Alternatively, if we place a constraint on the total transmitted energy and calculate the number of bits that can be sent during the round trip, a round trip energy constraint of the form $\int_0^T P(t) dt = E$ yields a simple water-filling solution for the optimal form of $P(t)$.

In Section 9, we apply the previously derived general results for time-varying doppler factors to the linear round trip or twin problem. This

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example illustrates the various asymmetries which result from the formulations of the communication problem in Sections 6, 7, and 8.

All asymmetries favor the stationary transmitter.

2. The Twin Paradox

Two identical spaceships sit side by side. One of them leaves at speed \( v = \beta c \) and travels a distance long compared to the distance traveled during acceleration. The traveler then reverses course, returning to the stationary spaceship at the same speed. It is found that the traveler has aged less by the factor \( \gamma = (1 - \beta^2)^{-1/2} \). Is this not a paradox? Could not either spaceship be considered the traveler, and the other stationary? The answer lies in the fact that although velocity is relative, acceleration is absolute, generating an apparent force in the accelerated frame. Familiar centrifugal effects come to mind. Acceleration at the turning point introduces the asymmetry in the twin problem and resolves the paradox.

We can use symmetry arguments to derive the relative aging factor (see Darwin [2] and Bondi [3] for alternative derivations of the aging factor). The following numerical example will make the concepts more clear.

Suppose spaceship \( A \) remains at rest while spaceship \( B \) travels 3 light years at \( 3/5 \) of the speed of light and returns. Additionally, suppose that \( A \) and \( B \) each carry clocks which emit one pulse per year. Figure 2.1a shows \( A \)'s clock pulses traveling to \( B \), while Figure 2.1b shows \( B \)'s clock pulses traveling to \( A \).

Invoking time reversal symmetry, we have immediately that \( B \)'s elapsed times on the outbound and inbound legs must be equal. We let \( T_B \) denote \( B \)'s total elapsed time for the round trip. We solve for \( T_B \) as follows.
Figure 2.1a shows that B receives 2 of A's clock pulses in the $T_B/2$ years of B's outbound trip. Figure 2.1b shows that A receives B's $T_B/2$ outbound clock pulses in 8 years of A's time. This is because of the 3 years necessary for light to travel to A from B's turning point. Spaceship A does not see B turn until the signals from the turning point reach A.

Figure 2.1a
A's Clock Pulses

Figure 2.1b
B's Clock Pulses

Space-time Diagrams of B's Trip with A at Rest, Showing Clock Pulses Traveling between A and B.

Pulses are received at rate $\alpha_-$ or $\alpha_+$. 

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We define $\alpha_-$ to be the receding doppler shift factor, the ratio of received pulse rate to transmitted rate when the receiver sees the transmitter receding. By the principle of relativity, A and B must receive pulses at the same rate when they see each other receding. Since the transmission rate is 1, this common reception rate is just $\alpha_-$. Using the pulse counts and time intervals above for each receiver, we have two ways to write the reception rate:

$$\alpha_- = 2/(T_B/2) = (T_B/2)/8.$$ 

This implies

$$T_B/2 = 4, \text{ B's outbound elapsed time},$$
and

$$\alpha_- = 1/2, \text{ the receding doppler factor}.$$

We can perform a similar analysis for B's inbound leg. The equality of reception rates gives

$$\alpha_+ = 8/(T_B/2) = (T_B/2)/2.$$ 

Thus

$$T_B/2 = 4, \text{ B's inbound elapsed time},$$
and

$$\alpha_+ = 2, \text{ the approaching doppler factor}.$$

Figure 2.2 shows a space-time diagram of the twin problem for an arbitrary value of $\beta$. The equality of outbound reception rates yields

$$\alpha_- = ((1-\beta)T_A/2)/(T_B/2) = (T_B/2)/((1+\beta)T_A/2),$$

implying

$$T_B/T_A = (1-\beta^2)^{1/2} = 1/\gamma,$$  \hspace{1cm} (2.1)

and

$$\alpha_- = (1-\beta)^{1/2}/(1+\beta)^{1/2} = (1-\beta)\gamma.$$  \hspace{1cm} (2.2)
The equality of inbound reception rates yields

\[ \alpha_+ = \frac{((1 + \beta)T_A/2)/(T_B/2)) = (T_B/2)/((1-\beta)T_A/2)}{\text{implying}} \]

\[ T_B/T_A = 1/\gamma \]

and

\[ \alpha_+ = (1 + \beta)^{1/2}/(1-\beta)^{1/2} = (1 + \beta)\gamma \quad \text{(2.3)} \]

Thus we see that the relative aging factor is

\[ T_A/T_B = \gamma = (1-\beta^2)^{-1/2} \quad \text{(2.4)} \]

![Space-Time Diagram for Velocity \( \beta c \)]

Figure 2.2

Space-Time Diagram for Velocity \( \beta c \)
3. The Additive White Gaussian Noise Channel with Doppler Shift

![Diagram of the channel with Doppler Shift]

- Power $P$
- Bandwidth $W$
- Rate $R$
- Signal power $P' = \alpha^2 P$
- Signal bandwidth $W' = \alpha W$
- Signal rate $R' = \alpha R$

Figure 3.1

The Additive White Gaussian Noise Channel with Doppler Shift

The AWGN channel with doppler shift is illustrated in Figure 3.1. The transmitter sends a signal process $X$ with average power $P$ and bandwidth $W$. Information is transmitted at rate $R$. These three quantities are all measured in the transmitter's frame. The noise process $Z$ is white Gaussian noise with power spectral density $N$, measured in the receiver's frame.

Suppose that the receiver sees the transmitter moving. Then the received signal is doppler-shifted by a factor $\alpha$. The exact form of the doppler factor $\alpha$ is unimportant here. We make the narrow-beam assumption that the entire signal from the transmitter is intercepted by the receiver. Path loss is assumed to be independent of the distance the signal travels. Without loss of generality, we assume the path loss is zero.

We need to know the characteristics of the doppler-shifted received signal in order to establish a constraint on the transmitted signal power,
rate, and bandwidth. First, the reception rate $R'$ is a frequency.
(Throughout this paper, we use primes to denote quantities pertaining to reception). Since the doppler factor $\alpha$ is the ratio of received to transmitted frequency, we have immediately that $R' = \alpha R$. Second, the component frequencies of the signal are all shifted by the factor $\alpha$, so that the received signal bandwidth is $W' = \alpha W$. Third, the received signal power is $P' = \alpha^2 P$. This can be shown from relativistic electromagnetic theory [4], but the following simple heuristic argument makes this plausible.

Suppose the transmitted signal is monochromatic. It then consists of a stream of photons of frequency $\nu$ being emitted at rate $\lambda$. The transmitted signal power $P$ equals the energy per photon $\hbar \nu$ times the photon emission rate $\lambda$. The received signal consists of a monochromatic stream of photons of frequency $\nu'$ arriving at rate $\lambda'$. Assuming that the doppler factor is $\alpha = \nu'/\nu = \lambda'/\lambda$, we can see that $P' = \lambda' h \nu' = \alpha^2 \lambda \hbar \nu = \alpha^2 P$.

This relationship is valid for each monochromatic component of any signal. Summing the components yields $P' = \alpha^2 P$ for an arbitrary signal process.

The received signal process $Y$ is thus the sum of a white Gaussian noise process with power spectral density $N$ and a signal process with power $P' = \alpha^2 P$, bandwidth $W' = \alpha W$, and information rate $R' = \alpha R$. We apply Shannon's [5] capacity result for the AWGN channel to conclude that the maximum reception rate at which the receiver can reliably decode messages is

$$C' = W' \log(1 + P'/NW') \quad \text{(Shannon capacity)}$$
$$= \alpha W \log(1 + \alpha^2 P/N\alpha W)$$
$$= \alpha W \log(1 + \alpha P/NW) \quad \text{bits/second.} \quad (3.1)$$
The corresponding transmission rate is

\[ C = C'/\alpha = W \log(1+\alpha P/NW) . \]

All logarithms have base two. To achieve this rate, the transmitter chooses a signaling interval \([0,T]\). He then chooses \(2^T\) signals at random from a white Gaussian random process with bandwidth \(W\) and power \((1-\varepsilon)P\). Then, for \(R < C\), and \(T \to \infty\), \(\varepsilon \to 0\), it can be shown that, with very high probability, this codebook achieves arbitrarily small probability of decoding error. Moreover, for any codebook, and for any rate \(R > C\), the probability of receiver error tends to 1.

Strictly speaking, all that we can say for short journeys and time-varying \(\alpha\) is that we expect the probability of error to be very small if the instantaneous rate \(R\) is less than \(C\) and to be very large if \(R\) exceeds \(C\).

It should be noted that, although the transmitter sends the signal over the time interval \([0,T]\), the receiver sees the signal over the interval \([0,\alpha T]\).

In summary, for reliable communication we require transmission at a rate satisfying

\[ R < C = C'/\alpha = \frac{(W'/\alpha) \log(1+P'/NW')}{W \log(1+\alpha P/NW)} . \]

Equivalently, if we wish to transmit information reliably at rate \(R\) and bandwidth \(W\), the required transmitted signal power is

\[ P = \alpha^{-1} NW (\exp_2(R/W) - 1) \Delta \frac{\alpha}{P_0} \]

where \(P_0 = NW(\exp_2(R/W) - 1)\) is the minimum transmitted power for reliable communication when there is no doppler shift.
4. The Circular Trajectory Problem

![Diagram](image)

**Figure 4.1**

Geometry of the Circular Trajectory Problem

In the previous section, we derived the transmission capacity for the AWGN channel with a doppler shift $\alpha$. In the next section, we will extend this result to allow a time-varying doppler factor $\alpha(t)$ like that found in the twin problem.

However, we can first use the results of Section 3 to solve steady-state communication problems in which the doppler shift factor is constant. The most interesting constant doppler shift example is the circular trajectory problem in which $B$ circles $A$ at a constant radius $r_B$ and at a constant speed $\omega r_B = \beta c$ (Figure 4.1). $B$ maintains this trajectory by continuous application of thrust; this analysis also applies to the case in which a gravitational mass is present. Since there is no preferred direction, the doppler factors must be constant. Using this fact, we can derive the relationship between the doppler shift factors and the relative aging factor.

Let $\alpha_A$ be the doppler factor for transmission from $A$ to $B$, and $\alpha_B$ be the factor for signals sent from $B$ to $A$. Let $T_A$ be the orbital
period as measured by A, and \(T_B\) be the elapsed time per orbit as measured by B. These times can be established by reference to a fixed object such as a distant star. Finally, let \(\gamma_0 = T_A/T_B\) be the relative aging factor. In this problem it turns out, not surprisingly, that \(\gamma_0 = \gamma = (1-B^2)^{-1/2}\).

We can derive the transmission doppler factors in terms of the aging factor \(\gamma_0\) as follows. Suppose A sends pulses to B at rate \(1/\alpha_A\) for one orbital period \(T_A\). These pulses will be received by B at rate \(\alpha_A/\alpha_A = 1\) for one orbital period \(T_B\). Since the number of pulses sent equals the number received, we have \((1/\alpha_A)T_A = (1)T_B\), or

\[
\alpha_A = T_A/T_B = \gamma_0 , \quad \text{the aging factor.} \quad (4.1)
\]

Similarly, suppose B sends pulses to A at rate \(1/\alpha_B\) for one orbital period \(T_B\). These will be received by A at rate \(\alpha_B/\alpha_B = 1\) for one orbital period \(T_A\). Pulse conservation yields \((1/\alpha_B)T_B = T_A\), or

\[
\alpha_B = T_B/T_A = 1/\gamma_0 , \quad \text{the inverse of the aging factor.} \quad (4.2)
\]

With these results and the capacity result of Chapter 3, we can write A's transmission capacity

\[
C_A = W \log(1+\gamma_0 P_A/NW) , \quad (4.3)
\]

and B's transmission capacity,

\[
C_B = W \log(1+(1/\gamma_0)P_B/NW) , \quad (4.4)
\]

where A and B transmit using signal powers \(P_A\) and \(P_B\) and bandwidth \(W\). For equal transmission powers, the received signal-to-noise ratio at
B is $\gamma_0^2$ times the signal-to-noise ratio at A. If A and B both wish to transmit at the same rate $C_A = C_B = C$ and with the same bandwidth $W$, B will need to use $\gamma_0^2$ times as much power as A. For each orbit B will use $\gamma_0$ times as much energy as A to send $1/\gamma_0$ times as much information. The asymmetry in efficiency is thus

$$\frac{E_B/N_B}{E_A/N_A} = \frac{(P_B T_B/P_A T_A)(C_A T_A/C_B T_B)}{P_B/P_A} = \gamma_0^2$$  \hspace{1cm} (4.5)

If A and B both transmit using the same power $P = P_A = P_B$, the ratio of transmission rates equals the ratio of energies per bit:

$$\frac{E_B/N_B}{E_A/N_A} = \frac{(P_B T_B)/(P_A T_A)}{(C_A T_A/C_B T_B)} = \frac{C_A}{C_B} = \frac{\log(1+\gamma_0 P/NW)}{\log(1+(1/\gamma_0)P/NW)}$$  \hspace{1cm} (4.6)

Note that the asymmetry in efficiency $(E_B/N_B)/(E_A/N_A)$ lies strictly between 1 and $\gamma_0^2$ for all $N$, $W$, and $P$, and approaches $\gamma_0^2$ as $P \to 0$. This supports the conclusion that the transmitter that ages less is less efficient.

5. The Fundamental Equations for Relativistic Capacity

We have found the channel capacity expression as a function of $\alpha$ for uniform motion. We now wish to consider a possibly time-varying transmitter power $P(t)$, transmitter rate $C(t)$, and integrate over transmitter time to calculate the total number of bits communicated from the transmitter to the receiver. To proceed, we introduce an anticipated doppler shift $\alpha(t)$.

First, let us agree on spacetime trajectories for A and for B. From the knowledge of these trajectories, we can calculate an anticipated
doppler shift factor $\alpha(t)$ as follows. We know that clock pulses sent at transmitter time $t$ will be received at some point in space-time by the receiver. Let $\alpha(t)$ be the observed doppler shift at the receiver of this signal sent at time $t$. This definition of $\alpha(t)$ is essential to the consideration of problems incorporating time-varying doppler factors.

We first derive the aging relationships from $\alpha(t)$ for later comparison to the information flow relationships. Suppose the entire journey for transmitter A takes time $T_A$. At the end of the journey A and B will be together, and suppose that B's clock then reads time $T_B$. Clearly, we have the trivial relationship

$$T_A = \int_0^{T_A} 1 \, dt.$$ 

Now we introduce an artifice. Let A transmit pulses to B at a time-varying rate $r(t) = 1/\alpha(t)$. Then the number of pulses sent by A will be given by

$$\int_0^{T_A} r(t) \, dt = \int_0^{T_A} \frac{1}{\alpha(t)} \, dt.$$ 

But $r(t)$ has been adjusted so that the received pulse rate by B is $r'(\tau) = \alpha(t)r(t) = \alpha(t)/\alpha(t) = 1$ pulse/second observed in B's time units. Thus B receives pulses at constant rate 1, and we can know B's age at the end of the journey by simply counting the pulses received by B. Finally, the number of pulses received by B is simply the number of pulses sent by A. Thus we have the useful result
\[ T_B = \int_0^{T_B} r'(\tau)d\tau = \int_0^{T_A} r(t)dt = \int_0^{T_A} (1/\alpha(t))dt. \quad (5.1) \]

We observe, by symmetry, that if we define the anticipated doppler factor \( \alpha_B(\tau) \) for B sending to A, now in B's time frame, we have

\[ \int_0^{T_B} (1/\alpha_B(\tau))d\tau = T_A \quad (5.2) \]

Now we investigate channel capacity using the anticipated doppler shift factors \( \alpha_A(t) \) and \( \alpha_B(\tau) \). If A sends power \( P(t) \) at A's time \( t \), then the received signal will have instantaneous power \( \alpha^2(t)P(t) \) and bandwidth \( \alpha(t)W \). Thus the instantaneous receiver capacity is

\[ C'(\tau) = \alpha(t)W \log(1+\alpha(t)P(t)/NW) \text{ bits/second}, \]

where we use \( \tau \) here to denote B's reception time of signals sent by A at time \( t \). Finally, correcting for the doppler factor, this corresponds to a transmission rate

\[ C(t) = C'(\tau)/\alpha(t) = W \log(1+\alpha(t)P(t)/NW) \text{ bits/second}, \quad (5.3) \]

measured in the transmitter's frame. This equation is the kernel of most of the results in this paper.

We now collect results. Transmitter A sends at power \( P_A(t) \), \( 0 \leq t \leq T_A \). His anticipated doppler shift factor is \( \alpha_A(t) \). Let \( N_A \) be the number of bits sent from A to B and let \( E_A \) be the total energy
expanded. Then we obtain the fundamental communication formulas:

\[ \int_0^{T_A} 1 \, dt = T_A ; \int_0^{T_B} 1 \, d\tau = T_B \]

\[ \int_0^{T_A} \left( \frac{1}{\alpha_A(t)} \right) dt = T_B ; \int_0^{T_B} \left( \frac{1}{\alpha_B(\tau)} \right) d\tau = T_A \]

\[ \int_0^{T_A} P_A(t) dt = E_A ; \int_0^{T_B} P_B(t) d\tau = E_B \]

\[ \int_0^{T_A} W \log(1 + \alpha_A(t)P_A(t)/NW) dt = N_A ; \quad (5.4) \]

\[ \int_0^{T_B} W \log(1 + \alpha_B(\tau)P_B(\tau)/NW) d\tau = N_B . \]

The reader should be aware that \( \alpha_A(t) \) and \( \alpha_B(\tau) \) are derived from the trajectories (and the presence of gravitational masses) and thus are not independent functions. We shall be especially interested in relations among \( T_A, T_B, E_A, E_B, N_A, N_B \) that are trajectory-independent, i.e., independent of \( \alpha_A(t) \), \( \alpha_B(\tau) \).

We are now prepared to investigate the following communication problems:

1) Constant communication rate:

\[ C(t) = W \log(1 + \alpha(t)P(t)/NW) = C_0 . \]

2) Constant transmission power:

\[ P(t) = P_0 , \text{ for all } t . \]

3) Energy constraint: Maximize the number bits sent (5.4) subject to the constraint \( \int P(t) dt = E \).
6. Constant Rate Transmission Yields Asymmetry $\gamma^2$

A particularly striking trajectory-independent asymmetry in information flow occurs when we demand that the transmitter send at constant rate

$$C_A = \log(1 + \alpha_A(t)P_A(t)/NW), \quad 0 \leq t \leq T_A.$$  \hspace{1cm} (6.1)

We can motivate this problem by supposing that the transmitter is sending a real time record of his life processes. Inspection of Equation (6.1) shows that the transmitter must vary his power $P_A(t)$ so that $P_A(t)\alpha_A(t) = P_A$, where $P_A$ satisfies $C_A = W \log(1 + P_A/NW)$. $P_A$ is the power required to communicate at rate $C_A$ between stationary terminals. Equations (5.4) now become

$$T_A = \int_0^{T_A} dt$$

$$T_B = \int_0^{T_A} \frac{1}{\alpha_A(t)} dt$$

$$E_A = \int_0^{T_A} P_A(t) dt = P_A \int_0^{T_A} \frac{1}{\alpha_A(t)} dt = P_A T_B$$

$$N_A = \int_0^{T_A} C_A dt = C_A T_A.$$ \hspace{1cm} (6.2)

Proceeding with the same calculations when $B$ is transmitting constant rate $C_B$, we find

$$E_B = P_B \int_0^{T_B} \frac{1}{\alpha_B(\tau)} d\tau = P_B T_A.$$
and
\[ N_B = \int_0^{T_B} C_B d\tau = C_B T_B, \] (6.3)

where \( P_B \) satisfies \( C_B = W \log(1+P_B/NW) \).

Notice that the energy \( E_B \) is proportional to \( T_A \) and not \( T_B \). Now define the overall aging factor \( \gamma_0 \) to be the ratio of ages
\[ \gamma_0 = \frac{T_A}{T_B} \] (6.4)

at the end of the arbitrary journeys of A and B (subject only to A and B beginning and ending their journeys together). Then A has expended energy per bit transmitted
\[ \frac{E_A}{N_A} = \frac{P_A}{C_A} \frac{T_B}{T_A} = \frac{P_A}{C_A} \frac{1}{\gamma_0}, \]

while B has expended energy per bit transmitted
\[ \frac{E_B}{N_B} = \frac{P_B}{C_B} \frac{T_A}{T_B} = \frac{P_B}{C_B} \gamma_0. \]

Thus the ratio of energies per bit transmitted for B and A is
\[ \frac{(E_B/N_B)/(E_A/N_A)}{P_B/C_B}/(P_A/C_A), \] (6.5)

which reduces to \( \gamma_0^2 \) if A and B transmit at the same rate. In fact, the entire trajectory dependence is contained in the factor \( \gamma_0^2 \). Setting \( \gamma_0 = 1 \) in (6.5) gives the correct result for communication between stationary terminals.

We conclude that the traveler who ages least (by the factor \( 1/\gamma_0 \), \( \gamma_0 > 1 \)) requires \( \gamma_0^2 \) as much energy per bit sent. The youngest traveler has the most difficult time communicating.
7. Round Trip Average Capacity with Constant Transmitter Power

We have derived the time-varying transmission capacity \( C(t) = W \log(1 + \alpha(t)P(t)/NW) \) and shown that for constant rate-transmission the younger traveler needs \( \gamma_0^2 \) more energy per bit sent. In this section and the next, we will investigate the average transmission capacity

\[
\bar{C} = \frac{1}{T} \int_0^T C(t)dt , \quad (7.1)
\]

and derive asymmetries in communication under constraints on the transmission power \( P(t) \).

The most natural constraint on transmission power is that it be constant. In this section, we will assume constant transmission powers \( P_A(t) = P_B(t) = P \) for \( A \) and \( B \), then compare achievable average transmission rates and efficiencies for \( A \) and \( B \). Let \( N_A \) and \( N_B \) denote the number of bits sent.

The ratio of transmitted energies is

\[
E_B/E_A = (PT_B)/(PT_A) = 1/\gamma_0 . \quad (7.2)
\]

During the round trip, \( A \) sends at an average rate

\[
\bar{C}_A = N_A/T_A = \frac{1}{T_A} \int_0^{T_A} W \log(1 + \alpha_A(t)P/NW)dt . \quad (7.3)
\]

Similarly, \( B \) sends a number of bits per unit time

\[
\bar{C}_B = N_B/T_B = \frac{1}{T_B} \int_0^{T_B} W \log(1 + \alpha_B(\tau)P/NW)d\tau \quad (7.4)
\]

\[
= \frac{1}{T_B} \int_0^{T_A} W \log(1 + \alpha_B(t')P/NW)(1/\alpha_A(t))dt . \quad (7.5)
\]
The last equality uses a change of variables identical to that derived in Section 5 by a pulse-counting argument. Here \( \tau'(t) \) is B's reception time of signals (or pulses) transmitted by A at time \( t \).

If terminal A remains fixed while terminal B moves along an arbitrary trajectory in flat space, we can show that the doppler shift factors are

\[
\alpha_A(t) = \gamma(\tau'(t))(1-\beta_r(\tau'(t)))
\]  

(7.6)

for signals transmitted by A, and

\[
\alpha_B(\tau) = \left[\gamma(\tau)(1+\beta_r(\tau))\right]^{-1}
\]  

(7.7)

for signals transmitted by B. In writing these expressions, we let \( \beta(\tau)c \) denote B's speed as measured in A's inertial frame at B's time \( \tau \). We define \( \gamma(\tau) = (1-\beta^2(\tau))^{-1/2} \), the instantaneous aging factor. We let \( \beta_r(\tau) \) denote the radial component (in A's frame) of \( \beta(\tau) \). Using Equations (7.6) and (7.7), we have

\[
\frac{\alpha_A(t)}{\alpha_B(\tau'(t))} = \gamma^2(\tau'(t))(1-\beta^2_r(\tau'(t))) = (1-\beta^2_r(\tau'(t)))/(1-\beta^2(\tau'(t))).
\]

Thus

\[
\alpha_A(t) \geq \alpha_B(\tau'(t)),
\]  

(7.8)

with equality if and only if B's motion at time \( \tau'(t) \) purely radial in A's inertial frame.

Applying this result to Equation (7.5), we have
\[ N_B/T_B \leq \frac{1}{T_B} \int_0^{T_A} W \log(1+\alpha_A(t)P/NW)(1/\alpha_A(t))dt. \]  

This yields the bound
\[
(N_A/T_A) - (N_B/T_B) \geq \frac{1}{T_A} \int_0^{T_A} (1-Y_\alpha\alpha_A(t))W \log(1+\alpha_A(t)P/NW)dt
\]
\[
= \frac{1}{T_A} \int_0^{T_A} (1-Y_\alpha\bar{y})W \log(1+(P/NW)/\bar{y})dt, \tag{7.10}
\]

where \( \bar{y} \triangleq 1/\alpha_A(t) \). The function inside the integral can be shown to be a convex function of \( \bar{y} \). Thus by Jensen's inequality
\[
N_A/T_A - N_B/T_B \geq (1-Y_\alpha\bar{y})W \log(1+(P/NW)/\bar{y}),
\]

where \( \bar{y} = \frac{1}{T_A} \int_0^{T_A} (1/\alpha_A(t))dt = T_B/T_A = 1/Y_\alpha \). Consequently, \((1-Y_\alpha\bar{y}) = 0\)

and
\[
N_A/T_A \geq N_B/T_B. \tag{7.11}
\]

Thus we can bound the asymmetry in efficiency
\[
\frac{E_A/N_A}{E_B/N_B} = \frac{P}{1} \frac{T_B/N_B}{T_A/N_A} \geq 1. \tag{7.12}
\]

We conclude that, if \( A \) remains stationary, then the traveler needs more energy per bit sent.

8. Transmission Under a Round Trip Energy Constraint

In Section 7, we analyzed constant power transmission. Another reasonable constraint on the transmission power \( P(t) \) is the round trip energy constraint
\[
\int_0^T P(t)dt = E. \tag{8.1}
\]
We can use straightforward variational techniques to maximize the average transmission capacity

\[ \bar{C} = \frac{1}{T} \int_0^T W \log(1 + \alpha(t)P(t)/NW) \, dt. \]  

(8.2)

The result is

\[ P(t) = \max[0, \lambda - NW/\alpha(t)], \]  

(8.3)

where \( \lambda \) is chosen to satisfy constraint (8.1). The form of \( P(t) \) can be determined by a "water-filling" technique as shown in Figure 8.1.

![Diagram showing the water-filling solution for the transmitted energy constraint]

**Figure 8.1**

Water Filling Solution for the Transmitted Energy Constraint
(Compare Shannon's use of the same technique in the frequency domain [5]).

First a graph of \( NW/\alpha(t) \) from \( t=0 \) to \( t=T \) is constructed. Then we can imagine pouring water over this graph and filling it with a total of \( E \) units of water, reaching a level \( \lambda \).

If the available energy \( E_A \triangleq P_A T_A \) for A's transmission is sufficiently large, the water filling solution will give \( \lambda > \max NW/\alpha_A(t) \).

Thus we will have \( P_A(t) > 0 \) for all \( t \), and we can reduce Equation (8.3) to

\[ P_A(t) = \lambda - NW/\alpha_A(t). \quad (8.4) \]

Since \( \int_0^{T_A} NW/\alpha_A(t)dt = NWT_B \), we obtain \( E_A = \int P_A(t)dt = \lambda T_A - NWT_B \).

Solving for \( \lambda \) and substituting in (8.4), we get

\[ P_A(t) = P_A + NW(T_B/T_A - 1/\alpha_A(t)) . \quad (8.5) \]

We can see immediately that this gives the correct average power \( P_A \), since the quantity in parentheses integrates to zero. The condition that \( P_A(t) > 0 \) reduces to

\[ 1/\alpha_A(t) < T_B/T_A + P_A/NW, \quad \text{for all} \ t. \quad (8.6) \]

Equation (7.5) is valid if and only if condition (8.6) is satisfied. Otherwise, the form of \( \lambda \) is more complicated, and \( P_A(t) = 0 \) for some interval of time.

We conjecture that with terminal A fixed, the traveler B needs more energy per bit sent under the round trip energy constraint. The pointwise result

\[ 1/\alpha_A(t) \leq 1/\alpha_B(\tau'(t)) , \quad (8.7) \]

together with the water-filling or "effective noise" interpretation of Fig. 8.1, may be useful tools in proving the asymmetry.
9. Examples for the Twin Problem

In this chapter, we consider communication in the linear round trip or twin problem introduced in Chapter 2. Traveler B leaves A at velocity \( v = \beta c \), travels for time \( T_B/2 \) and returns, rejoining A after a total time \( T_B \). A's elapsed time \( T_A \) is greater than \( T_B \) by the aging factor \( \gamma = (1-\beta^2)^{-1/2} \) derived in Section 2. This factor becomes infinite as \( \beta \to 1 \).

In the twin problem, the traveler B needs more energy per bit sent (although not always by the factor \( \gamma^2 \)) under the transmitter constraints:
1) constant rate
2) constant power
3) total energy.

Explicit expressions for these asymmetries will be developed.

9.1 Time Dependence of the Doppler Shift Factors

To analyze the problem of communication between A and B, we must first compute the doppler shift factors for transmitted signals. As in Section 5, \( \alpha(t) \) denotes the doppler shift experienced by signals transmitted at time \( t \). The derivation of \( \alpha_- \) and \( \alpha_+ \), the receding and approaching doppler factors for the twin problem is outlined in Chapter 2. The results are

\[
\alpha_- = \gamma = (1-\beta)/(1+\beta)/(1+\beta) \]

and

\[
\alpha_+ = \gamma = (1+\beta)/(1-\beta) \]

(9.1)
(9.2)

Since A's transmission at time \((1-\beta)T_A/2\) reaches B at the turning point, A's transmission doppler factor is

\[
\alpha_A(t) = \begin{cases} 
\alpha_- & , \quad 0 < t < (1-\beta)T_A/2 \\
\alpha_+ & , \quad (1-\beta)T_A/2 < t < T_A 
\end{cases}
\]

(9.3)
Signals sent by A before time \((1-\beta)T_A/2\) are received by B on the out-bound leg and experience a doppler shift \(\alpha_-\). Signals sent by A after this time are received on the inbound leg and experience a doppler shift \(\alpha_+\) (see Figure 2.2).

Since A is at rest, signals sent by B experience a doppler shift that depends solely on B's velocity at the time of transmission. Thus B's transmission doppler shift factor is

\[
\alpha_B(\tau) = \begin{cases} 
\alpha_- , & 0 < \tau < T_B/2 \\
\alpha_+ , & T_B/2 < \tau < T_B.
\end{cases}
\]

(9.4)

9.2 Constant Rate Transmission

Suppose A and B both wish to transmit reliably to each other at the constant transmission rate \(R_0\). From Section 6, we know that an asymmetry in efficiency of \(\gamma^2\) will result. It is nevertheless instructive to derive this asymmetry directly for the numerical example of the twin problem in Section 2 (see Figure 2.1). In this example, B travels 3 light years at velocity \(v = .6c\) and returns to A. From Section 9.1, we have the transmission doppler factors

\[
\alpha_A(t) = \begin{cases} 
\alpha_- = 1/2 , & 0 < t < 2 \\
\alpha_+ = 2 , & 2 < t < 10 ,
\end{cases}
\]

(9.5)

and

\[
\alpha_B(t) = \begin{cases} 
\alpha_- = 1/2 , & 0 < \tau < 4 \\
\alpha_+ = 2 , & 4 < \tau < 8 .
\end{cases}
\]

(9.6)

The transmission doppler factor for A anticipates B's turning by the 3
years needed for the signals from A to reach the turning point. Once the
doppler factors are known, we can apply Equation (5.3),
\[ C(t) = W \log(1 + \alpha(t)P(t)/NW) . \]
It is easy to see that \( C(t) = R_0 \) implies that the required transmission
power is
\[ P(t) = P_0/\alpha(t) \quad (9.7) \]
where \( P_0 \) satisfies
\[ R_0 = W \log(1 + P_0/NW) . \quad (9.8) \]

From Equations (9.7) and (9.5), we see that A's required transmission
power is \( P_0/\alpha_- = 2P_0 \) for A's first 2 years and \( P_0/\alpha_+ = P_0/2 \) for A's
remaining 8 years. This power requirement is illustrated in Figure 9.1.

B must transmit with power \( P_0/\alpha_- = 2P_0 \) for the first 4 years and
power \( P_0/\alpha_+ = P_0/2 \) for the remaining 4 years. This requirement is shown
in Figure 9.2.

We now integrate the transmission rate and power. Terminal A trans-
mits a total of 10 \( R_0 \) bits (\( R_0 \) bits per year for 10 years) using energy
\[ E_A = 2(2P_0) + 8(P_0/2) = 8P_0 . \]
Similarly, B sends 8 \( R_0 \) bits using energy
\[ E_B = 4(2P_0) + 4(P_0/2) = 10P_0 . \]
The energy ratio is
\[ E_B/E_A = 10P_0/8P_0 = 5/4 = \gamma . \]
The ratio of energies per bit transmitted is

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\[
\frac{(E_B/N_B)}{(E_A/N_A)} = \frac{10P_0/8R_0}{8P_0/10R_0} = (5/4)^2 = \gamma^2.
\]

These results agree, as they should, with the general results given in Section 6.

Figure 9.1

A's Transmission Power Requirements for Transmission at Constant Rate \( R_0 \)
Figure 9.2

B's Transmission Power Requirements for Transmission at Constant Rate $R_0$
9.3 Constant Power Transmission

Let us return to the linear round trip problem with a general velocity $v = \beta c$. Suppose $A$ and $B$ transmit using the constant powers $P_A(t) = P_A$ and $P_B(t) = P_B$, respectively.

To simplify the analysis and the result, we also make the infinite bandwidth assumption. Under this condition, the transmission capacities become

$$C_A(t) = \lim_{\omega \to \infty} W \log(1 + \alpha_A(t)P_A/Nw) = \alpha_A(t)(P_A/N) \log e$$

and

$$C_B(\tau) = \alpha_B(\tau)(P_B/N) \log e.$$ 

If $A$ transmits at this time-varying rate for time $T_A$ while $B$ leaves and returns at velocity $v = \beta c$, the total number of bits transmitted by $A$ is

$$N_A = \int_0^{T_A} C_A(t) dt = \int_0^{T_A} \alpha_A(t) dt = \left[ \alpha_-(1-\beta)T_A/2 + \alpha_+(1+\beta)T_A/2 \right] (P_A/N) \log e$$

$$= \left[ (1-\beta)^2 \gamma/2 + (1+\beta)^2 \gamma/2 \right] (P_A T_A/N) \log e$$

$$= (1+\beta^2) \gamma (P_A T_A/N) \log e. \quad (9.9)$$

We can derive an analogous result for $B$:
\[ N_B = \left( \frac{P_B}{N} \right) \log e \int_0^{T_B} \alpha_B(\tau) d\tau \]
\[ = \left( \frac{P_B}{N} \right) \log e \left( \frac{T_A}{2\gamma} \right) (\alpha_- + \alpha_+) \]
\[ = \left( \frac{P_B}{N} \frac{T_A}{\gamma} \right) \log e . \quad (9.10) \]

The ratio of required energies is

\[ \frac{E_B}{E_A} = \frac{P_B}{P_A} \frac{T_B}{T_A} = \frac{1}{\gamma} \left( \frac{P_B}{P_A} \right) . \quad (9.11) \]

Equations (9.9)-(9.11) yield the ratio of energies per bit transmitted,

\[ \left( \frac{E_B}{N_B} \right) / \left( \frac{E_A}{N_A} \right) = (1 + \beta^2) = \gamma^2 (1 - \beta^4) . \quad (9.12) \]

Note that the asymmetry \((1 + \beta^2)\), unlike \(\gamma^2\), is bounded by 2 for all \(-1 \leq \beta \leq 1\). We can see that for communication in the linear round trip problems, constant power transmission does not yield results as simple as those for constant rate transmission. Nevertheless, the relative efficiency of the traveler is strictly less than 1 in both cases.

9.4 Transmission Under an Energy Constraint

Suppose we allow \(A\) and \(B\) to vary their transmission power under the energy constraints

\[ E_A = \int_0^{T_A} P_A(t) dt = P_0 T_A \]
\[ E_B = \int_0^{T_B} P_B(t) dt = P_0 T_B . \]
Section 8 gives a water-filling solution for the optimal form of $P(t)$. The water-filling solutions for $P_A(t)$ and $P_B(t)$ are shown in Figure 9.3.

![Diagram showing water-filling solutions for transmission power in the twin problem.]

**Figure 9.3**

*Water-filling Solutions for Transmission Power in the Twin Problem*

If $P_0/NW > \beta(1+\beta)Y/2$, both transmission problems have nontrivial solutions. Both A and B transmit with positive power for the entire trip duration. Using Equations (8.2) and (8.5) and the twin problem results

$$\log \alpha_A = \beta \log(1+\beta)Y,$$

and

$$\log \alpha_B = 0,$$

we find that A's average transmission capacity is

$$C_A = W \log(1/Y + P_0/NW) + W \beta \log(1+\beta)Y.$$  (9.13)

Similarly, B's average transmission capacity is
\[ \mathcal{C}_B = W \log(Y + P_0/NW). \]  

(9.14)

Although the ratio of average rates has no simple form, we can prove from these results that once again the traveler needs more energy per bit sent, i.e.,

\[ \frac{E_B/N_B}{E_A/N_A} = \frac{\mathcal{C}_A}{\mathcal{C}_B} > 1. \]

10. **Summary and Conclusions**

The main result of this paper is that if terminals A and B transmit at the same constant rate and bandwidth, the asymmetry in efficiency is equal to the square of the relative aging factor \( \gamma_0 \). The traveler who ages less needs more energy by the factor \( \gamma_0 \) and more energy per bit sent by the factor \( \gamma_0^2 \). This result is independent of acceleration and gravitational fields.

Instead of demanding constant transmission rates and then computing required energies for A and B, we can demand fixed transmission powers and compare achievable average rates. With one terminal fixed, the traveler needs more energy per bit sent.

The analysis here supports the conjecture that under any symmetric constraints on transmission, the traveler needs more energy per bit sent.
References


