TIME VARYING VOLATILITY IN THE TREASURY MARKET

by

KENNETH C. BARON
STANFORD UNIVERSITY

TECHNICAL REPORT NO. 70
MAY 1989

PREPARED UNDER THE AUSPICES
OF
NATIONAL SCIENCE FOUNDATION
GRANT DC185-20136

DEPARTMENT OF STATISTICS
STANFORD UNIVERSITY
STANFORD, CALIFORNIA
TIME VARYING VOLATILITY IN THE TREASURY MARKET

by
KENNETH C. BARON
STANFORD UNIVERSITY

TECHNICAL REPORT NO. 70
MAY 1989

PREPARED UNDER THE AUSPICES
OF
NATIONAL SCIENCE FOUNDATION
GRANT DC185-20136

DEPARTMENT OF STATISTICS
STANFORD UNIVERSITY
STANFORD, CALIFORNIA
Time Varying Volatility in the Treasury Market

Kenneth C. Baron *

Department of Statistics
Stanford University
Stanford, California 94305
May 1989

Abstract

This paper examines two often used models in fixed income analysis: the normal and the log-normal models of yield fluctuations. We find that neither model accurately describes yield movements in the treasury market. Both models are outperformed by a model where volatility is allowed to make occasional jumps. One implication of this time varying volatility model is that volatility is independent of level. We find evidence supporting this proposition.

*I benefitted from the generous help of Joe Chang, Tom Cover, Tom DiCiccio, and Joe Romano. In particular, I would like to thank Ken Garbrade for his extensive comments on an earlier version and his constant encouragement.
This paper studies the distribution of changes in yields on U.S. Treasury Bonds. Knowing the distribution of changes in yields is important for many different areas of financial analysis.

One application is in the pricing of contingent claim contracts. Most option pricing models rely on an assumption about the distribution of price changes of the underlying security. In interest rate models, practitioners typically assume that interest rate changes follow either a normal or a log-normal process. (See, for example, Dyer and Jacob (1989).) Knowing the true distribution of yield changes allows for the appropriate pricing of an option.

Another area in which the distribution of yield changes is important is the field of capital adequacy. Consider, for example, an institution holding a position in bonds. Since the future value of that position is unknown, the institution is exposed to interest rate risk. An important question for the institution, then, is "how much capital must be on hand so that it can be confident that it can cover potential losses?" (See, for example, Board of Governors of the Federal Reserve System (1987).) To answer this question, the institution must have an accurate model for the distribution of yield changes.

Many papers have examined the distribution of stock returns. (See Cox and Rubinstein (1985) for an extensive and nearly up-to-date bibliography.) It is well known that daily stock returns exhibit positive kurtosis. Researchers have studied how increasing the length of the differencing interval (for example from daily to weekly intervals) affects the kurtosis and the distribution of returns. This work typically involves the central limit theorem.
The central limit theorem in its simplest form (See Billingsley (1985), for example.) can be stated as: if $X_i$ are independent and identically distributed with mean $\mu$ and finite variance $\sigma^2$, and if $S_n = X_1 + \cdots + X_n$ then $S_n$ converges weakly to the normal distribution with mean $n\mu$ and variance $n\sigma^2$. Thus, if stock returns are independent with identical distributions (hereafter i.i.d.) and finite variances, then the distribution of stock returns converges to the normal as we increase the length of the return interval. However, the i.i.d. hypothesis and finite variance hypothesis are strong conditions: it seems unlikely that they would strictly hold for stock returns.

Fortunately, all of the hypotheses in the central limit theorem have been weakened. Weakening the i.i.d. assumption typically leads to convergence to the normal distribution. However, if the hypothesis of finite variances is eliminated, then the normal distribution is no longer the unique limiting distribution. Under these circumstances, the distribution will converge to a member of the stable paretian family. (See Fama (1963) for the relevant issues.) The normal distribution is a special case of the stable paretian family and is the only member of the stable family with a finite variance.

Mandelbrot (1963) was the first to propose that the distribution of stock returns might not be well described by a finite variance process and therefore not converge to normality. He suggested that one needed to look to the more general stable family for the true distribution of returns. This thesis aroused significant interest in the academic community: many papers and serious debate followed. This debate was considered at least partially resolved by Blattberg and Gonedes (1974). They showed that, albeit slowly, the distribution of stock returns converges to normality.
In particular, they found that weekly returns look more normally distributed than daily returns. Later, Fama (1976) concluded that monthly stock returns exhibit only slight kurtosis and are well approximated by the normal distribution.

While the work on stocks has been extensive, little work has been done with yield fluctuations in the U.S. Treasury market. Recently, Baron and Garbade (1989) examined the distribution of yield fluctuations for on-the-run treasury bonds from the time period of 1984 to mid-1988. They concluded that yield changes look approximately normally distributed after differencing intervals of two weeks. This paper seeks to extend the analysis of Baron and Garbade using a data set that goes back to 1973.

Section 1 introduces two different and often used models in fixed income analysis. The first model says that, over time, changes in yields on a constant maturity bond are normally distributed with a constant variance. We refer to this as the “normal random walk model” of yield fluctuations. The second model - the “log-normal model” of yield fluctuations - states that changes in the natural logarithm of yields are normally distributed with a constant variance.

Section 2 examines the empirical evidence on these two models. We find that neither model performs altogether satisfactorily. First, sample skewness and kurtosis statistics suggest that both models require long differencing intervals for yield fluctuations to be normally distributed: the normal model requires five months and the log-normal model requires two months for approximate normality. Second, neither model explains the relationship between the level of yields and the volatility of yields over the data set.
Because of these deficiencies of the standard models, section 3 introduces a third model. In this model, we relax the assumption of a constant variance in a simple way: we allow volatility to make discrete jumps. We establish the time of the jumps by visual inspection and observe two such jumps in our data set. This "time varying volatility model" significantly outperforms the previous two models. First, over homogeneous time intervals, the distribution of yield changes are normally distributed at one month. Second, this model accurately describes the relationship between yield level and yield volatility.

The data set for this analysis is yields on a constant maturity series on the 10 year treasury bond. This data is provided by the U.S. Treasury and consists of daily closing yields from the period 1-03-73 to 8-31-88. To avoid day of the week effects, we work with weekly yields. Typically, yields are taken at the end of trading on Wednesday. In the somewhat rare event that a Wednesday is a holiday, the yield is taken at the close of the next business day. In this way, weekly intervals never include more than five trading days.

1 Two Models of Yield Fluctuations

We start with a series of weekly closing yields on a treasury bond. Let $R_0, R_1, \ldots$, $R_n$ denote our observations. We speak of the change in yield during week $t$ as the difference $R_t - R_{t-1}$, and we write the yield change during week $t$ as $e_t$. Thus,

$$e_t = R_t - R_{t-1}, \quad t = 1, 2, \ldots, n$$

(1)

A yield fluctuation model is simply a model for describing the $e_t$. 
Practitioners often assume that yield fluctuations follow a "normal random walk model." This model can be formulated as follows – (N1) $e_t$ is normally distributed; (N2) $e_s$ and $e_t$ are uncorrelated for $s \neq t$; (N3) $e_t$ has mean 0 and finite variance $\sigma^2$. This model says that yields follow a random walk through time with uncorrelated additive increments of finite variance.

A second model that is often used is the "log-normal model" of yield fluctuations. We define the quantity $f_t$ as follows -

$$ f_t = \ln R_t - \ln R_{t-1}, \quad t = 1, 2, \ldots, n $$(2)

The log-normal model has the following three assumptions— (LN1) $f_t$ is normally distributed; (LN2) $f_s$ and $f_t$ are uncorrelated for $s \neq t$; (LN3) $f_t$ has mean 0 and finite variance $\omega^2$. Thus, the log-normal model has the same assumptions as the normal model, except that we substitute $f_t$ for $e_t$ in (N1) - (N3). The log-normal model states that changes in the natural logarithm of yields are uncorrelated and normally distributed.

It is useful to reformulate the log-normal model in terms of the $e_t$ of equation (1). By (1),

$$ \ln R_t = \ln(R_{t-1} + e_t) $$ (3)

Using as Taylor series expansion about $R_{t-1}$, we get

$$ \ln R_t = \ln R_{t-1} + e_t/R_{t-1} - (1/2)(e_t/R_{t-1})^2 + \cdots $$ (4)

Since the ratio of a weekly yield change to the yield level is small, all terms higher than the first order term are small. So

$$ \ln R_t \approx \ln R_{t-1} + e_t/R_{t-1} $$ (5)
Rearranging, we have

$$\frac{e_t}{R_{t-1}} \approx \ln(R_t) - \ln(R_{t-1})$$  \hspace{1cm} (6)

Thus, under the assumption (LN1) and by equation (2), we have that

$$e_t \sim N(0, (wR_{t-1})^2) \hspace{0.5cm} t = 1, 2, \ldots, n$$  \hspace{1cm} (7)

We can now reformulate the log-normal model as – (LN1') $e_t$ is approximately normally distributed; (LN2') $e_s$ and $e_t$ are uncorrelated for $s \neq t$; (LN3') $e_t$ has mean 0 and finite variance $(wR_{t-1})^2$. (LN3') says that volatility (as measured by the standard deviation) is proportional to the level of yields. This implies, for example, if the yield level is high then volatility of yields will be high. Conversely, if the yield level is near 0, then volatility will be low.

We offer two criticisms of these models. First, we criticize these models on their first assumption – the assumption of normality. Neither model allows for the possibility of significant skewness or kurtosis, even though stock returns exhibit these characteristics. Second, both models are in a sense rigid. The normal model implies that yield volatility is independent of level and constant over time. The log-normal model implies that volatility is proportional to the level of rates. In neither model can volatility vary with respect to exogenous factors, such as monetary policy or time. Next in section 2, we examine the empirical significance of these criticisms.
2 An Empirical Examination of the Normal and Log-Normal Models

We now test the models over our 16 year data set on the 10 year treasury bond. We use three different approaches for testing. First, the likelihood ratio evaluates the relative performance of the normal versus the log-normal. Second, sample skewness and kurtosis statistics test the assumption of normality in both models. And last, a scatter plot determines if the level/volatility relationship is consistent with either model.

2.1 A Direct Comparison Between the Models

To introduce the likelihood ratio procedure, we first need some notation. Let $e_1, e_2, \ldots, e_n$ be observations from a sequence of $n$ i.i.d. random variables each with marginal density function $f_1$. In the case of i.i.d. random variables, the joint density $f_n$ is just the product of the marginal densities. The likelihood function of the $n$ observations is the joint density function evaluated at the $n$ observations. Thus, we can write the likelihood function as follows –

$$f_n(e_1, \ldots, e_n) = \prod_{i=1}^{n} f_1(e_i)$$

Next, consider comparing two different parametric models to describe the observations on the $e_i$. We seek to evaluate the relative performance of the model specified by $f_1$ versus the model specified by the density function $g_1$. (For example, $f_1$ might be the normal family and $g_1$ might be the log-normal family.) We define the likeli-
hood ratio \( L_n \) as follows:

\[
L_n(e_1, \ldots, e_n) = \frac{\max g_n(e_1, \ldots, e_n)}{\max f_n(e_1, \ldots, e_n)}
\]  

(9)

In the numerator, the maximum is taken over all possible parameter values from the family of \( g_1 \). In the denominator, the maximum is taken over all possible parameter values from the family of \( f_1 \). The likelihood ratio is then just the ratio of these two quantities. Simply put, the likelihood ratio \( L_n \) is the ratio of the likelihood functions \( f_n \) and \( g_n \) evaluated at their maximum likelihood estimators.

If \( f_1 \) and \( g_1 \) satisfy certain regularity conditions and \( n \) is large enough, then \( L_n \) measures the relative performance of the families \( f_1 \) and \( g_1 \). (See, for example, Zellner (1971).) If we take a priori that \( f_1 \) and \( g_1 \) are equally probable as models of the \( e_i \), then the ratio \( L_n \) expresses the asymptotic posterior odds of \( g_1 \) to \( f_1 \). For example, a value of 5 for \( L_n \) suggests odds of 5 to 1 in favor of \( g_1 \). A value of 19 for \( L_n \) is evidence in favor of \( g_1 \) at exactly the 5% significance level.

In our framework, the \( e_i \) are the weekly changes in yields on the 10 year bond. We take \( f_1 \) as the normal density and \( g_1 \) as the normal density affiliated with (LN1'), (LN3'). Then, \( L_n \) compares the normal and log-normal models over the data set. Table 1 lists \( \ln(L_n) \) for differencing intervals between one and ten weeks. The values are generally decreasing as we increase the differencing interval. However, the log-likelihood ratio is always positive and large. This is strong evidence in favor of the log-normal model over the normal model.
Table 1: $\ln(L_n)$ - the natural log of the likelihood ratio, where $g_n$ is the density for the log-normal model and $f_n$ is the density for the normal model.

<table>
<thead>
<tr>
<th>Differencing Interval</th>
<th>$n$</th>
<th>$\ln(L_n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>817</td>
<td>127.9</td>
</tr>
<tr>
<td>2</td>
<td>408</td>
<td>72.5</td>
</tr>
<tr>
<td>3</td>
<td>272</td>
<td>42.5</td>
</tr>
<tr>
<td>4</td>
<td>204</td>
<td>30.6</td>
</tr>
<tr>
<td>5</td>
<td>163</td>
<td>24.2</td>
</tr>
<tr>
<td>6</td>
<td>136</td>
<td>23.9</td>
</tr>
<tr>
<td>7</td>
<td>116</td>
<td>15.9</td>
</tr>
<tr>
<td>8</td>
<td>102</td>
<td>20.7</td>
</tr>
<tr>
<td>9</td>
<td>90</td>
<td>13.7</td>
</tr>
<tr>
<td>10</td>
<td>81</td>
<td>9.7</td>
</tr>
</tbody>
</table>
2.2 Normality in the Normal and Log-Normal Models

Next, we investigate the assumption of normality in both models. Many techniques are known to test for normality. (See Shapiro (1986) for an overview.) We choose sample skewness and kurtosis statistics to test for possible departures from normality. These statistics generally provide efficient detection of non-normality as well as an indication of the nature of the departure (if any).

Let \( x_1, x_2, \ldots, x_n \) represent \( n \) independent drawings from a random variable \( X \) with mean 0. Then define the sample skewness statistic \( S_n \) as

\[
S_n = \frac{\frac{1}{n} \sum_{i=1}^{n} x_i^3}{\left(\frac{1}{n} \sum_{i=1}^{n} x_i^2\right)^{3/2}}
\]

(10)

If \( X \) is symmetric about its mean, then the sample skewness statistic should be close to 0. A large positive value of \( S_n \) indicates that \( X \) has a right tail longer than its left tail. A large negative value of \( S_n \) indicates the reverse. Generally, the skewness statistic measures the degree of asymmetry of the distribution of \( X \).

For the same \( x_i \), define the sample kurtosis statistic \( K_n \) as

\[
K_n = \frac{\frac{1}{n} \sum_{i=1}^{n} x_i^4}{\left(\frac{1}{n} \sum_{i=1}^{n} x_i^2\right)^2} - 3
\]

(11)

The kurtosis statistic will be close to 0 if the observations are taken on a normal random variable. If the distribution is more peaked around its mean than the normal or has longer tails than the normal, then the sample kurtosis statistic will tend to be positive. Kurtosis measures departures from normality in the center and in the tails of the distribution.

A note on the kurtosis statistic. The measurement of extreme values has been called excess, leptokurtosis, kurtosis, thick tails, heavy tails, and long tails. In general.
all of these terms tend to describe the same thing - that the number of extreme values in the distribution is significantly larger than that predicted by the normal. (However, see Rosenberger and Gasko (1983) for a distinction between heavy tails and long tails.) To describe this characteristic, throughout this paper we use the term “kurtosis.”

If the random variable $X$ is normally distributed, then as we increase the number of samples $n$, the sample skewness and kurtosis statistics converge to 0. Skewness will decrease at the rate $\sqrt{1/n}$ and kurtosis will decrease at the rate $1/n$. (See Snedecor and Cochran (1980) for discussion and tables.) Thus, if skewness and kurtosis are of equal magnitude on weekly differencing intervals, we expect that kurtosis will disappear at a faster rate than skewness as we increase the length of the differencing interval.

Table 2 presents sample skewness and kurtosis statistics for the normal and log-normal models over intervals between one and 25 weeks. For the normal model, significant skewness seems to persist out to 15 weeks and significant kurtosis persists out to 20 weeks. In the log-normal model, the log transformation helps to eliminate skewness. By differencing intervals of three weeks, the skewness is no longer significant. For the log-normal model, it takes about eight weeks for the positive kurtosis to be wiped out. We see that for both models, convergence to normality is slow: it takes nearly five months in the normal model and two months in the log-normal model for normality to be a working approximation.
Table 2: Skewness and kurtosis statistics for the normal model and the log-normal model.

<table>
<thead>
<tr>
<th>Differencing Interval</th>
<th>Normal Model</th>
<th></th>
<th>Log-Normal Model</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>n</td>
<td></td>
<td>Skewness</td>
<td>Kurtosis</td>
<td>Skewness</td>
</tr>
<tr>
<td>1</td>
<td>817</td>
<td>-0.721 **</td>
<td>5.873 **</td>
<td>-0.237 **</td>
</tr>
<tr>
<td>2</td>
<td>408</td>
<td>-1.272 **</td>
<td>7.930 **</td>
<td>-0.484 **</td>
</tr>
<tr>
<td>3</td>
<td>272</td>
<td>-0.783 **</td>
<td>4.780 **</td>
<td>-0.207</td>
</tr>
<tr>
<td>4</td>
<td>204</td>
<td>-0.495 **</td>
<td>4.324 **</td>
<td>0.098</td>
</tr>
<tr>
<td>5</td>
<td>163</td>
<td>-0.500 **</td>
<td>2.218 **</td>
<td>0.041</td>
</tr>
<tr>
<td>6</td>
<td>136</td>
<td>-0.972 **</td>
<td>5.171 **</td>
<td>-0.105</td>
</tr>
<tr>
<td>7</td>
<td>116</td>
<td>-0.251</td>
<td>2.034 **</td>
<td>0.363 *</td>
</tr>
<tr>
<td>8</td>
<td>102</td>
<td>-0.793 **</td>
<td>2.367 **</td>
<td>-0.237</td>
</tr>
<tr>
<td>9</td>
<td>90</td>
<td>-0.827 **</td>
<td>1.921 **</td>
<td>-0.399</td>
</tr>
<tr>
<td>10</td>
<td>81</td>
<td>-0.777 **</td>
<td>2.947 **</td>
<td>0.045</td>
</tr>
<tr>
<td>15</td>
<td>54</td>
<td>-0.962 **</td>
<td>2.980 **</td>
<td>0.035</td>
</tr>
<tr>
<td>20</td>
<td>40</td>
<td>-0.496</td>
<td>1.209 *</td>
<td>0.169</td>
</tr>
<tr>
<td>25</td>
<td>32</td>
<td>0.235</td>
<td>0.692</td>
<td>0.957</td>
</tr>
</tbody>
</table>

* Significant at $\alpha=.10$

** Significant at $\alpha=.02$
2.3 The Relationship Between Volatility and Level

To examine the association between volatility and level, we divide up the data set into disjoint three month intervals. For example, the first interval is from 1-03-73 to 3-28-73. There are \( I = 63 \) such intervals in our data set. Let \( y_{i0}, y_{i1}, \ldots, y_{i,n_i-1} \) denote weekly yields within interval \( i \). Then define \( \overline{y_i} \) as

\[
\overline{y_i} = \frac{1}{n_i} \sum_{j=0}^{n_i-1} y_{ij}
\]

(12)

Here, \( \overline{y_i} \) measures the level of yields in interval \( i \). Define \( d_{ij} \) as

\[
d_{ij} = y_{ij} - y_{i,j-1} \quad j = 1, 2, \ldots, n_i - 1
\]

(13)

Similarly define \( \overline{d_i} \) as the mean of the \( d_{ij} \)'s over \( j \). Then define

\[
s_i = \sqrt{\frac{1}{n_i - 1} \sum_{j=1}^{n_i-1} (d_{ij} - \overline{d_i})^2}
\]

(14)

Here, \( s_i \) is a measure of the volatility of yields in interval \( i \).

In figure 1, a plot of \( s_i \) vs. \( \overline{y_i} \) is given. The normal model implies that volatility and level should be independent. Figure 1, however, suggests a strong positive relationship between level and volatility, thus refuting the normal model. The log-normal model implies that volatility against level should plot as a straight line through the origin. Note in figure 1, volatility versus level doesn't plot as a line through the origin. Figure 1 provides evidence against both of our models.

In summary, we find that the log-normal model significantly outperforms the normal model. The likelihood ratio strongly favors the log-normal model. Skewness and kurtosis statistics are of smaller magnitude and vanish faster in the log-normal model.
Figure 1: Volatility vs. level on the 10 year bond.
And further, figure 1 suggests a positive relationship between level and volatility as implied by the log-normal model. However, the log-normal model has significant shortcomings. The model needs a differencing interval of nearly two months before the distribution of yield fluctuations is approximately normal. Also, the log-normal model is unable to explain the precise relationship between level and volatility implied by figure 1.

3 A Time Varying Volatility Model of Yield Fluctuations

Because of the deficiencies of the normal and log-normal models, we introduce a third model in this section. First we motivate the model's assumptions by plotting volatility over time. Then, we examine the empirical performance of the model.

3.1 A Model for the Dynamics of Volatility

To investigate how volatility moves over time, we plot the $s_t$ against time in figure 2. In the period from 1-73 through 9-79, we observe levels of volatility around 10 basis points. From 10-79 through 12-82, average volatility is high and around 38 basis points. Finally, between 1-83 and 8-88, volatility is centered around 20 basis points. Roughly speaking, figure 2 suggests three significantly different volatility regimes—a period of low volatility, a period of high volatility, and most recently a period of moderate volatility.
Figure 2: Volatility over time on the 10 year bond.
We further examine the hypothesis of three volatility regimes in Table 3, which shows the sample standard deviations within each of the three time periods. Approximate standard errors for these estimates can be calculated using (See Miller (1986).)

$$SE(s_i) \approx \left( \frac{s_i}{2} \right) \sqrt{\frac{2 + K_{n_i}}{n_i}}$$  \hspace{1cm} (15)

Here, \(n_i\) is the number of data points within interval \(i\) and \(K_{n_i}\) is the sample kurtosis within interval \(i\). Table 3 shows that the sample standard deviation from period 2 is significantly different from the sample standard deviations of periods 1 and 3. This indicates that volatility is not constant and jumps in at least two places.

With this observation, we now relax the assumption of a constant variance and allow volatility to make discrete jumps. For this “time varying volatility model” of yield fluctuations, we assume that (TV1) \(e_t\) is normally distributed; (TV2) \(e_s\) and \(e_t\) are uncorrelated for \(s \neq t\); (TV3) \(e_t\) has mean 0 and finite variance \(v_t^2\), where \(i=1, 2, 3\) depending on whether \(t\) is in time period 1, 2, or 3. The third assumption incorporates the idea that volatility may be a function of time - specifically, we allow volatility to jump between intervals. We refer to the three volatility periods within our data set as regimes and describe intervals within a particular regime as homogeneous.

### 3.2 An Empirical Examination of the Time Varying Model

Before we appraise this new model, note that the normal model of yield fluctuations is just a special case of the time varying volatility model where \(v_1 = v_2 = v_3\). Thus, we expect the time varying volatility model to perform at least as well as the normal model.
Table 3: Estimates of s and SE(s) within each time period and over the entire interval.

<table>
<thead>
<tr>
<th>Period</th>
<th>Dates</th>
<th>n</th>
<th>Kurtosis</th>
<th>s</th>
<th>SE(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1/73-9/79</td>
<td>352</td>
<td>1.536</td>
<td>0.095</td>
<td>0.005</td>
</tr>
<tr>
<td>2</td>
<td>10/79-12/82</td>
<td>170</td>
<td>1.249</td>
<td>0.376</td>
<td>0.026</td>
</tr>
<tr>
<td>3</td>
<td>1/83-8/88</td>
<td>295</td>
<td>1.231</td>
<td>0.199</td>
<td>0.011</td>
</tr>
<tr>
<td>entire</td>
<td>1/73-8/88</td>
<td>817</td>
<td>5.873</td>
<td>0.218</td>
<td>0.011</td>
</tr>
</tbody>
</table>

Table 4: Levene's s test for equality of variances within each time period and over the entire interval.

<table>
<thead>
<tr>
<th>Period</th>
<th>Dates</th>
<th>I-1</th>
<th>N-I</th>
<th>F</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1/73-9/79</td>
<td>26</td>
<td>325</td>
<td>1.843*</td>
</tr>
<tr>
<td>2</td>
<td>10/79-12/82</td>
<td>12</td>
<td>157</td>
<td>1.105</td>
</tr>
<tr>
<td>3</td>
<td>1/83-8/88</td>
<td>22</td>
<td>272</td>
<td>1.712</td>
</tr>
<tr>
<td>entire</td>
<td>1/73-8/88</td>
<td>62</td>
<td>754</td>
<td>3.551**</td>
</tr>
</tbody>
</table>

* Significant at $\alpha=.05$
** Significant at $\alpha=.01$
The time varying volatility model makes two precise statements about volatility over the data set. First it says that volatility varies significantly across the 16 year interval. Second, the model says that volatility is constant within each of the three homogeneous time intervals. We now formally test these two hypothesis. When the data set is not normally distributed, the Levene's test gives accurate p-values for testing whether variances across different populations are equal. (See Miller (1986) for an excellent discussion of the issues.) Next we introduce the notation and the test.

Consider, for example, testing over the entire data set whether the variances over each three month interval are equal. Here, we test over \( I = 63 \) intervals. Let \( N \) be the total number of sample points; \( N \) is just the sum of the \( n_i \) and is equal to 754. Define \( z_{ij} \) as

\[
z_{ij} = (y_{ij} - \overline{y}_i)^2
\]  

(16)

Let \( \overline{z}_i \) be the mean of the \( z_{ij} \)'s over \( j \). And let \( \overline{z}_i \) be the grand mean of \( z_{ij} \)'s. The Levene's test treats the \( z_{ij} \)'s as if they are i.i.d. normal and performs the usual ANOVA \( F \) test for equality of population means. Specifically, to test whether the variance are equal at significance level \( \alpha \), compare an \( F_\alpha (I - 1, N - I) \) with

\[
F = \frac{I^{-1} \sum_{i=1}^{I} n_i (\overline{z}_i - \overline{z}_i)^2}{\sum_{i=1}^{I} \sum_{j=1}^{n_i} (z_{ij} - \overline{z}_i)^2}
\]  

(17)

The \( z_{ij} \) certainly aren't i.i.d. normal, but it has been shown that the Levene's test provides surprisingly robust results.

Table 4 shows the results of the Levene's test. Looking at the bottom line, we can reject the hypothesis of equal variances over the entire interval. This provides support for a time varying volatility model. Examining the first three lines of the table, we find
some evidence that variances are not constant within each of the three time periods. Indeed, in the first period the equal variance hypothesis can be rejected at $\alpha = .05$ though not at $\alpha = .01$. In periods 2 and 3 we cannot reject the hypothesis of equal variances. Though a more complicated model may better describe the process, we do not find strong evidence of this and proceed with further tests of this time varying volatility model.

Table 5 presents the log-likelihood ratio of the time varying volatility model with the normal and log-normal models. All entries are positive and quite large. This is evidence in favor of the time varying volatility model over both the normal and log-normal models.

Table 6 compares the skewness and kurtosis statistics of the time varying volatility model over the three intervals. Within each regime we see that after four weeks skewness and kurtosis are no longer significant. Thus, within each regime, the distribution of yield fluctuations is approximately normal after one month. This compares favorably to the normal model, which requires five months for normality, and to the log-normal model, which requires two months for normality.

Recall the results of Baron and Garbade (1989), who concluded that the normal model performs well between 1984 and 1988. This time interval is entirely contained in our third volatility regime. Table 6 confirms Baron and Garbade's results - that the normal model performs well within any particular homogeneous time period. However, as section 1 showed, the normal model performs poorly over non-homogeneous time intervals. In this case, a more general model is required.

Next, we reexamine the relationship between level and volatility. Note that the
Table 5: ln(L_n) of the time varying volatility model versus the log-normal model and the normal model.

<table>
<thead>
<tr>
<th>Differencing Interval</th>
<th>n</th>
<th>ln(L_n) of TVV vs. LN</th>
<th>ln(L_n) of TVV vs. N</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>817</td>
<td>85.5</td>
<td>213.4</td>
</tr>
<tr>
<td>2</td>
<td>408</td>
<td>56.1</td>
<td>128.6</td>
</tr>
<tr>
<td>3</td>
<td>272</td>
<td>27.1</td>
<td>69.6</td>
</tr>
<tr>
<td>4</td>
<td>204</td>
<td>30.0</td>
<td>60.6</td>
</tr>
<tr>
<td>5</td>
<td>163</td>
<td>34.2</td>
<td>58.4</td>
</tr>
<tr>
<td>6</td>
<td>136</td>
<td>16.6</td>
<td>40.5</td>
</tr>
<tr>
<td>7</td>
<td>116</td>
<td>27.6</td>
<td>43.5</td>
</tr>
<tr>
<td>8</td>
<td>102</td>
<td>10.3</td>
<td>31.0</td>
</tr>
<tr>
<td>9</td>
<td>90</td>
<td>11.4</td>
<td>25.1</td>
</tr>
<tr>
<td>10</td>
<td>81</td>
<td>12.3</td>
<td>22.0</td>
</tr>
</tbody>
</table>
Table 6: Skewness and kurtosis statistics over each time period.

<table>
<thead>
<tr>
<th>Differencing Interval</th>
<th>Period 1</th>
<th></th>
<th></th>
<th>Period 2</th>
<th></th>
<th></th>
<th>Period 3</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>n</td>
<td>Skewness</td>
<td>Kurtosis</td>
<td>n</td>
<td>Skewness</td>
<td>Kurtosis</td>
<td>n</td>
<td>Skewness</td>
<td>Kurtosis</td>
</tr>
<tr>
<td>1</td>
<td>352</td>
<td>0.258</td>
<td>* 1.536</td>
<td>**</td>
<td>170</td>
<td>-0.620</td>
<td>** 1.249</td>
<td>**</td>
<td>295</td>
</tr>
<tr>
<td>2</td>
<td>176</td>
<td>0.476</td>
<td>** 1.287</td>
<td>**</td>
<td>85</td>
<td>-1.030</td>
<td>** 1.949</td>
<td>**</td>
<td>147</td>
</tr>
<tr>
<td>3</td>
<td>117</td>
<td>0.460</td>
<td>* 0.830</td>
<td>**</td>
<td>56</td>
<td>-0.543</td>
<td>* 0.931</td>
<td></td>
<td>98</td>
</tr>
<tr>
<td>4</td>
<td>88</td>
<td>0.337</td>
<td>-0.254</td>
<td></td>
<td>42</td>
<td>0.305</td>
<td>0.561</td>
<td></td>
<td>73</td>
</tr>
<tr>
<td>5</td>
<td>70</td>
<td>0.497</td>
<td>* -0.110</td>
<td></td>
<td>34</td>
<td>-0.230</td>
<td>0.561</td>
<td></td>
<td>59</td>
</tr>
<tr>
<td>6</td>
<td>58</td>
<td>0.436</td>
<td>* -0.317</td>
<td></td>
<td>28</td>
<td>-0.246</td>
<td>-1.020</td>
<td></td>
<td>49</td>
</tr>
<tr>
<td>7</td>
<td>50</td>
<td>0.491</td>
<td>-0.614</td>
<td></td>
<td>24</td>
<td>-0.030</td>
<td>0.161</td>
<td></td>
<td>42</td>
</tr>
<tr>
<td>8</td>
<td>44</td>
<td>0.546</td>
<td>-0.545</td>
<td></td>
<td>21</td>
<td>-0.571</td>
<td>-0.615</td>
<td></td>
<td>36</td>
</tr>
<tr>
<td>9</td>
<td>39</td>
<td>0.531</td>
<td>-0.989</td>
<td></td>
<td>18</td>
<td>-0.345</td>
<td>-0.689</td>
<td></td>
<td>32</td>
</tr>
<tr>
<td>10</td>
<td>35</td>
<td>0.576</td>
<td>-0.128</td>
<td></td>
<td>17</td>
<td>-0.019</td>
<td>-1.164</td>
<td></td>
<td>29</td>
</tr>
</tbody>
</table>

* Significant at α = .10
** Significant at α = .02
time varying volatility model predicts that within homogeneous intervals, volatility should be constant and independent of level. Figure 3 plots the $s_i$ vs. the $\overline{y_i}$ as does figure 1. Again, note the strong positive relationship between volatility and level over the whole period. In addition, figure 3 plots each data point with a 1, 2, or 3 corresponding to the time period the point comes from. The relationship between volatility disappears within each time period. In any of the time intervals, there is no systematic relationship between volatility and level.

We can empirically test these observations with the Spearman correlation coefficient, a non-parametric measure of the association of two random variables $X$ and $Y$. (See, for example, Snedecor and Cochran (1980) for discussion and tables.) Let $x_1, \ldots, x_n$ be observations on $X$ and let $y_1, \ldots, y_n$ be observations on $Y$. Let $r(x_i)$ denote the rank of $x_i$ among the x’s and define $r(y_i)$ similarly. Define the Spearman correlation coefficient $R$ as

$$R = \frac{\sum_{i=1}^{n} (r(x_i) - \overline{r(x)})(r(y_i) - \overline{r(y)})}{\sqrt{\sum_{i=1}^{n} (r(x_i) - \overline{r(x)})^2 \sum_{i=1}^{n} (r(y_i) - \overline{r(y)})^2}}$$

(18)

Here, $\overline{r(x)} = \sum_{i=1}^{n} r(x_i)/n$ and $\overline{r(y)} = \sum_{i=1}^{n} r(y_i)/n$. One can show that $R$ varies between -1 and 1. If $R$ is close to 1, then there exists a strong positive relationship between $X$ and $Y$. If $R$ is large and negative, then there is a strong negative relationship between $X$ and $Y$. Under the hypothesis that $X$ and $Y$ are independent, $R$ has expected value 0 and variance $1/(n - 1)$.

Table 7 shows the Spearman coefficient for the pair $(\overline{d_i}, s_i)$ for the three periods and the entire data set. Over the entire interval, $R$ equals .944 and is significant at $\alpha = .01$, suggesting a strong positive relationship between level and volatility. Within
Figure 3: Volatility vs. level on the 10 year bond by period.
Table 7: Spearman's correlation coefficient $R$ within each time period and over the entire interval.

<table>
<thead>
<tr>
<th>Period</th>
<th>Dates</th>
<th>n</th>
<th>$R$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1/73-9/79</td>
<td>27</td>
<td>0.284</td>
</tr>
<tr>
<td>2</td>
<td>10/79-12/82</td>
<td>13</td>
<td>-0.049</td>
</tr>
<tr>
<td>3</td>
<td>1/83-8/88</td>
<td>23</td>
<td>0.236</td>
</tr>
<tr>
<td>entire</td>
<td>1/73-8/88</td>
<td>63</td>
<td>0.944**</td>
</tr>
</tbody>
</table>

* Significant at $\alpha=.05$
** Significant at $\alpha=.01$
each interval, R is small and we can’t reject the hypothesis of independence. Thus table 7 supports our earlier observations.

4 Summary and Implications

In this paper, we studied the performance of three models of yield fluctuations: the normal model, the log-normal model, and the time varying volatility model. We found that the time varying volatility model provides a superior description of our data set. This model implies that volatility follows a jump process. Jumps seem to occur rarely - we observe two in our 16 year data set. Another implication of the model is that within a homogeneous time interval, the volatility of yields should be independent of level. This implication was supported by the data.

If a data set does not span across more than one regime, one could incorrectly conclude that the normal model of yield fluctuations is appropriate. Indeed, Baron and Garbade (1989) concluded that the normal model provides a good fit over 1984 to 1988. As this illustrates, empirical results can be extremely dependent on the time interval from which the data was taken as well as the length of the data set. It is important to be aware of the caveat associated with not having a “long enough” data set. (The question of “how long is long enough?” is certainly a difficult question to address.) This fact, however, has been often overlooked in empirical studies.

For example, consider the the stable-paretian vs. normality debate for stock returns. Mandelbrot (1963, 1967) found evidence in favor of the stable paretian hypothesis through studying data sets as long as 80 years. His papers espouse the
view of the stable paretian hypothesis as a long term model. In testing the stable paretian hypothesis. Blattberg and Gonedes (1974) look at stock returns typically over periods of less than five years. It is not immediately clear to us whether this duration is sufficient to provide refutation of Mandelbrot's long term model for stock returns.

5 References


6. Dyer, L. and D. Jacob, 1989, "Guide to Fixed Income Option Pricing Mod-


