ON THE ASYMPTOTIC EXPANSIONS FOR THE MOMENTS
AND THE LIMITING DISTRIBUTIONS OF SOME
ADDITIVE ARITHMETIC FUNCTIONS

BY

ALI REJALI

TECHNICAL REPORT NO. 116
AUGUST 1978

PREPARED UNDER THE AUSPICES
OF
NATIONAL SCIENCE FOUNDATION GRANT MCS77-16974

DEPARTMENT OF STATISTICS
STANFORD UNIVERSITY
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In The Name Of
God

To
My Parents,
Brother and Sisters
ACKNOWLEDGMENTS

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CHAPTER I

I.1 Introduction and Summary.

In this thesis we consider the distribution of real-valued additive arithmetic functions. Note that a real-valued function \( f \) on the set of positive integers is additive if \( f(mn) = f(m) + f(n) \), whenever \( m \) and \( n \) are relatively prime. To estimate the distribution of \( f \), one can find approximations for its mean-value

\[
\lim_{x \to \infty} \frac{1}{x} \sum_{n < x} f(n)
\]

Then the asymptotic variances

\[
\left[ \lim_{x \to \infty} \frac{1}{x} \sum_{n < x} \left( f(n) - \frac{1}{x} \sum_{m < x} f(m) \right)^2 \right]
\]

is an indicator of how \( f \) is distributed around its mean-value. This will be most useful if the standard deviation (square root of the variance) of \( f \) is small compared to the mean-value.

The first results on this subject are due to Hardy and Ramanujan (1917). For example, they considered \( \omega(n) \), the number of distinct prime divisors of \( n \) (so \( \omega(12) = 2 \)) and proved that

\[
\frac{1}{x} \sum_{n < x} \omega(n) = \log \log x + B_1 + O\left( \frac{1}{\log x} \right), \text{ as } x \to \infty,
\]

where \( B_1 = \gamma + \sum_{p} \log(1 - \frac{1}{p}) + \frac{1}{p} \) and \( \gamma \) is Euler's constant.

(Throughout this thesis sums like \( \sum_{p} \) are over all primes \( p \).)
If we want to use an approximation such as this in an application, numerical accuracy becomes a problem. Define

\[ r(x) = \text{residual (x)} = (\text{true value (x)} - \text{fitted value (x)}). \]

Diaconis (1976) compared the residuals obtained by neglecting terms of order smaller than \( \frac{1}{\log x} \), that is

\[ \frac{1}{x} \sum_{n<x} \omega(n) = \log \log x + B_1 + \frac{\gamma-1}{\log x} + O\left(\frac{1}{\log^2 x}\right) \]

and the approximation due to Hardy and Ramanujan for two values, \( x = 10^2 \) and \( x = 10^6 \), as follows:

<table>
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<th>( r_x ) (first term only)</th>
<th>( r_x ) (with second term)</th>
</tr>
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<td>( 10^2 )</td>
<td>-.0787</td>
<td>.013</td>
</tr>
<tr>
<td>( 10^6 )</td>
<td>-.0336</td>
<td>-.003</td>
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In this example the residuals might be regarded as being sufficiently small, but in many other examples the convergence is very slow and one wants to get better approximations. For example, the residuals for \( \Omega \), the number of prime divisors counted with multiplicity are

<table>
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<tr>
<td>( 10^2 )</td>
<td>-.1718</td>
<td>-.0801</td>
</tr>
<tr>
<td>( 10^6 )</td>
<td>-.0338</td>
<td>-.0032</td>
</tr>
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Saffari (1966), Delange (1971) and Diaconis (1976) have shown that there are constants $a_k$ such that
\[
\frac{1}{x} \sum_{n<x} \omega(n) = \log \log x + \sum_{k=0}^{n} \frac{a_k}{(\log x)^k} + O\left(\frac{1}{(\log x)^{n+1}}\right) \quad \text{as} \quad x \to \infty,
\]
for arbitrary $n$ (see Theorem 1). In Section 1 of Chapter II we give a closed form expression for the $a_k$'s so that this expansion becomes usable.

Hardy and Ramanujan also found the expansion for the mean-value of $\Omega$ with an error of order $\frac{1}{\log x}$ . Theorem 3 of Chapter II states the expansion for $\frac{1}{x} \sum_{n<x} \Omega(n)$ with computed constants.

As an application, Knuth and Trabb Pardo (1976) used the distribution of $\Omega(n)$ for analyzing the running time of the "divide and factor" algorithm which is a standard method for discovering the prime factorization of an integer $n$.

The expansion of the variances of $\omega$ and $\Omega$ are the subject of Section 2 in Chapter II. There we carry out the details of explicitly finding the constants in these expansions.

The difference $\Omega - \omega$ which is also an additive arithmetic function is discussed in the rest of Chapter II. We have also presented the numerical values for the constants in the expansions of the means and variances of $\omega$ and $\Omega$ and their covariance, neglecting the terms of order $\frac{\log \log x}{(\log x)^2}$ or smaller.

The method that we use in Chapter II is called the analytic method and is based on the Riemann zeta function. In order to
use it we need to have the Dirichlet series corresponding to the functions that we work with. To derive these Dirichlet series we use a probability model which is the basic definition of zeta density on natural numbers (see Section 2 of this chapter for details).

We continue applying this method in order to test it for other functions. Since for a non-zero multiplicative arithmetic function \( g \), the well-defined function "log \( g \)" is additive, the study of this form of additive functions is interesting. One of the most commonly occurring multiplicative functions is \( \phi(n) \), the number of positive integers not exceeding \( n \) that are relatively prime to \( n \) (the Euler totient function). We find

\[
\mu_x(\log \phi) = \log x - 1 + \sum_{p} \frac{\log(1 - \frac{1}{p})}{p} + O\left(\frac{1}{(\log x)^{n+1}}\right) \quad \text{as } x \to \infty,
\]

for arbitrary non-negative \( n \), as the mean-value of \( \log \phi \) and

\[
1 + \sum_{p} \frac{\log^2(1 - \frac{1}{p})}{p} - \sum_{p} \frac{\log^2(1 - \frac{1}{p})}{p^2}
\]

as the asymptotic variance of \( \log \phi \) (see Theorem 15.)

The limiting distribution of these additive functions with different norming constants and a survey of known results about the limiting distributions are the subject of discussions in
Chapter V. A typical result is the following limit theorem for $\log \phi$:

$$\lim_{n \to \infty} P\{n \leq N : \log \phi(n) - \log n \leq x\} = H(x),$$

where $H(x)$ is the distribution of $X + \sum_{p} X_p$, $-X$ has exponential distribution with parameter 1 and is independent of the $X_p$'s for all $p$ and the $X_p$'s are themselves mutually independent with two point densities:

$$P(X_p = 0) = 1 - \frac{1}{p} \quad \text{and} \quad P(X_p = \log(1 - \frac{1}{p})) = \frac{1}{p},$$

(see Theorem 17).

We also study the asymptotic relation between two additive functions $f$ and $g$ by looking at the asymptotic correlation coefficient between them, i.e.,

$$R(f,g) = \lim_{x \to \infty} \frac{\text{Cov}_x(f,g)}{\sqrt{\text{Var}_x(f) \text{Var}_x(g)}}.$$

This subject is studied in Section 4 of Chapter III.

Rejali (1978) derived a generalization of Renyi's theorem (1955) for the differences of a completely additive function and its strongly additive contraction, in particular the limiting distribution of $\log n - \log \prod_{p|n}$ $p$. We use this result to get
\[
\lim_{n \to \infty} \mathbb{P}_n \left\{ \frac{m \leq n \cdot \log \prod_{p \mid m} p - \log n \leq x}{} \right\}
\]
\[
= \begin{cases} 
  e^x \left[ \sum_{k=1}^{[e^{-x}]} k \beta_k' + \sum_{k=[e^{-x}]+1}^{\infty} \beta_k' \right] & \text{if } x < 0 \\
  1 & \text{if } x \geq 0 
\end{cases}
\]

where \[ \beta_k' = \lim_{n \to \infty} \mathbb{P}_n \left\{ \frac{m \leq n \cdot \log m - \log \prod_{p \mid m} p = \log k}{} \right\} \]

with
\[
\sum_{k=1}^{\infty} \beta_k' k^it = \prod_{p} \left(1 - \frac{1}{p}\right) \left(1 + \frac{1}{p-p^it}\right).
\]

Finally, in Chapter IV, we state the hyperbolic method and some general formulas that can be used to derive the moments of some additive functions by the hyperbolic method. We also compare two methods of deriving the moments, the hyperbolic method and the analytic method in some problems where both can be used.

In the course of studying these problems we have derived several number theoretic formulas and estimates that we mention now.
1 - \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \zeta(s) \frac{x^s}{s} ds = [x], \text{ for } a > 1 \quad \text{(Lemma 5)}.

2 - \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \zeta'(s) \frac{x^s}{s} ds = x(1 - \log x), \text{ for } a > 1
\quad \text{(Corollary 2)}.

3 - \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \zeta''(s) \frac{x^s}{s} ds = x(\log^2 x - 2 \log x + 2), \text{ for } a > 1
\quad \text{(Corollary 3)}.

4 - \sum_{p<x} \frac{\log p}{p} = \log x + D + O\left(\frac{1}{(\log x)^n}\right) \text{ as } x \to \infty,
where \quad D = -\gamma - \sum_p \frac{\log p}{p(p-1)} \approx -1.33253
\quad \text{(Lemma 13)}

(\sum_{p<x} \frac{\log p}{p} = \log x + O(1) \text{ is already known, see Gelfand and Linnik (1965).})

5 - \sum_{p<x} \frac{\log^2 p}{p} = \frac{1}{2} \log^2 x + E + O\left(\frac{1}{(\log x)^n}\right), \text{ as } x \to \infty
\quad \text{where} \quad E = -2 \gamma_1 - \gamma^2 - \sum_p \frac{(2p-1) \log^2 p}{p(p-1)^2}
\quad \text{(Lemma 14)}.

(\gamma_1 = \lim_{N \to \infty} \left\{ \sum_{n=1}^{N} \frac{\log n}{n} - \frac{\log^2 N}{2} \right\} \text{ and } \gamma \text{ is Euler's constant}.)}
6 - Numerical values for some of the constants in Laurent's expansion of the Riemann zeta function, that is

\[ \zeta(s) = \frac{1}{s-1} + \sum_{k=0}^{\infty} d_k (s-1)^k \quad \text{for } \Re s > 0 \]

with

\[ d_0 \approx 0.57721 \quad d_1 \approx 0.05494 \]
\[ d_2 \approx 0.02092 \quad d_3 \approx -0.03715 \]
\[ d_4 \approx 0.06771 \quad d_5 \approx -0.12557 . \quad \text{(Lemma 1)} \]

To end this introductory section, we mention some notation that we use throughout this thesis.
Notation.

1 - $(m, n)$ is the greatest common divisor of the positive integers $m$ and $n$.

2 - $[m, n]$ is the least common multiple of the positive integers $m$ and $n$.

3 - By "log" we mean natural logarithm.

4 - $m | n$ (m divides n) whenever there exists a positive integer $k$ such that $n = mk$, otherwise $m \nmid n$.

5 - We usually denote prime numbers by $p$ or $q$.

6 - $p^k \nmid n$ whenever $p^k | n$ and $p^{k+1} \nmid n$.

7 - Let $a$ be any real number including $\pm \infty$, and $f(x)$, $g(x)$ be two functions defined in some neighborhood of $a$ and $g \neq 0$. We say $f(x) = O(g(x))$ if there exists a constant $K > 0$ and a neighborhood $N(a)$ of $a$ such that $|f(x)| \leq K|g(x)|$ for all $x \in N(a)$.

By definition, $f(x) = o(g(x))$ if $\lim_{x \to \infty} \left| \frac{f(x)}{g(x)} \right| = 0$.

8 - $f(x) \sim g(x)$ means that $\frac{f(x)}{g(x)}$ converges to 1 as $x$ goes to $\infty$. The notation $\sim$ is defined in Chapter III.

9 - $\int_{a-i\infty}^{a+i\infty} h(s) \, ds = \lim_{T \to \infty} \int_{a-iT}^{a+iT} h(s) \, ds$.

10 - $[x]$ is the greatest integer smaller than the real number $x$. 

9
11 - \( \sum_{p} \) means the summation of the terms inside \( \Sigma \) for all primes \( p \).

12 - \( \sum_{p^k \mid n} (\cdot) = \sum_{(p,k) : p^k \mid n} (\cdot) \) and \( \sum_{p \leq x} (\cdot) = \sum_{(p,k) : p \leq x} (\cdot) \).

Definition. A real-valued arithmetic function is called:

1 - Additive, if \( f(mn) = f(m) + f(n) \) whenever \( m \) and \( n \) are relatively prime \( (m,n) = 1 \).

2 - Strongly Additive, if it is additive and \( f(p^k) = f(p) \) for all primes \( p \) and all positive integers \( k \).

3 - Completely or Absolutely Additive, if \( f(mn) = f(m) + f(n) \) for all \( m,n \).

4 - Multiplicative, if \( f \neq 0 \) and \( f(mn) = f(m) f(n) \) whenever \( (m,n) = 1 \).

Properties.

a - If \( g \) is a positive multiplicative function, then \( f = \log g \) is an additive function.

b - If \( F \) is completely additive, then \( F(p^k) = k F(p) \) for all primes \( p \) and all positive integers \( k \).

c - By the fundamental theorem of arithmetic, if \( f \) is strongly additive then \( f(n) = \sum_{p \mid n} f(p) \) for all \( n \) and if \( F \) is completely additive then \( F(n) = \sum_{p \mid n} k_p F(p) \) for all \( n \) where \( p^k \parallel n \).
d - There is a 1-1 linear correspondence between the set of strongly additive and the set of completely additive functions. Say, \( f \leftrightarrow F \) whenever \( f(p) = F(p) \) for all primes \( p \). (\( f \) is strongly additive and \( F \) is completely additive.)

Note that we only work with real-valued arithmetic functions.

1.2 Probabilistic Model.

The following model gives a mathematically rigorous way to justify the classical heuristic computations used in number theory which depends on the independence of divisibility by distinct primes.

For a fixed real number \( s > 1 \) define a sequence of independent integer valued random variables \( \nu_p \) (\( p \) prime) as follows:

\[
P(\nu_p = k) = (1 - \frac{1}{p^s}) \frac{1}{p^{ks}}, \quad k = 0, 1, 2, \ldots
\]

Now define \( N = \prod_{p} \nu_p \) which is almost surely well-defined because \( \sum_p P(\nu_p > 0) = \sum_p \frac{1}{p^s} < \infty \) so that \( P(\nu_p > 0 \text{ i.o.}) = 0 \).

The random variable \( N \) has a zeta distribution, i.e.

\[
P(N=n) = \frac{1}{\zeta(s)} \frac{1}{n^s} \quad \text{for} \quad n = 1, 2, \ldots \quad \text{with} \quad \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad (\text{the zeta function}).
\]

This model is defined by Diaconis (1976) who gives references to earlier versions.
If \( f \) is an additive function, then the random variable
\[
f(N) = \sum f(p \ p^k)
\]
is a sum of independent random variables
\[
f(p \ p^k). \quad \text{For each prime } p, \ f(p \ p^k) \ \text{takes values } f(p^k) \ \text{with probability } \left(1 - \frac{1}{p^s}\right) \frac{1}{p^{sk}} \quad k = 0, 1, 2, \ldots.
\]
\[
E f(p \ p^k) = \sum_{k=1}^{\infty} f(p^k) \left(1 - \frac{1}{p^s}\right) \frac{1}{p^{sk}} = \left(1 - \frac{1}{p^s}\right) \sum_{k=1}^{\infty} \frac{f(p^k)}{p^{sk}}.
\]
(The first term vanishes since \( f(1) = 0 \) for any additive function.) Consequently
\[
E f(N) = \sum_{p} E(f(p \ p^k)) = \sum_{p} \left(1 - \frac{1}{p^s}\right) \sum_{k=1}^{\infty} \frac{f(p^k)}{p^{sk}}.
\]
On the other hand
\[
E f(N) = \sum_{n} f(n) \ \text{P}(N=n) = \frac{1}{\zeta(s)} \sum_{n=1}^{\infty} \frac{f(n)}{n^s}.
\]
So now we get another expression for the Dirichlet series of an additive function
\[
(I.1) \quad \sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \zeta(s) \left\{ \sum_{p} \left(1 - \frac{1}{p^s}\right) \sum_{k=1}^{\infty} \frac{f(p^k)}{p^{sk}} \right\}.
\]
If \( f \) and \( g \) are two additive functions, then
\[ f(N) g(N) = \sum_{p} \sum_{q} f(p^N) g(q^N) \]

\[ = \sum_{p} f(p^N) g(p^N) + \sum_{p \neq q} f(p^N) g(q^N) . \]

So

\[ E f(N) g(N) = \sum_{p} \sum_{k=1}^{\infty} f(p^k) g(p^k) \left( 1 - \frac{1}{p^s} \right) \frac{1}{p^{sk}} \]

\[ + \sum_{p} \sum_{q \neq p} \sum_{k=1}^{\infty} f(p^k) \left( 1 - \frac{1}{p^s} \right) \frac{1}{p^{ks}} \sum_{j=1}^{\infty} g(q^k) \left( 1 - \frac{1}{q^s} \right) \frac{1}{q^{js}} \]

\[ = \sum_{p} \left( 1 - \frac{1}{p^s} \right) \sum_{k=1}^{\infty} \frac{f(p^k)}{p^{ks}} g(p^k) \]

\[ - \sum_{p} \left( 1 - \frac{1}{p^s} \right)^2 \left( \sum_{k=1}^{\infty} \frac{f(p^k)}{p^{ks}} \right) \left( \sum_{j=1}^{\infty} \frac{g(p^k)}{p^{ks}} \right) \]

\[ + \left[ \sum_{p} \left( 1 - \frac{1}{p^s} \right) \left( \sum_{k=1}^{\infty} \frac{f(p^k)}{p^{ks}} \right) \right] \left[ \sum_{p} \left( 1 - \frac{1}{p^s} \right) \left( \sum_{k=1}^{\infty} \frac{g(p^k)}{p^{ks}} \right) \right] . \]

On the other hand

\[ E f(N) g(N) = \sum_{n=1}^{\infty} f(n) g(n) \frac{1}{n^s} \frac{1}{\zeta(s)} \]

so

13
\[
(I.2) \sum_{n=1}^{\infty} \frac{r(n)g(n)}{n^s} = \zeta(s) \left\{ \sum_{p} \left( 1 - \frac{1}{p^s} \right) \sum_{k=1}^{\infty} \frac{r(p^k)}{p^{ks}} \right. \\
\left. - \sum_{p} \left( 1 - \frac{1}{p^s} \right)^2 \left( \sum_{k=1}^{\infty} \frac{r(p^k)}{p^{ks}} \right) \left( \sum_{k=1}^{\infty} \frac{g(p^k)}{p^{ks}} \right) \\
+ \left[ \sum_{p} \left( 1 - \frac{1}{p^s} \right) \left( \sum_{k=1}^{\infty} \frac{r(p^k)}{p^{ks}} \right) \right] \left[ \sum_{p} \left( 1 - \frac{1}{p^s} \right) \left( \sum_{k=1}^{\infty} \frac{g(p^k)}{p^{ks}} \right) \right] \right\}
\]

For \( f = g \), (I.2) specializes to:

\[
(I.3) \sum_{n=1}^{\infty} \frac{r^2(n)}{n^s} = \zeta(s) \left\{ \sum_{p} \left( 1 - \frac{1}{p^s} \right) \sum_{k=1}^{\infty} \frac{r^2(p^k)}{p^{ks}} \\
+ \sum_{p} \left( 1 - \frac{1}{p^s} \right)^2 \left( \sum_{k=1}^{\infty} \frac{r(p^k)}{p^{ks}} \right)^2 \\
+ \left[ \sum_{p} \left( 1 - \frac{1}{p^s} \right) \left( \sum_{k=1}^{\infty} \frac{r(p^k)}{p^{ks}} \right) \right]^2 \right\}.
\]

So far (I.1), (I.2) and (I.3) have been proved for real values \( s > 1 \). We now show that they are also valid for certain complex values of \( s \). For example, the right hand side of (I.1) is the product of two Dirichlet series \( \sum_{n=1}^{\infty} \frac{1}{n^s} \) and \( \sum_{n=1}^{\infty} \frac{h(n)}{n^s} \), where

\[
h(n) = r(p^k) - r(p^{k+1}) \text{ if } n = p^k \text{ for } k = 2,3,\ldots,
\]

\[
= r(p) \quad \text{if } n = p
\]

\[
= 0 \quad \text{otherwise}.
\]

So for all values of \( s \) in the region of absolute convergence of \( \sum_{n=1}^{\infty} \frac{f(n)}{n^s} \) for which \( \text{re } s > 1 \), the right hand side is a
Dirichlet series, say, \( \sum_{n=1}^{\infty} \frac{h_1(n)}{n^s} \), (Theorem 11.5 on page 228 of Apostol (1976)). But since left and right hand sides are equal for all sufficiently large real numbers, then according to Theorem 11.3 on page 227 of Apostol (1976) \( f(n) = h_1(n) \) and so (I.1) is valid for all values of \( s \) in the region of absolute convergence of \( \sum_{n=1}^{\infty} \frac{f(n)}{n^s} \). The same argument also works for (I.2) and (I.3).

---

I.3 Special Cases.

Case a - \( f(p^k) = 1 \) if \( k \neq 0 \)

\[ = 0 \] if \( k = 0 \) for all primes \( p \),

i.e., \( f(n) = \omega(n) = \text{number of distinct prime divisors of } n \).

\[ (I.4) \quad \sum_n \frac{\omega(n)}{n^s} = \zeta(s) \sum_{p} \frac{1}{p^s} . \]

\[ (I.5) \quad \sum_n \frac{\omega^2(n)}{n^s} = \zeta(s) \left\{ \sum_{p} \frac{1}{p^s} - \sum_{p^2} \frac{1}{p^{2s}} + \left( \sum_{p} \frac{1}{p^s} \right)^2 \right\} . \]

Case b - \( F(p^k) = k \) for all primes \( p \) and all non-negative integers, i.e., \( F(n) = \Omega(n) = \text{number of prime divisors of } n \).

\[ (I.6) \quad \sum_{n=1}^{\infty} \frac{\Omega(n)}{n^s} = \zeta(s) \sum_{p} \frac{1}{p^{s-1}} \]
\[(I.7) \quad \sum_{n=1}^{\infty} \frac{\Omega^2(n)}{n^s} = \zeta(s) \left\{ \sum_{p} \frac{p^s}{(p^s-1)^2} + \left(\sum_{p} \frac{1}{p^{s-1}}\right)^2 \right\} \]

Also

\[(I.8) \quad \sum_{n=1}^{\infty} \frac{\omega(n) \Omega(n)}{n^s} = \zeta(s) \left(\sum_{p} \frac{1}{p^s}\right) \left(1 + \sum_{p} \frac{1}{p^{s-1}}\right) . \]

Note that in each of the above formulas the corresponding Dirichlet series is absolutely convergent for \( \text{re} \ s > 1 \) by the use of Theorem 11.5 on page 228 of Apostol (1976).

Case c - General case of \( f \) for strongly additive functions \( f \) \((f(p^k) = f(p), k \geq 1.)\)

\[(I.9) \quad \sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \zeta(s) \sum_{p} \frac{f(p)}{p^s} . \]

\[(I.10) \quad \sum_{n=1}^{\infty} \frac{f^2(n)}{n^s} = \zeta(s) \left\{ \sum_{p} \frac{f^2(p)}{p^s} - \sum_{p} \frac{f^2(p)}{p^{2s}} + \left(\sum_{p} \frac{f(p)}{p^s}\right)^2 \right\} . \]

Case d - General case of \( h \) for completely additive functions \( F \) \((F(p^k) = k F(p), k \geq 1.)\)

\[(I.11) \quad \sum_{n=1}^{\infty} \frac{F(n)}{n^s} = \zeta(s) \left\{ \sum_{p} \frac{F(p)}{p^{s-1}} \right\} . \]

\[(I.12) \quad \sum_{n=1}^{\infty} \frac{F^2(n)}{n^s} = \zeta(s) \left\{ \sum_{p} \frac{F^2(p)}{(p^s-1)^2} + \left(\sum_{p} \frac{F(p)}{p^{s-1}}\right)^2 \right\} . \]

Now if \( F(p) = f(p) \), that is, \( F \) and \( f \) are the corresponding
completely and strongly additive functions, then:

\[(I.13) \sum_{n=1}^{\infty} \frac{f(n) F(n)}{n^s} = \zeta(s) \left\{ \sum_p \frac{f^2(p)}{p^s} + \left( \sum_p \frac{f(p)}{p^s} \right) \left( \sum_p \frac{f(p)}{p^{s-1}} \right) \right\}.\]

\[(I.9)-(I.13)\) are valid for the domain of the absolute convergence of the corresponding Dirichlet series.

In Chapters II and III these formulas will be used to obtain the mean and variance of some specific additive functions and also the covariance between two of them.

We will end this chapter by stating some of the lemmas on the Riemann-zeta function that are going to be used later in this thesis.

I.4 Riemann Zeta Function \( \zeta(s) \). (For the definition and properties see Titchmarsh (1951).)

**Lemma 1.** (Briggs and Chowla (1955).)

\[\zeta(s) = \frac{1}{s-1} + \sum_{k=0}^{\infty} d_k (s-1)^k \text{ for } \Re s > 0\]

with

\[d_k = \frac{(-1)^k \gamma_k}{k!} \text{ where } \gamma_k = \lim_{N \to \infty} \left\{ \sum_{n=1}^{N} \frac{\log^k n}{n} - \frac{\log^{k+1} N}{k+1} \right\}.\]

(Note: \( d_0 = \gamma_0 = \gamma \) is Euler's constant.)
Numerical values for \( d_0, \ldots, d_5 \) in Laurent's expansion of the Riemann zeta function are given below. The values are known to be accurate to at least three figures past the decimal point:

\[
\begin{align*}
  d_0 & \approx 0.57721 \quad d_1 \approx 0.051494 \\
  d_2 & \approx 0.02092 \quad d_3 \approx -0.03715 \\
  d_4 & \approx 0.06771 \quad d_5 \approx -1.25571.
\end{align*}
\]

**Lemma 2.** For \( \text{re } s > 1 - \frac{a}{\log T} \), \( \text{Im } s < T \), when \( a \) is a positive constant and \( T \) arbitrarily large,

\[
\log[\zeta(s)(s-1)] = \sum_{k=1}^{\infty} A_k (s-1)^k \quad \text{with} \quad A_1 = d_0 = \gamma \quad \text{and} \quad A_k = \frac{k \sum_{j=1}^{k-1} d_k-1 \cdot A_j}{d_k} \quad k = 2, 3, \ldots.
\]

**Proof.** Theorem 21 on page 13 of Titchmarsh (1951) says that \( \zeta(s) \) is regular (analytic) for all values of \( s \) except \( s = 1 \), where there is a simple pole with residue 1. So according to Ahlfors (1966) \( \zeta(s)(s-1) \) is entire. By Theorem 19 on page 58 of Ingham (1932) there exists a positive constant \( \sigma \) such that \( \zeta(s) \) has no zeros in the domain \( \sigma > 1 - \frac{a}{\log(|t|+2)} \), for all \( t \), when \( s = \sigma + it \). Therefore, \( \log[\zeta(s)(s-1)] \) is analytic for \( \text{re } s > 1 - \frac{a}{\log T} \), \( \text{Im } s < T \) and \( T \) is an arbitrary large number. (See also Theorem 6.10 on page 81 of Ayoub (1963).) So in that region
\[
\log(\zeta(s)(s-1)) = \sum_{k=0}^{\infty} A_k (s-1)^k
\]

with

\[
A_k = \frac{1}{k!} \left[ \frac{d^k}{ds^k} \log(\zeta(s)(s-1)) \right]_{s=1}
\]

e.g., \( A_0 = \log 1 = 0 \). By Lemma 1, if \( s \neq 1 \),
\[
\zeta(s)(s-1) = 1 + \sum_{k=0}^{\infty} \frac{d^k}{ds^k} (s-1)^{k+1}
\]
and if \( s = 1 \), \( \lim_{s \to 1} \zeta(s)(s-1) = 1 \)
(Titchmarsh (1951), page 16). So in that region
\[
\zeta(s)(s-1) = 1 + \sum_{k=0}^{\infty} \frac{d^k}{ds^k} (s-1)^{k+1}.
\]

Then
\[
\sum_{k=1}^{\infty} A_k (s-1)^k = \log \left( 1 + \sum_{k=0}^{\infty} \frac{d^k}{ds^k} (s-1)^{k+1} \right)
\]

and both sides are analytic. By taking the derivative with respect to \( s \) we get
\[
\sum_{k=0}^{\infty} \frac{(k+1)d_k}{1 + \sum_{k=0}^{\infty} \frac{d_k}{ds^k} (s-1)^{k+1}} = \sum_{j=1}^{\infty} \frac{A_j}{j} (s-1)^{j-1}.
\]

The denominator is not zero, so
\[
\sum_{k=0}^{\infty} (k+1) d_k (s-1)^k = \sum_{k=0}^{\infty} (k+1) A_{k+1} (s-1)^k \\
+ \sum_{k=1}^{\infty} \sum_{j=1}^{k} j A_j d_{k-j} (s-1)^k.
\]

Therefore \( A_1 = d_0 \) and

\[(k+1) A_{k+1} + \sum_{j=1}^{k} j d_{k-j} A_j = (k+1)d_k \quad \forall k \geq 1.\]

By changing \( k+1 \) to \( k \) we get the result.

Finally we will combine these two lemmas and get:

**Lemma 3.** In the region of Lemma 2 we have:

\[(\zeta(s) - \frac{1}{s-1}) \log(\zeta(s)(s-1)) = \sum_{k=1}^{n-1} g'_k (s-1)^k + O((s-1)^n)\]

with \( g'_k = \sum_{m=1}^{k} d_{k-m} A_m \), \( k \geq 1 \), where notations are as in previous lemmas. In particular,

\[g'_1 = \gamma^2\]

\[g'_2 = d_1 A_1 + d_0 A_2 = -2 \gamma_1 \gamma - \gamma^2.\]
Lemma 4. If \( a > 0 \) and \( T \) is an arbitrary large positive number, then

\[
\frac{1}{2\pi i} \int_{a-iT}^{a+iT} \frac{y^s}{s} \, ds = \begin{cases} 
1 + O\left( \frac{y}{T|\log y|} \right) & y > 1 \\
O\left( \frac{y^a}{T|\log y|} \right) & 0 \leq y < 1.
\end{cases}
\]

Proof. If we argue as in Ayoub (1963), pp. 50 and 51, integration by parts gives us

\[
\frac{1}{2\pi i} \int_{a-iT}^{a+iT} y^s \frac{ds}{s} = \frac{y^{a+iT}}{2\pi i(a+iT) \log y} - \frac{y^{a-iT}}{2\pi i(a-iT) \log y}
\]

\[
+ \frac{1}{2\pi i \log y} \int_{a-iT}^{a+iT} y^s \frac{ds}{s}, \text{ when } y \neq 1.
\]

However,

\[
\frac{y^{a+iT}}{(a+iT) \log y} = O\left( \frac{y^a}{T|\log y|} \right).
\]

Now for computation of \( \int_{a-iT}^{a+iT} y^s \frac{ds}{s} \) consider two cases.

1. If \( y > 1 \), then replace the line of integration \( L_1 \) by the curve \( C_1 \), the left part of the circle centered at the origin and passing through the points \( a-iT, a+iT \) in Figure 1.
Inside the contour bounded by $C_1$ and 1, $\frac{y^s}{s^2}$ has a double pole at $s = 0$ with residue

$$\left. \frac{3}{s} \frac{s^2 y^s}{s^2} \right|_{s=0} = \left. \frac{3}{s} y^s \right|_{s=0} = \log y.$$

So by Cauchy's theorem:

$$\int_{a-iT}^{a+iT} y^s \frac{ds}{s^2} = 2\pi i \log y + \int_{C_1} y^s \frac{ds}{s^2}.$$

On $C_1$, $\text{Re} s \leq a$ and since $y > 1$, $|y^s| \leq y^a$ also

$$\int_{C_1} |ds| \leq 2\pi \sqrt{a^2 + T^2}$$

and thus

$$\int_{C_1} y^s \frac{ds}{s^2} = O\left(\frac{y^a}{T}\right).$$
II: If \( 0 \leq y < 1 \), replace the line of integration by the other part of the circle in Figure 1, say, \( C_2 \). Since \( \frac{y^s}{s^2} \) is analytic inside the contour bounded by \( C_2 \) and \( L \), by Cauchy's theorem

\[
\int_{a-iT}^{a+iT} y^s \frac{ds}{s^2} = \int_{C_2} y^s \frac{ds}{s^2},
\]

On \( C_2 \) re \( s > a \), \( 0 \leq y < 1 \) then \( |y^s| \leq y^a \) and also

\[
\int_{C_2} |ds| \leq 2\pi \sqrt{a^2 + T^2}
\]

so:

\[
\int_{C_2} y^s \frac{ds}{s^2} = O\left(\frac{y^a}{T}\right)
\]

and therefore we have the result for \( y \neq 1 \).

---

**Lemma 5.** If \( a > 1 \), then for non-integer positive real numbers \( x \) we have

\[
\frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \zeta(s) x^s \frac{ds}{s} = [x].
\]

**Proof.** Let \( T \) be a fixed positive real number and

\[
I(T) = \frac{1}{2\pi i} \int_{a-iT}^{a+iT} \zeta(s) x^s \frac{ds}{s}.
\]

On the line of integration re \( s > 1 \), so \( \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \), therefore

\[
I(T) = \frac{1}{2\pi i} \int_{a-iT}^{a+iT} \sum_{n=1}^{\infty} \frac{x^s}{n^s} \frac{ds}{s}.
\]
By Fubini's theorem (e.g., Theorem 7.8 on page 150 of Rudin (1974))

\[ I(T) = \sum_{n=1}^{\infty} \frac{1}{2\pi i} \int_{a-iT}^{a+iT} (\frac{x}{n})^s \frac{ds}{s}. \]

So

\[ I(T) = \sum_{n<x} 1 + \sum_{n<x} o\left(\frac{x^a}{T|\log \frac{x}{n}|}\right) + \sum_{n>x} o\left(\frac{x^a}{T|\log \frac{x}{n}|}\right), \]

but

\[ \sum_{n<x} o\left(\frac{x^a}{T|\log \frac{x}{n}|}\right) = o\left(\frac{1}{T}\right) \]

because we have only a finite summation of \( o\left(\frac{1}{T}\right) \)'s and

\[ \sum_{n>x} o\left(\frac{x^a}{T|\log \frac{x}{n}|}\right) = \sum_{x<n\leq xe} o\left(\frac{x^a}{T|\log \frac{x}{n}|}\right) + \sum_{n\geq xe} o\left(\frac{x^a}{T|\log \frac{x}{n}|}\right) \]

\[ = o\left(\frac{1}{T}\right) + o\left(\frac{x^a}{T} \sum_{n\geq xe} \frac{1}{n^a}\right) \]

\[ = o\left(\frac{1}{T}\right) \]

because if \( n \geq xe \), then \( |\log \frac{x}{n}| = \log \frac{n}{x} \geq 1 \) and since \( a > 1 \)

\[ \sum_{n\geq xe} \frac{1}{n^a} \]

converges to a finite number. So \( I(T) = [x] + o\left(\frac{1}{T}\right) \),

therefore,

\[ \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \zeta(s) \frac{x^s}{s} ds = [x]. \]
Lemma 6. Let \( T \geq e^2 \), \( \sigma_0 = 1 - \frac{1}{\log T} \) then in the rectangle with vertices \( \sigma_0 - iT, \sigma_0 + iT, 2 + iT \) and \( 2 - iT \), we have \( \zeta''(s) = O(\log^3 T) \).

Proof. Formula 4 on page 57 of Ayoub (1963) is:

\[
\Delta(s) = \zeta(s) - \frac{1}{s-1} = \sum_{k=1}^{n-1} \frac{1}{k^s} + \frac{1}{2} n^{-s} + \frac{n^{1-s} - 1}{s-1} - s \int_{n}^{\infty} x^{-s-1} (x - [x] - \frac{1}{2}) dx
\]

\[
= \sum_{k=1}^{n-1} \frac{1}{k^s} + \frac{1}{2} n^{-s} - \int_{1}^{n} x^{-s} dx - s \int_{n}^{\infty} x^{s-1} (x - [x] - \frac{1}{2}) dx.
\]

So the second derivative with respect to \( s \) of \( \Delta(s) \) is:

\[
D(s) = \zeta''(s) - \frac{2}{(s-1)^3}
\]

\[
= \sum_{k=1}^{n-1} k^{-s} \log^2 k + \frac{n^{-s}}{2} \log^2 n - \int_{1}^{n} x^{-s} \log^2 x dx
\]

\[
+ 2 \int_{n}^{\infty} x^{-1-s} (x - [x] - \frac{1}{2}) \log x dx
\]

\[
- s \int_{n}^{\infty} x^{-1-s} (x - [x] - \frac{1}{2}) \log^2 x dx.
\]

Hence for \( \sigma = \text{re} s \) we have:

\[
|D(s)| < \log^2 n \left\{ \sum_{k=1}^{n} k^{-\sigma} + \int_{1}^{n} x^{-\sigma} dx \right\} + \int_{n}^{\infty} x^{-1-\sigma} \log x dx
\]

\[
+ \frac{|s|}{2} \int_{n}^{\infty} x^{-1-\sigma} \log^2 x dx.
\]
Since $T \geq e^2$ and $\sigma_0 \leq \sigma$ then $\sigma > \frac{1}{2}$ and by integration by parts

$$\int_n^\infty x^{-1-\sigma} \log x \, dx = \frac{n^{-\sigma} \log n}{\sigma} + \frac{n^{-\sigma}}{\sigma^2} \leq c_2 \, n^{-\sigma} \log n$$

and

$$\int_n^\infty x^{-1-\sigma} \log^2 x \, dx = \frac{n^{-\sigma} \log^2 n}{\sigma} + \frac{2}{\sigma} \int_n^\infty x^{-1-\sigma} \log x \, dx \leq c_1 \, n^{-\sigma} \log^2 n.$$ 

Also

$$\sum_{k=1}^{n} k^{-\sigma} + \int_{1}^{n} x^{-\sigma} \, dx < 2 \frac{n^{1-\sigma}}{1-\sigma}.$$ 

And since $|s| < T + 2$, we have

$$|D(s)| < 2 \, n^{1-\sigma} \log T \log^2 n + c_1 \, n^{-\sigma} \log^2 n \left( \frac{T+2}{2} \right) + c_2 \, n^{-\sigma} \log n.$$ 

Now let $n = \lfloor T \rfloor$, then

$$|D(s)| < n^{1-\sigma} \log^2 n \left[ 2 \log T + c_1 (1 + \frac{2}{T}) + c_2 \right].$$

But $1 - \sigma < \log T$. So $n^{1-\sigma} < 1$, and therefore $D(s) = O(\log^3 T)$.

Also in that region $|s-1| \geq \frac{1}{\log T}$, so $\frac{1}{(s-1)^3} \leq \log^3 T$ and therefore $\zeta''(s) = D(s) + \frac{2}{(s-1)^3} = O(\log^3 T)$. 

26
II.1 Arithmetic Mean of $\omega$ and $\Omega$.

As we mentioned in Chapter I, the convergence rates of the asymptotic formulas due to Hardy and Ramanujan (1917) are very slow. Therefore, extra terms are needed. The following result gives the form of these terms for the mean of $\omega$.

**Theorem 1.** (Delange (1971), Diaconis (1976) and Saffari (1968).) For any non-negative integer $n$

$$\mu_x(\omega) = \frac{1}{x} \sum_{m \leq x} \omega(m) = \log \log x + \sum_{k=0}^{n} \frac{a_k}{(\log x)^k} + O\left(\frac{1}{(\log x)^{n+1}}\right) \text{ as } x \to \infty,$$

where

$$a_0 = H_1 = \gamma + \sum_{p} \left(\log(1 - \frac{1}{p}) + \frac{1}{p}\right), \quad a_1 = \gamma - 1,$$

$\gamma$ is Euler's constant and the other $a_k$'s are computable constants. We give the constants explicitly in the following theorem.

**Theorem 2.** In Theorem 1 $a_k = (k-1)! \left(\sum_{j=0}^{k-1} \frac{\gamma_j}{j!} - 1\right)$ for $k \geq 1$, where

$$\gamma_j = \lim_{N \to \infty} \left[ \frac{N}{\sum_{t=1}^{N} \frac{\log^j t}{t} - \frac{\log^{j+1} N}{j+1}} \right] \quad j = 0, 1, 2, \ldots,$$

(e.g., $a_1 = \gamma - 1 \approx 0.42279$ and $a_2 = \gamma + \gamma_1 - 1 \approx 0.36785$).
The parts of the proof of Theorem 1 which are useful for other theorems of this thesis will be presented. For further details see Diaconis (1976).

**Proof of Theorem 1.** For $\text{Re } s > 1$, we have from equation (1.4):

$$
\sum_{m=1}^{\infty} \frac{\omega(m)}{m^s} = \zeta(s) \sum_{p} \frac{1}{p^s} = \zeta(s) \{\log(\zeta(s)) - f(s)\} = \zeta(s) h(s)
$$

with $f(s) = \sum_{p} \sum_{k=2}^{\infty} \frac{1}{k p^s}$ which is analytic for $\text{Re } s > \frac{1}{2}$ and uniformly bounded in any compact set in $\text{Re } s \geq \frac{1}{4}$.

Theorem 3.1 on page 50 of Ayoub (1963) with $a = 1 + \frac{c}{(\log T)^{9}}$ where $c$ is a positive constant and $\log T = c(\log x)^{1/10}$ gives us

$$
I = \sum_{m \leq x} \omega(m) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} h(s) \frac{x^s}{s} \, ds = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \zeta(s) \log(\zeta(s)) \frac{x^s}{s} \, ds
$$

$$
- \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \zeta(s) f(s) \frac{x^s}{s} \, ds = I_1 + I_2,
$$

when $x$ is not an integer.

Diaconis (1976) has shown that $I_1 = I_1(1) + I_1(2) + \ldots + I_1(7)$ where $I_1(i)$ is the value of integral over the path $i$, indicated in Figure 2.
Figure 2

In Figure 2, \( b = 1 - \frac{c}{(\log T)^9} \), \( C \) is a closed curve around \( 1 \) with end points at \( b \).

The same method works for \( I_2 \) and

\[ I_2 = I_2(1) + I_2(2) + \ldots + I_2(7) \]. Diaconis (1976) has also shown that

\[
E = I_1(1) + I_2(1) + \ldots + I_1(6) + I_2(6)
\]

\[ = 0 \left( x e^{-c (\log x)^{1/10}} \right). \]

Now
\[
I_1(t) = \frac{1}{2\pi i} \int_C \zeta(s) \log(\zeta(s)) \frac{x^s}{s} \, ds
\]

\[
= \frac{1}{2\pi i} \int_C \frac{1}{s-1} \log(\frac{1}{s-1}) x^s \, ds + \frac{1}{2\pi i} \int_C (\zeta(s) - \frac{1}{s-1} - 1) \log(\frac{1}{s-1}) \frac{x^s}{s} \, ds
\]

\[
+ \frac{1}{2\pi i} \int_C \zeta(s) \log((s-1) \zeta(s)) \frac{x^s}{s} \, ds = I_1^1 + I_1^2 + I_1^3.
\]

where \( I_1^1 = x[\log \log x + \gamma + O(e^{-\left(\log T\right)^{-\delta}} \log x)] \) and \( I_1^3 = 0 \).

\( I_1^2 \): According to Lemma 1

\[
\zeta(s) - \frac{1}{s-1} - 1 = \sum_{j=0}^{n-1} d_j' (s-1)^j + O((s-1)^n)
\]

with \( d_0' = d_0 - 1 = \gamma - 1 \), \( d_j' = d_j \); \( j \geq 1 \). Let

\[
J_j' = \frac{1}{2\pi i} \int_C (s-1)^j \log(\frac{1}{s-1}) \frac{x^s}{s} \, ds, \quad j = 0, 1, 2, \ldots
\]

Then

\[
I_1^2 = \sum_{j=0}^{n-1} d_j' J_j' + O(J_n').
\]

(When \( n = 0 \), \( I_1^2 = O(J_0') \).)

\((s-1)^j \log(\frac{1}{s-1})\) has an analytic branch in a neighborhood of 1 and if we consider the integral through the path in Figure 3 and let \( \xi \to 0 \), then
\[ J'_j = (-1)^j \int_0^x (1 - \frac{\log u}{\log x})^j \frac{du}{\log u} + O(x^b) \]

\[ = (-1)^j j! \int_0^x \frac{du}{(\log u)^{j+1}} + O(x^b) \]

\[ = (-1)^j j! x \sum_{k=j+1}^{j+m} \frac{(k-1)!}{j! (\log x)^k} + O\left(\frac{x}{(\log x)^{j+m+1}}\right) + O(x^b) \]

for all \( m \geq 1 \).

![Figure 3](image)

Now let \( m = n - j \). For each \( j \) we have:

\[ J'_j = (-1)^j x \sum_{k=j+1}^n \frac{(k-1)!}{(\log x)^k} + O\left(\frac{x}{(\log x)^{n+1}}\right) + O(x^b) \]

So

\[ I^2 = x \sum_{j=0}^{n-1} d'_j (-1)^j \sum_{k=j+1}^n \frac{(k-1)!}{(\log x)^k} + O\left(\frac{x}{(\log x)^{n+1}}\right) + O(x^b) \]

\[ = x \left\{ \sum_{k=1}^n \sum_{j=0}^{k-1} (-1)^j d'_j \frac{1}{(\log x)^k} \right\} + O\left(\frac{x}{(\log x)^{n+1}}\right) + O(x^b) \]
or
\[
I_2 = x \left\{ \sum_{k=1}^{n} \frac{s_k}{(\log x)^k} + O\left( \frac{1}{(\log x)^{n+1}} \right) + O\left( e^{-(\log T)^{-9} \log x} \right) \right\}
\]

with \( a_1 = d'_0 = \gamma - 1 \) and for \( k > 1 \),

\[
a_k = (k-1)! \sum_{j=0}^{k-1} (-1)^j d'_j = (k-1)! \left[ d'_0 + \sum_{j=1}^{k-1} (-1)^j d'_j \right] = (k-1)! \left[ \gamma - 1 + \sum_{j=1}^{k-1} \frac{(-1)^j \gamma_j}{j!} \right] = (k-1)! \left[ \sum_{j=0}^{k-1} \frac{\gamma_j}{j!} - 1 \right], \text{ where } \gamma_j = \lim_{N \to \infty} \left[ \sum_{t=1}^{N} \frac{\log^j t}{t} - \frac{\log^{j+1} N}{j+1} \right], \text{ for } j = 0, 1, 2, \ldots .
\]

So
\[
I_1(7) = x \left\{ \log \log x + \gamma + \sum_{k=1}^{n} \frac{s_k}{(\log x)^k} + O\left( \frac{1}{(\log x)^{n+1}} \right) + O\left( e^{-(\log T)^{-9} \log x} \right) \right\}.
\]

Since the function \(-\zeta(s) f(s) \frac{x^s}{s}\) has a simple pole at \( s = 1 \) with residue \(-x f(1)\), \( I_2(7) = -x f(1) \). Also

\[
0\left( x e^{-(\log T)^{-9} \log x} \right) = 0\left( x e^{-c(\log x)^{1/10}} \right) = 0\left( \frac{x}{(\log x)^{n+1}} \right).
\]

So
\[ I = x \left\{ \log \log x + \gamma - f(1) + \sum_{k=1}^{n} \frac{a_k}{(\log x)^k} + O\left(\frac{1}{(\log x)^{n+1}}\right) \right\}. \]

But

\[ -f(1) = -\sum_{p} \sum_{k=2}^{\infty} \frac{1}{kp^k} = \frac{1}{p} \left[ \log(1 - \frac{1}{p}) \right], \]

therefore we have the results of Theorems 1 and 2.

Remarks.

1 - Saffari (1968) has proved Theorem 1 by the use of the hyperbolic method. (For details see Section 4 of Chapter IV.)

2 - In the proof of Theorem 1 we have shown that

\[ E = I_1(1) + I_2(1) + \ldots + I_1(6) + I_2(6) = O\left(\frac{x}{(\log x)^{n+1}}\right) \]

so

\[ I_1(1) + \ldots + I_1(6) = O\left(\frac{x}{(\log x)^{n+1}}\right) \]

and therefore

\[ I_1 = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \zeta(s) \log(\zeta(s)) \frac{x^s}{s} \, ds = x \left\{ \log \log x + \gamma + \sum_{k=0}^{n} \frac{a_k}{(\log x)^k} \right\} + O\left(\frac{1}{(\log x)^{n+1}}\right). \]

Also \[ I_2(1) + \ldots + I_2(6) = O\left(\frac{1}{(\log x)^{n+1}}\right). \]

Hardy and Ramanujan (1918) have shown that

\[ \mu_x(\Omega) = \frac{1}{x} \sum_{m \leq x} \Omega(m) = \log \log x + B_2 + O\left(\frac{1}{\log x}\right) \]

where \[ B_2 = B_1 + \sum_{p} \frac{1}{p(p-1)} \]. We shall prove

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Theorem 3. For any non-negative integer \( n \),

\[
\mu_x(\Omega) = \log \log x + \sum_{k=0}^{n} \frac{a'_k}{(\log x)^k} + O \left( \frac{1}{(\log x)^{n+1}} \right) \quad \text{as} \quad x \to \infty,
\]

with \( a'_0 = B_2 = B_1 + \sum_{p} \frac{1}{p(p-1)} \) and \( a'_k = a_k \) (as in Theorem 2) for \( k \geq 1 \).

Proof. For \( \text{re} \, s > 1 \), we have from equation (I.6)

\[
\sum_{m=1}^{\infty} \Omega(m) m^{-s} = \zeta(s) \sum_{p} \frac{1}{p^{-s-1}}
\]

also

\[
\zeta(s) = \prod_{p} \left( 1 - \frac{1}{p^s} \right)^{-1} = \prod_{p} \left( 1 + \frac{1}{p^{s-1}} \right)
\]

and therefore

\[
\log(\zeta(s)) = \sum_{p} \log \left( 1 + \frac{1}{p^{s-1}} \right) = \sum_{p} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k(p^{s-1})^k}
\]

or

\[
\log(\zeta(s)) = \sum_{p} \frac{1}{p^{s-1}} + g(s)
\]

with

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\[ g(s) = \sum_{p} \sum_{k=2}^{\infty} \frac{(-1)^{k-1}}{k(p^s-1)^k} \]

which is analytic for \( \text{Re } s > \frac{1}{2} \) and uniformly bounded in any compact set in \( \text{Re } s \geq \frac{1}{4} \). So

\[ \sum_{m=1}^{\infty} \frac{\Omega(m)}{m^s} = \zeta(s) \log(\zeta(s)) - \zeta(s) g(s) \]

with abscissa of convergence 1, so if \( a = 1 + \frac{c}{(\log T)^9} \) with \( \log T = c(\log x)^{1/10} \), then

\[ \sum_{m \leq x} \Omega(m) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \zeta(s) \log(\zeta(s)) \frac{x^s}{s} \, ds - \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \zeta(s) g(s) \frac{x^s}{s} \, ds \]

\[ = T_1 + T_2 . \]

\[ T_1 = I_1 \text{ (in Theorem 1) } = x \left\{ \log \log x + \gamma + \sum_{k=1}^{n} \frac{a_k}{(\log x)^k} \right\} + O\left(\frac{1}{(\log x)^{n+1}}\right) \]

for any fixed \( n \) as \( x \to \infty \). For \( T_2 \) break the integral into integrals over the paths indicated in Figure 2. So \( T_2 = T_2(1) + \ldots + T_2(7) \), because \( -\zeta(s) g(s) \frac{x^s}{s} \) is analytic inside the contour bounded by the lines 2, 3, 4, 5 and 8 and the boundary of the circle \( C \).
$$T_2(7) = -\frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \zeta(s) \ g(s) \ \frac{x^s}{s} \ ds = -x \ g(1),$$

because \(-\zeta(s) \ g(s) \ \frac{x^s}{s}\) has a simple pole at \(s = 1\) with residue \(-x \ g(1)\). Next by using the proof used to bound the integrals \(I_2(1), \ldots, I_2(6)\) in Theorem 1 we have

$$T_2(1) + \ldots + T_2(6) = o\left(\frac{x}{(\log x)^{n+1}}\right).$$

Therefore

$$T_1 + T_2 = x \left\{ \log \log x + \gamma - g(1) + \sum_{k=1}^{n} \frac{a_k}{(\log x)^k} + o\left(\frac{1}{(\log x)^{n+1}}\right) \right\}.$$  

But

$$-g(1) = \sum_{p} \sum_{k=2}^{\infty} \frac{(-1)^{k-1}}{k(p-1)^k} = \sum_{p} \sum_{k=1}^{\infty} \frac{(-1)^{k}}{k(p-1)^k} + \sum_{p} \frac{1}{p-1}$$

$$= \sum_{p} \left[\frac{1}{p-1} + \log(1 - \frac{1}{p})\right].$$

Substituting the value of \(-g(1)\) in the previous formula concludes the proof of Theorem 3.

Theorems 1 and 3 imply:

**Corollary 1.** For any non-negative integer \(n\),
\[ \mu_x(\Omega - \omega) = \mu_x(\Omega) - \mu_x(\omega) = \sum_{p} \frac{1}{p^{p-1}} + O\left(\frac{1}{(\log x)^{n+1}}\right) \]

as \( x \to \infty \).

Note. Diaconis, Mosteller and Onishi (1977) have shown this corollary for the case when \( n = 0 \).

II.2 Variances of \( \omega \) and \( \Omega \).

Hardy and Ramanujan (1917) showed that the variance of \( \omega(n) \) about the mean \( \mu_x(\omega) \) is asymptotically \( \log \log x \). Diaconis, Mosteller and Onishi (1977) found the first term of the asymptotic expansion of the variance and finally Diaconis (1976) proved the following theorem.

**Theorem** 1. For any non-negative integer \( n \),

\[
\text{Var}_x(\omega) = \frac{1}{x} \sum_{m \leq x} (\omega(m) - \mu_x(\omega))^2 = \log \log x \sum_{k=0}^{n} \frac{b_k}{(\log x)^k} + \frac{c_k}{(\log x)^k} + O\left(\frac{\log \log x}{(\log x)^{n+1}}\right) \text{ as } x \to \infty,
\]

where \( b_0 = 1, \ b_1 = 0, \ c_0 = B_3 = B_1 = \frac{\pi^2}{6} - \sum_{p} \frac{1}{p^2} \) and the other \( b_i, c_i \) are computable constants.

Here again our goal is to find the constants.
Theorem 5. In Theorem 4 we have:

\[ b_0 = 1 , \]

\[ b_1 = 0 , \]

\[ c_0 = B_1 = B_1 - \sum_{p} \frac{1}{p^2} - \frac{\pi^2}{6} , \]

\[ c_1 = 3 \gamma - 1 + 2 \sum_{p} \frac{\log x}{p(p-1)} , \]

\[ b_k = 2(k-1)! \ a_1 - 2 \ a_k \quad \text{and} \]

\[ c_k = m'_k + h_k - 2 \ f'_k + \ a_k - \sum_{j=0}^{k} \ a_{k-j} \ a_j , \]

where

\[ m'_k = 2(k-1)! \left( \gamma - \sum_{j=1}^{k-1} \frac{1}{j} \right) a_1 , \]

\[ h_k = 2(k-1)! \left( \sum_{j=1}^{k-1} (-1)^j (A_{j+1} + \varepsilon'_j) + \gamma \right) \]

and

\[ f'_k = \sum_{j=0}^{k-1} \binom{k-1}{j} (-1)^j a_{k-j} f^{(j)}(1) + \frac{(-1)^{k-1} f^{(k)}(1)}{k} \quad \text{for} \ k \geq 2 \]

with

\[ a_0 = B_1 = \gamma + \sum_{p} \left( \log(1 - \frac{1}{p}) + \frac{1}{p} \right) , \]

\[ a_j = (j-1)! \left( \sum_{m=0}^{j-1} \frac{\gamma}{m!} - 1 \right) \quad \text{for} \ j = 1, 2, \ldots , \]
also $A_\perp = \gamma$ and for $j = 1,2,3,\ldots$

$$A_{j+1} = \frac{(-1)^j \gamma_j}{j!} - \frac{1}{j+1} \sum_{m=1}^{j} \frac{m(-1)^j}{(j-m)!} \gamma_{j-m} A_m,$$

$$g_j = \frac{1}{j} \frac{(-1)^j \gamma_j}{(j-m)!} A_m$$

and finally

$$f(j)(1) = \lim_{s \to 1} \frac{\alpha_j}{s^j} \left( \sum_{p=2}^{\infty} \sum_{k=1}^{\infty} \frac{1}{kp^s} \right),$$

$$\gamma_j = \lim_{N \to \infty} \left[ \sum_{t=1}^{N} \frac{\log^j t}{t} - \frac{\log^{j+1} N}{j+1} \right] \text{ for } j = 0,1,2,\ldots$$

$$= \gamma_0 \text{ (Euler's constant).}$$

The parts of the proof of Theorem 4 which are used for the theorems will be presented. For further details see Riss (1976).

**Proof of Theorem 4.** For $re s > 1$, we have from equation:

$$\sum_{m=1}^{\infty} \frac{\omega^2(m)}{m^s} = \zeta(s) \left\{ \sum_{p} \frac{1}{p^s} - \sum_{p} \frac{1}{p^{2s}} + \left( \sum_{p} \frac{1}{p^s} \right)^2 \right\}.$$ 

Again by Theorem 3.1 on page 50 of Ayoub (1963) for

$$1 + \frac{c}{(\log T)^9} \text{ where } c \text{ is a positive constant and } \log T = c(\log x)^{1/10}$$

we have

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\[
\sum_{m \leq x} \omega^2(m) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \zeta(s) \left( \sum \frac{1}{p^s} \right) \frac{x^s}{s} \, ds \\
- \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \zeta(s) \left( \sum \frac{1}{p^{2s}} \right) \frac{x^s}{s} \, ds + \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \zeta(s) \left( \sum \frac{1}{p^s} \right)^2 \frac{x^s}{s} \, ds
\]

\[= J_1 + J_2 + J_3.\]

\[J_1 = \mathcal{I} \text{ (in Theorem 1)} = x \log \log x + \sum_{k=0}^{n} \frac{a_k}{(\log x)^k} \]

\[+ O \left( \frac{1}{(\log x)^{n+1}} \right),\]

\[J_2 = -x \sum \frac{1}{p^2} + O \left( x e^{-c(\log x)^{1/10}} \right) \quad \text{and} \]

\[J_3 = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \zeta(s) \{ \log(\zeta(s)) - f(s) \}^2 \frac{x^s}{s} \, ds \]

where \( f(s) \) is defined in the proof of Theorem 1, so

\[J_3 = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \zeta(s) \log^2(\zeta(s)) \frac{x^s}{s} \, ds \\
- \frac{2}{2\pi i} \int_{a-i\infty}^{a+i\infty} \zeta(s) f(s) \log(\zeta(s)) \frac{x^s}{s} \, ds \]

\[+ \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \zeta(s) f^2(s) \frac{x^s}{s} \, ds = J_3(1) + J_3(2) + J_3(3).\]

The argument that is used for computation of \( T_2 \) in Theorem 3 also works for \( J_3(3) \) and gives us
\[ J_3(3) = x f^2(1) + O \left( x e^{-c \log x}^{1/10} \right). \]

The integral over the paths \( l \) through \( 6 \) (of Figure 2) of \( J_3(2) \) is \( O \left( x e^{-c \log x}^{1/10} \right) \). For details see Diaconis (1976).

Now use Lemma 1 and the expansion of \( f(s) \) i.e.,

\[ f(s) = \sum_{j=0}^{\infty} B_j''(s-1)^j \quad \text{with} \quad B_j'' = \frac{1}{j!} \left[ \frac{d^j f(s)}{d s^j} \right]_{s=1} = \frac{f^{(j)}(1)}{j!} \]

to get

\[ f(s) \zeta(s) = \sum_{j=0}^{\infty} B_j''(s-1)^{j-1} + \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} d_n B_j''(s-1)^{j+n} \]
\[ = \frac{B_0''}{(s-1)} + \sum_{k=0}^{\infty} \left( B_{k+1}'' + \sum_{j=0}^{k} d_{k-j} B_j'' \right)(s-1)^k, \]

or

\[ f(s) \zeta(s) = \frac{f(1)}{(s-1)} + \sum_{k=0}^{n-1} e_k (s-1)^k + O((s-1)^n) \]

with

\[ e_k = \frac{f^{(k+1)}(1)}{(k+1)!} + \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} \gamma_{k-j} f^{(j)}(1) \]

(e.g., \( e_0 = f'(1) + \gamma f(1) \)).

Since \( f(s) \zeta(s) \log(\zeta(s) (s-1)) \) is analytic, then
$$L_3 = \frac{1}{2\pi i} \int_C \zeta(s) f(s) \log(\zeta(s)) \frac{x^s}{s} ds$$

$$= \frac{1}{2\pi i} \int_C \zeta(s) f(s) \log(\zeta(s)(s-1)) \frac{x^s}{s} ds$$

$$+ \frac{1}{2\pi i} \int_C \zeta(s) f(s) \log\left(\frac{1}{s-1}\right) \frac{x^s}{s} ds$$

$$= \frac{1}{2\pi i} \int_C \zeta(s) f(s) \log\left(\frac{1}{s-1}\right) \frac{x^s}{s} ds$$

$$= \frac{f(1)}{2\pi i} \int_C \frac{1}{s-1} \log\left(\frac{1}{s-1}\right) \frac{x^s}{s} ds + \sum_{k=0}^{n-1} \frac{e_k}{2\pi i} \int_C (s-1)^k \log\left(\frac{1}{s-1}\right) \frac{x^s}{s} ds$$

$$+ O\left(\left(\int_C (s-1)^n \log\left(\frac{1}{s-1}\right) \frac{x^s}{s} ds\right)\right).$$

But

$$\int_C \frac{1}{s-1} \log\left(\frac{1}{s-1}\right) \frac{x^s}{s} ds = \int_C \log\left(\frac{1}{s-1}\right) \frac{x^s}{(s-1)} ds - \int_C \log\left(\frac{1}{s-1}\right) \frac{x^s}{s} ds.$$

Now using the notation in the proof of Theorem 1, we get

$$\frac{1}{2\pi i} \int_C \frac{1}{s-1} \log\left(\frac{1}{s-1}\right) \frac{x^s}{s} ds = I' - J'_0$$

$$= x \left\{ \log \log x + \gamma + \frac{1}{9} \log x \right\} - J'_0.$$

So
\[ L_3 = f(1) x \left\{ \log \log x + \gamma + o \left( e^{- (\log T)^{-9} \log x} \right) \right\} \]

\[ - f(1) J'_0 + e_0 J'_0 + \sum_{k=1}^{n-1} e_k J'_k + o(J'_n) \]

or

\[ L_3 = f(1) x \left\{ \log \log x + \gamma + o \left( e^{- (\log T)^{-9} \log x} \right) \right\} \]

\[ + \sum_{k=0}^{n-1} \frac{(-1)^k}{(\log x)^j} \cdot \left( \frac{\log x}{(\log x)^n+1} \right) + o(x^b) \]

with \( d''_0 = -f(1) + e_0 = f'(1) + (\gamma - 1) f(1) \), \( d''_k = e_k \) for \( k \geq 1 \) and \( b = 1 - \frac{c}{(\log t)^9} \). So

\[ L_3 = x \left\{ f(1) \log \log x + \gamma f(1) + o \left( e^{- (\log T)^{-9} \log x} \right) \right\} \]

\[ + \sum_{k=0}^{n-1} \frac{(-1)^k \cdot (j-1)!}{(\log x)^j} \cdot \left( \frac{1}{(\log x)^n+1} \right) \]

\[ = x \left\{ f(1) \log \log x + \gamma f(1) + \sum_{j=1}^{J_{1-1}} \frac{(j-1)! \cdot (-1)^k d''_k}{(\log x)^j} \cdot \left( \frac{1}{(\log x)^n+1} \right) \right\} \]

and therefore

\[ J_3(2) = x \left\{ -2 f(1) \log \log x - \sum_{k=0}^{n} \frac{2f_k}{(\log x)^k} + o \left( \frac{1}{(\log x)^{n+1}} \right) \right\} \]

with
\[ f_k = (k-1)! \sum_{j=0}^{k-1} (-1)^j d_j \] for \( k \geq 1 \) and \( f_0 = \gamma f(1) \).

Substituting for the \( d_j \)'s and using the definition of the \( a_k \)'s gives us \( f_0 = \gamma f(1), \ f_1 = f'(1) + (\gamma - 1) f(1) \) and

\[ f_k = \sum_{\ell=0}^{k-1} \binom{k-1}{\ell} (-1)^\ell a_{k-\ell} f(\ell)(1) + \frac{(-1)^{k-1} f(1)}{k} \] for \( k \geq 2 \).

Finally for

\[ J_3(1) = \frac{1}{2\pi i} \int_{s-i\infty}^{s+i\infty} \zeta(s) \log^2(\zeta(s)) \frac{x^s}{s} \, ds \]

let

\[ L_3' = \frac{1}{2\pi i} \int_{\mathcal{C}} \zeta(s) \{ \log(\zeta(s)) \}^2 \frac{x^s}{s} \, ds \]

\[ \quad = \frac{1}{2\pi i} \int_{\mathcal{C}} \zeta(s) \left\{ \log(\zeta(s)(s-1)) + \log\left(\frac{1}{s-1}\right) \right\}^2 \frac{x^s}{s} \, ds \]

\[ \quad = \frac{1}{2\pi i} \int_{\mathcal{C}} \zeta(s) \log^2(\zeta(s)(s-1)) \frac{x^s}{s} \, ds \]

\[ \quad + \frac{2}{2\pi i} \int_{\mathcal{C}} \zeta(s) \log(\zeta(s)(s-1)) \log\left(\frac{1}{s-1}\right) \frac{x^s}{s} \, ds \]

\[ \quad + \frac{1}{2\pi i} \int_{\mathcal{C}} \zeta(s) \log^2\left(\frac{1}{s-1}\right) \frac{x^s}{s} \, ds = K_1 + K_2 + K_3. \]

Here, \( K_1 = 0 \) because \( \zeta(s) \log^2(\zeta(s)(s-1)) \frac{x^s}{s} \) is analytic inside the contour \( C \).
\[ K_2 = \frac{2}{2\pi i} \int_c \left( \zeta(s) - \frac{1}{s-1} \right) \log \frac{1}{s-1} \log(\zeta(s)(s-1)) \frac{x^s}{s} \, ds \]

\[ + \frac{2}{2\pi i} \int_c \frac{1}{s-1} \log \frac{1}{s-1} \log(\zeta(s)(s-1)) \frac{x^s}{s} \, ds \]

\[ = K_2(1) + K_2(2). \]

For \( K_2(1) \) use Lemma 3

\[ K_2(1) = \sum_{k=1}^{n-1} 2 \, g'_k \, J'_{k-1} + O(J'_n) \]

\[ = x \sum_{k=2}^{n} \frac{h'_k}{(\log x)^k} + O(\frac{x}{(\log x)^{n+1}}) \]

with

\[ h'_k = 2(k-1)! \sum_{j=1}^{k-1} (-1)^j g'_j = 2(k-1)! \sum_{j=1}^{k-1} (-1)^j A_j \sum_{m=0}^{k-j-1} \frac{\gamma_m}{m!} \]

for \( k \geq 2 \), e.g., \( h'_2 = -4 \gamma^2 \).

Now for \( K_2(2) \) use Lemma 2 to get

\[ K_2(2) = \sum_{k=1}^{n} 2 \, A_k \, J'_k_{k-1} + O(J'_n) = \sum_{j=0}^{n-1} (2 \, A_{j+1}) J'_j + O(J'_n) \]

\[ = x \sum_{k=1}^{n} \frac{h''_k}{(\log x)^k} + O(\frac{x}{(\log x)^{n+1}}) \]

with
\[ h_k'' = 2(k-1)! \sum_{j=0}^{k-1} (-1)^j A_{j+1}. \]

For example,

\[ h_1'' = 2A_1 = 2\gamma \quad \text{and} \quad h_2'' = 2(A_1 - A_2) = 2\left(\gamma + \gamma_1 + \frac{\gamma^2}{2}\right). \]

Therefore

\[ K_2 = x \sum_{k=1}^{n} \frac{h_k}{(\log x)^k} + O\left(\frac{x}{(\log x)^{n+1}}\right) \]

with \[ h_1 = h_1'' = 2\gamma \quad \text{and} \]

\[ h_k = h_k' + h_k'' = 2(k-1)! \left(\sum_{j=1}^{k-1} (-1)^j (A_{j+1} + \gamma_j') + \gamma\right) \quad \text{for} \quad k \geq 2, \]

(e.g., \[ h_2 = 2(\gamma + \gamma_1 - \frac{\gamma^2}{2})\]).

For \( K_3 \), use Lemma 1,

\[ K_3 = \frac{1}{2\pi i} \int_{C} \zeta(s) \log^2 \left(\frac{1}{s-1}\right) \frac{X^s}{s} ds \]

\[ = \frac{1}{2\pi i} \int_{C} \frac{1}{s-1} \log^2 \left(\frac{1}{s-1}\right) \frac{X^s}{s} ds + \sum_{k=0}^{\infty} d_k \frac{1}{2\pi i} \int_{C} (s-1)^k \log^2 \left(\frac{1}{2-1}\right) \frac{X^s}{s} ds \]

\[ = K_3(-1) + \sum_{k=0}^{\infty} d_k K_3(k), \]

where
\[ K_3(k) = \frac{1}{2\pi i} \int_C (s-1)^k \log \left( \frac{1}{s-1} \right) \frac{x^s}{s} \, ds \]

for \( k = -1, 0, 1, 2, \ldots \).

Now
\[ K_3(-1) = \frac{1}{2\pi i} \int_C \frac{1}{s-1} \log \left( \frac{1}{s-1} \right) \frac{x^s}{s} \, ds = \frac{1}{2\pi i} \int_C \frac{1}{s-1} \log \left( \frac{1}{s-1} \right) x^s \, ds \]
\[ - \frac{1}{2\pi i} \int_C \log \left( \frac{1}{s-1} \right) \frac{x^s}{s} \, ds = K_3'(-1) + K_3''(-1) \]

On page 20 of Diaconis (1976), it has been shown that

\[ \frac{1}{x} K_3'(-1) = (\log \log x)^2 + 2 \gamma \log \log x + \gamma^2 - \frac{\pi^2}{6} + O\left(e^{-c(\log x)^{1/10}}\right). \]

Also

\[ \frac{1}{x} K_3''(-1) = \frac{2}{\log x} \int_0^\delta \log y \, e^{-y(1 - \frac{y}{\log x})^{-1}} \, dy \]
\[ - \frac{2 \log \log x}{\log x} \int_0^\delta \log y \, e^{-y(1 - \frac{y}{\log x})^{-1}} \, dy, \]

where \( \delta = c(\log x)^{-9/10} \). Substituting \((1-t)^{-1} = \sum_{k=0}^{n-1} t^k + O(t^n)\)

for \( t = \frac{y}{\log x} \) implies that

\[ \frac{1}{x} K_3''(-1) = \frac{2}{\log x} \sum_{k=0}^{n-1} \frac{j_k'''}{\log x} \sum_{k=0}^{n-1} \frac{j_k'''}{(\log x)^k} + 0\left(\frac{j_n'''}{\log x^{n+1}}\right) + O\left(\frac{\log \log x}{(\log x)^{n+1}}\right) \]

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with

\[ J''_k = \int_0^\delta \log x \log y e^{-y} y^k \, dy \quad \text{and} \quad J'''_k = -\int_0^\delta \log x e^{-y} y^k \, dy \]

for \( k = 0, 1, 2, \ldots \).

\[ J'''_0 = \int_0^\delta \log x - e^{-y} \, dy = e^{-c(\log x)^{1/10}} - 1 \]

and in general by using the corresponding formula on page 310 of Gradshteyn and Ryzhik (1965) with \( u = 1 \) and \( u = \delta \log x \) we will get

\[ J'''_k = -\int_0^\delta \log x e^{-y} y^k \, dy = -k! + e^{-\delta \log x} \sum_{j=0}^k \frac{k!}{j!} (\delta \log x)^j \]

\[ = e^{-c(\log x)^{1/10}} \sum_{j=0}^k \frac{k!}{j!} (c)^j (\log x)^{j/10} - k! \]

We will also get

\[ \frac{2 \log \log x}{\log x} \sum_{k=0}^{n-1} \frac{J'''_k}{(\log x)^k} = \log \log x \left( \sum_{k=0}^{n-1} \frac{-2k!}{(\log x)^{k+1}} + O(e^{-c(\log x)^{1/10}}) \right) \]

\[ = \log \log x \left( \sum_{k=1}^n \frac{l''_k}{(\log x)^k} + O\left( \frac{1}{(\log x)^{n+1}} \right) \right) \]

with \( l''_k = -2(k-1)! \) for \( k \geq 1 \) (e.g., \( l''_1 = -2 \)). Also for
\[ J''_k = \int_0^\delta \log y \, e^{-y} \, y^k \, dy, \text{ use the corresponding formula on page 576 of Gradshteyn and Ryzhik (1965) with } \mu = 1: \]

\[
\Gamma'(k+1) = \int_0^\infty \log y \, e^{-y} \, y^k \, dy = k! \left[ \sum_{j=1}^{k} \frac{1}{j} - \gamma \right] \text{ for } k \geq 1
\]

\[ = -\gamma \quad \text{for } k = 0. \]

Now use Theorem 1 on page 281 of Feller (1970) for slowly varying function \( \log y \, e^{-y} \) to get

\[
-\int_0^\infty \log y \, e^{-y} \, y^k \, dy \approx \frac{(\delta \log x)^{k+1}}{k+1} \left( \log \delta + \log \log x \right) e^{-\delta \log x}
\]

and therefore

\[
J''_k = \Gamma'(k+1) + \frac{(\delta \log x)^{k+1}}{k+1} \left( \log \delta + \log \log x \right) e^{-\delta \log x}
\]

as \( x \to \infty \),

or

\[
\frac{2}{\log x} \sum_{k=0}^{n-1} \frac{J''_k}{(\log x)^k} = \frac{2}{\log x} \sum_{k=1}^{n} \frac{\Gamma'(k)}{(\log x)^k} + \sum_{k=1}^{n} \frac{2 \delta^k \log \delta}{k} e^{-\delta \log x} + \log \log x \sum_{k=1}^{n} \frac{2 \delta^k}{k} e^{-\delta \log x}
\]

\[ = \sum_{k=1}^{n} \frac{\ell'_k}{(\log x)^k} + O\left( e^{-c(\log x)^{1/10}} \right) + O\left( \log \log x \, e^{-c(\log x)^{1/10}} \right)\]
with $k'_k = 2 \Gamma'(k)$ i.e., $k'_1 = -2 \gamma$ and $k'_k = 2(k-1)! \left( \sum_{j=1}^{k-1} \frac{1}{j} - \gamma \right)$ for $k = 2, 3, \ldots$.

Also we have

$$0 \left( \frac{J''}{(\log x)^{n+1}} \right) = O \left( \frac{1}{(\log x)^{n+1}} \right) \quad \text{and}$$

$$0 \left( \frac{J'''}{(\log x)^{n+1}} \right) = O \left( \frac{1}{(\log x)^{n+1}} \right).$$

So

$$\frac{1}{x} K_3''(-1) = \sum_{k=1}^{n} \frac{k'_k}{(\log x)^k} + \log \log x \sum_{k=1}^{n} \frac{k''_k}{(\log x)^k}$$

$$+ O \left( \frac{\log \log x}{(\log x)^{n+1}} \right),$$

and

$$K_3''(-1) = x \left\{ (\log \log x)^2 + \sum_{k=0}^{n} \frac{k'_k}{(\log x)^k} + \log \log x \sum_{k=0}^{n} \frac{k''_k}{(\log x)^k} \right\}$$

$$+ O \left( \frac{\log \log x}{(\log x)^{n+1}} \right)$$

with

$$k'_0 = \gamma^2 - \frac{\pi^2}{6}, \quad k'_1 = -2 \gamma, \quad k'_k = 2(k-1)! \left( \sum_{j=1}^{k-1} \frac{1}{j} - \gamma \right) \quad \text{for} \ k = 2, 3, \ldots$$

and $k''_0 = 2 \gamma, \ k''_k = -2(k-1)! \quad \text{for} \ k = 1, 2, 3, \ldots$.
But
\[
K_3(0) = \frac{1}{2\pi i} \int_c \log^2(\frac{1}{s-1}) \frac{x^s}{s} \, ds = -K_3''(-1).
\]

Similar arguments show that \( K_3(k) = o\left(\frac{x \log \log x}{(\log x)^2}\right) \) for \( k = 1, 2, 3, \ldots \), therefore we have

\[
K_3 = x \left\{ (\log \log x)^2 + \log \log x \sum_{k=0}^{n} \frac{m_k''}{(\log x)^k} + \sum_{k=0}^{n} \frac{m_k'}{(\log x)^k} \right. \\
+ o\left(\frac{\log \log x}{(\log x)^{n+1}}\right) \left\}\right.
\]

and

\[
J_3(1) = x \left\{ (\log \log x)^2 + \log \log x \sum_{k=0}^{n} \frac{m_k''}{(\log x)^k} + m_0' + \sum_{k=1}^{n} \frac{m_k'}{h_k} \right. \\
+ o\left(\frac{\log \log x}{(\log x)^{n+1}}\right) \left.\right\},
\]

with

\[
m_0'' = \ell_0'' = 2 \gamma
\]

\[
m_k'' = \ell_k'' (1-\gamma) = 2(k-1)! (\gamma-1) \quad \text{for} \quad k = 1, 2, \ldots
\]

and

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\[ m_0' = l_0' = \gamma^2 - \frac{\pi^2}{6} \]

\[ m_1' = l_1'(1-\gamma) = 2\gamma(\gamma-1) \]

and

\[ m_k' = l_k'(1-\gamma) = 2(k-1)! \left( \gamma - \sum_{j=1}^{k-1} \frac{1}{j} \right) (\gamma-1) \]

for \( k = 2,3,\ldots \).

So

\[
\frac{1}{x} \sum_{m \leq x} \omega^2(m) = \left\{ \begin{array}{l}
(\log \log x)^2 + \log \log x \sum_{k=0}^{n-1} \frac{n_k}{(\log x)^k} \\
+ \sum_{k=0}^{n-1} \frac{n_k'}{(\log x)^k} + O\left( \frac{\log \log x}{(\log x)^{n+1}} \right) \end{array} \right\}
\]

with

\[ n_0'' = m_0'' - 2f(1) + 1 = 2\gamma - 2f(1) + 1 \]

\[ n_k'' = m_k'' = 2(k-1)! (\gamma-1) \quad \text{for} \quad k = 1,2,3,\ldots \]

and

\[ n_0' = m_0' - 2f_0 + f^2(1) + a_0 - \sum_{p \leq \sqrt{x}} \frac{1}{p} = (\gamma-f(1))^2 + b_1 - \sum_{p \leq \sqrt{x}} \frac{1}{p} - \frac{\pi^2}{6} \]

\[ n_1' = m_1' + h_1 - 2f_1 + a_1 = 2\gamma^2 - 2f(1)(\gamma-1) - 2f'(1) + \gamma - 1 \]

\[ n_k' = m_k' + h_k - 2f_k + a_k \quad \text{for} \quad k = 2,3,4,\ldots . \]
Now subtract \( \mu_x(\omega) \) from \( \frac{1}{x} \sum_{m \leq x} \omega^2(m) \) to get the results of Theorem 4 and Theorem 5.

Diaconis, Mosteller and Onishi (1977) have shown that

\[
\text{Var}_x(\Omega) = \frac{1}{x} \sum_{m \leq x} (\Omega(m) - \mu_x(\Omega))^2 = \log \log x + B_4
\]

\[+ O\left(\frac{\log \log x}{\log x}\right)\]

where \( B_4 = B_4 - \frac{\pi^2}{6} + \sum \frac{(2 \cdot p - 1)}{p \cdot p(p-1)^2} \).

We shall prove:

**Theorem 6.** For any non-negative integer \( n \),

\[
\text{Var}_x(\Omega) = \log \log x \sum_{k=0}^{n} \frac{b'_k}{(\log x)^k} + \sum_{k=0}^{n} \frac{c'_k}{(\log x)^k} + O\left(\frac{\log \log x}{(\log x)^{n+1}}\right)
\]

as \( x \to \infty \), where

\[b'_0 = 1\]

\[b'_1 = 0\]

\[c'_0 = B_4 = B_4 - \frac{\pi^2}{6} + \sum \frac{(2 \cdot p - 1)}{p \cdot p(p-1)^2}\]

\[c'_1 = 3 \gamma - 1 - 2 \sum \frac{\log p}{p \cdot (p-1)^2}\]

and
\[ b'_k = 2(k-1)! \ a_1 - 2 \ a_k, \]
\[ c'_k = m'_k + h_k - 2 \ g_k + a_k - \sum_{j=1}^{k-1} a_{k-j} \ a_j - 2 \ a'_0 \ a_k \]

with
\[ m'_k = 2(k-1)! \left( \gamma - \sum_{j=1}^{k-1} \frac{1}{j} \right) a_1 \]
\[ h_k = 2(k-1)! \left( \sum_{j=1}^{k-1} (-1)^j (A_{j+1} + e'_j) + \gamma \right) \]

and
\[ g_k = \sum_{j=0}^{k-1} (-1)^{k-1} a_{k-j} g^{(j)}(1) + \frac{(-1)^{k-1}}{k} g^{(k)}(1) \text{ for } k \geq 2 \]

with
\[ a'_0 = B_2 = \gamma + \sum_p \left( \log(1 - \frac{1}{p}) + \frac{1}{p-1} \right) \]

and
\[ a_j = j! \left( \frac{1}{\sum_{m=0}^{j-1} \frac{\gamma_m}{m!}} - 1 \right) \text{ for } j = 1, 2, \ldots \]

also \( A_1 = \gamma \) and for \( j = 1, 2, 3, \ldots \)
\[ A_{j+1} = \frac{(-1)^j \gamma_j}{j!} - \frac{1}{j+1} \sum_{m=1}^{j-m} \frac{m(-1)^{j-m} \gamma_{j-m}}{(j-m)!} A_m, \]
\[ e'_j = \sum_{m=1}^{j} \frac{(-1)^{j-m} \gamma_{j-m} A_m}{(j-m)!} \]
and finally

\[ g^{(j)}(1) = \lim_{s \to 1} \frac{g^j}{s^j} \left( \sum_{p \geq 2} \sum_{k=2}^{\infty} \frac{(-1)^{k-1}}{k(p^s-1)^k} \right) \]

\[ \gamma_j = \lim_{N \to \infty} \left[ \log^j t - \log^j N \right] \]

for \( j = 0, 1, 2, \ldots \)

and \( \gamma = \gamma_0 \) (Euler's constant).

Proof.

\[ \sum_{\Omega^2(m)/m^s} = \zeta(s) \left\{ \sum_{p} \frac{1}{p} \left( \frac{1}{p^s-1} \right)^2 + \left( \sum_{p} \frac{1}{p^s-1} \right)^2 \right\} \]

\[ = \zeta(s) \sum_{p} \frac{1}{p^s-1} + \zeta(s) \frac{1}{p} \left( \frac{1}{p^s-1} \right)^2 + \zeta(s) \left( \sum_{p} \frac{1}{p^s-1} \right)^2. \]

By Theorem 3.1 on page 50 of Ayoub (1963) for \( a = 1 + \frac{c}{(\log T)^9} \)

where \( c \) is a positive constant and \( \log T = c(\log x)^{1/10} \) we have

\[ \sum_{m \leq x} \Omega^2(m) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \zeta(s) \left( \sum_{p} \frac{1}{p^s-1} \right) \frac{x^s}{s} ds \]

\[ + \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \zeta(s) \sum_{p} \frac{1}{p} \left( \frac{1}{p^s-1} \right)^2 \frac{x^s}{s} ds \]

\[ + \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \left( \sum_{p} \frac{1}{p^s-1} \right)^2 \zeta(s) \frac{x^s}{s} ds = S_1 + S_2 + S_3. \]

\[ S_1 + S_2 \text{ (in Theorem 3)} = x \left\{ \log \log x + \sum_{k=0}^{n} \frac{a_k}{(\log x)^k} \right\} + O \left( \frac{1}{(\log x)^{n+1}} \right). \]
For $S_2$, break the integral over the paths 1 through 7 as indicated in Figure 2.

Since $\zeta(s) \sum \frac{1}{p} \frac{1}{(p^s-1)^2}$ has a simple pole at $s = 1$ with residue $\sum \frac{1}{p} \frac{1}{(p-1)^2}$ and other parts of the integral are $O(x e^{-c(\log x)^{1/10}})$ (see the proof of Theorem 4), then

$$S_2 = x \left\{ \sum \frac{1}{p} \frac{1}{(p-1)^2} + O\left( e^{-c(\log x)^{1/10}} \right) \right\}.$$  

$$S_3 = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \zeta(s) \left\{ \log(\zeta(s)) - g(s) \right\}^2 \frac{x^s}{s} \, ds$$

$$= \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \zeta(s) \log^2(\zeta(s)) \frac{x^s}{s} \, ds$$

$$- \frac{2}{2\pi i} \int_{a-i\infty}^{a+i\infty} \zeta(s) \log(\zeta(s)) \frac{x^s}{s} \, ds$$

$$+ \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \zeta(s) \frac{x^s}{s} \, ds$$

$$= S_3(1) + S_3(2) + S_3(3).$$

Since $\zeta(s) g(s)$ has a simple pole at $s = 1$ with residue $g^2(1)x$ then

$$S_3(3) = x \left\{ g^2(1) + O\left( e^{-c(\log x)^{1/10}} \right) \right\}.$$
Computation of \( S_3(2) \) is the same as \( J_3(2) \) (in Theorem 4), but instead of \( f(s) \), we have \( g(s) = \sum p \sum \frac{(-1)^{k-1}}{k(p-1)^k} \) as defined in Theorem 3. So

\[
S_3(2) = x \left\{-2 g(1) \log \log x - \sum_{k=0}^{n} \frac{2 g_k}{(\log x)^k} + O\left(\frac{1}{(\log x)^{n+1}}\right)\right\}
\]

with

\[
g_0 = \gamma g(1), \quad g_1 = g'(1) + (\gamma-1) g(1)
\]

and

\[
g_k = \sum_{k=0}^{k-1} \binom{k-1}{\ell} (-1)^{\ell} a_{k-\ell} g^{(\ell)}(1) + \frac{(-1)^{k-1}}{k} g^{(k)}(1) \quad \text{for } k \geq 2.
\]

Also

\[
S_3(1) = J_3(1) \quad \text{(in Theorem 4)}
\]

\[
= x \left\{(\log \log x)^2 + \log \log x \sum_{k=0}^{n} \frac{m''_k}{(\log x)^k} + \left(\sum_{k=1}^{n} \frac{m'_k + h_k}{(\log x)^k} + O\left(\frac{\log \log x}{(\log x)^{n+1}}\right)\right)\right\}
\]

where \( m''_k, m'_k, h_k \) are defined in the proof of Theorem 4.

Now substituting \( S_3(1), S_3(2) \) and \( S_3(3) \) in the formula for \( \sum \theta^2(m) \) gives us:
\[
\frac{1}{x} \sum_{m \leq x} \Omega^2(m) = \left\{ (\log \log x)^2 + \log \log x \sum_{k=0}^{n} \frac{t_k}{(\log x)^k} + \sum_{k=0}^{n} \frac{t'_k}{(\log x)^k} + O\left(\frac{\log \log x}{(\log x)^{n+1}}\right) \right\}
\]

with

\[
t''_0 = m''_0 - 2 \cdot g(1) + 1 = 2 \gamma - 2 \cdot g(1) + 1
\]

\[
t''_k = m''_k = 2^{(k-1)!} (\gamma-1) \quad \text{for} \quad k = 1, 2, 3, \ldots
\]

and

\[
t'_0 = m'_0 - 2 \cdot g_0 + g^2(1) + a'_0 + \frac{1}{p \cdot (p-1)^2} = (\gamma-g(1))^2 + b_h,
\]

\[
t'_1 = m'_1 + h_1 - 2 \cdot g_1 + a_1 = 2 \gamma^2 - 2 \cdot g(1)(\gamma-1) - 2 \cdot g'(1) + \gamma - 1 \quad \text{and}
\]

\[
t'_k = m'_k + h'_k - 2 \cdot g_k + a_k \quad \text{for} \quad k = 2, 3, 4, \ldots
\]

Now subtract \( \mu_x^2(\Omega) \) (obtained in Theorem 3) from \( \frac{1}{x} \sum_{m \leq x} \Omega^2(m) \)
to get the result.

II.3 \( \Omega - \omega \).

In Section 1 of this chapter we have shown that for any non-negative integer \( n \),

\[
\mu_x(\Omega - \omega) = \frac{1}{x} \sum_{m \leq x} (\Omega(m) - \omega(m)) = \sum_{p} \frac{1}{p(p-1)} + O\left(\frac{1}{(\log x)^{n+1}}\right)
\]

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Diaconis, Mosteller and Onishi (1977) have shown that:

\[
\text{Var}_x(\Omega - \omega) = \frac{1}{x} \sum_{m \leq x} \left( \Omega(m) - \omega(m) - \mu_x(\Omega - \omega) \right)^2 \]

\[
= \sum \frac{p^2 + p - 1}{p^2(p-1)^2} + o\left( \frac{\log \log x}{\log x} \right).
\]

We shall prove:

**Theorem 7.** For any non-negative integer \( n \),

\[
\text{Var}_x(\Omega - \omega) = \sum \frac{p^2 + p - 1}{p^2(p-1)^2} + o\left( \frac{\log \log x}{(\log x)^{n+1}} \right)
\]

as \( x \to \infty \).

In order to prove Theorem 7, we need to compute the covariance between \( \omega \) and \( \Omega \).

Diaconis, Mosteller and Onishi (1977) have shown that

\[
\text{Cov}_x(\Omega, \omega) = \frac{1}{x} \sum_{m \leq x} \left( \Omega(m) - \mu_x(\Omega) \right) \left( \omega(m) - \mu_x(\omega) \right)
\]

\[
= \log \log x + B_1 - \frac{\pi^2}{6} + o\left( \frac{\log \log x}{\log x} \right).
\]

They have also used this formula to show that \( \Omega \) and \( \omega \) are asymptotically linearly dependent by showing that

\[
R_x(\omega, \Omega) = \frac{\text{Cov}_x(\omega, \Omega)}{[\text{Var}_x(\omega) \text{Var}_x(\Omega)]^{1/2}} \text{ is asymptotically } 1,
\]

(see Chapter III).
We will prove a stronger result than the above formula for the covariance between $\Omega$ and $\omega$.

**Lemma 6.** For any non-negative integer $n$,

$$\text{Cov}_x(\Omega, \omega) = \left\{ \log \log x \sum_{k=0}^{n} \frac{b_k''}{(\log x)^k} + \sum_{k=0}^{n} \frac{c_k''}{(\log x)^{k+1}} \right\} + O\left(\frac{\log \log x}{(\log x)^{n+1}}\right) \quad \text{as} \quad x \to \infty,$$

with

$$b_0'' = 1, \quad b_1'' = 0, \quad b_k'' = 2(k-1)! (\gamma - 1) - 2 a_k \quad k = 2, 3, \ldots$$

and

$$c_0'' = B_1 - \frac{\pi^2}{6}, \quad c_1'' = 3 \gamma - 1 - f'(1) - g'(1),$$

$$c_k'' = a_k (1 - 2\gamma + f(1) + g(1)) + \sum_{k=1}^{k-1} a_{k-j} a_j$$

for $k = 2, 3, \ldots$.

**Proof.** For $\Re s > 1$, we have

$$(I.8) \quad \sum_{m=1}^{\infty} \frac{\omega(m) \Omega(m)}{m^s} = \zeta(s) \left\{ \sum_{p} \frac{1}{p^s} + \left( \sum_{p} \frac{1}{p^s} \right)^2 \right\}.$$
So by the use of Theorem 3.1 on page 50 of Ayoub (1963) and the notation introduced in the proofs of Theorems 1, 3, 4 and 6 and Figure 2, we have

\[ \sum_{m \leq x} \omega(m) \Omega(m) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \zeta(s) \left( \sum_{p} \frac{1}{p} \right) \frac{x^s}{s} \, ds \]

\[ + \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \zeta(s) \left( \sum_{p} \frac{1}{p^s} \right) \left( \sum_{p \leq \sqrt{x}} \frac{1}{p-1} \right) \frac{x^s}{s} \, ds \]

\[ = I + \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \frac{x^s}{s} \zeta(s) \{ \log \zeta(s) - f(s) \} \{ \log \zeta(s) - g(s) \} \, ds \]

\[ = I + \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \zeta(s) \log^2(\zeta(s)) \frac{x^s}{s} \, ds - \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \log(\zeta(s)) f(s) \zeta(s) \frac{x^s}{s} \, ds \]

\[ - \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \zeta(s) \log(\zeta(s)) g(s) \frac{x^s}{s} \, ds + \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \zeta(s) f(s) g(s) \frac{x^s}{s} \, ds \]

\[ = I + J_3(1) + \frac{1}{2} J_3(s) + \frac{1}{2} S_3(2) + P \]

where

\[ P = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \zeta(s) f(s) g(s) \frac{x^s}{s} \, ds. \]

Using the same method that we have used for obtaining \( I_2 \) in Theorem 1 gives us

\[ P = x \left\{ f(1) g(1) + O \left( \frac{1}{(\log x)^{n+1}} \right) \right\} \]

and therefore,
\[
\frac{1}{x} \sum_{m \leq x} \omega(m) \Omega(m) = \left\{ \right.
\begin{align*}
& (\log \log x)^2 + \log \log x \left( m''_0 + 1 - f(1) - g(1) \right) \\
& + \log \log x \sum_{k=1}^{n} \frac{m_k''}{(\log x)^k} + \left( m'_0 + a_0 + f(1) g(1) - f_0 - g_0 \right) \\
& + \sum_{k=1}^{n} \frac{a_k + m_k' + h_k - f_k - g_k}{(\log x)^k} + o\left( \frac{\log \log x}{(\log x)^{n+1}} \right) \Bigg\}.
\end{align*}
\]

Now subtracting \( \frac{1}{x} \sum_{m \leq x} \omega(m) \Omega(m) \) from \( \frac{1}{x} \sum_{m \leq x} \omega(m) \Omega(m) \) concludes the result.

For the notation see the statements of the theorems 4 and 6.

Note that \( b''_k = -2(k-1)! \sum_{j=1}^{k-1} \frac{\gamma_j}{j!} \).

**Proof of Theorem 5.**

\[\text{Var}(\Omega-\omega) = \text{Var } \Omega + \text{Var } \omega - 2 \text{ Cov}(\Omega, \omega), \] use Theorems 4, 6 and Lemma 6 to get the result.

**Final Note.** Throughout this chapter:

1 - \( c \) is a positive constant which may be different in different formulas so that the conditions of the theorems we are using show that the formulas are satisfied.

2 - \( x \) is not an integer because of the Theorem 3.1 on page 50 of Ayoub (1963).

3 - Fubini's theorem frequently is used to show that the remainder terms are of the order \( \frac{1}{(\log x)^{n+1}} \). (See Theorem 7.8 on page 150 of Rudin (1974).)
4 - T satisfies \( \log T = c(\log x)^{1/10} \) in order that the remainder terms are of the order \( \frac{1}{(\log x)^{n+1}} \). For details see page 13 of Diaconis (1976).

To end this chapter we will give approximate numerical values for constants appearing in the theorems of this chapter, neglecting terms of order smaller than \( \frac{1}{(\log x)^2} \). The values are known accurate to at least three figures past the decimal point.

\[
\mu_x(\omega) = \log \log x + .2615 + \frac{.4228}{\log x} + O\left(\frac{1}{(\log x)^2}\right)
\]

\[
\mu_x(\Omega) = \log \log x + 1.0346 + \frac{.4228}{\log x} + O\left(\frac{1}{(\log x)^2}\right)
\]

\[
\mu_x(\Omega-\omega) = .7731 + O\left(\frac{1}{(\log x)^n}\right), \text{ for all } n
\]

\[
\text{Var}_x(\omega) = \log \log x - 1.8357 + \frac{2.2322}{\log x} + O\left(\frac{\log \log x}{(\log x)^2}\right)
\]

\[
\text{Var}_x(\Omega) = \log \log x + .7648 - \frac{2.1164}{\log x} + O\left(\frac{\log \log x}{(\log x)^2}\right)
\]

\[
\text{Cov}_x(\omega, \Omega) = \log \log x - 1.3834 + \frac{0.0578}{\log x} + O\left(\frac{\log \log x}{(\log x)^2}\right)
\]

\[
\text{Var}_x(\Omega-\omega) = 3.8377 + O\left(\frac{\log \log x}{(\log x)^n}\right), \text{ for all } n.
\]
CHAPTER III

III.1 Logarithm Function.

For a positive (real-valued) multiplicative function \( g \), \( f = \log g \) is additive and therefore the analytic method that we introduced in previous chapters can be applied to get some information about the distribution of \( \log g \).

One of the most commonly occurring multiplicative functions is the Euler totient function \( \phi(n) \), which is the number of positive integers not exceeding \( n \) and relatively prime to \( n \). As we will see in Section 5, for deriving any moment of \( \log \phi \), we need to have the same moment of \( F(n) = \log n \) which itself is a well known completely additive arithmetic function. The mean of \( \log n \) is known from Stirling's formula

\[
\frac{1}{n} \sum_{j \leq n} \log j = \log n - 1 + \log \frac{n^2}{2n} + O\left(\frac{1}{\sqrt{n}}\right), \quad \text{as } n \to \infty,
\]

(page 358 of Ayoub (1963)). We apply the analytic method in order to compare it with other techniques. This method does not give us the best known reminder terms for \( \log n \).

For a classical approach to Stirling's formula via complex analysis see Ahlfors (1966), pages 199-204.
To find the mean of \( F \) take the derivative of \( \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \) with respect to \( s \), for \( \text{Re } s > 1 \) (Theorem 11.12 on page 263 of Apostol (1976)) to get

\[
(\text{III.1}) \quad \sum_{n=1}^{\infty} \frac{\log n}{n^s} = -\zeta'(s)
\]

[This formula can also be derived by using Formula (I.11).] Now use Theorem 3.1 on page 50 of Ayoub (1963).

If \( x \) is a non integer real number, then

\[
\log[x]! = \log \prod_{n \leq x} n = \sum_{n \leq x} \log n = -\frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \zeta'(s) \frac{x^s}{s} \, ds
\]

for any \( a > 1 \). Let \( a = 1 + \frac{1}{\log T} \) and \( b = 1 - \frac{1}{\log T} \) where \( \log T = \sqrt{2} \log x \) and consider Figure 2.

\[
J = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \zeta'(s) \frac{x^s}{s} \, ds = \frac{1}{2\pi i} \int_{a-i\infty}^{a-iT} \zeta'(s) \frac{x^s}{s} \, ds + \frac{1}{2\pi i} \int_{a-iT}^{b-iT} \zeta'(s) \frac{x^s}{s} \, ds
\]

\[
+ \frac{1}{2\pi i} \int_{b-iT}^{b-iT} \zeta'(s) \frac{x^s}{s} \, ds + \frac{1}{2\pi i} \int_{c}^{b+iT} \zeta'(s) \frac{x^s}{s} \, ds + \frac{1}{2\pi i} \int_{b}^{b+i\infty} \zeta'(s) \frac{x^s}{s} \, ds
\]

\[
+ \frac{1}{2\pi i} \int_{b+iT}^{a+i\infty} \zeta'(s) \frac{x^s}{s} \, ds + \frac{1}{2\pi i} \int_{a+iT}^{a+i\infty} \zeta'(s) \frac{x^s}{s} \, ds
\]

\[
= J_6 + J_5 + J_4 + J_7 + J_3 + J_2 + J_1.
\]
(Note that \( \zeta'(s) \frac{x^s}{s} \) is analytic inside the contour bounded by the lines 2, 3, 4, 5, 8 and the circle C of Figure 2.)

Now \( \zeta'(s) \frac{x^s}{s} \) has a pole at \( s = 1 \) with residue

\[
\lim_{s \to 1} \frac{3}{6} (s-1)^2 \zeta'(s) \frac{x^s}{s} = -x \log x + x,
\]

\[
(s-1)^2 \zeta'(s) = -1 + \sum_{n=1}^{\infty} n \frac{d_n}{n} (s-1)^{n+1},
\]

so \( J_7 = x - x \log x \).

Let \( E' = J_1 + J_2 + J_3 + J_4 + J_5 + J_6 \), we will show that

\[
E' = O\{x \log^2 x e^{-c\sqrt{\log x}}\} \text{ as } x \to \infty.
\]

\[
J_1 = \lim_{U \to \infty} \frac{1}{2\pi i} \int_{a+iT}^{a+iU} \zeta'(s) \frac{x^s}{s} ds = \lim_{U \to \infty} \frac{1}{2\pi i} \int_{a+iT}^{a+iU} -\sum_{n=1}^{\infty} \frac{\log n}{n} \frac{x^s}{s} ds
\]

\[
= \lim_{U \to \infty} -\sum_{n=1}^{\infty} \frac{\log n}{n} \frac{x^s}{s} \int_{a+iT}^{a+iU} (\frac{x}{n})^{s} ds
\]

here we have used \( \zeta'(s) = -\sum_{n=1}^{\infty} \frac{\log n}{n} \) (because \( \text{Re } s = a > 1 \)) and Fubini's theorem

\[
\left( \sum_{n=1}^{\infty} \log n \int_{T}^{U} \frac{x^n a}{\sqrt{a^2 + t^2}} dt \right) < \frac{x}{a} \sum_{n=1}^{\infty} \frac{\log n}{n} (U-T) < \infty
\]

According to page 66 of Ayoub (1963)
\[
\int_{a+iT}^{a+iU} \left( \frac{x^n}{n} \right)^s \frac{ds}{s} = O\left( \frac{x^n}{T \log \frac{x}{n}} \right)
\]

and by the arguments on pages 12 and 13 of Diaconis (1976) and the use of Fubini's theorem we have

\[
J_1 = 0\left( \frac{x^n}{T} \sum_{n=1}^{\infty} \frac{n^{-a} \log n}{|\log \frac{x}{n}|} \right) = 0\left\{ \frac{x^n}{T} \left( \frac{\log x}{a-1} + (\log x)^2 x^{1-a} \right) \right\}.
\]

Also

\[
J_6 = 0\left\{ \frac{x^n}{T} \left( \frac{\log x}{a-1} + (\log x)^2 x^{1-a} \right) \right\}.
\]

Now by using Theorem 4.3 and its corollary on pages 59 and 60 of Ayoub (1963) we have \( \zeta'(s) = O(\log^2 T) \) in the rectangle with vertices \( b-iT, b+iT, 2+iT, 2-iT \) when \( T \geq e^2 \). So

\[
J_2 = \frac{1}{2\pi i} \int_{b+iT}^{a+iT} \zeta'(s) \frac{x^s}{s} ds
\]

\[
= O\left( \int_b^a |\zeta'(\sigma+iT)| \left| \frac{x^{\sigma+iT}}{\sigma+iT} \right| d\sigma \right)
\]

\[
= O\left( \log^2 T \int_b^a \frac{x^\sigma}{T} d\sigma \right) = O\left( \frac{x^a}{T} \log^2 T \right).
\]

Also \( J_5 = O(\log^2 T \frac{x^a}{T}) \).
And finally

\[ J_3 + J_4 = \int_{b-iT}^{b+iT} \zeta'(s) \frac{x^s}{s} \, ds \]

\[ = 0 \left( \log^2 T x^b \int_0^T \frac{dt}{\sqrt{b^2 + t^2}} \right) \]

\[ = 0(x^b \log^3 T). \]

So:

(III.2) \[ E' = O \left\{ \frac{x^a \log x}{T(a-1)} + \frac{x \log^2 x}{T} + \frac{x^a \log^2 T}{T} + \log^3 T x^b \right\}. \]

Now we want to select \( T \) as a function of \( x \) so as to make the order of magnitude of \( E' \) as small as possible when \( x \to \infty \).

Since \( a - 1 = \frac{1}{\log T} \) then \( \frac{x^a \log x}{T(a-1)} = \frac{x^a \log T \log x}{T} \), therefore the dominant terms are \( \frac{x^a \log^2 T}{T} \) and \( \log^3 T x^b \). We balance their effect by letting \( \frac{x^a \log^2 T}{T} = \log^3 T x^b \) or \( T = x^{a-b} \). As on page 70 of Ayoub (1963), let \( \log x = \frac{\log^2 T}{2} \) or \( \log T = \sqrt{2} \log x \), so \( E' = O(x \log^2 x e^{-c \sqrt{\log x}}) \) with \( c = \frac{\sqrt{2}}{2} \).

Therefore we get

**Theorem 3.**

\[ \mu_x(\log) = \frac{1}{x} \sum_{n \leq x} \log n = \log x - 1 + O \left\{ \log^2 x e^{-c(\log x)^{1/2}} \right\} \]

as \( x \to \infty \).
Also if in (III.2) we fix \( x \) and let \( T \to \infty \) then \( E' \to 0 \) and therefore:

**Corollary 2.** For \( a' > 1 \) and any non-integer positive real number \( x \)

\[
\frac{1}{2\pi i} \int_{a'-i\infty}^{a'+i\infty} \zeta'(s) \frac{x^s}{s} \, ds = x(1 - \log x).
\]

[Note that:

\[
\int_{a'-i\infty}^{a'+i\infty} \zeta'(s) \frac{x^s}{s} \, ds = \int_{a-i\infty}^{a+i\infty} \zeta'(s) \frac{x^s}{s} \, ds
\]

because \( \zeta'(s) \frac{x^s}{s} \) is analytic inside the contour bounded by lines \( \text{re } s = a, \text{ re } s = a' \).]

**Remark.** For Theorem 8 one can use Euler's summation formula (page 54 of Apostol (1976)) to get

\[
\frac{1}{x} \sum_{1 < n \leq x} \log n = \log x - 1 + \log x \left( \left[ \frac{x}{x} \right] - x \right) + \frac{1 + \int_1^x \frac{[t-[t])dt}{t}}{x}.
\]

\[
= \log x - 1 + O \left( \frac{\log x}{x} \right).
\]

This shows that using an elementary method gives us a better order for the reminder term in this case. Also
\[ \frac{1}{x} \sum_{1 \leq n \leq x} \log^2 n = \log^2 x - 2 \log x + 2 + O\left(\frac{\log^2 x}{x}\right) \]

and therefore we have:

**Theorem 9.**

\[ \text{Var}_x(\log) = \frac{1}{x} \sum_{n \leq x} \left( \log n - \frac{1}{x} \sum_{m \leq x} \log m \right)^2 \]

\[ = 1 + O\left(\frac{\log^2 x}{x}\right), \quad \text{as} \quad x \to \infty. \]

We can also use the analytic method to get the variance of \( \log n \).

Take the derivative with respect to \( s \) of (III.1) to get

\[(III.3) \quad \sum_{n=1}^\infty \frac{\log^2 n}{n^s} = \zeta''(s) \quad \text{for} \quad \text{re} \ s > 1 .\]

And therefore for a non-integer \( x \),

\[ \sum_{n \leq x} \log^2 n = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \zeta''(s) \frac{x^s}{s} \, ds , \quad \text{for any} \quad a > 1. \]

Let \( a = 1 + \frac{1}{\log T} \) and \( b = 1 - \frac{1}{\log T} \) where \( \log T = \sqrt{2 \log x} \)

and consider Figure 2. Again we can break this integral to seven parts say, \( M_1, M_2, \ldots, M_7 \), where \( M_1 \) is the integral over the ith path in Figure 2. \( \zeta''(s) \frac{x^s}{s} \) has a pole at \( s = 1 \) with residue
\[
\lim_{s \to 1} \left( s^{-1} \frac{d^2}{ds^2} \zeta(s) \right) = x(\log^2 x - 2 \log x + 1)
\]

\[
\left[ (s-1)^3 \zeta''(s) = 2 + \sum_{n=2}^{\infty} n(n-1) d_n (s-1)^{n+1} \right],
\]

so: \( M_1 = x(\log^2 x - 2 \log x + 2) \). Now let \( M' = M_1 + M_2 + \ldots + M_6 \),

we will show that \( M' = O(x \log^3 x e^{-c\sqrt{\log x}}) \) as \( x \to \infty \).

\[
M_1 = \lim_{U \to \infty} \frac{1}{2\pi i} \int_{a+iT}^{a+iU} \zeta''(s) \frac{x^s}{s} \, ds = \lim_{U \to \infty} \frac{1}{2\pi i} \int_{a+iT}^{a+iU} \sum_{n=1}^{\infty} \frac{\log^2 n x^s}{s} \, ds
\]

\[
= \lim_{U \to \infty} \frac{1}{2\pi i} \sum_{n=1}^{\infty} \frac{\log^2 n}{2\pi i} \int_{a+iT}^{a+iU} \frac{x^s}{s} \, ds
\]

\[
= O\left( \left( \frac{x}{T} \right)^a \sum_{n=1}^{\infty} \frac{n^{-a} \log^2 n}{|\log \frac{x}{n}|} \right),
\]

for details see the proof of Theorem 8.

Now break the summation into three pieces \( n < \frac{x}{2} \), \( n > 2x \)

and \( \frac{x}{2} < n \leq 2x \) and follow the arguments as on page 13 of

Diaconis (1976) to get

\[
M_1 = O\left( \frac{x^a}{T} \left( \frac{(\log x)^2}{a-1} + (\log x)^3 \log x \right) \right).
\]

The same is true for \( M_6 \).

Now use Lemma 6 for the \( M_2 \) and \( M_5 \), for example,
\[ M_2 = \frac{1}{2\pi i} \int_{b+it}^{a+it} \zeta''(s) \frac{x^s}{s} \, ds = 0 \left( \int_{b}^{a} \frac{\zeta''(s)}{\sigma + iT} \frac{x^\sigma}{\sigma + iT} \, d\sigma \right) \]
\[ = 0 \left( \log^3 T \int_{b}^{a} \frac{x^\sigma}{T} \, d\sigma \right) = 0 \left( \frac{\log^3 T}{T} x^a \right). \]

Finally
\[ M_3 + M_4 = \frac{1}{2\pi i} \int_{b-it}^{b+it} \zeta''(s) \frac{x^s}{s} \, ds \]
\[ = \frac{x^b}{\log^3 T} \int_{0}^{T} \frac{dt}{\sqrt{b^2 + t^2}} = O(\log^4 T x^b). \]

[for details see Diaconis (1976)].

\[ M' = 0 \left\{ \frac{x}{T} \frac{(\log x)^2}{a-1} + \frac{x}{T} (\log x)^3 + \frac{\log^3 T}{T} x^a + x^b \log^4 T \right\}. \]

Now let \( x \) be fixed and \( T \to \infty \) to get

**Corollary 3.** For any \( a' > 1 \) and any non-negative positive real number \( x \);

\[ \frac{1}{2\pi i} \int_{a'-i\infty}^{a'+i\infty} \zeta''(s) \frac{x^s}{s} \, ds = x(\log^2 x - 2 \log x + 2). \]

[Note that \( \zeta''(s) \frac{x^s}{s} \) is analytic inside the contour bounded by lines \( \text{re } s = a', \text{re } s = a \).]
Now back to the variance of $\log n$. As in the proof of
Theorem 8, the dominant terms are $\frac{x^a \log^3 T}{T}$ and $\log^4 T \times^b$,
we balance their effect by letting $\frac{x^a \log^3 T}{T} = \log^4 T \times^b$ and
again $\log T = \sqrt{2 \log x}$ works and $M' = O\{x \log^3 x \ e^{-c\sqrt{\log x}}\}$.
Now subtract $\mu_x^2 (\log)$ from:

$$\frac{1}{x} \sum_{n<x} \log^2 n = (\log^2 x - 2 \log x + 2) + O\{\log^3 x \ e^{-c\sqrt{\log x}}\}.$$ 

to get

$$\text{Var}_x (\log) = 1 + O(\log^4 x \ e^{-c\sqrt{\log x}}) \text{ as } x \to \infty,$$

which has a larger order for the reminder term than Theorem 9.

III.2 Product of Prime Divisors of $n$.

In order to test the analytic method for another additive
arithmetic function let us introduce

$$f_1(n) = \sum_{p|n} \log p = \log \prod_{p|n} p = \log \ell(n)$$

where $\ell(n)$ is the product of the prime divisors of $n$. $f_1(n)$
is the strongly additive contraction of $\log n$ with $f_1(p) = \log p$.

The function $\ell(n)$ is a well known multiplicative function
and $f_1(n) = \sum_{p|n} \Lambda(p)$, where $\Lambda$ is the Mangoldt function, i.e.,
\[ \Lambda(n) = \begin{cases} 
\log p & \text{if } n = p^k \text{ for some prime } p \text{ and some } k \geq 1 \\
0 & \text{otherwise.} 
\end{cases} \]

thus the study of the moments of \( f_1(n) \) is of independent interest.

Now we show

\textbf{Theorem 10.}

\[ \mu_x(\log \ell) = \frac{1}{x} \sum_{n < x} \log \ell(n) = \log x - 1 - \sum_{p} \frac{\log p}{p(p-1)} \]
\[ + \mathcal{O}\left( \log^2 x \mathrm{e}^{-c(\log x)^{3/4}} \right), \quad \text{as } x \to \infty. \]

And

\textbf{Theorem 11.}

\[ \text{Var}_x(\log \ell) = \frac{1}{x} \sum_{n < x} \left( \log \ell(n) - \frac{1}{x} \sum_{m < x} \log \ell(m) \right)^2 \]
\[ = \mathcal{O}\left( \frac{(p^2 + p - 1) \log^2 p}{p^2(p-1)^2} \right) + 1 + \mathcal{O}\left( \frac{1}{\log^4 x \mathrm{e}^{-c(\log x)^{3/4}}} \right), \quad \text{as } x \to \infty. \]

To prove these theorems we use the same argument that has been used in previous theorems.

\textbf{Proof of Theorem 10.} Formula (1.9) gives us

\[ \sum_{n=1}^{\infty} \frac{\log \ell(n)}{n^s} = \zeta(s) \sum_{p} \frac{\log p}{p^s}, \quad \text{for } \Re s > 1. \]
According to Formula 1.1.8 on page 4 of Titchmarsh (1951)

\[(III.4) \quad \sum_{p} \frac{\log p}{p^s} = -\frac{\zeta'(s)}{\zeta(s)} - \sum_{p} \sum_{k=2}^{\infty} \frac{\log p}{p^{ks}}\]

therefore

\[\sum_{n=1}^{\infty} \frac{\log \ell(n)}{n^s} = -\zeta'(s) + \zeta(s) f'(s)\]

(see the proof of Theorem 1). Therefore, the argument of

Theorem 8 gives us the result because \(f'(s)\) is analytic for

\(\text{re} \ s \geq \frac{1}{2}\) and uniformly bounded in any compact set in \(\text{re} \ s \geq \frac{1}{2}\)

and \(f'(1) = -\sum_{p} \frac{\log p}{p(p-1)}\).

---

**Proof of Theorem 11.** By Formula (I.10) we have

\[\sum_{n=1}^{\infty} \frac{\log^2 \ell(n)}{n^s} = \zeta(s) \left\{ \sum_{p} \frac{\log^2 p}{p^s} - \sum_{p} \frac{\log p}{p^{2s}} + \left( \sum_{p} \frac{\log p}{p^s} \right)^2 \right\}\]

for \(\text{re} \ s > 1\).

Now use the argument of the proof of Theorem 8 to get
\[
\sum_{n \leq x} \log^2 \zeta(n) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \zeta(s) \left\{ \sum_{p} \frac{\log^2 p}{p^s} \right\} \frac{x^s}{s} \, ds \\
- \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \zeta(s) \sum_{p} \frac{\log^2 p}{2s} \frac{x^s}{s} \, ds \\
+ \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \zeta(s) \left( \sum_{p} \frac{\log p}{p^s} \right)^2 \frac{x^s}{s} \, ds \\
= N_1 + N_2 + N_3.
\]

For \( N_1 \) note that

\[
\frac{\xi''(s)}{\xi(s)} - \left( \frac{\xi'(s)}{\xi(s)} \right)^2 = \sum_{p} \sum_{k=1}^{\infty} \frac{k \log^2 p}{p^{ks}}
\]

so

\[
N_1 = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \zeta''(s) \frac{x^s}{s} \, ds - \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \zeta(s) \left( \frac{\xi'(s)}{\xi(s)} \right)^2 \frac{x^s}{s} \, ds \\
- \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \zeta(s) \xi''(s) \frac{x^s}{s} \, ds,
\]

where

\[
\xi''(s) = \sum_{p} \sum_{k=2}^{\infty} \frac{k \log^2 p}{p^{ks}} = \sum_{p} \frac{(2p^{s-1}) \log^2 p}{(p^s-1)^2 p^s}.
\]

\[
N_1 = x(\log^2 x - 2 \log x + 2) - \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \zeta(s) \left( \frac{\xi'(s)}{\xi(s)} \right) \frac{x^s}{s} \, ds \\
- \sum_{p} \frac{(2p^{s-1}) \log^2 p}{(p^s-1)^2 p^s} x + O \left( x \log^2 x e^{-c (\log x)^{3/2}} \right).
\]
And also

\[ N_2 = -\sum \frac{\log^2 p}{p^2} x + O\left( x \log^2 x \ e^{-c\left(\log x\right)^{1/2}} \right). \]

Finally \( N_3 \): according to (III.4)

\[ \left( \sum \frac{\log p}{ps} \right)^2 = \left( \frac{\zeta'(s)}{\zeta(s)} \right)^2 + (f'(s))^2 - 2 \frac{f'(s)}{\zeta(s)} \frac{\zeta'(s)}{\zeta(s)}. \]

So

\[
N_3 = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \zeta(s) \left( \frac{\zeta'(s)}{\zeta(s)} \right)^2 \frac{x^s}{s} \, ds + \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \zeta(s) (f'(s))^2 \frac{x^s}{s} \, ds
\]

\[ - \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \left( 2 \frac{f'(s)}{\zeta(s)} \right) \frac{x^s}{s} \, ds \]

\[ = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \zeta(s) \left( \frac{\zeta'(s)}{\zeta(s)} \right)^2 \frac{x^s}{s} \, ds + (f'(1))^2 \ x \]

\[ + 2 \ x[f''(1) + f'(1) \log x - f'(1)] + O(x \log^3 x \ e^{-c\sqrt{\log x}}) \]

because

\[
\lim_{s \to 1} \frac{\partial}{\partial s} \left[ (s-1)^2 \left( 2 \frac{f'(s)}{\zeta(s)} \right) \frac{x^s}{s} \right] = -2[f''(1) + f'(1) \log x - f'(1)] x.
\]

Therefore
\[
\frac{1}{x} \sum_{n \leq x} \log^2 \ell(n) = \log^2 x + \log x(-2 + 2 f'(1)) + 2 + f''(1)
\]
\[
- \sum_{p} \frac{\log \frac{p}{p^2}}{p^2} + (f'(1))^2 - 2 f'(1) + O\left(\log^3 x \cdot e^{-c \sqrt{\log x}}\right)
\]

Now subtract \( (\mu_x(\log \ell))^2 \) from \( \frac{1}{x} \sum_{n \leq x} \log^2 \ell(n) \) to get the result.

III.3 \( \log n - \log \ell(n) \).

As an immediate corollary of Theorems 8 and 10 we have

Corollary 4.

\[
\mu_x(\log - \log \ell) = \frac{1}{x} \sum_{n \leq x} (\log n - \log \ell(n))
\]
\[
= \frac{1}{x} \sum_{n \leq x} \log \left(\prod_{p \mid n} p^{k-1}\right) = \sum_{p \mid n} \frac{\log p}{p(p-1)} + O\left(\log^2 x \cdot e^{-c(\log x)^{1/2}}\right)
\]
as \( x \to \infty \).

We show:

Theorem 12.

\[
\text{Var}_x(\log - \log \ell) = \frac{1}{x} \sum_{n \leq x} \left(\log n - \log \ell(n) - \frac{1}{x} \sum_{m \leq x} (\log m - \log \ell(n))^2\right)
\]
\[
= \sum_{p} \frac{(p^2 + p - 1) \log \frac{p}{p^2}}{p(p-1)^2} + O\left(\log^4 x \cdot e^{-c(\log x)^{1/2}}\right), \quad \text{as} \ x \to \infty .
\]
In order to prove this theorem we need to have:

**Lemma 7.**

\[
\text{Cov}_x(\log, \log \lambda) = 1 + O\left(\log^\frac{1}{2} x \ e^{-c(\log x)^{\frac{3}{2}}}\right),
\]

as \( x \to \infty \).

**Proof.** Formula (I.13) gives us:

\[
\sum_{n=1}^{\infty} \frac{\log n \log \lambda(n)}{n^s} = \zeta(s) \left\{ \frac{\log^2 p}{p} + \left( \sum_{p} \frac{\log p}{p^s} \right) \left( \sum_{p} \frac{\log p}{p^{s-l}} \right) \right\}
\]

\[
= \zeta(s) \sum_{p} \frac{\log^2 p}{p^s} + \zeta(s) \left[ -\frac{\zeta'(s)}{\zeta(s)} + r'(s) \right] \left[ -\frac{\zeta'(s)}{\zeta(s)} \right],
\]

(because according to Formula (I.11) we have

\[
\sum_{n=1}^{\infty} \frac{\log n}{n^s} = -\zeta'(s) = \zeta(s) \sum_{p} \frac{\log p}{p^{s-1}}.
\]

So

\[
\sum_{n=1}^{\infty} \frac{\log n \log \lambda(n)}{n^s} = \zeta(s) \sum_{p} \frac{\log^2 p}{p^s} + \zeta(s) \frac{\zeta'(s)}{\zeta(s)} - r'(s) \zeta'(s).
\]

Now by using the notation that has been used in previous sections

and the same methods we have:
\begin{align*}
\sum_{n \leq x} \log n \log \varrho(n) &= \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \zeta(s) \sum_{p} \frac{\log^2 p}{p} \frac{x^s}{s} \, ds \\
&\quad + \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \frac{\zeta'(s)}{s} \frac{x^s}{s} \, ds \\
&\quad - \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} f'(s) \zeta'(s) \frac{x^s}{s} \, ds \\
&= N_1' + N_2' + N_3'.
\end{align*}

N_1' = N_1 \quad \text{(as in Theorem 11)}

\begin{align*}
&= x(\log^2 x - 2 \log x + 2) - N_2' - f''(1) x \\
&\quad + 0\left( x \log^3 x \, e^{-c(\log x)^{3/2}} \right).
\end{align*}

N_3' is also computed in Theorem 11 and

\begin{align*}
N_3' &= +x[f''(1) + f'(1) \log x - f'(1)] + 0(x \log^3 x \, e^{-c\sqrt{\log x}}).
\end{align*}

Therefore

\begin{align*}
\frac{1}{x} \sum_{n \leq x} \log n \log \varrho(n) &= \log^2 x + (f'(1) - 2) \log x \\
&\quad + 2 - f'(1) + 0\left( \log^3 x \, e^{-c(\log x)^{3/2}} \right).
\end{align*}

Now subtract

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\[
\left(\frac{1}{x} \sum_{n \leq x} \log n\right) \left(\frac{1}{x} \sum_{n \leq x} \log \Lambda(n)\right) = \log^2 x + (f'(1) - 2) \log x + 1 - f'(1) + O\left(\log^4 x e^{-c(\log x)^{1/2}}\right)
\]

from \(\frac{1}{x} \sum_{n \leq x} \log n \log \Lambda(n)\) to get the result.

Now here is the proof of Theorem 12.

\[
\text{Var}_x(\log - \log \Lambda) = \text{Var}_x \log \Lambda + \text{Var}_x \log \Lambda - 2 \text{Cov}_x(\log, \log \Lambda)
\]

substituting from Theorems 9, 11 and Lemma 7 gives us the result.

III.4 Correlation Coefficient.

Definition. Let \(f\) and \(g\) be two arithmetic functions.

For a real number \(x\), put

\[
R_x(f,g) = \text{Cov}_x(f,g)/[(\text{Var}_x f)(\text{Var}_x g)]^{1/2}
\]

where

\[
\text{Cov}_x(f,g) = \frac{1}{x} \sum_{n \leq x} \left(f(n) - \frac{1}{x} \sum_{m \leq x} f(m)\right)\left(g(n) - \frac{1}{x} \sum_{m \leq x} g(m)\right)
\]

\[
= \frac{1}{x} \sum_{n \leq x} f(n) g(n) - \left(\frac{1}{x} \sum_{n \leq x} f(n)\right)\left(\frac{1}{x} \sum_{n \leq x} f(n)\right)
\]
and

\[ \text{Var}_X(f) = \text{Cov}_X(f, f) = \frac{1}{X} \sum_{n \leq X} f^2(n) - \left( \frac{1}{X} \sum_{n \leq X} f(n) \right)^2 \]

then

\[ R(f, g) = \lim_{X \to \infty} R_X(f, g) \]

is called the asymptotic correlation coefficient between \( f \) and \( g \).

Diaconis, Mosteller and Onishi (1977) have mentioned that

\( R_X(\omega, \Omega) \) is asymptotically 1 and so \( \Omega \) and \( \omega \) are "linearly dependent".

We show:

**Theorem 13.** If \( \omega(n) \) is the number of distinct prime divisors, \( \Omega(n) \), the number of prime divisors and \( \ell(n) \), the product of distinct prime divisors of \( n \), then:

\[ a - R(\omega, \Omega) = 1, \]

\[ b - R(\log, \log \ell) = \left( 1 + \sum_{p} \frac{(p^2 + p - 1) \log^2 p}{p^2 (p-1)^2} \right)^{-\frac{1}{2}}, \]

\[ c - R(\omega, \log \ell) = 0, \]

\[ d - R(\omega, \log) = 0, \]

\[ e - R(\Omega, \log) = 0 \quad \text{and finally} \]

\[ f - R(\Omega, \log \ell) = 0. \]
That is \( \omega, \Omega \) and \( \log, \log \ell \) are asymptotically linearly dependent while \( \omega \) is asymptotically uncorrelated with \( \log, \log \ell \) and \( \Omega \) is asymptotically uncorrelated with \( \log, \log \ell \).

To prove Theorem 13 we need to have approximate covariances between the mentioned functions.

(In this chapter we introduce the notation

\[
f(x) \asymp g(x) \quad \text{for} \quad f(x) = g(x) + O\left(\frac{1}{(\log x)^{\gamma}}\right), \quad \text{as} \quad x \to \infty
\]

to avoid complicated calculation for formulas which are going to be used in asymptotic identities.)

First of all as a result of Lemma 6 and Lemma 7 we have

\[
\text{Cov}_x(\Omega, \omega) \asymp \log \log x + B_1 - \frac{\pi^2}{6} \quad \text{and} \quad \text{Cov}_x(\log, \log \ell) \asymp 1.
\]

We will also show:

**Lemma 8.**

1. \( 1 - \text{Cov}_x(\omega, \log \ell) \asymp \gamma_1 - \gamma + 1 - \sum \frac{\log p}{p^2} \)
2. \( 2 - \text{Cov}_x(\omega, \log) \asymp \gamma_1 - \gamma + 1 \)
3. \( 3 - \text{Cov}_x(\Omega, \log) \asymp \gamma_1 - \gamma + 1 \)
4. \( 4 - \text{Cov}_x(\Omega, \log \ell) \asymp \gamma_1 - \gamma + 1 - \sum \frac{\log p (2p-1)}{p p(p-1)^2} \),
as \( x \to \infty \), with

\[
\gamma_1 = \lim_{N \to \infty} \left[ \sum_{n=1}^{N} \frac{\log n}{n} - \frac{\log^2 N}{2} \right] \text{ and }
\]

\[
\gamma = \lim_{N \to \infty} \left[ \sum_{n=1}^{N} \frac{1}{n} - \log N \right] \text{ (Euler's constant).}
\]

**Proof of Lemma 8.**

1. Use Formula (I.2) to get:

\[
\sum_{n=1}^{\infty} \frac{\omega(n) \log \ell(n)}{n^s} = \zeta(s) \left\{ \sum \frac{\log p}{p^s} - \sum \frac{\log p}{p^{2s}} + \left( \sum \frac{1}{p} \right) \left( \sum \frac{\log p}{p^s} \right) \right\}
\]

and therefore, with the notation used in Theorem 8 and Lemma 7:

\[
\sum_{n \leq x} \omega(n) \log \ell(n) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \zeta(s) \left( \sum \frac{\log p}{p^s} \right) \frac{x^s}{s} \, ds
\]

\[
- \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \zeta(s) \left( \sum \frac{\log p}{p^{2s}} \right) \frac{x^s}{s} \, ds
\]

\[
+ \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \zeta(s) \left( \sum \frac{1}{p^s} \right) \left( \sum \frac{\log p}{p^s} \right) \frac{x^s}{s} \, ds
\]

\[
= P_1 + P_2 + P_3.
\]

\( P_1 \sim x \{ \log x - 1 + f'(1) \} \) (Theorem 10.) and \( P_2 \sim -x \sum \frac{\log p}{p^{2}} \).

Now for \( P_3 \):
\[
\sum_{p \leq x} \frac{1}{p^s} = \log(\zeta(s)) - f(s), \quad \text{(Theorem 1.)}
\]

\[
\sum_{p \leq x} \frac{\log p}{p^s} = -\frac{\zeta'(s)}{\zeta(s)} + f'(s).
\]

So

\[
\zeta(s) \left( \sum_{p \leq x} \frac{1}{p^s} \right) \left( \sum_{p \leq x} \frac{\log p}{p^s} \right)
\]

\[
= (\log(\zeta(s)) - f(s)) (-\zeta'(s) + f'(s) \zeta(s)).
\]

Therefore:

\[
P_3 = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \zeta'(s) \log \zeta(s) \frac{x^s}{s} \, ds
\]

\[
+ \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} f'(s) \zeta(s) \log(\zeta(s)) \frac{x^s}{s} \, ds
\]

\[
+ \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} f(s) \zeta'(s) \frac{x^s}{s} \, ds
\]

\[
- \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} f(s) f'(s) \zeta(s) \frac{x^s}{s} \, ds
\]

\[
= P' + P^2 + P^3 + P^4, \quad \text{where}
\]

\[
P^4 \equiv x \{-f(1) f'(1)\} \quad \text{and} \quad P^3 \text{ is the same as } N_3' \text{ in Lemma 7, but}
\]
instead of $f'(s)$ we have $-f(s)$ with similar properties so

$$P^3 \sim x(-f'(1) - f(1) \log x + f(1)).$$

For $P^1$ and $P^2$ we use the notation in Theorems 1 and 4 and since all the functions are analytic between the lines: \[ \text{re } s = 1 + \frac{1}{\log T} \text{ and } \text{re } s = 1 + \frac{C}{(\log T)^9} \] then according to Cauchy's theorem the transformation of the lines of integration does not affect the values of the integrals. $P^2$ is the same as $J_3(2)$ in Theorem 4 [instead of $(-2 f(s))$ we have $f'(s)$]. So

$$P^2 \sim x(f'(1) \log \log x + \gamma f'(1)).$$

Now for $P'$. The integrals over the paths 1, 2, ..., 6 in Figure 2 are of order $x e^{-\{\log T\}^{-9} \log x}$ for this integrand. (The method of Theorem 1 works here too.) So what we need to calculate is

$$P' = P'_{c} = \frac{-1}{2\pi i} \int_{C} \zeta'(s) \log(\zeta(s)) \frac{x^s}{s} \, ds.$$ 

Again by the use of Cauchy's theorem
\[
\frac{1}{2\pi i} \int_C \zeta'(s) \log(\zeta(s) (s-1)) \frac{x^s}{s} \, ds = 0
\]

so

\[
P'_C = \frac{-1}{2\pi i} \int_C \zeta'(s) \log(\frac{1}{s-1}) \frac{x^s}{s} \, ds.
\]

Now according to Lemma 1

\[
\zeta(s) = \frac{1}{(s-1)} + \sum_{k=0}^{\infty} d_k (s-1)^k
\]

and so

\[
\zeta'(s) = -\frac{1}{(s-1)^2} + \sum_{k=0}^{\infty} e'_k (s-1)^k
\]

with

\[
e'_k = (k+1) d_{k+1} = \frac{(-1)^{k+1} \gamma_{k+1}}{k!}
\]

for \( k = 0,1,2,\ldots \).

Therefore

\[
P'_C = \frac{1}{2\pi i} \int_C \frac{1}{(s-1)^2} \log(\frac{1}{s-1}) \frac{x^s}{s} \, ds - \sum_{k=0}^{n-1} \frac{e'_k}{2\pi i} \int_C (s-1)^k \log(\frac{1}{s-1}) \frac{x^s}{s} \, ds
\]

\[+ \bigg( \int_C (s-1)^n \log(\frac{1}{s-1}) \frac{x^s}{s} \, ds \bigg).\]
\[
\frac{1}{s(s-1)^2} = \frac{1}{(s-1)^2} - \frac{1}{s-1} + \frac{1}{s}
\]

so

\[
P_C' = \frac{1}{2\pi i} \int_C \frac{1}{(s-1)^2} \log\left(\frac{1}{s-1}\right)x^s \, ds - I' + J_0' - \sum_{k=0}^{n-1} e_k' J_k' + O(J_n')
\]

(using the notation defined in Theorem 1). However, integration by parts and Cauchy's theorem give us

\[
P_C' = \frac{-1}{2\pi i} \int_C \frac{x^s}{(s-1)^2} \, ds + \frac{\log x}{2\pi i} \int_C \frac{x^s}{s-1} \log\left(\frac{1}{s-1}\right) \, ds
\]

\[
- I' + J_0' - \sum_{k=0}^{n-1} e_k' J_k' + O(J_n')
\]

\[
= -x \log x + (\log x - 1) I' + (1 - e_0') J_0' + \sum_{k=1}^{n-1} e_k' J_k' + O(J_n')
\]

or

\[
P' = x\{\log x \log \log x - \log \log x + (\gamma-1) \log x + \gamma_1 - \gamma + 1\}
\]

Then:

\[
P_3 = \{\log x \log \log x + (f'(1)-1) \log \log x + (\gamma-f(1)-1) \log x
\]

\[
+ \gamma f'(1) - f'(1) - f(1) f'(1) + f(1) + \gamma_1 - \gamma + 1\}
\]
So

\[
\frac{1}{x} \sum_{n \leq x} w(n) \log \ell(n) \gtrsim \left\{ \log x \log \log x + (f'(1)-1) \log \log x \\
+ (\gamma-f(1)) \log x + \gamma f'(1) - f(1) f'(1) + f(1) + \gamma_1 - \gamma - \sum_{p} \frac{\log p}{p^2} \right\}.
\]

Now subtract \( \left( \frac{1}{x} \sum_{n \leq x} w(n) \right) \left( \frac{1}{x} \sum_{n \leq x} \log \ell(n) \right) \) from this to get

\[
\text{Cov}_x(\omega, \log \ell) \gtrsim \gamma_1 - \gamma - \sum_{p} \frac{\log p}{p^2} + 1.
\]

2 - Take derivatives on both sides of (I.4) to get

\[
\sum_{n=1}^{\infty} \frac{w(n) \log n}{n^s} = -\zeta'(s) \sum_{p} \frac{1}{p} + \zeta(s) \sum_{p} \frac{\log p}{p^s}.
\]

\[
= -\zeta'(s) \log \zeta(s) + \zeta'(s) f(s) - \zeta'(s) + \zeta(s) f'(s).
\]

So

\[
\sum_{n \leq x} w(n) \log n = \frac{-1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \zeta'(s) \log(\zeta(s)) \frac{x^s}{s} \, ds
\]

\[
+ \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \zeta(s) f(s) \frac{x^s}{s} \, ds - \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \zeta'(s) \frac{x^s}{s} \, ds
\]

\[
+ \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \zeta(s) f'(s) \frac{x^s}{s} \, ds = p' \ (\text{Part 1}) + p^3 \ (\text{Part 1})
\]

\[
- J \ (\text{Theorem 8}) + \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \zeta(s) f'(s) \frac{x^s}{s} \, ds
\]

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\[ x \{ \log x \log \log x - \log \log x + (\gamma - f(\gamma)) \log x + \gamma_1 - \gamma + f(\gamma) \} \]

Now subtract \( \left( \frac{1}{x} \sum_{n \leq x} w(n) \right) \left( \frac{1}{x} \sum_{n \leq x} \log n \right) \) from \( \frac{1}{x} \sum_{n \leq x} w(n) \log n \)

to get

\[ \text{Cov}_x(w, \log) \approx \gamma_1 - \gamma + 1. \]

3 - Use Formula (1.6), i.e.,

\[ \sum_{n=1}^{\infty} \frac{\Omega(n)}{n^s} = \zeta(s) \{ \log(\zeta(s)) - g(s) \} \quad \text{[Theorem 3.]} \]

to get

\[ \sum_{n=1}^{\infty} \frac{\Omega(n) \log n}{n^s} = -\zeta'(s) \log(\zeta(s)) + \zeta'(s) g(s) - \zeta'(s) + \zeta(s) g'(n), \]

so

\[ \sum_{n \leq x} \frac{\Omega(n) \log n}{n^s} = \frac{-1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \zeta'(s) \log(\zeta(s)) \frac{x^s}{s} ds \]

\[ + \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \zeta'(s) g(s) \frac{x^s}{s} ds - \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \zeta'(s) \frac{x^s}{s} ds \]

\[ + \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \zeta(s) g'(s) \frac{x^s}{s} ds. \]
All of the integrals are computed for $f$, which has similar properties to $g$, therefore

\[
\frac{1}{x} \sum_{n \leq x} \Omega(n) \log n \cdot \left( \log x \log \log x - \log \log x + (\gamma - \gamma_1) \log x + \gamma_1 - \gamma + g(1) \right).
\]

And finally

\[
\text{Cov}_x(\Omega, \log) = \gamma_1 - \gamma + 1.
\]

4 - Use Formula (1.2) to get

\[
\sum_{n=1}^{\infty} \frac{\Omega(n) \log f(n)}{n^s} = \zeta(s) \left\{ \sum \frac{\log p}{p^s} + \left( \sum \frac{1}{p^{s-1}} \right) \left( \sum \frac{\log p}{p^s} \right) \right\}
\]

So

\[
\sum_{n \leq x} \Omega(n) \log f(n) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \zeta(s) \left( \sum \frac{\log p}{p^s} \right) \frac{x^s}{s} \, ds
\]

\[
+ \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \zeta(s) \left( \sum \frac{1}{p^{s-1}} \right) \left( \sum \frac{\log p}{p^s} \right) \frac{x^s}{s} \, ds
\]

\[
= Q_1 + Q_2.
\]

$Q_1 = P_1$ [as in part 1] $\approx x(\log x - 1 + f'(1))$. 

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Since
\[ \sum_{p} \frac{1}{p^{s} - 1} \log \zeta(s) - g(s) \sum_{p} \frac{\log p}{p^{s}} = \frac{-\zeta'(s)}{\zeta(s)} + f'(s) \]
then
\[ \zeta(s) \left( \sum_{p} \frac{1}{p^{s} - 1} \right) \left( \sum_{p} \frac{\log p}{p^{s}} \right) = -\zeta'(s) \log \zeta(s) + f'(s) \zeta(s) \log \zeta(s) \]
\[ + g(s) \zeta'(s) - g(s) f'(s) \zeta(s) \]

Therefore
\[ Q_2 = -\frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \zeta'(s) \log(\zeta(s)) \frac{x^s}{s} \, ds \]
\[ + \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} f'(s) \zeta(s) \log(\zeta(s)) \frac{x^s}{s} \, ds \]
\[ + \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} g(s) \zeta'(s) \frac{x^s}{s} \, ds \]
\[ - \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \zeta(s) g(s) f'(s) \frac{x^s}{s} \, ds \]
\[ = Q_1 + Q_2 + Q_3 + Q_4 \]

\[ Q_4 \sim x \{-f'(1) g(1)\} \]

\[ Q_3 \]

is the same as \( P^3 \) in part 1, but
instead of \( f(s) \) we have \( g(s) \). So

\[
Q^3 = x \{-g'(1) - g(1) \log x + g(1)\}
\]

\[
Q^2 = p^2 = x \{f'(1) \log \log x + \gamma f'(1)\}
\]

Also

\[
Q' = p' = x \{\log x \log \log x - \log \log x + (\gamma - 1) \log x + \gamma_1 - \gamma + 1\}
\]

therefore

\[
Q_2 = x \{\log x \log \log x + (f'(1) - 1) \log \log x
\]

\[+ (\gamma - g(1) - 1) \log x - f'(1) g(1) - g'(1) + g(1) + \gamma f'(1) + \gamma_1 - \gamma + 1\}
\]

and

\[
\frac{1}{x} \sum_{n < x} \Omega(n) \log \mathcal{L}(n) = \{\log x \log \log x + (f'(1) - 1) \log \log x
\]

\[+ (\gamma - g(1)) \log x - f'(1) g(1) + g(1) + \gamma f'(1)
\]

\[+ \gamma_1 - \gamma + f'(1) - g'(1)\}
\]

So
\[
\text{Cov}_x(\Omega, \log \ell) = \frac{1}{x} \sum_{n<x} \Omega(n) \log \ell(n) - (\mu_x(\Omega))(\mu_x(\log \ell))
\]

\[
\approx \left( \gamma + 1 + f'(1) - g'(1) \right),
\]

but

\[
f(s) - g(s) = \sum_{p} \frac{1}{p \cdot s(p-1)},
\]

so

\[
f'(s) - g'(s) = -\sum_{p} \frac{\log p (2 \cdot p^{s-1})}{p \cdot p^{s(p-1)^2}}
\]

and therefore

\[
f'(1) - g'(1) = -\sum_{p} \frac{\log p (2 \cdot p-1)}{p \cdot (p-1)^2},
\]

this concludes the proof of Lemma 8.

Now by the use of Lemmas 6, 7 and 8, Theorems 4, 6, 9 and 11 the proof of Theorem 13 is obvious.

To finish this section we will state another example for which the correlation coefficient is neither zero nor one.

**Theorem 14.**

\[
R(\Omega-\omega, \log - \log \ell) = \frac{\sum \frac{\log p \cdot (p^2+p-1)}{p \cdot p^2 \cdot (p-1)^2}}{\sqrt{\sum \frac{(p^2+p-1) \log p \cdot p^2 \cdot (p-1)^2}{p \cdot p^2 \cdot (p-1)^2}}}
\]

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Proof.

\[ \text{Cov}_x(\Omega - \omega, \log - \log \lambda) = \text{Cov}_x(\Omega, \log) \]

\[ - \text{Cov}_x(\omega, \log) - \text{Cov}_x(\Omega, \log \lambda) + \text{Cov}_x(\omega, \log \lambda) \]

\[ = \sum_{p \mid n} \frac{\log p}{p} \frac{(p^2 + p - 1)}{(p - 1)^2} \cdot \]

Now Theorems 7 and 12 give us the result.

III.5 Euler Totient Function \( \phi \).

**Definition.** For each \( n \), \( \phi(n) \) is defined to be the number of positive integers not exceeding \( n \) which are relatively prime to \( n \). It is well known that \( \phi(n) = n \prod_{p \mid n} (1 - \frac{1}{p}) \) and \( \phi \) is multiplicative. So \( \log \phi \) is additive and \( \log \phi = \log + \log \lambda_1 \), where \( \lambda_1(n) = \prod_{p \mid n} (1 - \frac{1}{p}) \). (See Apostol (1976), pp. 25-28 for properties of \( \phi \).) Note that

\[ (\text{III.5}) \quad \mu_x(\log \phi) = \frac{1}{x} \sum_{m < x} \log \phi(m) = \mu_x(\log) + \mu_x(\log \lambda_1) \]

and

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(III.6) \( \text{Var}_x(\log \phi) = \frac{1}{x} \sum_{m \leq x} (\log \phi(m) - \mu_x(\log \phi))^2 = \text{Var}_x(\log) + \text{Var}_x(\log \ell_1) - 2 \text{Cov}_x(\log, \log \ell_1) \).

And if \( g \) is another additive function, then

(III.7) \( \text{Cov}_x(\log \phi, g) = \text{Cov}_x(\log, g) + \text{Cov}_x(\log \ell_1, g) \).

So for finding the expansions of the mean and variance of \( \log \phi \) and its covariance with other additive arithmetic functions we need to have the mean and variance of \( \log \ell_1 \) and its covariance with them. First we show:

Lemma 9.

\[ \mu_x(\log \ell_1) = \sum_{p} \frac{\log(1 - \frac{1}{p})}{p} + O\left(\frac{1}{(\log x)^{n+1}}\right) \text{ as } x \to \infty, \]

for an arbitrary non-negative \( n \).

Proof. Formula I.9 gives us

\[ \sum_{n=1}^{\infty} \frac{\log \ell_1(n)}{n^s} = \zeta(s) \sum_{p} \frac{\log(1 - \frac{1}{p})}{p^s}. \]

But since

\[ \log(1 - \frac{1}{p}) = -\sum_{m=1}^{\infty} \frac{1}{p^m}, \]
then

\[ \sum_{p \leq x} \frac{\log(1 - \frac{1}{p})}{p} = -\sum_{p} \sum_{m=1}^{\infty} \frac{1}{p^{s+m}} \]

which is analytic for \( \text{re } s > 0 \) and uniformly bounded in any compact set in \( \text{re } s > 0 \). The method of calculating \( I_2 \) in Theorem 1 gives us the result.

Looking at the proof of Theorem 1, one can see that the reminder term in Lemma 9 is of the order of \( e^{-c\sqrt{\log x}} \), but we state a weaker order to have a unique error term for all of the theorems in this section. (Let \( \log T = (\log x)^{1/16} \), which satisfies the condition of the proof of Theorem 1.)

**Lemma 10.**

\[
\text{Var}_x(\log L_1) = \sum_{p} \frac{\log^2(1 - \frac{1}{p})}{p} - \sum_{p} \frac{\log^2(1 - \frac{1}{p})}{p^2} \\
+ O\left(\frac{1}{(\log x)^{n+1}}\right) \quad \text{as } x \to \infty,
\]

for an arbitrary non-negative \( n \).

**Proof.** According to Formula (I.10) we have

\[
\sum_{n=1}^{\infty} \frac{\log^2 L_1(n)}{n^s} = \zeta(s) \left\{ \sum_{p} \frac{\log^2(1 - \frac{1}{p})}{p^s} - \sum_{p} \frac{\log^2(1 - \frac{1}{p})}{p^{2s}} \\
+ \left( \sum_{p} \frac{\log(1 - \frac{1}{p})}{p^s} \right)^2 \right\}.
\]
Since the function in parentheses is analytic for \( \text{re } s > 0 \) again we have:

\[
\frac{1}{x} \sum_{n<x} \log^2 \varphi_1(n) = \sum_{p} \frac{\log^2(1 - \frac{1}{p})}{p} - \sum_{p} \frac{\log^2(1 - \frac{1}{p})}{p^2} \\
+ \left( \sum_{p} \frac{\log(1 - \frac{1}{p})}{p} \right)^2 + O\left( e^{-c(\log x)^{1/10}} \right) \quad \text{as } x \to \infty,
\]

Now subtracting \( \left( \frac{1}{x} \sum_{n<x} \log \varphi_1(n) \right)^2 \) from the above expansion gives us the result.

**Lemma 11.**

1 - \( \text{Cov}_x(\omega, \log \varphi_1) \approx \sum_{p} \frac{\log(1 - \frac{1}{p})}{p} - \sum_{p} \frac{\log(1 - \frac{1}{p})}{p^2} \)

2 - \( \text{Cov}_x(\Omega, \log \varphi_1) \approx \sum_{p} \frac{\log(1 - \frac{1}{p})}{p} \)

3 - \( \text{Cov}_x(\log \varphi, \log \varphi_1) \approx -\sum_{p} \frac{\log p \log(1 - \frac{1}{p})}{p^2} \)

4 - \( \text{Cov}_x(\log, \log \varphi_1) \approx 0 \)

as \( x \to \infty \).

**Proof.**

1 - According to Formula (I.2) we have
\[
\sum_{n=1}^{\infty} \frac{\omega(n) \log \ell_1(n)}{n^s} = \zeta(s) \left\{ \sum_{p} \frac{\log(1 - \frac{1}{p^s})}{p^s} - \sum_{p} \frac{\log(1 - \frac{1}{p})}{p^{2s}} \right\} \\
+ (\log \zeta(s) - f(s)) \left\{ \sum_{p} \frac{\log(1 - \frac{1}{p})}{p^s} \right\},
\]

from the notation and method of Theorem 1. We have:

\[
\sum_{n<x} \omega(n) \log \ell_1(n) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \zeta(s) \left( \sum_{p} \frac{\log(1 - \frac{1}{p^s})}{p^s} - \sum_{p} \frac{\log(1 - \frac{1}{p})}{p^{2s}} \right) \frac{x^s}{s} \, ds \\
+ \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \zeta(s) \log \zeta(s) \left( \sum_{p} \frac{\log(1 - \frac{1}{p})}{p^s} \right) \frac{x^s}{s} \, ds \\
- \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \zeta(s) f(s) \left( \sum_{p} \frac{\log(1 - \frac{1}{p})}{p^s} \right) \frac{x^s}{s} \, ds.
\]

The first and third integral can be derived as \( I_2 \) in Theorem 1.

The second integral is similar to \( J_3(2) \) in the proof of Theorem 5, but instead of \(-2f(s)\) in \( J_3(2) \), we have \( \frac{\log(1 - \frac{1}{p})}{p} \) with similar properties. Therefore

\[
\sum_{n<x} \omega(n) \log \ell_1(n) = x \left\{ \sum_{p} \frac{\log(1 - \frac{1}{p})}{p} - \sum_{p} \frac{\log(1 - \frac{1}{p})}{p^2} - f(1) \sum_{p} \frac{\log(1 - \frac{1}{p})}{p} \right\} \\
+ x \left\{ \sum_{p} \frac{\log(1 - \frac{1}{p})}{p} \log \log x + \sum_{k=0}^{n} \frac{\alpha_k}{(\log x)^k} + \Theta \left( \frac{1}{(\log x)^{n+1}} \right) \right\}
\]

as \( x \to \infty \),
with

\[
\alpha_0 = \gamma \sum_p \frac{\log(1 - \frac{1}{p})}{p},
\]

\[
\alpha_1 = -\sum_p \frac{\log p}{p} \log(1 - \frac{1}{p}) + (\gamma - 1) \sum_p \frac{\log(1 - \frac{1}{p})}{p},
\]

and

\[
\alpha_k = \left[ \sum_{j=0}^{k-1} \alpha_{k-j} \left( \sum \frac{\log^j p \log(1 - \frac{1}{p})}{p} \right) \right] - \frac{1}{k} \sum_p \frac{\log^k p \log(1 - \frac{1}{p})}{p},
\]

where

\[
a_0 = \gamma + \sum_p \left[ \log(1 - \frac{1}{p}) + \frac{1}{p} \right], \quad a_k = (k-1)! \left( \sum_{j=0}^{k-1} \frac{\gamma_j}{j!} - 1 \right) \text{ for } k \geq 1,
\]

with

\[
\gamma_j = \lim_{N \to \infty} \left( \sum_{t=1}^{N} \frac{\log^j t}{t} - \frac{\log^{j+1} N}{j+1} \right).
\]

Note that \(f(s) = \sum_p \sum_{k=2}^{\infty} \frac{1}{kp^s} \) and so

\[
f(1) = -\sum_p \left[ \frac{1}{p} + \log(1 - \frac{1}{p}) \right] = \gamma - B_1,
\]

that is
\[
\sum_{n < x} \omega(n) \log \frac{\log x}{p} = x \left\{ \sum_{p} \frac{\log(1 - \frac{1}{p})}{p} \log \log x - \sum_{p} \frac{\log(1 - \frac{1}{p})}{p^2} \right. \\
+ \sum_{p} \frac{\log(1 - \frac{1}{p})}{p} (B_1 + 1) + \sum_{k=1}^{n} \frac{\alpha_k}{(\log x)^k} + O\left(\frac{1}{(\log x)^{n+1}}\right) \right\}
\]
as \( x \to \infty \).

Subtracting the product of \( \mu_x(\log \ell_1) \) [Lemma 9] and \( \mu_x(\omega) \) [Theorem 1] from \( \frac{1}{x} \sum_{n < x} \omega(n) \log \ell_1(n) \) gives us:

\[
\text{Cov}_x(\omega, \log \ell_1) = \sum_{p} \frac{\log(1 - \frac{1}{p})}{p} + \sum_{k=1}^{n} \frac{\alpha_k - \sum_{p} \frac{\log(1 - \frac{1}{p})}{p} a_k}{(\log x)^k} \\
- \sum_{p} \frac{\log(1 - \frac{1}{p})}{p^2} + O\left(\frac{\log \log x}{(\log x)^{n+1}}\right) \text{ as } x \to \infty.
\]

2 - Formula (I.2) gives us:

\[
\sum_{n=1}^{\infty} \frac{\Omega(n) \log \ell_1(n)}{n^s} = \zeta(s) \left\{ \sum_{p} \frac{\log(1 - \frac{1}{p})}{p^s} \\
+ (\log \zeta(s) - g(s)) \sum_{p} \frac{\log(1 - \frac{1}{p})}{p^s} \right\}
\]

where

\[
g(s) = \sum_{p} \sum_{k=2}^{\infty} \frac{(-1)^{k-1}}{k(p^{s-1}k^k}
\]

[notation of Theorem 3].
The method that has been used in part 1 also works here and gives us:

\[
\sum_{n \leq x} \Omega(n) \log \ell_1(n) = x \left\{ \sum_{p} \frac{\log(1 - \frac{1}{p})}{p} - g(1) \sum_{p} \frac{\log(1 - \frac{1}{p})}{p} \right. \\
+ \left. \sum_{p} \frac{\log(1 - \frac{1}{p})}{p} \log \log x + \sum_{k=0}^{n} \frac{a_k}{(\log x)^k} + O\left(\frac{1}{(\log x)^{n+1}}\right) \right\}
\]

as \( x \to \infty \).

But since \( g(1) = -\sum_{p} \frac{1}{p-1} + \sum_{p} \log(1 - \frac{1}{p}) \) and \( a_0 = \gamma \sum_{p} \frac{\log(1 - \frac{1}{p})}{p} \)
then

\[
\frac{1}{x} \sum_{n \leq x} \Omega(n) \log \ell_1(n) = \left\{ \sum_{p} \frac{\log(1 - \frac{1}{p})}{p} \log \log x + \sum_{p} \frac{\log(1 - \frac{1}{p})}{p} \right\} (1 + B_2) \\
+ \sum_{k=1}^{n} \frac{a_k}{(\log x)^k} + O\left(\frac{1}{(\log x)^{n+1}}\right) \text{ as } x \to \infty .
\]

Now subtracting the product of \( \mu_x(\Omega) \) [Theorem 3] and \( \mu_x(\log \ell_1) \) [Lemma 9] from the above expansion gives us

\[
\text{Cov}_x(\Omega, \log \ell_1) = \sum_{p} \frac{\log(1 - \frac{1}{p})}{p} + \sum_{k=1}^{n} \frac{a_k}{(\log x)^k} \\
+ O\left(\frac{\log \log x}{(\log x)^{n+1}}\right) \text{ as } x \to \infty .
\]
3 - Again according to Formula (I.2)

\[ \sum_{n=1}^{\infty} \frac{\log \frac{\zeta(n)}{\zeta(1)}}{n^s} = \zeta(s) \left\{ \sum_{p} \frac{\log p \log(1 - \frac{1}{p})}{p^s} \right\} \]

\[ - \sum_{p} \frac{\log p \log(1 - \frac{1}{p})}{p^{2s}} + \left( \sum_{p} \frac{\log p}{p^s} \right) \left( \sum_{p} \frac{\log(1 - \frac{1}{p})}{p^s} \right) \]

\[ = \zeta(s) \left\{ \sum_{p} \frac{\log p \log(1 - \frac{1}{p})}{p^s} - \sum_{p} \frac{\log p \log(1 - \frac{1}{p})}{p^{2s}} \right\} \]

\[ - \zeta'(s) \sum_{p} \frac{\log(1 - \frac{1}{p})}{p^s} + \zeta(s) f'(s) \sum_{p} \frac{\log(1 - \frac{1}{p})}{p^s} \]

(see the proof of Theorem 10).

Since

\[ \sum_{p} \frac{\log p \log(1 - \frac{1}{p})}{p^s} = -\frac{3}{\zeta(s)} \sum_{p} \frac{\log(1 - \frac{1}{p})}{p^s}, \]

then the left side is analytic for \( \text{re } s > 0 \) and uniformly bounded in any compact set in \( \text{re } s > 0 \).

The same argument holds for

\[ \sum_{p} \frac{\log p \log(1 - \frac{1}{p})}{p^{2s}} = -\frac{3}{\zeta(s)} \sum_{p} \frac{\log(1 - \frac{1}{p})}{2p^{2s}}. \]

We now use the method that has been used to derive \( I_2 \) in Theorem 1 and \( N'_3 \) in Lemma 7 to get the result. Note that
\[ N_3' = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} r'(s) \zeta'(s) \frac{x^s}{s} \, ds, \]

but here we have

\[ \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \left( \sum_p \frac{\log(1 - \frac{1}{p})}{p^s} \right) \zeta'(s) \frac{x^s}{s} \, ds \]

with

\[ \left[ \frac{\partial}{\partial s} \sum_p \frac{\log(1 - \frac{1}{p})}{p^s} \right]_{s=1} = -\sum_p \frac{\log p \log(1 - \frac{1}{p})}{p} \]

and

\[ \left[ \sum_p \frac{\log(1 - \frac{1}{p})}{p^s} \right]_{s=1} = \sum_p \frac{\log(1 - \frac{1}{p})}{p} \]

so

\[ \sum_{n<x} \log \lambda(n) \log \lambda_1(n) \approx x \left\{ \sum_p \frac{\log p \log(1 - \frac{1}{p})}{p} \right\} \]

\[ - \sum_p \frac{\log p \log(1 - \frac{1}{p})}{p^2} + r'(1) \sum_p \frac{\log(1 - \frac{1}{p})}{p} \]

\[ - \sum_p \frac{\log p \log(1 - \frac{1}{p})}{p} + \sum_p \frac{\log(1 - \frac{1}{p})}{p} \log x \]

\[ - \sum_p \frac{\log(1 - \frac{1}{p})}{p} \]

or

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\[
\sum_{n < x} \log \zeta(n) \log \varphi_1(n) \approx x \left\{ \sum_{p} \frac{\log(1 - \frac{1}{p})}{p} \log x + \sum_{p} \frac{\log(1 - \frac{1}{p})}{p} \left( f'(1) - 1 \right) \right\} + \sum_{p} \frac{\log(1 - \frac{1}{p})}{p} \log p \log(1 - \frac{1}{p}) \right\}.
\]

Therefore, by the use of Theorem 10 and Lemma 9 we have the result.

Finally

\[
\frac{\log n \log \varphi_1(n)}{n} = -\frac{\partial}{\partial s} \sum_{n=1}^{\infty} \frac{\log \varphi_1(n)}{n^s} = -\zeta'(s) \sum_{p} \frac{\log(1 - \frac{1}{p})}{p} + \zeta(s) \sum_{p} \frac{\log p \log(1 - \frac{1}{p})}{p^s}.
\]

Therefore

\[
\sum_{n < x} \log n \log \varphi_1(n) \approx x \left\{ -\sum_{p} \frac{\log p \log(1 - \frac{1}{p})}{p} \right\} + \sum_{p} \frac{\log(1 - \frac{1}{p})}{p} \log x - \sum_{p} \frac{\log(1 - \frac{1}{p})}{p} + \sum_{p} \frac{\log p \log(1 - \frac{1}{p})}{p} \right\}.
\]

Now the statements of Theorem 8 and Lemma 9 give us the result.

Use Formulas III.5, III.6, III.7, Lemmas 9, 10, 11, Theorems 8, 9 and parts 2 and 3 of Lemma 8 to get

Theorem 15. If \( \phi(n) \) is the Euler totient function at \( n \), \( \Omega(n) (\omega(n)) \), the number of (distinct) prime divisors and \( \ell \) the
product of distinct prime divisors of \( n \) then we have:

\[
a - \mu_x(\log \phi) = \log x - 1 + \sum_{p} \frac{\log(1 - \frac{1}{p})}{p} + O(\log^2 x \cdot e^{-c\sqrt{\log x}}),
\]

as \( x \to \infty \).

\[
b - \text{Var}_x(\log \phi) \approx 1 + \sum_{p} \frac{\log^2(1 - \frac{1}{p})}{p} - \sum_{p} \frac{\log^2(1 - \frac{1}{p})}{p^2}.
\]

\[
c - \text{Cov}_x(\log \phi, \omega) \approx \gamma_1 - \gamma + 1 + \sum_{p} \frac{\log(1 - \frac{1}{p})}{p} - \sum_{p} \frac{\log(1 - \frac{1}{p})}{p^2}
\]

\[
d - \text{Cov}_x(\log \phi, \Omega) \approx \gamma_1 - \gamma + 1 + \sum_{p} \frac{\log(1 - \frac{1}{p})}{p}
\]

\[
e - \text{Cov}_x(\log \phi, \log \ell) \approx 1 - \sum_{p} \frac{\log p \log(1 - \frac{1}{p})}{p^2}
\]

\[f - \text{Cov}_x(\log \phi, \log) \approx 1 \quad \text{as} \quad x \to \infty,
\]

where \( \gamma_1 = \lim_{N \to \infty} \left[ \frac{1}{N} \sum_{n=1}^{N} \frac{\log n}{n} - \frac{1}{2} \log N \right] \) and \( \gamma = \lim_{N \to \infty} \left[ \frac{1}{N} \sum_{n=1}^{N} \frac{1}{n} - \log N \right] \)

(Euler's constant).

And finally, Theorems 4, 6, 9, 11 and 15 give us:

**Theorem 16.** If \( \phi(n) \) is the Euler totient function at \( n \), \( \Omega(n) (\omega(n)) \), the number of (distinct) prime divisors and \( \ell \) the product of distinct prime divisors of \( n \), then
\[ 1 - R(\log \phi, \omega) = 0 \]

\[ 2 - R(\log \phi, \Omega) = 0 \]

\[ 3 - R(\log \phi, \log \lambda) = \left( 1 - \sum_{p} \frac{\log p \log(1 - \frac{1}{p})}{p^2} \right) \left( 1 + \sum_{p} \frac{\log^2(1 - \frac{1}{p})}{p^2} \right)^{-\frac{1}{2}} \left( 1 + \sum_{p} \frac{(p^2 + p - 1) \log^2 p}{p^2 (p-1)^2} \right)^{-\frac{1}{2}} \]

\[ 4 - R(\log \phi, \log) = \left( 1 + \sum_{p} \frac{\log^2(1 - \frac{1}{p})}{p^2} \right)^{-\frac{1}{2}} \left( 1 - \sum_{p} \frac{\log^2(1 - \frac{1}{p})}{p^2} \right)^{-\frac{1}{2}} \]

that is, \( \log \phi \) is asymptotically uncorrelated with \( \omega \) and \( \Omega \), but asymptotically depends on \( \log, \log \lambda \).

**Final Note.** Throughout this chapter again \( x \) is assumed to be a non-integer positive real number, \( c \) is a constant which may have different positive values in different places and \( C \) is the circle in Figure 2, which depends on the values of \( a \) and \( b \). \( a = 1 + \frac{1}{\log T} \), \( b = 1 - \frac{1}{\log T} \) are the values of \( a \) and \( b \) in most of the cases, but only in section 4 can they take \( 1 + \frac{c}{(\log T)^9} \) and \( 1 - \frac{c}{(\log T)^9} \). (See the proof of Lemma 8.)
IV.1 Summary of Techniques.

So far we have studied an analytic method to obtain expansions for the moments of some additive arithmetic functions. In this section we summarize the techniques that we have used and compare them to a non-analytic method.

a - Analytic method.

Suppose we want to find the asymptotic expansion for the mean-value of a strongly additive arithmetic function \( f \). According to Formula (1.9) the Dirichlet series with coefficients \( \{f(n)\} \) is

\[
\sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \zeta(s) \sum_{p} \frac{f(p)}{p^s}.
\]

Assume the convergence of this Dirichlet series for \( \Re(s) > 1 \).

This assumption holds if \( |f(p)| \leq c \log^k p \), for all primes \( p \) (\( c \) and \( k \) are non-negative constants). Note that this condition is satisfied for all the examples we have considered.

Now use Theorem 3.1 on page 50 of Ayoub (1963) [the standard approach to the sum of the coefficients of a Dirichlet series] to get

\[
\sum_{n \leq x} f(n) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \zeta(s) \sum_{p} \frac{f(p)}{p^s} \frac{x^s}{s} \, ds, \text{ for some } a > 1.
\]

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Let \( a = 1 + e(T) \), where \( e(T) \) is a non-negative function which goes to zero as \( T \) goes to infinity. \( T \) itself is a function of \( x \) and goes to infinity as \( x \) goes to infinity. (The choice of \( e \) as a function of \( T \) and \( T \) as a function of \( x \) depend on the problem and the more skillfully we choose \( T \), the better the error term. In Chapter II for the number of prime divisors we choose \( a = 1 + \frac{c}{(\log T)^{1/10}} \) and in Chapter III for \( \log \varrho(n) \), we choose \( a \) to be \( 1 + \frac{1}{\log T} \)) Let \( b = 1 - e(T) \) and deform the path of integration to be as in Figure 2. The integrations over parts 1, 2, 3, 4, 5 and 6 are considered to be reminder terms, i.e., we have

\[
E' = \frac{1}{2\pi i} \sum_{n=1}^{\infty} \frac{f(n)}{n} \int_{a-iT}^{\infty} \frac{x^s}{s} \, ds + \frac{1}{2\pi i} \sum_{n=1}^{\infty} \frac{f(n)}{n} \int_{a-iT}^{b-iT} \frac{x^s}{s} \, ds
\]

\[
+ \frac{1}{2\pi i} \sum_{n=1}^{\infty} \frac{f(n)}{n} \int_{b-iT}^{b+iT} \frac{x^s}{s} \, ds + \frac{1}{2\pi i} \sum_{n=1}^{\infty} \frac{f(n)}{n} \int_{b+iT}^{a+iT} \frac{x^s}{s} \, ds
\]

\[
+ \frac{1}{2\pi i} \sum_{n=1}^{\infty} \frac{f(n)}{n} \int_{b+iT}^{a+iT} \frac{x^s}{s} \, ds + \frac{1}{2\pi i} \sum_{n=1}^{\infty} \frac{f(n)}{n} \int_{a+iT}^{a-iT} \frac{x^s}{s} \, ds
\]

\[= I_6' + I_5' + I_4' + I_3' + I_2' + I_1'.\]

For getting the error terms let us assume \( |f(p)| \leq c \log^k p \) for all primes \( p \) (\( c, k \) non-negative constants).
For $I_1' + I_6'$ we may argue as in Titchmarsh [1951, page 53] about $a(n) = f(n) \leq c n \log^k n$ to show that:

$$I_1' + I_6' = 0 \left( \frac{x}{T} \sum_{n=1}^{\infty} n^{-a} \log^k \frac{x}{n} \right).$$

Then argue as on page 13 of Diaconis (1976) to get

$$I_1' + I_6' = 0 \left( \frac{x}{T} \left\{ \log^k \frac{x}{a-1} + (\log x)^{k+1} x^{1-a} \right\} \right).$$

Now consider the estimation of

$$I_3' + I_4' = \frac{1}{2\pi i} \int_{b-iT}^{b+IT} \sum_{n=1}^{\infty} \frac{f(n) x^s}{n^s} \frac{x^s}{s} ds.$$

Note first that

$$\left| \sum_{p} \frac{f(p)}{p^s} \right| \leq c \left| \sum_{p} \log^k \frac{p}{p^s} \right| = c \left| \frac{\partial^k}{\partial s^k} \sum_{p} \frac{1}{p^s} \right|$$

$$= c \frac{\partial^k}{\partial s^k} \left( \log \zeta(s) - \sum_{p}^{\infty} \frac{1}{p^k s} \right).$$

(See Theorem 1.) Following Titchmarsh (1951, page 42), Lemma 6 and also Titchmarsh (1951, page 44) one can get a bound of the form $O((\log T)^{\beta})$ for $\sum_{n=1}^{\infty} \frac{f(n)}{n^s}$ on the line $\text{Re } s = 1 - \epsilon(T)$ when $\beta$ is a positive number. So
\[ I_3' + I_4' = O(x^b (\log T)^{6+\ell}) \]

(This argument is similar to the argument on page 12 of Diaconis (1976).)

Also

\[ I_2' = \frac{1}{2\pi i} \int_{b+iT}^{a+iT} \sum_{n=1}^{\infty} \frac{f(n)}{n^s} \frac{x^s}{s} \, ds \]

\[ = O\left( \log T \left( \frac{x^b}{T} \right) \right) \]

similarly

\[ I_5' = O\left( \log T \left( \frac{x^a}{T} \right) \right) \]

Now combine the various error terms and balance the effect of dominant terms by letting them be equal (for example, see Theorem 8) to get the error term as small as possible.

We can calculate

\[ I_7 = \int_C \zeta(s) \left\{ \sum_{p} \frac{f(p)}{p^s} \right\} \frac{x^s}{s} \, ds \]

only if one of the following four cases occur.
\[ 1 - \sum \frac{f(p)}{p^s} \] converges in a neighborhood of 1.

(Example: \( f(n) = \log \frac{\phi(n)}{n} = \sum_{p|n} \log(1 - \frac{1}{p}) \), Lemma 9.)

In this case \( \zeta(s) \sum \frac{f(p)}{p^s} s \) has a simple pole at \( s = 1 \)

with residue \( \frac{\zeta(s)}{s} \sum f(p) p \) and therefore

\[ I_\gamma' = \frac{\zeta(s)}{s} \sum f(p) p \text{ and } \frac{1}{x} \sum_{n<x} f(n) \ll \sum \frac{f(p)}{p} \cdot \]

\[ 2 - \sum \frac{f(p)}{p^s} \] has a pole of order \( d \) at 1.

(Example: \( f(n) = \log n \), with \( d = 1 \), Theorem 8.)

In this case \( \zeta(s) \sum \frac{f(p)}{p^s} s \) has a pole of order \( d + 1 \) at 1 with residue

\[ \lim_{s \to 1} \frac{1}{d!} \frac{\partial^d}{\partial s^d} \left\{ \left( (s-1) \zeta(s) \right) \left[ (s-1) \left( \sum \frac{f(p)}{p^s} s \right) \right] \right\} = R(x) \]

and \( I_\gamma' = R(x) \), therefore

\[ \frac{1}{x} \sum_{n<x} f(n) \ll \frac{R(x)}{x} \cdot \]

\[ 3 - \sum \frac{f(p)}{p^s} \] can be written as a summation of other

Dirichlet series, where each new series belongs to one of the
above cases.

(Example, in Theorem 10: \( f(n) = \log \lambda(n) \) and

\[
\sum_{p} \frac{1}{p^s} = -\frac{\zeta'(s)}{\zeta(s)} - \sum_{p} \sum_{k=2}^{\infty} \frac{\log p}{p^{ks}}.
\]

The first term has a simple pole at 1 and the second one is analytic at 1.) In this case we calculate the integral of each term separately and add up the results.

- For some functions such as \( \omega \) (Theorem 1) some terms of the combination in the third case do not belong to either cases 1 or 2. In the case of \( \omega \) we have:

\[
\sum_{p} \frac{1}{p^s} = \log \zeta(s) - \sum_{p} \sum_{k=2}^{\infty} \frac{1}{kp^{ks}}
\]

(Theorem 1). The second term belongs to case 1, but for

\[
\frac{1}{2\pi i} \int_{C} \zeta(s) \log \zeta(s) \frac{x^{s}}{s} \, ds
\]

none of the above methods work.

Sometimes, like in this example, the Laurent's series expansion helps us to break the integral into some calculatable integrals and some more error terms (for details see the proof of Theorem 1).

From the above discussion one can see that for some strongly additive arithmetic functions this analytic method helps us to find the expansions for their mean-values. The expansions for the mean of some completely additive arithmetic functions can also be calculated by the above method (see Theorem 3 for
details in a special case). The variances and covariances of some additive arithmetic functions can be discussed the same way, but instead of \( f \), we have powers of \( f \) or a product of \( f \) with another function (see Theorems 4 and 6 and Lemma 6 for details in special cases).

This method has some disadvantages, namely:

1. Some of the Dirichlet series arising during the application of the method are multiple-valued (like \( \log \zeta(s) \) in Theorem 1) and we have to avoid the singularity at \( s = 1 \).

2. Some of the integrals are not absolutely convergent and need careful treatment (for example, \( I_1(x) \) in the proof of Theorem 1).

In some special cases we can avoid the difficulties, but not in general.

With all the trouble in this method, sometimes other approaches are better. For example, for deriving Stirling's formula (Theorem 8), a very easy calculation based on Euler's summation formula:

\[
\sum_{y<n\leq x} f(n) = \int_y^x f(t)dt + \int_y^x (t-[t]) f'(t)dt + f(x) ([x]-x) - f(y)([y]-y)
\]

when \( f \) is a continuously differentiable function on the interval \([y,x]\), \( 0 < y < x \) with derivative \( f' \) (Apostol (1976))
gives us a better result than the analytic approach (remark after Corollary 2 of Chapter III). To use Euler's summation formula, we must have a well-defined continuously differentiable extension of the function $f$ on some intervals $[a, \infty)$. In a case like $\omega(n)$, where we cannot generalize $\omega$ to a well-defined function on real numbers this method does not work.

**b - Hyperbolic Method.**

There is another method for finding mean-values of additive arithmetic functions called the "hyperbolic method". The first use of this method was Dirichlet's geometric approach to the divisor problem:

$$\sum_{m \leq n} d(n) = n \log n + (2 \gamma - 1) n + O(\sqrt{n})$$

where $d(n)$ is the number of divisors of $n$ (Theorem 320 on pages 264-265 of Hardy and Wright (1975)).

In the rest of this chapter we want to explain this method and show by some examples how it works.

In order to use the hyperbolic method for expansions of the first and second moments we derive some general formulas by using simple arguments (see Sections 2 and 3). One can apply Lemma 12 in Section 4 to those formulas to get the expansions in some cases (see examples at the end of Section 4).
Before starting the derivation of these general formulas let us talk about this method in a special case, i.e., the mean-value of a strongly additive function $f$.

$$
\sum_{n \leq x} f(n) = \sum_{n \leq x} \sum_{p | n} f(p) = \sum_{p \leq x} f(p) \frac{x}{p} \quad \text{(Formula IV.7)}.
$$

This formula can be derived by applying Theorem 3.1 on page 50 of Ayoub (1963) to the Dirichlet series with coefficients \{f(n)\}, i.e.,

$$
\sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \zeta(s) \sum_{p} \frac{f(p)}{p^s} \quad \text{(Formula I.9),}
$$

then

$$
\sum_{n \leq x} f(n) = \int_{a-i\infty}^{a+i\infty} \zeta(s) \sum_{p} \frac{f(p)}{p^s} \frac{x^s}{s} \, ds
$$

$$
= \sum_{p \leq x} f(p) \int_{a-i\infty}^{a+i\infty} \zeta(s) \left( \frac{x}{p} \right)^s \frac{ds}{s}
$$

$$
+ \int_{a-i\infty}^{a+i\infty} \sum_{p \leq x} \left( \frac{f(p)}{p^s} \right) \zeta(s) \frac{x^s}{s} \, ds
$$

$$
= \sum_{p \leq x} f(p) \left[ \frac{x}{p} \right] + \lim_{T \to \infty} \int_{a-iT}^{a+iT} \sum_{p > x} \frac{f(p)}{p^s} \zeta(s) \frac{x^s}{s} \, ds
$$

$$
= \sum_{p \leq x} f(p) \left[ \frac{x}{p} \right]
$$

for some $a > a_0$ (the abscissa of convergence of $\sum \frac{f(n)}{n^s}$).
Now Fubini's theorem can be applied to the second part and then the result follows.

We can also apply Lemma 5 (i.e.,

\[ [x] = \int_{a-i\infty}^{a+i\infty} \zeta(s) \frac{x^s}{s} \, ds \quad \text{for} \quad a > 1 \]

and \( x \) non-integer positive real number) to this formula. Then Fubini's theorem yields:

\[
\sum_{n < x} f(n) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \zeta(s) \sum_{p} \frac{f(p)}{p^s} \frac{x^s}{s} \, ds .
\]

This is the starting point of the analytic method. (See remarks at the end of Section 2.) So the starting points of analytic and hyperbolic methods are actually the same.

The basic idea for the hyperbolic method is Lemma 12. To apply this lemma for our problem let

\[
g(n) = \begin{cases} 
  f(n) \text{ if } n \text{ is prime} \\
  0 \quad \text{otherwise}
\end{cases}
\]

and \( h(n) = 1 \) \( \forall n \), so \( G(x) = \sum_{p<x} f(p) \) and \( H(x) = [x] \).

For any \( 1 < y \leq x \):
\[
\sum_{n \leq x} f(n) = \sum_{p \leq x} f(p) \left[ \frac{x}{p} \right] = \sum_{p \leq x} f(p) \left[ \frac{x}{p} \right] + \sum_{\frac{x}{y} < p \leq \frac{x}{\sqrt{y}}} f(p) \\
- \sum_{p < y} f(p) \left[ \frac{x}{y} \right] \\
= x \sum_{p < y} \frac{f(p)}{p} + \sum_{\frac{x}{y} < p \leq \frac{x}{\sqrt{y}}} f(p) - \left[ \frac{x}{y} \right] \sum_{p < y} f(p) \\
+ \sum_{p < y} f(p) \left( \left[ \frac{x}{p} \right] - \frac{x}{p} \right).
\]

Now use the prime number theorem

\[
\left[ \pi(t) = \text{li} t + \eta(t) \text{ where } \text{li} t = \int_{2}^{t} \frac{du}{\log u} + O(1) \right]
\]

to get

\[
\sum_{p < y} \frac{f(p)}{p} = \int_{a}^{y} \frac{f(t)}{t} \, d\pi(t) \text{ for some } 1 < a < 2
\]

if \{f(p)\} has a continuously differentiable extension on the real line.

[So far, the advantage of this method over the application of Euler's summation formula is that here we only need the extensions of \{f(p)\} not \{f(n)\}. For example, for \omega we can not work with Euler's summation, but the hyperbolic method works (see example 1 of Section 4).]

Integration by parts gives us
\[ \sum_{p < y} \frac{f(p)}{p} = \pi(y) \frac{f(y)}{y} - \int_a^y \left( \frac{f'(t)}{t} - \frac{f(t)}{t^2} \right) \pi(t) \, dt \]

\[ = \text{li} \, y \frac{f(y)}{y} - \int_a^y \left( \frac{f'(t)}{t} - \frac{f(t)}{t^2} \right) \text{li} \, t \, dt \]

\[ + \eta(y) \frac{f(y)}{y} - \int_a^y \left( \frac{f'(t)}{t} - \frac{f(t)}{t^2} \right) \eta(t) \, dt \]

\[ = \int_a^y \frac{f(t)}{t \log t} \, dt + A_1 + \eta_1(y). \]

Also

\[ \sum_{p < t} f(p) = \int_a^t f(u) \, d\pi(u) \]

\[ = \pi(t) f(t) - \int_a^t \pi(u) f'(u) \, du \]

\[ = \text{li} \, t \, f(t) - \int_a^t \text{li} \, u \, f'(u) \, du \]

\[ + \eta(t) f(t) - \int_a^t \eta(u) f'(u) \, du \]

\[ = \int_a^t \frac{f(u) \, du}{\log u} + A_2 + \eta_2(t) \]

where

\[ \eta_1(t) = \eta(y) \frac{f(y)}{y} + \int_y^\infty \left( \frac{f'(t)}{t} - \frac{f(t)}{t^2} \right) \eta(t) \, dt \]

\[ \eta_2(t) = \eta(t) f(t) - \int_y^\infty \eta(u) f'(u) \, du \]
\[ A_1 = -\frac{\text{li} \ a \ f(u)}{a} - \int_a^\infty \left( \frac{f'(t)}{t} - \frac{f(t)}{t} \right) (\pi(t) - \text{li}(t)) \, dt \]

and

\[ A_2 = -\frac{\text{li} \ a \ f(a)}{a} - \int_a^\infty f'(u)(\pi(u) - \text{li}(u)) \, du . \]

Now substitute in \( \sum_{n<x} f(n) \) to get

\[ \sum_{n<x} f(n) = x \int_a^x \frac{f(t)}{t \log t} \, dt + A_1 + \eta_1(y) \]

\[ + \sum_{\frac{x}{y} < \frac{x}{y}} \int_a^{\frac{x}{y}} \frac{f(u) \, du}{\log u} + A_2[\frac{x}{y}] \]

\[ + \sum_{\frac{x}{y} > \frac{x}{y}} \eta_2(\frac{x}{y}) - [\frac{x}{y}] \int_a^{\frac{x}{y}} \frac{f(u) \, du}{\log u} - [\frac{x}{y}] A_2 \]

\[ - [\frac{x}{y}] \eta_2(y) + \sum_{p<y} \tau(p)([\frac{x}{p}] - \frac{x}{p}) . \]

So

\[ \sum_{n<x} f(n) = x \int_a^x \frac{f(t) \, dt}{t \log t} + \sum_{\frac{x}{y} < \frac{x}{y}} \int_a^{\frac{x}{y}} \frac{f(u) \, du}{\log u} \]

\[ - [\frac{x}{y}] \int_a^{\frac{x}{y}} \frac{f(u) \, du}{\log u} + A_1 + \eta_3(y) \]

where
\[ \eta_3(y) = \eta(y) \frac{f(y)}{y} + \int_{y}^{\infty} \left( \frac{f'(t)}{t} - \frac{f(t)}{t^2} \right) \eta(t) \, dt \]

\[ + \sum_{\ell < \frac{X}{y}} \eta\left(\frac{X}{\ell}\right) f\left(\frac{X}{\ell}\right) - \sum_{\ell < \frac{X}{y}} \int_{\frac{X}{\ell}}^{\infty} \eta(u) f'(u) \, du \]

\[ - \left[ \frac{X}{y} \right] \eta(y) f(y) - \left[ \frac{X}{y} \right] \int_{y}^{\infty} \eta(u) f'(u) \, du \]

\[ + \sum_{p \leq y} f(p) \left( \left[ \frac{X}{p} \right] - \frac{X}{p} \right). \]

The evaluation of the integrals, the constants and the error term depends on the additive function \( f \). \( y \) is an increasing function of \( x \), \( y \rightarrow \infty \) as \( x \rightarrow \infty \) and \( y \leq x \). We choose \( y \) so that we get a small reminder term as \( x \rightarrow \infty \).

In Section 4, we apply this method to \( \omega \) and \( \log \ell \) and we get the same results as in the analytic method. Note that in general this method works for additive arithmetic functions of the form \( \sum_{p | n} \log^k n \), where \( k \) is a non-negative integer (for example, for \( k = 0 \); \( \omega \) and \( k = 1 \); \( \log \ell \)).

So in order to apply the hyperbolic method we need:

1 - a continuously differentiable extension of \( \{f(p)\} \) on the real line,

2 - the integrals in the above formula for \( \sum_{n < x} f(n) \) to be computable,

3 - some bounds for \( \{f(p)\} \),

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4 - bounds on the behavior of \( \sum_{p \leq x} \frac{f(p)}{p} \) which generally have to be proved by use of the prime number theorem. For example, for \( \omega(n) \) we need Formula (IV.15) and for \( \log \lambda(n) \) we had to prove Lemma 13.

For example, if we look at the application of the hyperbolic method for \( \log \lambda(n) \), we will see the extension is \( f(t) = \log t \) with derivative \( \frac{1}{t} \). The integrals are obtained either in Lemma 10 or during the course of the proof.

We can see that the prime number theorem helps us to evaluate these integrals, but the prime number theorem with exponential errors can only be proved by analytic methods (see Theorems 2.7 and B on pages 48 and 49 of Ayoub (1963)), so roughly speaking this method is also based on an analytic method.

IV.2 General Formulas For Additive Arithmetic Functions.

In this section we will find some general formulas which are useful for the derivation of the mean and variance of additive arithmetic functions and also for the covariances of two of them.

Let \( f \) and \( g \) be two additive arithmetic functions. Use the notation 6 of Chapter I and let \( x \) be a non-integer positive real number. Then

\[
\sum_{n \leq x} f(n) = \sum_{n \leq x} \sum_{p \parallel n} f(p^k) = \sum_{n \leq x} \left[ \sum_{p \parallel n} f(p^k) - \sum_{p \parallel n} f(p^{k+1}) \right].
\]

But
\[
\sum_{n \leq x} \sum_{p^k \mid n} f(p^k) = \sum_{p^k \leq x} \sum_{n = p^k m} f(p^k) = \sum_{p^k \leq x} f(p^k) \sum_{m \leq \frac{x}{p^k}} 1
\]
\[
= \sum_{p^k \leq x} f(p^k) \left\lfloor \frac{x}{p^k} \right\rfloor,
\]
and
\[
\sum_{n \leq x} \sum_{p^{k+1} \mid n} f(p^{k+1}) = \sum_{p^{k+1} \leq x} \sum_{n = p^{k+1} m} f(p^{k+1}) = \sum_{p^{k+1} \leq x} f(p^{k+1}) \sum_{m \leq \frac{x}{p^{k+1}}} 1
\]
\[
= \sum_{p^{k+1} \leq x} f(p^{k+1}) \left\lfloor \frac{x}{p^{k+1}} \right\rfloor.
\]
So
\[
(\text{IV.1)} \quad \sum_{n \leq x} f(n) = \sum_{p^k \leq x} f(p^k) \left\lfloor \frac{x}{p^k} \right\rfloor - \sum_{p^{k+1} \leq x} f(p^{k+1}) \left\lfloor \frac{x}{p^{k+1}} \right\rfloor.
\]
Now by letting \(K_p(x) = \left\lfloor \frac{\log x}{\log p} \right\rfloor = \max\{k : p^k \leq x\}\) and since for \(p > \sqrt{x}\) \(K_p(x) = 1\), we have
\[
\sum_{p^k \leq x} f(p^k) \left\lfloor \frac{x}{p^k} \right\rfloor = \sum_{p \leq \sqrt{x}} \sum_{k=1}^{K_p(x)} f(p^k) \left\lfloor \frac{x}{p^k} \right\rfloor
\]
\[
= \sum_{p \leq \sqrt{x}} \sum_{k=1}^{K_p(x)} f(p^k) \left\lfloor \frac{x}{p^k} \right\rfloor + \sum_{\sqrt{x} < p \leq x} f(p) \left\lfloor \frac{x}{p} \right\rfloor
\]
\[
= \sum_{p \leq \sqrt{x}} f(p) \left\lfloor \frac{x}{p} \right\rfloor + \sum_{\sqrt{x} < p \leq x} f(p^{k+1}) \left\lfloor \frac{x}{p^{k+1}} \right\rfloor
\]

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and

\[ \sum_{p^{k+1} < x} f(p^k) \left[ \frac{x}{p^{k+1}} \right] = \sum_{p < \sqrt{x}} \sum_{k=1}^{K_p(x)-1} f(p^k) \left[ \frac{x}{p^{k+1}} \right]. \]

So

\[ (IV.2) \sum_{n < x} f(n) = \sum_{p < x} f(p) \left[ \frac{x}{p} \right] + \sum_{p < \sqrt{x}} \sum_{k=1}^{K_p(x)-1} \left( f(p^{k+1}) - f(p^k) \right). \]

Similar formulas for the product of two additive functions can be derived.

\[ \sum_{n < x} f(n) g(n) = \sum_{n < x} \sum_{p^k | n} f(p^k) \sum_{q^j | n} g(q^j) \]

\[ = \sum_{n < x} \sum_{p^k | n} \sum_{q^j | n} f(p^k) g(q^j) - \sum_{n < x} \sum_{p^k | n} \sum_{q^{j+1} | n} f(p^k) g(q^j) \]

\[ - \sum_{n < x} \sum_{p^{k+1} | n} \sum_{q^j | n} f(p^k) g(q^j) + \sum_{n < x} \sum_{p^{k+1} | n} \sum_{q^{j+1} | n} f(p^k) g(q^j) \]

\[ = \sum_{p^k < x} \sum_{q^j < x} \sum_{n=p^k q^j m \atop n < x} f(p^k) g(q^j) - \sum_{p^k < x} \sum_{q^{j+1} < x} \sum_{n=p^k q^{j+1} m \atop n < x} f(p^k) g(q^j) \]

\[ - \sum_{p^{k+1} < x} \sum_{q^j < x} \sum_{n=p^{k+1} q^j m \atop n < x} f(p^k) g(q^j) \]

\[ + \sum_{p^{k+1} < x} \sum_{q^{j+1} < x} \sum_{n=p^{k+1} q^{j+1} m \atop n < x} f(p^k) g(q^j). \]
\begin{align}
&\sum_{n \leq x} f(n) g(n) = \sum_{p^k \leq x} f(p^k) g(q^j) \left[ \frac{x}{p^k \cdot q^j} \right] \\
&\quad - \sum_{p^{k+1} \leq x} f(p^k) g(q^j) \left[ \frac{x}{p^{k+1} \cdot q^j} \right] \\
&\quad - \sum_{p^{k+1} \leq x} f(p^k) g(q^j) \left[ \frac{x}{p^{k+1} \cdot q^j+1} \right] \\
&\quad + \sum_{p^{k+1} \leq x} f(p^k) g(q^j) \left[ \frac{x}{p^{k+1} \cdot q^j+1} \right].
\end{align}

We can also write (IV.3) in the following form

\begin{align}
&\sum_{n \leq x} f(n) g(n) = \sum_{p^k \leq x} f(p^k) g(q^j) \left[ \frac{x}{p^k \cdot q^j} \right] \\
&\quad - \sum_{p^{k+1} \leq x} f(p^k) g(q^j) \left[ \frac{x}{p^{k+1} \cdot q^j} \right] \\
&\quad - \sum_{p^{k+1} \leq x} f(p^k) g(q^j) \left[ \frac{x}{p^{k+1} \cdot q^j+1} \right] \\
&\quad + \sum_{p^{k+1} \leq x} f(p^k) g(q^j) \left[ \frac{x}{p^{k+1} \cdot q^j+1} \right].
\end{align}

But for a fixed prime $p$ and a fixed positive integer $k$, we have

\begin{align}
&\sum_{q \leq \sqrt{x}} g(q^j) \left[ \frac{x}{p^k \cdot q^j} \right] - \sum_{q \leq \sqrt{x}} g(q^j) \left[ \frac{x}{p^k \cdot q^j+1} \right] \\
&= \sum_{q \leq x} g(q) \left[ \frac{x}{p^k \cdot q} \right] - \sum_{q \leq \sqrt{x}} g(q^j) \left[ \frac{x}{p^k \cdot q^{j+1}} \right] (g(q^{j+1}) - g(q^j)).
\end{align}
(by the same method that we used to obtain (IV.2).) Also

\[ K_p(x) \sum_{q<x} \sum_{j=1}^{K_q(x)-1} g(q^j) \left[ \frac{x}{p^{k+1} q^{j+1}} \right] - \sum_{q<x} \sum_{j=1}^{K_q(x)-1} g(q^j) \left[ \frac{x}{p^{k+1} q^{j+1}} \right] \]

\[ = \sum_{q<x} f(q) \left[ \frac{x}{p^{k+1} q} \right] + \sum_{q<x} \sum_{j=1}^{K_q(x)-1} g(q^j) \left( g(q^{j+1}) - g(q^j) \right) \cdot \]

So

\[ \sum_{n<x} f(n) \ g(n) = \sum_{p<x} \sum_{k=1}^{K_p(x)} \sum_{q<x} f(p^k) g(q) \left[ \frac{x}{p^k q} \right] \]

\[ + \sum_{p<x} \sum_{k=1}^{K_p(x)} \sum_{q<x} \sum_{j=1}^{K_q(x)-1} f(p^k) \left[ \frac{x}{p^k q^{j+1}} \right] \left( g(q^{j+1}) - g(q^j) \right) \]

\[ - \sum_{p<x} \sum_{k=1}^{K_p(x)} \sum_{q<x} \sum_{j=1}^{K_q(x)-1} f(p^{k+1}) \left[ \frac{x}{p^{k+1} q^{j+1}} \right] \left( g(q^{j+1}) - g(q^j) \right) \]

Now use the same argument for \( p, k \) and consider the first and third expression together and also the other two together to get
(IV.4)

\[ \sum_{n < x} f(n) g(n) = \sum_{p < x} \sum_{q < x} f(p) g(q) \left[ \frac{x}{pq} \right] \\
+ \sum_{p < \sqrt{x}} \sum_{q < \sqrt{x}} g(q) \left[ \frac{x}{p^{k+1} q} \right] (f(p^{k+1}) - f(p^k)) \\
+ \sum_{p < x} \sum_{q < \sqrt{x}} f(p) \left[ \frac{x}{p q^{j+1}} \right] (g(q^{j+1}) - g(q^j)) \\
+ \sum_{p < \sqrt{x}} \sum_{q < \sqrt{x}} k_{p(x)} - 1 \sum_{k=1}^{k_{q(x)} - 1} \sum_{j=1}^{q_{j+1}} \left[ \frac{x}{p^{k+1} q^{j+1}} \right] (f(p^{k+1}) - f(p^k))(g(q^{j+1}) - g(q^j)). \]

Now for \( f = g \), we have

(IV.5) \[ \sum_{n < x} f^2(n) = \sum_{p < x} \sum_{q < x} f(p) f(q) \left[ \frac{x}{p^k q^j} \right] \\
- 2 \sum_{p^{k+1} < x} \sum_{q^j < x} f(p) f(q) \left[ \frac{x}{p^{k+1} q^j} \right] \\
+ \sum_{p^{k+1} < x} \sum_{q^{j+1} < x} f(p) f(q) \left[ \frac{x}{p^{k+1} q^{j+1}} \right] \text{ or} \]

(IV.6)

\[ \sum_{n < x} f^2(n) = \sum_{p < x} \sum_{q < x} f(p) f(q) \left[ \frac{x}{pq} \right] \\
+ 2 \sum_{p < \sqrt{x}} \sum_{q < \sqrt{x}} f(p) \left[ \frac{x}{p^{k+1} q} \right] (f(p^{k+1}) - f(p^k)) \\
+ \sum_{p < \sqrt{x}} \sum_{q < \sqrt{x}} k_{p(x)} - 1 \sum_{k=1}^{k_{q(x)} - 1} \sum_{j=1}^{q_{j+1}} \left[ \frac{x}{p^{k+1} q^{j+1}} \right] (f(p^{k+1}) - f(p^k))(f(q^{j+1}) - f(q^j)). \]
Remarks.

1 - We can derive Formula (IV.1) by applying Theorem 3.1 on page 50 of Ayoub (1963) (that is, the theorem for partial summation of coefficients of Dirichlet series) to Formula (I.1) of Chapter 1, i.e.,

\[ \sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \zeta(s) \left( \sum_{p} \left(1 - \frac{1}{p^s} \right) \sum_{k=1}^{\infty} \frac{f(p^k)}{p^{ks}} \right) \].

The same argument is useful for deriving (IV.3) from (I.2) and (IV.5) from (I.3).

2 - By applying Lemma 5 [i.e., \[ x = \int_{a-i\infty}^{a+i\infty} \zeta(s) \frac{x^s}{s} ds \] for \( a > 1 \) and \( x \) non-integer positive real number] to Formula (IV.1) and also by the use of Fubini's theorem (e.g., Theorem 7.8 on page 150 of Rudin (1974)) we get

\[ \sum_{n<x} f(n) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \zeta(s) \sum_{p} \sum_{k=1}^{\infty} \frac{f(p^k)}{p^{ks}} \left(1 - \frac{1}{p^s} \right) \frac{x^s}{s} ds \]

for \( a > a_0 \), where \( a_0 \) is the abscissa of absolute convergence of \( \sum_{n=1}^{\infty} \frac{f(n)}{n^s} \). Also from equation (IV.3) we have

\[ \sum_{n<x} f(n) g(n) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \zeta(s) \left( \sum_{p} \frac{f(p^k)}{p^{ks}} \right) \left( \sum_{k=1}^{\infty} \frac{g(p^k)}{p^{ks}} \right) \]

\[ - \sum_{p} \left( \sum_{k=1}^{\infty} \frac{f(p^k)}{p^{ks}} \right) \left( \sum_{k=1}^{\infty} \frac{g(p^k)}{p^{ks}} \right) \]

\[ + \left( \sum_{p} \left( \sum_{k=1}^{\infty} \frac{f(p^k)}{p^{ks}} \right) \right) \left( \sum_{p} \left( \sum_{k=1}^{\infty} \frac{g(p^k)}{p^{ks}} \right) \right) \frac{x^s}{s} ds \]
for $a > a'_0$, where $a'_0$ is the abscissa of absolute convergence of $\sum_{n=1}^{\infty} \frac{f(n)g(n)}{n^s}$. And finally from equation (IV.5)

$$\sum_{n<x}^{} f^2(n) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \zeta(s) \left\{ \sum_p (1 - \frac{1}{p^s}) \sum_{k=1}^{\infty} \frac{f^2(p^k)}{p^{ks}} 
- \sum_p (1 - \frac{1}{p^s})^2 \left( \sum_{k=1}^{\infty} \frac{f(p^k)}{p^{ks}} \right)^2 
+ \left[ \sum_p (1 - \frac{1}{p^s}) \left( \sum_{k=1}^{\infty} \frac{f(p^k)}{p^{ks}} \right) \right] \frac{x^s}{s} \, ds \right\},$$

for $a > a''_0$, where $a''_0$ is the abscissa of absolute convergence of $\sum_{n=1}^{\infty} \frac{f^2(n)}{n^s}$.

IV.3 Special Cases.

Case 1 - If $f$ is strongly additive, i.e., $f(p^k) = f(p)$ for $k \geq 1$. Since $f(p^{k+1}) = f(p^k)$ for all $k \geq 1$, then from Formulas (IV.2) and (IV.6) we have

(IV.7) $\sum_{n<x} f(n) = \sum_{p<x} f(p) \left[ \frac{x}{p} \right]$ and

(IV.8) $\sum_{n<x} f^2(n) = \sum_{p<x} \sum_{q<x} f(p) f(q) \left[ \frac{x}{p q} \right].$

Case 2 - If $F$ is completely additive, i.e., $F(p^k) = k F(p)$ for $k \geq 1$. Since $F(p^{k+1}) - F(p^k) = F(p)$, we have
\[(IV.9) \sum_{n<x} F(n) = \sum_{p<x} F(p) \left[ \frac{x}{p} \right] + \sum_{p<\sqrt{x}} F(p) \sum_{k=1}^{K_p(x)-1} \left[ \frac{x}{p^k+1} \right] \text{ and} \]

\[(IV.10) \sum_{n<x} F^2(n) = \sum_{p<x} \sum_{q<x} F(p) F(q) \left[ \frac{x}{pq} \right] \]

\[+ 2 \sum_{p<\sqrt{x}} \sum_{q<\sqrt{x}} F(p) F(q) \sum_{k=1}^{K_p(x)-1} \sum_{j=1}^{K_q(x)-1} \left[ \frac{x}{p^k q^j+1} \right].\]

Case 3 - If \( f \) and \( F \) are respectively the corresponding strongly additive and completely additive functions, i.e.,

\[f(p^k) = f(p), \quad F(p^k) = k f(p) \quad \text{for all primes } p\]

and all integers \( k \geq 1 \), then

\[(IV.11) \sum_{n<x} f(n) F(n) = \sum_{p<x} \sum_{q<x} f(p) f(q) \left[ \frac{x}{pq} \right] \]

\[+ \sum_{p<x} \sum_{q<\sqrt{x}} f(p) f(q) \sum_{j=1}^{K_q(x)-1} \left[ \frac{x}{p q^j+1} \right].\]

Case 4 - If \( f \) and \( g \) are strongly additive arithmetic functions (i.e., \( f(p^k) = f(p), g(p^k) = g(p) \ \forall k \geq 1 \)), then

\[(IV.12) \sum_{n<x} f(n) g(n) = \sum_{p<x} \sum_{q<x} f(p) g(p) \left[ \frac{x}{p} \right].\]
Case 5 - If \( F \) and \( G \) are completely additive arithmetic functions (i.e., \( F(p^k) = k F(p) \), \( G(p^k) = k G(p) \ \forall k \geq 1 \)), then

\[
\sum_{n \leq x} F(n) G(n) = \sum_{p \leq \sqrt{x}} \sum_{q \leq x} F(p) G(q) \frac{x}{pq} + \sum_{p < \sqrt{x}} \sum_{q < x} F(p) G(q) \sum_{k=1}^{K_p(x)-1} \frac{x}{p^{k+1} q} \\
+ \sum_{p < x} \sum_{q < \sqrt{x}} F(p) G(q) \sum_{j=1}^{K_q(x)-1} \frac{x}{p \cdot q^{j+1}} \\
+ \sum_{p < \sqrt{x}} \sum_{q < \sqrt{x}} F(p) G(q) \sum_{k=1}^{K_p(x)-1} \sum_{j=1}^{K_q(x)-1} \frac{x}{p^{k+1} q^{j+1}}.
\]

Finally

Case 6 - If \( f \) is strongly additive and \( G \) is completely additive (i.e., \( f(p^k) = f(p) \), \( G(p^k) = k G(p) \ \forall k \geq 1 \)), then

\[
\sum_{n \leq x} f(n) G(n) = \sum_{p \leq \sqrt{x}} \sum_{q \leq x} f(p) G(p) \frac{x}{pq} + \sum_{p < \sqrt{x}} \sum_{q < x} f(p) G(q) \sum_{j=1}^{K_q(x)-1} \frac{x}{p \cdot q^{j+1}}.
\]

IV.4 Hyperbolic Method and Its Application.

For using the above formulas to get the expansion for the moments of additive arithmetic functions there is the hyperbolic method (see Section 1) which is based on the following lemma.
**Lemma 12.** Let \( f \) be the convolution of two arithmetic functions, say \( f(n) = (g * h)(n) = \sum_{k|n} f(k) h(n/k) \) then for any pair of real numbers \( x, y \) such that \( 1 < y \leq x \) we have:

\[
F(x) = \sum_{n < x} f(n) = \sum_{k < y} g(k) \mathcal{H}(\frac{x}{k}) + \sum_{\ell < \frac{x}{y}} h(\ell) G(\frac{x}{\ell}) - G(y) H(\frac{x}{y})
\]

where

\[
G(t) = \sum_{k < t} g(k) \quad \text{and} \quad H(t) = \sum_{\ell < t} h(\ell).
\]

**Proof.**

\[
F(x) = \sum_{n < x} f(n) = \sum_{n < x} \sum_{k|n} g(k) h(n/k) = \sum_{n < x} \sum_{n = k} g(k) h(n/k) = \sum_{k < x} g(k) \sum_{\ell < \frac{x}{k}} h(\ell) = \sum_{k < x} g(k) \mathcal{H}(\frac{x}{k}).
\]

But

\[
\sum_{y \leq k < x} g(k) \mathcal{H}(\frac{x}{k}) = \sum_{y \leq k < x} g(k) \sum_{\ell < \frac{x}{k}} h(\ell) = \sum_{\ell < \frac{x}{y}} h(\ell) \sum_{k < \frac{x}{\ell}} g(k)
\]

\[
= \sum_{\ell < \frac{x}{y}} h(\ell) \sum_{k < \frac{x}{\ell}} g(k) - \sum_{\ell < \frac{x}{y}} h(\ell) \sum_{k < y} g(k)
\]

\[
= \sum_{\ell < \frac{x}{y}} h(\ell) G(\frac{x}{\ell}) - G(y) H(\frac{x}{y}).
\]

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And finally

\[ F(x) = \sum_{k \leq y} g(k) H \left( \frac{x}{k} \right) + \sum_{\ell \leq \frac{\sqrt{x}}{y}} h(\ell) G \left( \frac{x}{\ell} \right) - G(y) H \left( \frac{x}{y} \right). \]

**Examples.**

1 - Saffari (1968) has used this lemma to show that for each \( n \geq 1 \)

\[ \sum_{m \leq x} \omega_{r,\ell}(n) = \frac{x \log \log x}{\phi(r)} + B_{r,\ell} x + \sum_{k=1}^{n} \frac{a_k}{\phi(r)} \frac{x}{(\log x)^k} \]

\[ + O \left( \frac{x}{(\log x)^{n+1}} \right) \text{ as } x \to \infty, \]

when \( \omega_{r,\ell}(n) \) is the number of distinct prime divisors \( p \) of \( n \) satisfying \( p \equiv \ell (\text{mod } r) \) for \( r \geq 1 \), with

\[ a_k = \int_{1}^{\infty} \frac{[t]-t}{t^2} (\log t)^{k-1} dt = \frac{(-1)^{k-1}}{k} \frac{d^k}{ds^k} \left[ \frac{(s-1) \zeta(s)}{s} \right]_{s=1} \]

for \( k \geq 1 \)

and \( B_{r,\ell} \) are computable constant and \( \phi \) is the Euler totient function. Now by putting \( r = 1, \ell = 0 \), we get Theorem 1.

Note that

\[ (IV.15) \quad \sum_{p \leq y} \frac{1}{p} = \log \log y + B_1 + O \left( \frac{1}{(\log x)^n} \right) \text{ as } y \to \infty. \]
[Theorems 427 and 428 on page 351 of Hardy and Wright (1975) and Lemma 2 on page 47 of Diaconis (1974)]. Also \( \lim_{y \to \infty} \frac{\pi(y)}{y} = 0 \) [e.g., Theorem 2.1 on page 39 of Ayoub (1963)]. Therefore, if \( a \) is any real number between 1 and 2, then

\[
B_{1,0} = \int_{a}^{\infty} \frac{\pi(t) - \text{li} t}{t^2} \, dt + \frac{\text{li} a}{a} - \log \log a
\]

\[
= \lim_{y \to \infty} \left( \int_{a}^{y} \frac{\pi(t) - \text{li} t}{t^2} \, dt + \frac{\text{li} a}{a} - \log \log a \right)
\]

\[
= \lim_{y \to \infty} \left( -\frac{\pi(y)}{y} + \sum_{p < y} \frac{1}{p} + \frac{\text{li} y}{y} - \log \log y \right)
\]

\[
= B_1 = a_0 \quad \text{(see Theorem 1)}.
\]

Also, according to Lemma 1

\[
h_1(s) = \zeta(s)(s-1) = \sum_{k=1}^{\infty} d_{k-1} (s-1)^k + 1
\]

with \( h_1(1) = 1 \) and \( h_1^{(i)}(1) = i! \, d_{i-1} \) for \( i = 1, 2, \ldots \). So for \( s_1(s) = \frac{\zeta(s)(s-1)}{s} = \frac{h_1(s)}{s} \) we have

\[
g_1(s) = \frac{\xi(s)(s-1)}{s} = \frac{h_1(s)}{s}
\]

\[
g_1(1) = 1, \quad g_1^{(i)}(1) = i! \left[ \sum_{j=1}^{i} (-1)^{i-j} \, d_{j-1} + (-1)^i \right]
\]

and

\[
a_k = (-1)^{k-1} k \int_{0}^{1} g_1^{(k)}(t) \, dt = (k-1)! \left[ \sum_{i=0}^{k-1} \frac{Y_i}{i!} - 1 \right]
\]
that is, the same values as in Theorem 2.

2 - As an example of a new result we will show that the hyperbolic method also works for \( \log \ell(n) = \sum_{p | n} \log p \). It yields

\[
\mu_x(\log \ell) = \frac{1}{x} \sum_{n < x} \log \ell(n) = \log x - 1 - \sum_{p} \frac{\log p}{p(p-1)} + \mathcal{O}\left(\frac{1}{(\log x)^\alpha}\right) \quad \text{as} \quad x \to \infty,
\]

for any arbitrary \( n \). (See Theorem 3.16 on page 68 of Apostol (1976).) But first note that an easy consequence of the prime number theorem and integration by parts is the following formula (see page 52 of Gelfand and Linnik (1965)):

\[
\sum_{p < x} \frac{\log p}{p} = \log x + \mathcal{O}(1), \quad \text{as} \quad x \to \infty.
\]

In order to use the hyperbolic method to obtain the mean of \( \log \ell \) we must strengthen this result. We now show:

**Lemma 13.** For any non-negative integer \( n \),

\[
\sum_{p < x} \frac{\log p}{p} = \log x + D + \mathcal{O}\left(\frac{1}{(\log x)^\alpha}\right) \quad \text{as} \quad x \to \infty,
\]

where \( D = -\gamma - \sum_{p} \frac{\log p}{p(p-1)} \cdot \)
Proof of Lemma 13. Let \( D(x) = \sum_{p<x} \frac{\log p}{p} \), then for some \( 1 < a < \inf\{2, x\} \):

\[
D(x) = \int_a^x \frac{\log t}{t} \, d\pi(t) = \frac{\pi(x) \log x}{x} - \int_a^x \frac{\pi(t) \, dt}{t^2} + \int_a^x \frac{\pi(t) \log t}{t^2} \, dt.
\]

By using the prime number theorem [e.g., page 65 of Ingham (1932), i.e., \( \pi(t) = \text{li} \, t + \eta(t) \) where for any positive integer \( n \), \( \eta(t) = o\left(\frac{t}{(\log t)^n}\right) \) as \( t \to \infty \),

\[
\text{li} \, t = \lim_{\varepsilon \to 0} \left( \int_0^{1-\varepsilon} \frac{du}{\log u} + \int_{1+\varepsilon}^t \frac{du}{\log u} \right) = \int_2^t \frac{du}{\log u} + O(1) \text{ for } t > 2
\]

we get

\[
D(x) = \frac{\text{li} \, x \log x}{x} - \int_a^x \frac{\text{li} \, t}{t^2} \, dt + \int_a^x \frac{\text{li} \, t \log t}{t^2} \, dt
\]

\[
+ \frac{\eta(x) \log x}{x} - \int_a^x \frac{\eta(t) \, dt}{t^2} + \int_a^x \frac{\eta(t) \log t}{t^2} \, dt
\]

\[= \log x + D + \eta_1(x)\]

with

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\[
D = -\frac{\log a}{a^2} - \int_a^\infty \frac{\log t}{t^2} \, dt + \int_a^\infty \frac{\pi(t) - \log t}{t^2} \, dt
\]

\[
\eta_1(x) = \frac{x}{x} \log x + \int_x^\infty \frac{\eta(t)}{t} \, dt - \int_x^\infty \frac{\log t}{t^2} \, dt
\]

\[
= o\left(\frac{1}{(\log x)^n}\right) \text{ as } x \to \infty ,
\]

for arbitrary \( n \). Now use the method on page 351 and 352 of Hardy and Wright (1975), that is, let

\[
G(\delta) = \sum \frac{\log p}{p^{1+\delta}} \frac{1}{p^{1+\delta}-1}
\]

which is uniformly convergent for all \( \delta > 0 \) and so

\[
G(\delta) \to G(0) = \sum \frac{\log p}{p(p-1)} \text{ as } \delta \to 0 .
\]

But \( G(\delta) = h(\delta) - \frac{\zeta'(1+\delta)}{\zeta(1+\delta)} \) with \( h(\delta) = -\sum \frac{\log p}{p^{1+\delta}} \). In Theorem 421 on page 346 of Hardy and Wright (1975) put \( C_p = \frac{\log p}{p} \), \( C_n = 0 \) if \( n \) is not prime and \( f(t) = t^{-\delta} \) to get

\[
-\sum_{p<x} \frac{\log p}{p^{1+\delta}} = -\frac{D(x)}{x^\delta} - \delta \int_2^x t^{-1-\delta} D(t) \, dt .
\]

Since \( \frac{D(x)}{x^\delta} \to 0 \) as \( x \to \infty \), we have
\[ h(\delta) = -\delta \int_2^\infty t^{-1-\delta} D(t) \, dt \]

\[ = -\delta \int_2^\infty t^{-1-\delta} \log t \, dt - \delta D \int_2^\infty t^{-1-\delta} \, dt \]

\[ + o\left(\delta \int_2^\infty t^{-1-\delta} \frac{dt}{(\log t)^n}\right). \]

But

\[ \int_1^\infty t^{-1-\delta} \log t \, dt = \int_0^\infty e^{-\delta u} u \, du = \frac{1}{\delta^2}, \]

\[ \int_1^\infty t^{-1-\delta} \, dt = \frac{1}{\delta}, \]

\[ \int_2^\infty \frac{t^{-1-\delta}}{(\log t)^n} \, dt = \int_{\log 2}^\infty \frac{du \, e^{-\delta u}}{u^n} \leq \frac{2^{-\delta} \delta (\log 2)^{-(n-1)}}{n-1} \]

and

\[ \int_1^2 t^{-1-\delta} (\log t + D) \, dt < \infty. \]

So \( h(\delta) + \frac{1}{\delta} \rightarrow -D \) as \( \delta \rightarrow 0 \). Also

\[ \frac{1}{\delta} + \frac{\zeta'(1+\delta)}{\zeta(1+\delta)} = \frac{\delta \zeta'(1+\delta)}{\delta} \frac{\zeta(1+\delta)}{\zeta(1+\delta)} \rightarrow A_0 = \gamma \] as \( \delta \rightarrow 0 \).

Finally
\[ G(\delta) = h(\delta) + \frac{1}{\delta} - \left( \frac{1}{\delta} + \frac{\zeta'(1+\delta)}{\zeta(1+\delta)} \right) \]

\[ + D - \gamma \text{ as } \delta \to 0. \]

So \[ \sum_p \frac{\log p}{p(p-1)} = -D - \gamma \text{ or } D = -\gamma - \sum_p \frac{\log p}{p(p-1)}. \]

Now back to the mean of \( \log \ell \). According to Formula (IV.7)

\[ \sum_{n < x} \log \ell(n) = \sum_{p < x} \log \frac{x}{p}. \]

In Lemma 12 put

\[ g(n) = \begin{cases} 
\log n & \text{if } n \text{ is prime} \\
0 & \text{elsewhere} \end{cases} , \]

and \( h(n) = 1 \forall n \) then \( G(x) = \sum_{p < x} \log p \) and \( H(x) = [x] \). So for any \( 1 < y \leq x \):

\[ \sum_{p < x} \log \frac{x}{p} = \sum_{p < y} \log \frac{x}{p} + \sum_{y < p \leq x} \log p - \sum_{p < y} \log \frac{x}{p}. \]

But

\[ x \sum_{p < y} \frac{\log p}{p} = x(\log y + D + \eta_1(y)) \] (Lemma 13)

and

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\[
\sum_{p < t} \log p = \int_\alpha^t \log u \, d\pi(u) = \pi(t) \log t - \int_\alpha^t \frac{\pi(u) \, du}{u} \\
= \operatorname{li} t \log t - \int_\alpha^t \operatorname{li} u \, \frac{du}{u} + \eta(t) \log t \\
- \int_\alpha^t \eta(u) \, \frac{du}{u} \\
= t + \log \alpha \operatorname{li} \alpha - \alpha - \int_\alpha^\infty \frac{\pi(u) - \operatorname{li} u \, du}{u} \\
+ \eta(t) \log t + \int_t^\infty \frac{\eta(u) \, du}{u} .
\]

[\eta(t) = \pi(t) - \operatorname{li} t , \; 1 < \alpha < \min\{2, t\}.] \text{ Also}

(IV.16) \quad \sum_{\frac{x}{y} < \frac{t}{x}} \frac{1}{x} = \int_{1/y}^{x/y} \frac{d[t]}{t} = \left[\frac{t}{y} \right] \frac{y}{x} + \int_1^{x/y} \frac{[t]}{t} \, dt \\
= \left[\frac{x}{y} \right] \frac{y}{x} + \int_1^\infty \frac{[t] - t}{t^2} \, dt + \int_{\frac{x}{y}}^\infty \frac{t - [t]}{t^2} \, dt \\
+ \log x - \log y \\
= \left[\frac{x}{y} \right] \frac{y}{x} + \log x - \log y + (\gamma - 1) + \int_{\frac{x}{y}}^\infty \frac{t - [t]}{t^2} \, dt .
\]

So
\[ \sum_{\frac{x}{y} < p} \sum_{\frac{x}{y} < \ell} \log p = x \sum_{\frac{x}{y} < \ell} \frac{1}{\ell} + \left[ \frac{x}{y} \right] \left( \log a \operatorname{li} a - a - \int_{a}^{\infty} \frac{\pi(u) - li u}{u} \, du \right) \]

\[ + \sum_{\frac{x}{y} < \ell} \eta \left( \frac{x}{\ell} \right) \log \frac{x}{\ell} + \sum_{\frac{x}{y} < \ell} \int_{\frac{x}{y}}^{\infty} \eta(u) \frac{du}{u} \]

\[ = \left[ \frac{x}{y} \right] y + x \log x - x \log y + (\gamma - 1)x + x \int_{\frac{x}{y}}^{\infty} \frac{t - \left[ \frac{t}{y} \right]}{t^2} \, dt \]

\[ + \left[ \frac{x}{y} \right] \left( \log a \operatorname{li} a - a - \int_{a}^{\infty} \frac{\pi(u) - li u}{u} \, du \right) \]

\[ + \sum_{\frac{x}{y} < \ell} \eta \left( \frac{x}{\ell} \right) \log \frac{x}{\ell} + \sum_{\frac{x}{y} < \ell} \int_{\frac{x}{y}}^{\infty} \eta(u) \frac{du}{u} . \]

Finally

\[ \sum_{n \leq x} \log \lambda(n) = x(\log x + D + \gamma - 1) + \eta_3(y) \]

with

\[ \eta_3(y) = x \frac{\eta(y) \log y}{y} + x \int_{y}^{\infty} \frac{\eta(t)}{t^2} \frac{dt}{t^2} - x \int_{y}^{\infty} \frac{\eta(t) \log t}{t^2} \, dt \]

\[ + \sum_{\frac{x}{y} < \ell} \eta \left( \frac{x}{\ell} \right) \log \frac{x}{\ell} + \sum_{\frac{x}{y} < \ell} \int_{\frac{x}{y}}^{\infty} \eta(u) \frac{du}{u} \]

\[ - \left[ \frac{x}{y} \right] \int_{y}^{\infty} \eta(u) \frac{du}{u} - \left[ \frac{x}{y} \right] \eta(y) \log y \]

\[ + x \int_{\frac{x}{y}}^{\infty} \frac{t - \left[ \frac{t}{y} \right]}{t^2} \, dt + \sum_{p \leq y} \log p \left( \frac{x}{y} - \left[ \frac{x}{p} \right] \right) . \]
But

\[ D + \gamma - 1 = -\sum_{p} \frac{\log p}{p(p-1)} - 1 \quad \text{and} \quad \eta_3(y) = O\left(\frac{x}{(\log x)^n}\right) \quad \text{as} \quad x \to \infty. \]

with \( y = \sqrt{x} \). (For details see Saffari (1968).)

Note that the argument in Lemma 13 also works for

\[ \sum_{p < x} \frac{\log^2 p}{p} \]

and gives us:

**Lemma 14.** For any non-negative integer \( n \),

\[ \sum_{p < x} \frac{\log^2 p}{p} = \frac{1}{2} \log^2 x + E + O\left(\frac{1}{(\log x)^n}\right) \quad \text{as} \quad x \to \infty, \]

where

\[ E = -2 \gamma_1 - \gamma^2 - \sum_{p} \frac{(2p-1) \log^2 p}{p(p-1)^2}. \]

**Proof.** Let \( E(x) = \sum_{p < x} \frac{\log^2 p}{p} \), use the method that has been used in proving Lemma 13 to get

\[ E(x) = \frac{1}{2} \log^2 x + E + \eta_2(x) \]

with

\[ E = \int_{a}^{\infty} \frac{\pi(t) - \text{li} t}{t^2} (\log^2 t - 2 \log t) dt - \frac{1}{2} \log^2 a \]

and

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\[ n_2(x) = \log x x \cdot n(x) + 2 \int_x^\infty n(t) \frac{\log t}{t^2} \, dt - \int_x^\infty n(t) \frac{\log^2 t}{t^2} \, dt \]

\[ 1 < a < \inf\{2, x\} . \]

\[ n_2(x) = O\left(\frac{1}{\sqrt{\log x}}\right) \quad \text{as} \quad x \to \infty . \]

To find \( E \), consider

\[ g_1(\delta) = -\sum p \frac{\log^2 p (2 \frac{p^{\delta+1}}{\delta+1} - 1)}{p^{\delta+1} (p^{\delta+1} - 1)^2} , \]

\[ h'(\delta) = \sum p \frac{\log p}{p^{\delta+1}} , \]

then

\[ g_1(\delta) = h_1(\delta+1) - \frac{3}{\delta} \left( \log \zeta(\delta+1) \right) . \]

Now use the argument of Lemma 13 to get

\[ h_1(\delta) - \frac{1}{\delta^2} \to E \]

as \( \delta \to 0 \). Also from Lemma 2 we have

\[ \frac{3}{\delta^2} \log \zeta(1+\delta) - \frac{1}{\delta^2} \to 2 A_2 = -2 \gamma_1 - \gamma^2 , \quad \text{as} \quad \delta \to 0 . \]

But

\[ g_1(0) = -\sum p \frac{\log^2 (2p-1)}{p (p-1)^2} , \]

and so
\[ E = -2 \gamma_1 - \gamma^2 - \sum_{p} \frac{(2p-1) \log^2 p}{p(p-1)^2} \]

where \( \gamma \) is Euler's constant and

\[ \gamma_1 = \lim_{N \to \infty} \left[ \sum_{n=1}^{N} \frac{\log n}{n} - \frac{\log^2 N}{2} \right]. \]

This Lemma is useful for deriving the second moment of \( \log \ell \).

So far we have studied the expansion formula for the moments of some additive arithmetic functions. In the next chapter we will discuss the limiting distribution of those functions.
CHAPTER V

V.1 Natural Density.

Diaconis (1974) has discussed some of the probability densities on the natural numbers. Zeta density was mentioned in Chapter I. In this chapter we define the natural density and determine the limiting distributions of the additive arithmetic functions that we study in this thesis.

Definitions.

1 - For a set of positive integers \( A \) and a positive integer \( n \), let \( N_n(A) \), be the number of positive integers \( m \leq n \) such that \( m \in A \). The natural density of \( A \) is defined to be

\[
D(A) = \lim_{n \to \infty} \frac{1}{n} N_n(A),
\]

provided this limit exists. Equivalently, for each positive integer \( n \), let \( P_n \) be the probability measure, that places mass \( \frac{1}{n} \) at each point of the set \( \{1, 2, \ldots, n\} \) then

\[
D(A) = \lim_{n \to \infty} P_n(A).
\]

Diaconis (1974) has studied the relation between different densities, for example, he has shown that if a set \( A \) has natural density, then it has the same zeta density. Therefore, the study of natural density seems to be more important. Galambos (1970) has a complete survey on this subject. In Galambos (1970) natural density is called asymptotic density.

2 - Let \( f \) be a real-valued arithmetic function, then

\[
D[f](x) = \lim_{n \to \infty} P_n \{m \leq n : f(m) \leq x\}
\]

is called the limiting or
asymptotic distribution of $f$, provided the limit exists for every continuity point of $D[f]$ and $D[f]$ is a univariate distribution function. (Notation: $f_n(m) \xrightarrow{d} D[f]$.)

3 - If $f$ and $g$ are two real-valued arithmetic functions, then:

$$D[f,g](x,y) = \lim_{n \to \infty} P\{m \leq n : f(m) \leq x, g(m) \leq y\}$$

is called the bivariate limiting distribution of $f$ and $g$, provided the limit exists for every continuity point of $D[f,g]$ and $D[f,g]$ is a bivariate distribution function. (Notation: $(f_n(m), g_n(m)) \xrightarrow{d} D[f,g]$.) For definitions of distribution functions see pages 152 and 161 of Gnedenko (1963).

4 - Two real-valued arithmetic functions $f$ and $g$ are asymptotically independent if $D[f,g](x,y) = D[f](x) \cdot D[g](y)$, for all real numbers $x$ and $y$.

V.2 Known Results.

Some of the additive functions which have been studied in previous chapters have no limiting distribution without some sort of normalization. These include $\omega$ and $\Omega$. But Erdos and Kac (1940) have shown that:

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(V.1) \[ \lim_{n \to \infty} P_n \left\{ m \leq n : \frac{\omega(m) - \log \log n}{\sqrt{\log \log n}} \leq x \right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2} du = \phi(x) \]

for all \( x \).

That is, \( \frac{\omega - \log \log n}{\sqrt{\log \log n}} \) has the standard normal as its limiting distribution. Billingsley (1969) has a probabilistic proof for this so called "central limit theorem for the prime divisor function".

From the Erdos and Kac theorem we can get Hardy and Ramanujan's (1917) theorem, i.e., the typical integer \( m \) has approximately \( \log \log m \) distinct prime divisors. (For details, see Billingsley (1969) and also Theorem 1.)

Erdos and Kac (1940) have also shown that:

(V.2) \[ \lim_{n \to \infty} P_n \left\{ m \leq n : \frac{\Omega(m) - \log \log n}{\sqrt{\log \log n}} \leq x \right\} = \phi(x) \]

for all \( x \).

A summary of some generalizations of (V.1) and (V.2) is in Galambos (1970), pp. 297-299.

As another result on the limiting distribution of additive arithmetic functions Renyi (1955) has shown that:

\[ \lim_{n \to \infty} P_n \{ m \leq n : \Omega(m) - \omega(m) = k \} = \beta_k \]

with:

\[ \beta_k = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left( \frac{1}{e} \right)^n k^n \]
\[
\sum_{k=0}^{\infty} \beta_k z^k = \Pi \left( 1 - \frac{1}{p} \right) \left( 1 + \frac{1}{p - z} \right).
\]

(See also Kac (1955).)

A generalization of Renyi's theorem can be found in Rejali (1978). That general form also gives us the limiting distribution of \( \log \log \ell \), i.e.,

\[(V.3) \quad \lim_{n \to \infty} \mathbb{P}_{n} \left\{ m \leq n : \log m - \log \Pi_{p|m} p = \log k \right\} = \beta'_k,
\]

with

\[
\sum_{k=1}^{\infty} \beta'_k k^it = \Pi \left( 1 - \frac{1}{p} \right) \left( 1 + \frac{1}{p - p^it} \right)
\]

as the characteristic function of this limiting distribution.

In the last section we use this result to find the limiting distribution of \( \log \Pi_{p|m} p \).

V.3 The Limiting Distribution of \( \log \phi \).

As we have mentioned in Chapter III,

\[\log \phi(m) = \log m + \log \ell_{1}(m) , \text{ where } \ell_{1}(m) = \Pi_{p|m} \left( 1 - \frac{1}{p} \right) , \text{ so in order to study the limiting distribution of } \log \phi , \text{ first we consider the limiting distribution of } \log m , \text{ that is,}\]
\[
\lim_{n \to \infty} P_{n} \{m \leq n : \log m - \log n \leq x\} = \begin{cases} 
\frac{e^x}{x} & \text{if } x < 0 \\
1 & \text{if } x \geq 0 
\end{cases}
\]

i.e., \(-\log m + \log n\) has an exponential limiting distribution, which is a non-singular distribution (i.e., the statement on page 67 of Kubilius (1968) which says "It is easy to show that for the function \(\log m\), no normalization can lead to a non-singular limiting distribution law" is not valid).

Now look at the limiting distribution of \(\log l_1\). If \(f(m) = \log l_1(m)\), then \(f\) is a real-valued additive arithmetic function with \(f(p) = \log(1 - \frac{1}{p})\). For example, \(-\log 2 < f(p) < 0\) so, \(|f(p)| < \log 2 < 1\), therefore, \(\|f(p)\| = \log(1 - \frac{1}{p})\) and

\[
\sum_{p} \|f(p)\| = \sum_{p} \frac{\log(1 - \frac{1}{p})}{p} = -\sum_{p} \sum_{k=1}^{\infty} \frac{1}{kp^{k+1}}
\]

which converges as well as

\[
\sum_{p} \frac{\|f(p)\|^2}{p} = \sum_{p} \frac{\log^2(1 - \frac{1}{p})}{p}.
\]

Thus the conditions of the Erdos and Wintner (1939) theorem are in force [see also Theorem 4.5 on page 74 of Kubilius (1968)], so that \(\lim_{n \to \infty} \{m \leq n : f(m) \leq x\} = G(x)\), for continuity points of \(G\). The characteristic function of the limiting distribution \(G\) is:
\[ \Pi(1 - \frac{1}{p}) \left( 1 + \sum_{\alpha=1}^{\infty} \frac{\zeta^{\alpha} \log(1 - \frac{1}{q})}{\alpha p^{\alpha}} \right) = \Pi \left( 1 - \frac{1}{p} \right) \left( 1 + \frac{(1 - \frac{1}{p})i \frac{1}{p}}{1 - \frac{1}{p}} \right) \]

and \( G \) is singular, (see Erdos (1939)).

Consider independent random variables \( X_p \) (on some probability space, one variable for each prime \( p \)) such that

\[ P(X_p = 0) = 1 - \frac{1}{p}, \quad P(X_p = \log(1 - \frac{1}{p})) = \frac{1}{p} \]

then \( G(x) \) is the distribution of \( \sum_{p} X_p \).

Now, consider the bivariate distribution

\[ P_n \{ m \leq n : \log m - \log n \leq x, \log \log_1(m) \leq y \} = P_n \{ m \leq n, m \leq n e^x, \log \log_1(m) \leq y \} \]

\[ = \begin{cases} 
\left\lfloor \frac{\left\lfloor n e^x \right\rfloor}{n} \right\rfloor \{ m \leq \left\lfloor n e^x \right\rfloor, \log \log_1(m) \leq y \} & \text{if } x < 0 \\
\{ P_n \{ m \leq n, \log \log_1(m) \leq y \} \} & \text{if } x \geq 0.
\end{cases} \]

So

(V.4)

\[ \lim_{n \to \infty} P_n \{ m \leq n : \log m - \log n \leq x, \log \log_1(m) \leq y \} = \begin{cases} 
e^x G(y) & \text{if } x < 0 \\
G(y) & \text{if } x \geq 0
\end{cases} \]
i.e., \((\log - \log n, \log l_1)\) converges to a distribution with

characteristic function \(\frac{1}{1+is} \prod_{p} \left(1 - \frac{1}{p} + \frac{(1 - \frac{1}{p})it}{p}\right)\).

(Because \(\int_{-\infty}^{0} e^{isx} e^x dx = \frac{1}{is + 1}\). But since \(\log \phi - \log n = \log - \log n + \log l_1\), then

\[
\log \phi - \log n \text{ has the convolution of } \begin{cases} e^x & x < 0 \\ 1 & x \geq 0 \end{cases}
\]

and \(G(x)\) as its limiting distribution or:

**Theorem 17.** If \(\phi\) is the Euler, totient function (i.e., \(\phi(n)\), is the number of positive integers not exceeding \(n\) which are relatively prime to \(n\)), then

\[
\lim_{n \to \infty} \prod_{m \leq n : \log \phi(m) - \log n \leq x} = H(x)
\]

for all continuity points of \(H\), where the characteristic function corresponding to \(H\) is:

\[
\frac{1}{1+it} \prod_{p} \left(1 - \frac{1}{p} + \frac{(1 - \frac{1}{p})it}{p}\right).
\]

**Remarks.** From Formula (V,4) we have:
\[
\lim_{n \to \infty} P_n \{ \frac{\omega(m) - \log \log n}{\sqrt{\log \log n}} \leq x, \log m - \log n \leq y \}
\]

\[
= \lim_{n \to \infty} P_n \{ m \leq n : \frac{\omega(m) - \log \log n}{\sqrt{\log \log n}} \leq x, \log \omega(n) \leq y \}
\]

i.e., \( \log - \log n \) and \( \log \omega(n) \) are asymptotically independent.

(See Lemma 11.)

V.4 Some More Bivariate Forms.

Theorem 18. If \( \Omega(n) \) and \( \omega(n) \) are the number of prime divisors of \( n \), counted with and without multiplicity, then:

1) \[
\lim_{n \to \infty} P_n \left\{ m \leq n : \frac{\Omega(m) - \log \log n}{\sqrt{\log \log n}} \leq x, \log m - \log n \leq y \right\} = \begin{cases} 
 e^y \phi(x) & \text{if } y < 0 \\
 \phi(x) & \text{if } y \geq 0
\end{cases}
\]

2) \[
\lim_{n \to \infty} P_n \left\{ m \leq n : \frac{\Omega(m) - \log \log n}{\sqrt{\log \log n}} \leq x, \log m - \log n \leq y \right\} = \begin{cases} 
 e^y \phi(x) & \text{if } y < 0 \\
 \phi(x) & \text{if } y \geq 0
\end{cases}
\]

for all \( x \), where \( \phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2} \, du \).
Proof.

\[ P_n \left\{ m \leq n : \frac{\omega(m) - \log \log n}{\sqrt{\log \log n}} \leq x, \log m - \log n \leq y \right\} = \]

\[ P_n \left\{ m \leq n, m \leq \lceil e^y n \rceil, \frac{\omega(m) - \log \log n}{\sqrt{\log \log n}} \leq x \right\}. \]

Now if \( y < 0 \), put \( N = \lceil n e^y \rceil \) which is less than \( n \) and

\[ \lim_{n \to \infty} \frac{\lceil n e^y \rceil}{n} = e^y. \] Since

\[ \log \log N = \log \log n + O(\sqrt{\log \log n}) \] and also

\[ \sqrt{\log \log N} = \sqrt{\log \log n + O(\sqrt{\log \log n})}, \] then

\[ \lim_{n \to \infty} P_n \left\{ m \leq n : \frac{\omega(m) - \log \log n}{\sqrt{\log \log n}} \leq x, \log m - \log n \leq y \right\} \]

\[ = e^y \lim_{N \to \infty} P_N \left\{ m \leq N, \frac{\omega(m) - \log \log N}{\sqrt{\log \log N}} \leq x \right\} \]

\[ = e^y \phi(x). \]

In the last step the Erdős, Kac theorem is used. Now if \( y \geq 0 \),
then \( \lceil n e^y \rceil \geq n \), so

\[ \lim_{n \to \infty} P_n \left\{ m \leq n, \frac{\omega(m) - \log \log n}{\sqrt{\log \log n}} \leq x, \log m - \log n \leq y \right\} \]

\[ = \lim_{n \to \infty} P_n \left\{ m \leq n, \frac{\omega(m) - \log \log n}{\sqrt{\log \log n}} \leq x \right\} = \phi(x). \]

The second part of the theorem has the same proof.
The results of Theorem 18, Erdos and Kac's theorem (1940) and the limiting distribution of \( \log - \log n \) indicate that suitably normalized \( \omega \) and \( \log \) as well as \( \Omega \) and \( \log \) are asymptotically independent.

Finally, use the result of Corollary 2 of Rejali (1978) (Formula V.3) to get:

**Theorem 19.**

\[
\lim_{n \to \infty} P_n \left( m \leq n : \log \prod_{p|m} p - \log n \leq x \right) = \begin{cases} 
\sum_{k=1}^{\infty} k \beta_k' \frac{e^{-x}}{k + 1} & \text{if } x \leq 0 \\
1 & \text{if } x > 0
\end{cases}
\]

with

\[
\sum_{k=1}^{\infty} \beta_k' k^i = \Pi(1 - \frac{1}{p})(1 + \frac{1}{p - p^i}).
\]

**Proof.**

\[
P_n \left( m \leq n : \log \prod_{p|m} p - \log n \leq x \right) = \sum_{k=1}^{\infty} P_n \left( \log \prod_{p|m} p - \log n \leq x \right) \log m - \log \prod_{p|m} p = \log k
\]
\[
\begin{align*}
&= \sum_{k=1}^{\infty} P_n \left( m \leq n, m \leq nke^{\log k}, \log m - \log p | m \right) \\
&= \sum_{k=1}^{\left[ nke^{\log k} \right]} P_n \left( m \leq nke^{\log k}, \log m - \log p | m \right) + \sum_{k=\left[ e^{-x} \right]+1}^{\infty} P_n \left( m \leq n; \log m - \log p | m \right).
\end{align*}
\]

And so

\[\lim_{n \to \infty} P_n \left( m \leq n : \log p | m \leq x \right) = \begin{cases} \\
\sum_{k=1}^{\left[ e^{-x} \right]} k \beta'_k + \sum_{k=\left[ e^{-x} \right]+1}^{\infty} \beta'_k & \text{if } x < 0 \\
1 & \text{if } x \geq 0 \\
\end{cases}\]

where \( \beta'_k = \lim_{n \to \infty} P_n \left( m \leq n : \log m - \log p | m = \log k \right) \) and

\[\sum_{k=1}^{\infty} \beta'_k k^{it} = \Pi(1 - \frac{1}{p})(1 + \frac{1}{p - p^{it}}). \]

(V.5) is the limiting distribution of \( \log \frac{\mu(m)}{n} \) (see Section 2 of Chapter III).
References


