POWER AND EFFICIENCY OF A CLASS OF
GOODNESS-OF-FIT TESTS

BY
CHRISTOPHER S. WITHERS

TECHNICAL REPORT NO. 16
DECEMBER 15, 1970

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NONTECHNICAL SUMMARY

The two main purposes of goodness-of-fit tests seem to be (1) as a preliminary test to see if the distribution underlying the data is close to a particular distribution so that tests designed for that distribution can then be applied (for example, the exponential distribution is usually assumed in reliability) or (2) in testing to see if there has been a change in the distribution underlying the data; (accumulated data may have been used to estimate the distribution - is the latest batch of data consistent with this?)

Traditional criteria governing the choice of the goodness-of-fit test to be used (i.e., power and efficiency) have been almost unavailable (a) because the theory has been largely restricted to simple (or parametric) alternatives and (b) because the theoretical tools needed have not been available.

This paper synthesizes ideas of Hoadley, Abrahamson, Bahadur, Chernoff, Hodges and Lehmann, and others with the methods of the calculus of variations, and differential and integral equations, in order to develop the theory of, and compute the efficiencies of, a wide range of goodness-of-fit tests when the underlying distribution is continuous and univariate.

Also generalized are ideas of Hajek, Anderson, and Darling to find the power of some goodness-of-fit tests, but the results of practical interest here are limited.

Often, in medical and industrial fields, only a minimum specification is required to be met (e.g., cure rate of a drug, life expectancy of machine parts, reliability of a system). In such cases the appropriate alternative to a close fit is "one-sided" and the non-parametric results presented mostly pertain to this situation.

Although the results are for "large" samples, other work indicates that often a sample size of ten is adequately large.
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§O. GLOSSARY OF NOTATION AND ABBREVIATIONS

4. (2) refers to equation (2) of §4

gof

goodness-of-fit

AP

asymptotic power

For $\omega_i$, $\mu_i$, $I(F,G)$, $I_i(r,F)$, $I_i(r,\omega)$, $i = 0, 1$, $e_{\alpha}$, $e_\beta$, $e_{\gamma}$, $J(\omega_0, \omega_1)$ see §3

\[
U(x) = \begin{cases} 
0, & x < 0 \\
0, & 0 \leq x \leq 1 \\
1, & 1 < x 
\end{cases}
\]

\[
\|G\|_{F,m} = \begin{cases} 
\exp\left[\int G \, dF\right] & \text{if } G > 0, m = 0 \\
\left(\int G^m \, dF\right)^{1/m} & \text{if } 0 < m < \infty \\
\sup G & \text{if } G \geq 0, m = \infty 
\end{cases}
\]

\[
\|G\|_m = \|G\|_{U,m}
\]

\[
\psi_0(t) = (t-t^2)^{-1/2}
\]

\[
\varphi(t) = (2\pi)^{-1/2} e^{-t^2/2}
\]

\[
\phi(t) = \int_{-\infty}^{t} \varphi(u) \, du
\]

\[
z_\alpha = \phi^{-1}(1-\alpha)
\]

\[
s \wedge t = \min(s,t)
\]

\[
s \vee t = \max(s,t)
\]
\( F_n \) is the empirical cdf of a sample \( X_1, \ldots, X_n \) iid \( F \).

\( U_n \) is \( F_n \) when \( F = U \).

\( \psi \) is a non-negative function on \([0,1]\).

\( T_{F_0}^0 (F) = \| F - F_0 \|_{\psi(F_0)}^{F_0, m} \), \( 0 \leq m \leq \infty \)

\( T_{F_0}^+ (F) = \| (F - F_0) \psi(F_0) \|_{F_0}^{m} \), \( m = 1, 3, 5, \ldots \)

\( D_{F_0} (F) = T_{F_0}^0 (F) \) at \( m = \infty \)

\( D_{F_0}^+ (F) = \sup (F - F_0) \psi(F_0) \)

\( D_{F_0}^- (F) = \sup (F_0 - F) \psi(F_0) \)

\( V_{F_0} (F) = D_{F_0}^+ (F) + D_{F_0}^- (F) \)

\( T_{n, m} (\psi) = T_{n, m} (F_n) \) \hspace{1cm} (1)

\( T_{n, m}^+ (\psi) = T_{n, m}^+ (F_n) \) \hspace{1cm} (2)

\( D_{n} (\psi) = D_{F_0} (F_n) \) \hspace{1cm} (3)

\( D_{n}^+ (\psi) = D_{F_0}^+ (F_n) \) \hspace{1cm} (4)

\( D_{n}^- (\psi) = D_{F_0}^- (F_n) \) \hspace{1cm} (5)

\( V_{n} (\psi) = V_{F_0} (F_n) \) \hspace{1cm} (6)

\( X_n \xrightarrow{P} X \) for all \( x \)

\( \Pr(X_n \leq x) \rightarrow \Pr(X \leq x) \) as \( n \rightarrow \infty \)
LR

\[ S_{n,m}(\psi) = \|F_n - F_0\|_F^m, \quad 0 \leq m \leq \infty \]  \hspace{1cm} (7)

\[ S_{n,m}^+(\psi) = \|(F_n - F_0) \psi(F_0)\|_F^m, \quad m = 1, 3, 5, \ldots \]  \hspace{1cm} (8)

\[ a(r,x) = \begin{cases} (x+r) \ln(1 + \frac{x}{x-r}) + (1-x-r) \ln(1 - \frac{r}{1-x}) & \text{for } 0 < x < 1-r \\ \infty & \text{otherwise} \end{cases} \]

\[ T_{n,1}^- = -T_{n,1}^+ \]

\[ \psi_1(t) = \begin{cases} \psi_0(t), & \text{for } .005 < t < .995 \\ \psi_0(.005), & \text{otherwise} \end{cases} \]

r(u) is defined by 2.1

p(u) is defined by 2.2

\[ I = \int_0^1 \frac{t^2}{p}, \text{ the Fisher information} \]

B = r \cdot \psi

y(\cdot) is the zero mean normal process with covariance \( s \cdot t - st \)

\[ W_0, W_2 \]

2.4

\[ S(t), K_k(x,y) \]

see Theorem 3, § 2.

K(x,y : \lambda)

e(\alpha, \theta) page 13

e(\alpha) page 14

G(x) (in §2) is the asymptotic null distribution of the statistic under consideration

\[ f = 0^*(g) \] means that \( f/g \) is bounded away from 0 and \( \infty \)

\[ x_\alpha = G^{-1}(1-\alpha) \]
§1. INTRODUCTION AND SUMMARY

Let $X_1, \ldots, X_n$ be iid $F$ a continous cdf on $R = (-\infty, \infty)$. We wish to test whether $F = F_0$ a given continuous cdf on $R$. As candidates we consider $T_n, m(\psi)$ and $V_n(\psi)$ defined by (1) and (6) of §0. This class includes the Cramer-von Mises, Kolmogorov-Smirnov and Kuiper statistics, $T_n, 2(1), T_n, \infty(1)$ and $V_n(1)$ whose asymptotic null distributions are given in [3], [31] and [48].

§2 can be read independently of §§ 3-9. §2 shows how the asymptotic power (AP) of $T_n, 2(\psi)$ may be computed and compares the AP of $T_n, 2(1), D_n(1), V_n(1)$ to the envelope power function for a particular example. Such comparisons are difficult and only local in nature.

Three non-local types of efficiency $(e_\alpha, e_\beta, e_\perp)$ are introduced in §3, for the general two-hypothesis testing situation. These are computed in §4, §5, and §8, for gof tests of the type $T_n, m(\psi)$ or $V_n(\psi)$ for parametric and non-parametric alternatives.

In §3 we argue that locally $T_n, \infty(\psi)$ is preferable to $T_n, m(\psi)$ if $m < \infty$, in testing $F = F_0$ against "$F$ is not close to $F_0".

An exact local comparison is the local Bahadur efficiency which usually equals the Pitman efficiency, given in §7.

For $\alpha$-level tests the Hodges-Lehmann efficiency [23] or its generalization the fixed-$\alpha$ efficiency (§3) is appropriate, but is shown in §4 to tend to one under suitable conditions, for the statistics we consider, when testing $F \in \omega_0$ against $F \in \omega_\perp$ as $\omega_0$ shrinks to $\{F_0\}$.

† Their percentiles are conveniently given for all $n$ in a table of [36].
§5 gives the Bahadur efficiency [6] and §§ the Chernoff efficiency [13] for some common parametric examples, using large deviation results derived in part from the work of Hoadley [22] and Abrahamson [1].

More interesting is a comparison of the statistics when testing whether $F_0$ is close to $F_\psi$ (sup $|F-F_0| = a_0$, say) or distant from $F$, (sup $|F-F_0| = a_1$, say). This is carried out by computing $e_\psi$ in §5 when $a_0 = 0$, for the statistics $T_{n,1}(1), T_{n,2}(1), V_n(1)$ and $D_n(\psi)$ for certain $\psi$. For the one sided version of the gof problem this is done in §9 for these and other statistics.

Proofs and most of the tables (as indicated in the Contents) are in §10.
§2. ASYMPTOTIC POWER

Consider a sequence of problems in which one tests $F = F_0$ against $F = F_1$ where the alternative varies with $n$, say $F_1 = H_n$. If $H_n \rightarrow F_0$ sufficiently rapidly, then the power of a fixed $\alpha$ level test will tend to a number less than one called the asymptotic power. We shall assume

\[ (1) \quad H_n(F_0^{-1}(u)) = u + h^{-1/2} r(u) + o(n^{-1/2}) \quad \text{uniformly in } u \in [0,1] \]

where $r(u)$ is bounded.

For example if $H_n(x) = F(x, \theta_n^{-1/2})$ and there is a $K$ such that for all $x$, and all $a$ in $[0, \theta]$

\[ \left| \frac{\partial}{\partial a} F(x,a) \right| < K, \quad \left| \frac{\partial^2}{\partial a^2} F(x,a) \right| < K \]

where $F(x,0) = F_0(x)$, then (1) holds with $r(u) = \theta p(u)$ where

\[ (2) \quad p(u) = \left[ \frac{\partial}{\partial \theta} F(F_0^{-1}(u), \theta) \right]_{\theta=0} \]

Let $y(\cdot)$ be the Brownian Bridge on $[0,1]$; i.e., the normal process such that

\[ Ey(s) = 0, \quad Ey(s) y(t) = s - st \quad \text{for } 0 \leq s \leq t \leq 1. \]

Then it is well known (cf [37]) that

\[ (3) \quad (n^{1/2}(F_n(H_0^{-1}(u_i)) - u_i))_{i=1, \ldots, p} \rightarrow (y(u_i))_{i=1, \ldots, p} \]

for $0 \leq u_i \leq 1$, $i = 1, \ldots, p$. ($F_n$ is the "empirical" or sample cdf.)
The techniques of [3] are extended in §10 to prove the following theorems.

Theorem 1. Suppose \( F = H_n \) satisfies (1).

(i) Suppose \( \psi \) is continuous on \([a, b]\) and zero elsewhere, and \(0 \leq a < b \leq 1\). Then

\[
\begin{align*}
\frac{1}{n^{1/2}} T_{n,m}(\psi) & \xrightarrow{L} \|r+y\cdot \psi\|_m, & 0 < m \leq \infty \\
\frac{1}{n^{1/2}} T^+_{n,m}(\psi) & \xrightarrow{L} \|(r+y)\cdot \psi\|_m & m = 1, 3, 5, \ldots \\
\frac{1}{n^{1/2}} D^+_n(\psi) & \xrightarrow{L} \sup (r+y)\psi \\
\frac{1}{n^{1/2}} D^-_n(\psi) & \xrightarrow{L} \sup -(r+y)\psi \\
\frac{1}{n^{1/2}} V_n & \xrightarrow{L} \sup (r+y)\psi + \sup -(r+y)\psi
\end{align*}
\]

(ii) Suppose \( \psi \) is continuous in \((0,1)\). Suppose (a) there exist finite \( a, b, d, e \) such that for \( 0 < t < 1 \), \( n = 1, 2, \ldots \)

\[
h_n - h_n^2 < a(t-t^2) + b|r|^{K(m)}
\]

\[
\frac{1}{n^{1/2}} |h_n-t| < d|r| + e(t-t^2)
\]

where \( h_n = H_n(F^{-1}_0(t)) \) and \( K(m) \) is the smallest even integer \( \geq m \);

(b) \( I_1 = \int_0^1 (t-t^2) \psi(t)^{K(m)} < \infty \)

\[
I_2 = \int_0^1 |r|^{K(m)} \cdot \psi(t)^{K(m)} < \infty
\]

Then

\[
\begin{align*}
\frac{1}{n^{1/2}} T_{n,m}(\psi) & \xrightarrow{L} \|r+y\cdot \psi\|_m, & 0 < m < \infty \\
\frac{1}{n^{1/2}} T^+_{n,m}(\psi) & \xrightarrow{L} \|(r+y)\cdot \psi\|_m, & m = 1, 3, 5, \ldots 
\end{align*}
\]
Note. Quade [37] proved (i) for $T_n$, $n^2(1)$, and gave bounds on the
correct function for the sequence of composite alternatives

$\{H_n : n^{1/2} \sup |H_n - F_0| \to \Delta\} = \{H_n : \sup |r| = \Delta\}$.

**Theorem 2.** Let $\psi$ be bounded in $(0,1)$. Let $g(u) = [2(u-u^2) \ln u - u^2]^{1/2}$.

(i) $\|g \cdot \psi\|_m < \infty$, $\|r \cdot \psi\|_m < \infty \Rightarrow \|r+y \cdot \psi\|_k$ exists a.s. for

$k = 1, 3, 5, \ldots$, $k \leq m$, and $\|r+y \cdot \psi\|_k$ exists a.s. for $0 < k \leq m$.

(ii) If $\sup_{(0,1/2)} g \cdot \psi = \infty$ and if for some $\varepsilon > 0$,

\[ (0,\varepsilon) \sup \begin{cases} r \geq 0, & \text{then } \sup (r+y) \cdot \psi = \infty \text{ a.s.} \\ r \leq 0, & \text{then } \sup -(r+y) \cdot \psi = \infty \text{ a.s.} \end{cases} \]

If $\sup_{(1/2,1)} g \cdot \psi = \infty$ and if for some $\varepsilon > 0$,

\[ (1-\varepsilon,1) \sup \begin{cases} r \geq 0, & \text{then } \sup (r+y) \cdot \psi = \infty \text{ a.s.} \\ r \leq 0, & \text{then } \sup -(r+y) \cdot \psi = \infty \text{ a.s.} \end{cases} \]

Let

\[ W_2 = \int_0^1 (r+y)^2 \cdot \psi^2, \quad W_0 = \int_0^1 y^2 \psi^2, \]

(4)

\[ B = r \cdot \psi, \quad z = y \cdot \psi, \quad G(x) = P(W_0 \leq x), \quad x_\alpha = G^{-1}(1-\alpha). \]
Note. \( x_\alpha \) is tabulated in [3] for \( \psi = 1 \).

**Theorem 3.**

\[
E e^{-tW_2} = \prod_{j=1}^{\infty} (1 + 2t/\lambda_j)^{-1/2} e^{-S(t)}
\]

where

\[
S(t) = t \sum_{j=1}^{\infty} b_j^2 (1 + 2t/\lambda_j)^{-1}
= t \int_0^1 B^2 - 2t^2 \int_0^1 \int_0^1 B(x) B(z) K(x, z : -2t) \, dx \, dz,
\]

\[
b_j = \int_0^1 B \cdot f_j \, dz,
\]

\( \{\lambda_j, f_j\} \) are the evals and efuns of

\[
K_1(x, z) = (\min(x, z) - xz) \psi(x) \psi(z),
\]

\( K(x, z : \lambda) \) is the 'resolvent kernel' \( \sum_{k=1}^{\infty} \lambda^{k-1} K_k(x, z) \)

(see p. 141 of [15] for alternative expressions),

\[
K_k(x, z) = \sum_{j=1}^{\infty} \lambda_j^{-k} f_j(x) f_j(z) = \int_0^1 K_{k-1}(x, u) K_1(u, z) \, du,
\]

the 'kth iterated kernel', \( k = 2, 3, \ldots \) and we assume that \( \psi \) satisfies either (E) or (F) (cf. p. 198 of [3]) so that (see [3])

\[
[z] = \mathbb{E} \left( \sum_{i=1}^{\infty} Y_i \cdot f_i \cdot \lambda_i^{-1/2} \right) \quad \text{where } Y_i \text{ are iid } \mathcal{N}(0, 1).
\]
(E): $\psi$ is continuous on $(0,1)$; for all $\varepsilon > 0$, $\psi(t) t[(l-s) \psi'(s) - \psi(s)]$ is continuous for $t \vee \varepsilon \leq s \leq l - \varepsilon$ and $\psi(t) (1-t) [s\psi'(s) + \psi(s)]$ is continuous for $\varepsilon \leq s \leq t \vee (1-\varepsilon)$.

(F): $\psi$ is continuous in $[a,b]$ and zero elsewhere, where $0 \leq a < b \leq 1$.

(b) $\mathbb{E} W_2 = \int_{0}^{1} [B(x)^2 + K(x,x)] \, dx$,

$\text{var} W_2 = 2 \int_{0}^{1} \int_{0}^{1} [2B(x) B(z) + K(x,z)] K(x,z) \, dx \, dz$.

**Example (a).** Suppose $\psi = 1$. Let $z = (2t)^{1/2}$. Then

$$\prod_{j=1}^{\infty} (1 + 2t/\lambda_j)^{-1/2} = (z/\sinh z)^{1/2}$$

and

$$K(a, b; -2t) = \frac{\sinh(bz) \cdot \sinh(z-az)}{z \sinh z} \quad \text{for } 1 \geq a \geq b \geq 0.$$ 

**Example (b).** When $\psi = 1$ and $F_0(x) = \frac{1}{2} e^{-|x|}$ and $H_n(x) = F_0(x-\theta_n^{-1/2})$ then the AP of $T_{n,2}(1)$ is

$$P(W_2 > x_0) = 1 - z^{3/2} e^{-z^2/2} \sum_{n=0}^{\infty} \frac{\theta_n^{2n}}{n!} \sum_{k=0}^{\infty} z^{-(n/2)-(3/4)} a_{kn} \mathcal{G}(A_k, x_0)$$

where

(6) $a_{kn} = \sum_{i=0}^{k} \left( \frac{n}{2} \right)^{-n - \frac{1}{2}} \left( \frac{k}{2} \right)^{-k - \frac{1}{2}} (-1)^i$. 

10
\( A_k = (k + \frac{1}{2}) 2^{1/2} \),

\[
G_n(A, x) = A x^{n/2-3/4} \cdot 2^{-n/2} \frac{\Gamma\left(\frac{3}{4} - \frac{n}{2}\right)}{\Gamma\left(\frac{3}{4} + \frac{n}{2}\right)} \cdot e^{-c} \]

\[
\sum_{i=0}^{n} \frac{n!}{(n-i)!} \sum_{k=0}^{[i/2]} \frac{(-1)^{i+k+n}}{(1-2k)!} \frac{1}{k!} \cdot 2^{-k} \cdot c^{n/2-k-1/4} \cdot K_{1/4 + 1 - k - n/2}(c)
\]

where \( c = A^2 / 8x \) and \( K_{\alpha}(c) \) is the modified Bessel function (eg., p. 5 of [9]) and \( x_\alpha \) is tabulated in Table 1, p. 203 [3].

Theorem 4. Local AP. (i) When \( H_n(x) = P_0 (x - \theta_n^{-1/2}) \), \( p, G(x), x_\alpha \)

are defined by (2) and (4), \( b = \psi \cdot p \), then as \( \theta \to 0 \)

\[
P(W_2 > x_\alpha) = \alpha + \theta^2 \cdot A(x_\alpha) + o(\theta^2)
\]

where

\[
A(x) = \hat{g}(x) \int_0^1 b^2 + \sum_{k=1}^{\infty} (-2)^k G^{(k+1)}(x) \int_0^1 b(u) \, du
\]

\[
\cdot \int_0^1 b(v) K_{\alpha}(u,v) \, dv
\]

(ii) If \( \psi = 1 \) then (i) is true and

\[
A(x) = \hat{g}(x) \int_0^1 b^2 - 2 \int_0^1 b(u) \int_0^1 b(v) G(x,u,v) \, dv
\]

where

11
\[ G(x,1-u,v) = \sum_{j=1}^{4} a_j \sum_{i=0}^{\infty} (-3)^i (-1)^i c_{ij}(u,v) \frac{1}{u} x^{-9/4} \cdot f(u) \] 

\[ f(c) = e^{-c} [c^{5/4} K_{5/4}(c) + 3(c-1) c^{1/4} K_{1/4}(c) + 3(c^2-1) c^{-3/4} K_{3/4}(c) + c^{5/4} K_{7/4}(c)] , \]

\[ d_1 = d_4 = -d_2 = -d_3 = 1 , \]

\[ c_{ij}(u,v) = \left( \frac{3}{2} - u + v + 2i \right) \cdot 2^{1/2} + 2^{3/2} . \]

\[ \begin{cases} 
0, & j = 1 \\
u, & j = 2 \\
v, & j = 3 \\
u+v, & j = 4. 
\end{cases} \]

**Example 2.** If \( \psi = 1 \) and \( \Phi_0(x) = \frac{1}{2} e^{-|x|} \), then (i) is true and

\[ A(x_\alpha) = \frac{1}{2} (1-\alpha) - 2^{-3/4} \sum_{k=0}^{\infty} a_{kl} G_1(A_k, x_\alpha) \]

where (6) and (7) give \( A_k \) and \( a_{kl} \), and

\[ G_1(A,x) = \left( \frac{A^2}{2x} \right)^{1/2} x^{-1} \cdot \exp \left[ - \frac{A^2}{8x} \right] \cdot [K_{3/4}(\frac{A^2}{8x}) - K_{1/4}(\frac{A^2}{8x})] . \]

Examples 1 and 2 have been used to tabulate the AP and the 'approximate AP', defined to be

\[ \phi(-z_\alpha/2 + i^{1/2} \cdot e(\alpha)^{1/2} \cdot \theta) + \phi(-z_\alpha/2 - e(\alpha)^{1/2} \cdot i^{1/2} \cdot \theta) \]

(see below) where I is given by (9) and \( e(\alpha) = A(x_\alpha)/z_\alpha/2 \cdot \phi(z_\alpha/2) \)
for the double-exponential case above, and \( z_\alpha = \Phi^{-1}(1-\alpha) \). See Tables 1a, 1b, and 1c.

Local AP of \( D_n^+(1), D_n(1), V_n(1) \).

Note. This method extends to approximating the AP by a polynomial in \( \theta \) arbitrarily well but with the number of integrals increasing with the degree of the polynomial; it also extends to \( \psi \) which are piecewise of the form \( \psi(t) = 1/(a+bt) \); when \( \psi = 1 \) it extends to give the A.P. for \( H_n \) such that \( p(x) \) is piecewise of the form \( a+bx \), (e.g., the double-exponential example for which Hajek found the AP of \( D_n^+(1) \) on p. 272 of [20].)

In the remainder of §2 we assume \( H_n(x) = F(x, \theta n^{-1/2}) \). We now give some generalizations of VI.4.5, p. 230 [20] and [2] which gave the local AP of \( D_n^+(1) \) and \( D_n^+(\psi_1) \) when \( F(x, \theta) = F_0(x-\theta) \) and

\[
\psi_1(t) = \begin{cases} 
\frac{1}{t} & \text{in } [a,1] \\
0 & \text{in } [0,a)
\end{cases}
\]

Let \( e(\alpha, \theta) = \lim_{n \to \infty} n_2/n \) where \( n_2 \) is the sample size required by the most powerful test of \( \theta = 0 \) versus \( \theta > 0 \) in the one-sided case or \( \theta \neq 0 \) in the two-sided case in order to achieve the power that the test statistic has with a sample of \( n \).

Let \( \theta_2 = (n_2/n)^{1/2} \cdot \theta \). The most powerful test of \( F(x,0) \) versus \( F(x, \theta n^{-1/2}) = F(x, \theta_2 n^{-1/2}) \) is the simple likelihood ratio test which has AP

\[
\Phi(1/2 \theta - z_\alpha) = \alpha + 1/2 \theta \Phi(z_\alpha) + o(\theta)
\]

13
In the 2-sided case asymptotically the composite LR test (and the test based on
\( \prod_{1}^{n} p(F_{0}(x_{i})) \)) are locally most powerful with AP

\[
\phi(I^{1/2} \cdot \theta - z_{\alpha/2}) + \phi(-I^{1/2} \cdot \theta - z_{\alpha/2}) = \alpha + I \cdot \theta^{2} \cdot z_{\alpha/2} \cdot \phi(z_{\alpha/2}) + o(\theta^{2})
\]

where

\( (9) \quad I = \int_{0}^{1} \frac{d}{d} \)

the Fisher information.

Along the lines of Hajek p. 273 [26] this suggests approximating the AP on the one-sided case by

\[
\phi(e(\alpha)^{1/2} \cdot I^{1/2} \cdot \theta - z_{\alpha})
\]

where

\[
e(\alpha) = \lim_{\theta \to 0} e(\alpha, \theta), \quad \text{the local asymptotic efficiency}
\]

since if \( \theta_{2} = e(\alpha)^{1/2} \theta \) then \( F(x, \theta_{2}^{-1/2}) \approx F(x, \theta^{-1/2}) \).

Let \( e = \lim_{\alpha \to 0} e(\alpha) \).

**Theorem 5.** For \( D_{n}^{+}(1), x_{\alpha} = (-\frac{1}{2} \ln \alpha)^{1/2} \) and

(i) The AP is

\[
P(\sup(y + \theta p) \geq x_{\alpha}) = \alpha - 2 \alpha x_{\alpha} \int_{0}^{1} \frac{d}{d} \psi(\alpha, u) du \cdot \theta + o(\theta)
\]

where \( \psi(\alpha, u) = 2\phi(x_{\alpha} \cdot (2u-1) (u-u^{2})^{-1/2}) - 1 \), and

\[14\]
(11) \[ e(\alpha) = \frac{4\pi \alpha^2 \ln \alpha^{-1}}{\alpha^2} \cdot \frac{1}{\frac{1}{\int_0^1 \frac{1}{p} \psi(u) \psi(\alpha, u) \, du}} \quad \text{and} \quad e = \frac{4}{p(\frac{1}{2})^2} \int_0^1 \frac{1}{p} \, dp. \]

Note. In §7 we show that the Pitman efficiency of \( D_{11}^+(1) \) is \( \frac{4}{\sup p} \int_0^1 \frac{1}{p} \, dp \). Hence Hajek's local efficiency or its limit as \( \alpha \to 0 \) need not equal the Pitman efficiency.

Theorem 6. Let \( G(x) = P(\sup |y| \leq x), \lambda = G^{-1}(1-\alpha) \). The AP of \( D_n(1) \) is

\[
P(\sup |y+\theta p| \geq \lambda) = \lambda + \frac{\theta^2}{2} \left[ (1-\alpha) \int_0^1 \frac{1}{p} \, dp - 2 \int_0^1 \frac{1}{p} \psi(u) \, du \int_0^1 \frac{1}{p} \psi(v) \, dv \right]
\]

where

\[ Q(x, z, u, v) = \frac{1}{2\pi \Sigma} \exp\left( -\frac{1}{2} \left( xz \right) \Sigma^{-1} \left( xz \right) \right) \]

(10) \[ f_{uv}(xz\lambda) = K(u, 0, x, \lambda) K(v-u, x, z, \lambda) K(1-v, z, 0, \lambda) \]

(11) \[ A_m = [(2m-1)\lambda\alpha] [(2m-1)\lambda\alpha - 2] \]

(12) \[ B_m = [(2m-1)\lambda\alpha + x] [(2m-1)\lambda\alpha + z] \]

C_m = [2m\lambda\alpha - x] [2m\lambda\alpha + z] + xz

D_m = [2m\lambda\alpha + x] [2m\lambda\alpha - z] + xz

15
\[ \Sigma = \begin{pmatrix} u-u^2 & u-uv \\ u-uv & v-v^2 \end{pmatrix}. \]

**Theorem 7.** Let \( G(x) = P(\sup y + \sup -y \leq x) \), \( \lambda = G^{-1}(1-\alpha) \). The AP of \( V_n(1) \) is

\[
P(\sup(y+\theta_2) + \sup(-y+\theta_2) \geq \lambda) \\
= \alpha + \frac{\sigma^2}{2} \left[(1-\alpha) \int_0^1 \frac{1}{p^2(u)} du \int_0^1 \frac{1}{v} \frac{\partial p(u)}{\partial u} dv \int xdx \int zdz \right] \\
\cdot Q(xzuv) + o(\sigma^2)
\]

where \( Q \) is defined by (10), where

\[
f_{uv}(x^2; \lambda) = \frac{\partial^2}{\partial x_1 \partial x_2} \int_{z_1^2 + z_2^2 \leq \lambda} H(u, x, x_1, x_2, x_3, z_1, z_2) \\
\cdot H(1-v, z_1, x_3, z_2) \prod_{i=1}^{3} dx_i \, dz_i,
\]

\[
H(u,v,x,z) = \frac{8}{u^2} \left[ \sum_{m=1}^{\infty} \left( m^2 \left( \exp(-2C_m/u) + \exp(-2D_m/u) \right) \\
- (m^2 - m) \left( \exp(-2A_m/u) + \exp(-2B_m/u) \right) \right) \right],
\]

\( A_m, B_m, C_m, D_m \) are given by problem 15, p. 199 [20] with \( a = B+x, \) \( b = B+z, \alpha = A-x, B = A-z. \)
Example 3. \( H_n(x) = \mathcal{F}_0(x - \theta_n^{-1/2}), \mathcal{F}_0(x) = \frac{1}{2} e^{-|x|}, \) then

(i) AP of \( D_n(l) \) is

\[
1 - \int_{-d}^{d} H(x, d)^2 \, d\phi(2^{1/2} \cdot x + \theta) \]

where

\[
H(x, z) = \begin{cases} 
0 & \text{if } |x| \geq z \\
1 - \sum_{m=1}^{\infty} (-1)^{m+1} \exp(-2m^2 z^2) \left( \exp(-2mxz) + \exp(2mxz) \right) & \text{if } |x| < z,
\end{cases}
\]

\[
G(z) = H(0, z), \quad \lambda = G^{-1}(1-\alpha), \quad d = 2^{1/2} \lambda.
\]

(ii) AP of \( V_n(l) \) is

\[
1 - \int_{-d}^{d} \Phi(2^{1/2} \cdot x + \theta) \int_{\min(d,d-f)}^{\min(d,d-f)} \mathcal{F}(a, a+f, d-a, d-a-f) \, da
\]

where

\[
G(x) = \mathcal{P}(\sup \{ y + \sup -y \geq x \}), \lambda = G^{-1}(1-\alpha), \quad d = 2^{1/2} \lambda,
\]

\[
H(a,b,\alpha,\beta) = \mathcal{P}(-a(l-t) - bt < y(t) < \alpha(l-t) + \beta t \text{ in } (0,1))
\]

(given by problem 15, p. 199 [20]),

\[
\mathcal{F}(a,b,\alpha,\beta) = 2 \frac{\partial H}{\partial a} + \frac{\partial H}{\partial b} \cdot H.
\]
TABLE 1

Asymptotic Power (AP) for the double-exponential
with shift alternative

(a) .1% level

<table>
<thead>
<tr>
<th>LR test of ( \theta = 0 )</th>
<th>( T_n,2(1) ) i.e., Cramer-von Mises</th>
<th>( D_n(1) ) i.e., Kolmogorov-Smirnov</th>
<th>( V_n(1) ) i.e., Kulper</th>
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†This uses Examples 1, 2, 3. The approximate AP is defined by (8). The exact small-sample power can be calculated using [46]. The small-sample power has been found for some examples by Monte Carlo methods: see [45], [50], [51].

*For the one-sided analogs see p. 274 of [20].
**TABLE 1**

(a) .1% level (continued)

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# TABLE 1

(b) 1% level

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<th>$D_n^{(1)}$ i.e., Kolmogorov-Smirnov</th>
<th>$V_n^{(1)}$ i.e., Kuiper</th>
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Table 1

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TABLE I

(c) 5% level (continued)

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Remarks. 1. While $e(\alpha) = .6$ for $T_{n-2}(1)$ and $D_n(1)$, $V_n(1)$ is only about half as efficient--in contrast with Bahadur efficiency and Pitman efficiency for which $V_n(1)$ is always as least as efficient as $D_n(1)$--see later. However $e(\alpha)$ depends on $\alpha$, unlike Bahadur or Pitman efficiency.

2. The approximation for asymptotic power is worst for $V_n(1)$ for which it is accurate to the first significant figure until $\theta = 1.0, (1.1, 1.8)$ for $\alpha = .001, (.01, .05)$

23
§3. THREE TYPES OF EFFICIENCY

Let $X_1, \ldots, X_n$ be iid $F$, a continuous cdf on $R$. Let $\omega_0, \omega_1$ be two disjoint sets of cdfs on $R$. Suppose we test $H_0 : F \in \omega_0$ against $H_1 : F \in \omega_1$ rejecting $H_0$ when $T(F_n) > r_n$.

(For simplicity of presentation we exclude randomized tests.) Here $T(\cdot)$ is some functional, and $F_n$ is the empirical cdf.

Suppose $T(F_n) = T(F) + o_p(1)$ and

$$ r_n \to r \in [\mu_0, \mu_1] \quad \text{as } n \to \infty $$

where

$$ \mu_0 = \mu_0(\omega_0) = \sup_{F \in \omega_0} T(F) $$

$$ \mu_1 = \mu_1(\omega_1) = \inf_{F \in \omega_1} T(F) $$

and we suppose $\mu_0 < \mu_1$. (If $\mu_0 > \mu_1$ the statistic cannot discriminate between $\omega_0$ and $\omega_1$.)

Let

$$ \alpha_n(r_n, F) = P_F(T(F_n) > r_n) $$

$$ \beta_n(r_n, F) = P_F(T(F_n) \leq r_n) $$

$$ \alpha_n(r_n) = \sup_{F \in \omega_0} \alpha_n(r_n, F), \quad \text{the maximum type 1 error} $$

$$ \beta_n(r_n) = \sup_{F \in \omega_1} \beta_n(r_n, F), \quad \text{the maximum type 2 error} $$
\[ \Omega_r = \{ \text{cdfs } Q \text{ on } R : T(Q) > r \} \]
\[ \Omega_r^c = \{ \text{cdfs } Q \text{ on } R : T(Q) \leq r \} \]

\[ I(F,G) = \int \ln \left( \frac{dF}{dG} \right) dF \quad \text{if } F,G \text{ are a.c. cdfs}, \]

\[ I(A,B) = \inf_{F \in A} \inf_{G \in B} I(F,G) \quad \text{for sets of cdfs } A \text{ and } B, \]

\[ I_1(r,\omega_1) = I(\Omega_r^c, \omega_1) \]

\[ I_0(r,\omega_0) = I(\Omega_r, \omega_0) \]

\[ I_i(r, F) = I_i(r, \{F\}), \quad i = 0, 1. \]

Hoadley [22] has shown that for "regular" \( T(\cdot) \), (in particular for \( T(\cdot) \) uniformly continuous w.r.t. the "usual" metric)

(1) \[ \alpha_n(r_n, F) = \exp(-nI_0(r, F) + o(n)) \]

and

(2) \[ \beta_n(r_n, F) = \exp(-nI_1(r, F) + o(n)) \]

if \( I_0(r, F) \) and \( I_1(r, F) \) are continuous at \( r \).

Suppose now that (1) holds uniformly in \( F \in \omega_0 \). (This is certainly true if \( \omega_0 \) and \( \omega_1 \) are finite sets.) Then

\[ \alpha_n(r_n) = \exp(-nI_0(r, \omega_0) + o(n)) \]

and

\[ \beta_n(r_n) = \exp(-nI_1(r, \omega_1) + o(n)) \].
(Even without uniformity we have \( \lim_{n} -\frac{1}{n} \ln \alpha_n(r_n) \leq I_0(r, \omega_0) \) and
\( \lim_{n} -\frac{1}{n} \ln \beta_n(r_n) \leq I_1(r, \omega_1) \).)

If the maximum type 1 error is fixed, i.e., \( \alpha_n(r_n) = \alpha \), or
if \( 0 < \alpha_1 \leq \alpha_n(r_n) \leq \alpha_2 \) for all \( n \) then \( I_0(r, \omega_0) = 0 \) so that
\( r \leq \mu_0 \).

If \( r \leq \mu_0 \) then \( r = \mu_0 \) minimizes asymptotically the maximum
type 2 error \( \beta_n(r_n) \), so that if (2) is uniform in \( F \in \omega_1 \),

\( \beta_n(r_n) = \exp(-n I_1(\mu_0, \omega_1) + o(n)) \).

Now
\[
(3) \quad I_1(\mu_0, \omega_1) \leq I(\omega_0, \omega_1),
\]
with equality if \( \omega_0 = \{ T(F) \leq a_0 \} \) for some \( a_0 \). Further in the parametric
case when \( T(F_n) \) is the fixed \( \alpha \)-level LR (likelihood-ratio) test, then
under suitable conditions (\( \star \)) (see p. 22)

\( \beta_n(r_n, F) = \exp(-n I(\omega_0, F) + o(n)) \),

so that if this holds uniformly for \( F \in \omega_1 \), then equality is obtained
in (3).

These considerations lead us to define the fixed-\( \alpha \)

**efficiency of** \( T(F_n) \) as

\[
\varepsilon_\alpha = \frac{I_1(\mu_0, \omega_1)}{I(\omega_0, \omega_1)}.
\]

For similar reasons we define the fixed-\( \beta \) efficiency of \( T(F_n) \) as

\[
\varepsilon_\beta = \frac{I_0(\mu_1, \omega_0)}{I(\omega_1, \omega_0)}.
\]
([7], [8], and [10] show that under suitable conditions for the LR test in the parametric case, (1) holds at \( r = \mu \) and \( I_0(\mu, \omega) = I(\omega, \mu) \); (*) above is the dual of these conditions.)

When \( \omega_0 \) and \( \omega_1 \) are simple, \( e_\alpha \) is the Hodges-Lehmann efficiency relative to the LR test, and \( e_\beta \) is the Bahadur efficiency relative to the LR test.

Bahadur's interpretation of \( e_\beta \) in terms of level attained can be extended to \( \omega_0 \) and \( \omega_1 \) composite.

Between the two extremes of fixing the maximum type 1 error and fixing the maximum type 2 error, is the course of choosing \( r_n \) to minimize \( \alpha_n(r_n) + \lambda \beta_n(r_n) \) for some \( \lambda > 0 \).

Uniformity in (1) and (2) implies that

\[
\alpha_n(r_n) + \lambda \beta_n(r_n) = \exp(-n \min(I_0(r, \omega_0), I_1(r, \omega_1)) + o(n))
\]

so that \( r_n \to \mu_2 \), the root of

\[
I_0(r, \omega_0) = I_1(r, \omega_1)
\]

which exists and is unique if \( I_i(r, \omega_i), i = 0, 1 \) are continuous and strictly monotone in \( [\mu_0, \mu_1] \).

In the parametric case, one can show from [10] and Lemma 8 of [13] that under suitable conditions (1) holds in \( \omega_0 \), (2) holds in \( \omega_1 \), and

\[
I_0(\mu_2, \omega_0) = I_1(\mu_2, \omega_1) \leq J(\omega_0, \omega_1)
\]

where
\[ J(\omega_0, \omega_1) = \inf_{F_0 \in \omega_0} \inf_{F_1 \in \omega_1} \sup_{0 < t < 1} \ln \int \frac{dF_0}{d\nu} (1-t) \frac{dF_1}{d\nu} \, d\nu \]

with equality for the LR test of \( \omega_0 \) against \( \omega_1 \). Hence we define the information efficiency as

\[ e_I = I_0(\mu_2, \omega_0) / J(\omega_0, \omega_1) \]

For \( \omega_0 \) and \( \omega_1 \) simple this is the Chernoff efficiency relative to the LR test.
§ 4. FIXED-\(\alpha\) EFFICIENCY

Here we show that under suitable conditions for many gof tests
\(e_\alpha \to 1\) as \(\omega \to \{F_0\}\). Consider testing \(F \in \omega\), a set of cdfs containing a cdf \(F_0\), against \(F \in \omega\), another set of cdfs. Suppose that we consider statistics \(T(F_n)\) such that

(a) \(T(F_0) = 0\)

(b) \(T(F) \leq 0 \implies F = F_0\)

(For example the gof tests \(T_n(\psi), V_n(\psi)\) satisfy these conditions for \(0 < m \leq \infty, \psi\) positive and bounded.)

Suppose also that

(c) \(I_1(r, \omega)\) is right-continuous at \(r = 0\)

(d) \(\mu_0 \to 0\) as \(\omega \to \{F_0\}\).

Then

\[
\lim_{r \downarrow 0} I_1(r, \omega) = I_1(0, \omega) = \inf_{T(F) \leq 0} I(F, \omega) = I(F_0, \omega).
\]

Hence \(e_\alpha \to 1\) as \(\omega \to \{F_0\}\).

However in order for \(e_\alpha\) to be a measure of efficiency when the type-one error is fixed, we require that \(3 \cdot (2)\) hold uniformly in \(F \in \omega\).

For example, by Hoadley's Theorem 1, sufficient additional conditions are

(e) \(\omega\) is a finite set (since then \(3 \cdot (2)\) holds uniformly if it holds pointwise)

(f) all cdfs in \(\omega\) are continuous

(g) \(\omega \to \{F_0\}\) in such a way that \(\mu_0 > 0\)

(h) For some \(\delta > 0\) for all \(F\) in \(\omega\), \(I_1(\cdot, F)\) is continuous in \((0, \delta)\)

(i) \(T(\cdot)\) is uniformly continuous w.r.t. the "usual" metric, \(\sup |F-G|\).

Note: In testing \(F_0\) against \(F_1\), (a) and (h) \(\implies e_\alpha = 1\); however this is of no interest in itself since \(3 \cdot (2)\) need not hold (with \(F = F_1\)) at \(r=0\).

In §§ 4-9 we confine our attention to gof tests.
§5. FIXED-$\beta$ EFFICIENCY

According to the definition in §3, in order to calculate $e_\beta$ in testing $F = F_0$ against $F \in \omega_1$ we need to find $I(0, F_0)$, $\mu_1$ and $I(\omega_1, F_0)$.

**Theorem 1.** For $T_{n,m}(1)$, $T_{n,m}^+(1)$, $r > 0$

$$I_0(r, F_0) = m^{-1} \lambda^{-1/m} \int_0^\gamma y e^y (\mu + y - e^y)^{1/m-1} dy$$

where $\lambda, \mu, \gamma, \epsilon$ are determined by

$$\mu = \epsilon \frac{e^\epsilon - \epsilon}{\epsilon} = e^\gamma - \gamma, \quad \epsilon < \gamma$$

$$m \lambda^{1/m} = \int_\epsilon^{\gamma} (\mu + y - e^y)^{1/m-1} dy$$

$$\lambda^{1+1/m \cdot m} = \int_\epsilon^{\gamma} (\mu + y - e^y)^{1/m} dy.$$

**Theorem 2.** For $T_{n,2}(\psi_0)$ with $r > 0$, $I_0(r, F_0) \equiv r^2$.

**Theorem 3.** Suppose $[(x-x^2) \psi(x)]^{1/\psi(x)} \to 0$ as $x \to 0, 1$, and that $\psi$ is positive and continuous in $(0,1)$. Then for $V_n(\psi)$ if $\psi = 1$, and for $D_n(\psi)$, $I_0(r, F_0)$ is continuous for $0 \leq r < \max(\delta_1, \delta_2)$ and

$$I_0(r, F_0) = \inf_{x} \min(a(x, r/\psi(x)), a(1-x, r/\psi(x)))$$

where

$$a(x, r) = \begin{cases} (x+r) \ln(1 + \frac{r}{x}) + (1-x-r) \ln(1 - \frac{r}{1-x}) & , 0 < x < 1-r \\ \infty & , \text{otherwise} \end{cases}$$

and where $\delta_1 = \sup x \psi(x), \delta_2 = \sup(1-x) \psi(x)$. For $D_n^+(\psi)$, $I_0(r, F_0) = \inf_{x} a(x, r/\psi(x))$ if $0 \leq r < \delta_2$ and $I_0(r, F_0)$ is continuous in this range. For $D_n^- (\psi)$, $I_0(r, F_0) = \inf_{x} a(1-x, r/\psi(x))$ if $0 \leq r < \delta_2$ and $I_0(r, F_0)$ is continuous in this range.
\[ I_0(r, F_0) \text{ is tabulated in } \S 10 \text{ for } T_{n,1}(1) \text{ (and so for } T_{n,1}^+(1), T_{n,1}^-(1)), T_{n,2}(1), D_n(1) \text{ (and so for } D_n^+(1), D_n^-(1), V_n(1)) \text{ and for } \\
D_n(\psi) \text{ (and so for } D_n^+(\psi), D_n^-(\psi)) \text{ where } \\
\psi = \begin{cases} \psi_0 & \text{in } [0.005, 0.995] \\
\psi_0(0.005) & \text{otherwise} \end{cases} \\
\text{When } \psi \text{ equals } \psi_0 \text{ one can show that } I_0(r, F_0) = 0 \text{ and } \\
\mu_1 = \lim_{n \to \infty} T(F_n) = \infty \text{ for } D_n(\psi_0), V_n(\psi_0), D_n^+(\psi_0), D_n^-(\psi_0), \text{ so that } e_\beta \text{ cannot be calculated using these methods; however } e_\beta \text{ becomes arbitrarily small as } \psi \text{ remains bounded but approaches } \psi_0. \\
\text{Theorem 4. Let } \omega_1 = \{F : \sup|F-F_0| \psi_A(F_0) = a_1\} \text{ where } \psi_A \text{ is a non-negative bounded function, and } \psi, \psi_A \text{ satisfy } C(\psi(x), \psi_A(x)) = (1) \text{ of } \\
\S 9 \text{ and the conditions of Theorem 1(b), } \S 9 \text{ and } C(\psi(1-x), \psi_A(1-x)). \\
(a) I(\omega_1, F_0) = I_0(a_1, F_0) \text{ for } D_n(\psi). \\
(b) \text{ For } D_n(\psi), \mu_1 = a_1, \inf_{\psi(x)/\psi_A(x)} (\max(x,1-x) \psi_A(x) \geq a_1, 0 \leq x \leq 1) \text{ and } \\
\text{Using these results and Lemma (iii),(iv) of } \S 6 \text{ it follows that } \\
\text{when } \omega_1 = \{F : \sup|F-F_0| = a_1\}, \\
e_\beta \text{ for } \begin{cases} D_n(\psi) (\psi(x) = 1, (x-x^2)^{-1/2}, x^{-1/2}) \\
T_{n,1}(1), T_{n,2}(1), V_n(1) \end{cases} \\
equal e_\beta \text{ for } \begin{cases} D_n^-(\psi) (\psi(x) = 1, (x-x^2)^{-1/2}, x^{-1/2}) \\
T_{n,1}^-(1), T_{n,2}(1), V_n(1) \end{cases} \text{ calculated for } \omega_1 = \{F \leq F_0 : \sup(F_0-F) = a_1\}, \text{ which are given in } \\
Table 1, \S 9. \\
These results were used to find } e_\beta \text{ for the following examples. }
TABLE 2

(a) Table of $e_\beta$ for $F_\theta(x) = \phi(x-\theta)$

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>$T_{n,1}(1)$</th>
<th>$T_{n,2}(1)$</th>
<th>$D_n(1)$, $V_n(1)$</th>
<th>$T_{n,2}(\psi_0)$</th>
<th>$D_n(\psi_1)$</th>
</tr>
</thead>
<tbody>
<tr>
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<td>.6366</td>
<td>.9611</td>
<td>.64</td>
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<td>1/16</td>
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<td></td>
<td></td>
<td>.9610</td>
<td></td>
</tr>
<tr>
<td>1/8</td>
<td>.953</td>
<td></td>
<td>.635</td>
<td>.9610</td>
<td>.46</td>
</tr>
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<td></td>
<td>.635</td>
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<tr>
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<td>.817</td>
<td>.635</td>
<td>.9608</td>
<td>.38</td>
</tr>
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<td>.629</td>
<td>.9598</td>
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<td>.9582</td>
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<td>.608</td>
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<td>.26</td>
</tr>
<tr>
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<td>.864</td>
<td>.712</td>
<td>.592</td>
<td>.9553</td>
<td>.26</td>
</tr>
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<td>.574</td>
<td>.9511</td>
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<td>.668</td>
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<td>.9342</td>
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<td></td>
<td></td>
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<td></td>
<td></td>
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<td></td>
</tr>
</tbody>
</table>

Note: Gaps occur when the cubic interpolation formula used is inaccurate.
TABLE 2

(b) Table of $e_{B}$ for $F_{\theta}(x) = (1 + e^{-x + \theta})^{-1}$

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>$T_{n,1}(1)$</th>
<th>$T_{n,2}(1)$</th>
<th>$D_{n}(1, V_{n}(1)$</th>
<th>$T_{n,2}(\psi_{0})$</th>
<th>$D_{n}(\psi_{1})$</th>
</tr>
</thead>
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<td>.7500</td>
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<td>.7500</td>
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</tr>
<tr>
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<td>.998</td>
<td>.748</td>
<td>1</td>
<td>.596</td>
<td></td>
</tr>
<tr>
<td>3/16</td>
<td>.999</td>
<td>.748</td>
<td>1</td>
<td>.547</td>
<td></td>
</tr>
<tr>
<td>1/4</td>
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<td>.508</td>
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</tr>
<tr>
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<td>.409</td>
</tr>
<tr>
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<td>.299</td>
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<td>.864</td>
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</table>

For all $\theta$, $T_{n,2}(\psi_{0})$ is just as good as the LR test, in the sense of $e_{A}$ or $e_{B}$.
### Table 2

(c) Table of $e_\theta$ for $F_\theta(x) = F_0(x-\theta)$; $\hat{F}_\theta(x) = \frac{1}{2} e^{-|x|}$

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>$T_{n \cdot 1}(1)$</th>
<th>$T_{n \cdot 2}(1)$</th>
<th>$D_n(1)$, $V_n(1)$</th>
<th>$T_{n \cdot 2}(\psi_0)$</th>
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\begin{table}
\centering
\begin{tabular}{ccccccc}
\hline
$\theta$ & \text{T}_{n\cdot1}(1) & \text{T}_{n\cdot2}(1) & \text{D}_n(1), \text{V}_n(1) & \text{T}_{n\cdot2}(\psi_0) & \text{D}_n(\psi_1) \\
\hline
0 & .7500 & .7311 & .5413 & .808 & .6476 \\
1/16 & .765 & .552 & .814 & .54 \\
1/8 & .779 & .562 & .819 & .46 \\
3/16 & .791 & .572 & .823 & .42 \\
1/4 & .803 & .581 & .827 & .38 \\
1/2 & .841 & .736 & .611 & .8421 & .31 \\
3/4 & .872 & .753 & .636 & .8536 & .28 \\
1 & .895 & .772 & .657 & .8629 & .26 \\
1 1/4 & .912 & .784 & .673 & .8706 & .24 \\
1 1/2 & .925 & .791 & .688 & .8771 & .23 \\
1 3/4 & .937 & .802 & .700 & .8826 & .23 \\
2 & .946 & .810 & .711 & .8875 & .225 \\
2 1/4 & .954 & .815 & .720 & .8917 & .222 \\
2 1/2 & .960 & .818 & .728 & .8954 & .219 \\
2 3/4 & .965 & .823 & .735 & .8988 & .216 \\
3 & .969 & .828 & .741 & .9018 & .217 \\
3 1/4 & .973 & .831 & .747 & .9045 & .216 \\
3 1/2 & .976 & .833 & .752 & .9070 & .216 \\
3 3/4 & .979 & .835 & .757 & .9093 & .216 \\
4 & .981 & .838 & .761 & .9114 & .216 \\
5 & .987 & .848 & .774 & .9133 & .213 \\
6 & .991 & .853 & .783 & .9236 & .21 \\
7 & .994 & .860 & .790 & .9273 & .22 \\
8 & .996 & .866 & .794 & .9312 & .21 \\
9 & .996 & .869 & .797 & .9341 & .22 \\
10 & .997 & .872 & .800 & .9366 & .24 \\
\hline
\end{tabular}
\caption{(a) Table of $e_\beta$ for $F_\theta = F_0^{\theta+1}$}
\end{table}
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<tr>
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<th>( T_{n\cdot1}(1) )</th>
<th>( T_{n\cdot2}(1) )</th>
<th>( D_n(1), V_n(1) )</th>
<th>( T_{n\cdot2}(\psi_0) )</th>
<th>( D_n(\psi_1) )</th>
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<td>.7500</td>
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<td>.9987</td>
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<td></td>
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<td>1/8</td>
<td>1</td>
<td>.749</td>
<td>.99(50)</td>
<td>.72</td>
<td></td>
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<tr>
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<td>.750</td>
<td>.9997</td>
<td>.64</td>
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<tr>
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<td>.750</td>
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<tr>
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<td>.9513</td>
<td>.21</td>
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That \( T_{n\cdot1}(1) \) is just as good as the LR test (in the sense of \( e_\alpha \) or \( e_\beta \)) is not surprising since the LR test is equivalent to \( T_{n\cdot1}(1) \).
§ 6. LOCAL INEFFICIENCY OF $T_{n,m}(\psi)$

Suppose that we test $F = F_0$ against the alternative that $F$ is not close to $F_0$, in the sense that

$$F \in \omega_1 = \{ F \text{ continuous cdf on } \mathbb{R}, \| F - F_0 \|_{A,F_0} \geq a_1 \}$$

where $a_1 > 0$ and $\psi_A$ is some non-negative function, and $0 \leq k \leq \infty$.

**Lemma.** Let $f(x) = \begin{cases} x & \text{in } [0, \frac{1}{2}] \\ 1-x & \text{in } [rac{1}{2}, 1] \end{cases}$.

(i) If $\| f \cdot \psi_A \|_{k} < \infty$, $1 \leq k \leq m$ then for $T_{n,m}(\psi)$,

$$a_1 \inf \psi_A \leq \mu_1 \leq a_1 \| f \cdot \psi_A \|_m / \| f \cdot \psi_A \|_k$$

(ii) If $0 < \inf \psi_A$, $1 \leq m \leq k$ then for $T_{n,m}(\psi)$

$$\mu_1 \leq \sup \psi A \left( \frac{a_1}{\inf \psi_A} \right)^{k/(k+1) \cdot (m+1)/m} \cdot (k+1)^{(m+1)/(k+1) \cdot (m+1)/m - 1/m}$$

(iii) If $\omega_1 = \{ F : \sup |F - F_0| \geq a_1 \}$, then for $T_{n,m}(1)$,

$$\mu_1 = a_1^{1+1/m \cdot (m+1) - 1/m} \text{ (i.e., equality is obtained in (ii)).}$$

(iv) For $V_n$, if $k = \infty$, $\inf \psi_A \cdot a_1 \leq \mu_1 \leq \sup \psi_A \cdot a_1$.

**Theorem.** Let $\psi, \psi_A$ be bounded away from 0, \infty. Suppose $\alpha_n(r_n, F_0)$, $\beta_n(r_n)$, (defined in §3) are $O^*(1)$, by allowing $a_1$ to decrease to 0 as $n \to \infty$, where by $f = O^*(g)$ we mean that $f/g$ is bounded away from 0 and \infty.

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Then $T_{n,m}(\psi)$ needs $\mathcal{O}(a_1^{-c})$ observations where

$$c = \begin{cases} 
2, & k \leq m \leq \infty \\
\frac{k}{k+1} \cdot \frac{m+1}{m}, & m < k \leq \infty 
\end{cases}$$

while $V_n(\psi)$ needs $\mathcal{O}(a_1^{-2})$ observations.

Hence in order to ensure that the local efficiency (in the sense implicit in the theorem -- a generalization of the idea of Pitman efficiency) is positive for all $k$ and $\psi$ bounded, one must take $m = \infty$ when using $T_{n,m}(\psi)$.  

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§7. PITMAN EFFICIENCY

In testing $F_0$ against $F_\theta$, if we keep the type 1 and type 2 errors fixed then $\theta$ varies with $n$ and under general conditions the Pitman efficiency will equal the limit of the fixed-$\beta$ efficiency (see Appendix 2 of [6]). Thus in order to find the Pitman efficiency we need the behavior of $I_0(r, F_0)$ near $r = 0$.

**Theorem 1.** (1) Let $\psi$ be continuous and non-negative, $0 < m < \infty$. For $T_{n,m}(\psi), T^+_{n,m}(\psi), S_{n,m}(\psi), S^+_{n,m}(\psi)$, defined in §0,

\[
I_0(r, F_0) = \lambda_1 \frac{r^2}{2} + o(r^2) \quad \text{as } r \to 0
\]

where $\lambda_1 = \lambda_1(m, \psi)$ is the minimum positive $\lambda$ such that $H + \lambda \psi H^{m-1} = 0$ has a non-negative solution $H$ in $[0, 1]$ such that $H(0) = H(1) = 0$, and $\int_0^1 \psi H^m = 1$, provided that such a solution is unique. Further

\[
\lambda_1^{-1} = \sup \left\{ \frac{1}{f^m} \int_0^1 \int_0^1 \left[ \min(s,t) - st \right] \psi(s) \psi(t) f(s)^{m-1} f(t)^{m-1} ds dt : f^m = 1, f \geq 0 \right\}
\]

where inf replaces sup for $m > 1$, and

\[
\lambda_1(m, l) = \frac{2}{m} \left( 1 + \frac{m}{2} \right)^{2/m-1} B \left( \frac{1}{2}, \frac{1}{m} \right)^2
\]

Further (1) holds for $\psi = \psi_0$ with
\[
\lambda_1(1, \psi_0) = \frac{8}{\pi^2 - 8}
\]
and
\[
\lambda_1(2, \psi_0) = 2
\]

(ii) Let \( \psi \) be bounded. Then for \( D_n(\psi), D^+_n(\psi), D^-_n(\psi) \), (1) holds with \( \lambda_1 = \lambda_1(\infty, \psi) \) given by
\[
\lambda_1^{-1} = \sup(x - x^2)^2 \psi(x)^2.
\]

(iii) Let \( \psi \) be bounded. Then for \( V_n(\psi) \), (1) holds with
\[
\lambda_1^{-1} = \sup_{0 < x < y < 1} [(x - x^2)^2 \psi(x)^2 + (y - y^2)^2 \psi(y)^2 - 2x(1-y) \psi(x) \psi(y)]
\]
\[
= \lambda_\psi^{-1},
\]
say.

**TABLE 3**

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<tr>
<th>( m )</th>
<th>0⁺</th>
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<th>.5</th>
<th>1</th>
<th>2</th>
<th>4</th>
<th>(-\infty)</th>
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<td>7.93</td>
<td>4</td>
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<tr>
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<td>4.27</td>
<td>2</td>
<td></td>
<td>1*</td>
<td>.5*</td>
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</table>

*Indicates the value for \( \psi \) bounded but arbitrarily close to \( \psi_0 \).
TABLE 4
Pitman efficiency at $\theta = 0$

(a) $\varphi(x-\theta)$: normal with shift alternative
(b) $(1 + e^{-x+\theta})^{-1}$: logistic with shift alternative
(c) \[
\begin{cases} 
  e^{x-\theta/2}, x \leq \theta; \\
  1 - e^{-x+\theta/2}, x \geq \theta
\end{cases}
\]
  double-exponential with shift alternative
(d) $F_{\theta+1}^0$:
Lehmann alternative
(e) $(e^{\theta F_0} - 1)/(e^\theta - 1)$: Cauchy with shift alternative
(f) $\frac{1}{2} + \frac{1}{\pi} \tan^{-1}(x-\theta)$: normal with scale alternative
(g) $\Phi(xe^{-\theta})$:

<table>
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<th>(b); (e)</th>
<th>(c)</th>
<th>(d)</th>
<th>(f)</th>
<th>(g)</th>
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<td>.7500</td>
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<td>1</td>
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<tr>
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\[
p(x) = -\varphi(y)^* -x+x^2 |x-\frac{1}{2}| -\frac{1}{2} x \ln x - \frac{1}{\pi} \cos^2 \pi(x - \frac{1}{2}) -y\varphi(y)^*\]

\[
\int_0^1 p^2 \, dp = \frac{1}{1/3} 1 \quad 1 \quad 1 \quad 1/2 \quad 2
\]

*For (e), $p(x) = -\frac{1}{2} (x-x^2)$.  \[^*y = \Phi^{-1}(x)\]
Theorem 2. Let $\psi$ be bounded, $F = F_\theta$, $p(x) = [\frac{\partial}{\partial \theta} F_\theta(F_0^{-1}(x))]_{\theta=0}$ and suppose $\frac{\partial^2 F_\theta}{\partial \theta^2}$ exists a.e. in $x$ for $\theta$ in a neighborhood of 0. Then the Pitman efficiency of

(a) $T_{n,m}(\psi)$ is $\lambda_1(m,\psi) \|p\|_m^2 / \int_0^1 p^2$, \hspace{1cm} 0 \leq m \leq \infty.

(b) $T_{n,m}^+(\psi)$ is $\lambda_1(m,\psi) \|p\|_m^2 / \int_0^1 p^2$, \hspace{1cm} m = 1, 3, 5, ...

(c) $D_n^+(\psi)$ is $\lambda_1(\infty,\psi) (\sup p\psi)^2 / \int p^2$

(d) $D_n^-(\psi)$ is $\lambda_1(\infty,\psi) (\sup -p\psi)^2 / \int p^2$

(e) $V_n(\psi)$ is $\lambda_V(\psi) (\sup p\psi + \sup -p\psi)^2 / \int p^2$

where $\lambda_1(m,\psi)$, $\lambda_V(\psi)$ are defined in Theorem 1.
§8. INFORMATION EFFICIENCY, $e_I$

To find $e_I$, according to §3 we need $I_1(r,F)$. Our results are confined to $D_n(\psi), V_n(\psi)$ with the exception of Theorem 1.

Theorem 1. Suppose $F(F_0^{-1}(x)) = f(x) = (e^{\theta x} - 1)/(e^\theta - 1), \theta > 0, m = 1, 2, 3, \ldots$ and $0 < r < \|f(x) - x\|_m$. Then for $\|F_n - F_0\|_{m,F_0}$

$$I_1(r,F) = -\frac{\theta}{2} - \ln\left(\frac{\theta}{e - 1}\right) + \sum_{i=1,2} \int_0^\infty \frac{J_0(\theta y + A_i(y))}{A_i(y) - 1} \, dy$$

where $e^x - x$ has positive inverse $K_1(\cdot)$, negative inverse $K_2(\cdot)$ and $A_i(y) = K_i(R(y)), i = 0, 1$ where $R(y) = \mu - \lambda y^m + \theta y$, and $\mu, \lambda, J_0$ are determined by

$$l = R(J_0)$$

$$l = \int_0^{J_0} (e^{A_1(y)} - 1)^{-1} - (e^{A_2(y)} - 1)^{-1} \, dy$$

$$r^m = \sum_{i=1,2} \int_0^{J_0} y^m \, dy$$

Definition. (i) Let $C_{ab} = \{G : cdf on [0,1], a(x) \leq G(x) \leq b(x) \}$ in $[0,1]$ where $a, b$ are given non-decreasing functions on $[0,1]$ such that $0 \leq a(x) \leq b(x) \leq 1$ in $[0,1]$. 

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Then the envelope cdf of $C_{ab}$, $H$, is constructed from the convex portions of $y = a(x)$, say $A_1, \ldots, A_p$, and the concave portions of $y = b(x)$, say $B_1, \ldots, B_p$ as follows: draw the longest tangent between $A_1$ (if $0 \in A_1$) or $B_1$ (if $0 \in B_1$) or $0$ (if $0 \notin A_1 \cup B_1$) and $T = (UA_1) \cup (UB_1) \cup \{(1,1)\}$ which lies in $C_{ab}$; suppose this lies between $(x_1, a_1)$ and $(x_2, a_2)$; set

$$H(x) = \begin{cases} 
    a(x) & \text{if } 0 \leq x < x_1, \ 0 \in A_1 \\
    b(x) & \text{if } 0 \leq x < x_1, \ 0 \in B_1 \\
    a_1 + \frac{x-x_1}{x_2-x_1} \cdot (a_2-a_1) & \text{if } x_1 \leq x \leq x_2; 
\end{cases}$$

if $x_2 < 1$ draw the longest tangent in $C_{ab}$ in a positive direction between $A_1$ (if $x_2 \in A_1$) or $B_1$ (if $x_2 \in B_1$) and $T$; suppose this lies between $(x_3, a_3)$ and $(x_4, a_4)$; then set

$$H(x) = \begin{cases} 
    a(x) & x_2 \in A_1 \\
    b(x) & x_2 \in B_1 \\
    a_3 + \frac{x-x_3}{x_4-x_3} \cdot (a_4-a_3) & \text{if } x_3 < x \leq x_4. 
\end{cases}$$

proceed thus until $H$ is defined on $[0,1]$.

(ii) When $a(x)$, $b(x)$ are not increasing functions on $[0,1]$ but $0 \leq A(x) \leq B(x) \leq 1$ in $(0,1)$ where

$$A(x) = \sup_{0 \leq y \leq x} a(y), \quad B(x) = \inf_{x \leq y \leq 1} b(y)$$

then the envelope cdf on $C_{ab}$ is defined to be the envelope cdf on $C_{AB}$. 

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Theorem 2. Suppose \( 0 \leq a(x) < b(x) \leq 1 \) in \((0,1)\) where \(a(x), b(x)\) are non-decreasing functions in \((0,1)\). Let

\[
C = C(a,b) = \{ G : \text{cdf on } [0,1], a(x) \leq G(x) \leq b(x) \text{ in } (0,1) \}
\]

Suppose \( I(C(a(x)+\varepsilon_1, b(x)+\varepsilon_2), U) \) is continuous at \( \varepsilon_1 = \varepsilon_2 = 0 \). Suppose there is a positive \( \delta \) such that

\[
(1) \quad a(\delta) = 0, \quad b(1-\delta) = 1.
\]

Then

\[
(2) \quad P(a(x) + o(n) \leq U_n(x) \leq b(x) + o(n) \text{ in } (0,1)) = \exp(-nI(C,U) + o(n))
\]

and

\[
(3) \quad I(C,U) = I(H,U) \quad \text{where } H \text{ is the envelope cdf of } C.
\]

Theorem 3. Let \( H_0 = F_0(F^{-1}) \) be a continuous cdf on \([0,1]\).

(a) For \( D^+(n) \) if \( \psi(1-) < \infty \) then \( I_{1}(r,F) = I(H_1,U) \) where \( H_1 \) is the (convex) envelope cdf of \( \{ G : G \leq H_0 + r/\psi(H_0) \} \).

(b) For \( D^-(n) \) if \( \psi(0+) < \infty \) then \( I_{1}(r,F) = I(H_2,U) \) where \( H_2 \) is the (concave) envelope cdf of \( \{ G : G \geq H_0 - r/\psi(H_0) \} \).

(c) For \( D(n) \) if \( \psi(0+) < \infty, \psi(1-) < \infty \), then \( I_{1}(r,F) = I(H_3,U) \) where \( H_3 \) is the envelope cdf of \( \{ G : H_0 - r/\psi(H_0) \leq G \leq H_0 + r/\psi(H_0) \} \).

45
(d) For $V_n(\psi)$ if $\psi(0^+) < \infty$, $\psi(1^-) < \infty$,

$$I_1(r, F) = \inf_{0 < t < r} \inf_{H_0(s) \cdot \psi(H_0(s)) > t} \inf_{s < 1} I(H_{st}, U)$$

where $H_{st}$ is the envelope cdf of

\[ G : H_0 - t/\psi(H_0) \leq G \leq H_0 + (r-t)/\psi(H_0), \]

\[ H_0(s) - t/\psi(H_0(s)) = G(s) \] .

(e) If $H_0$ is concave (or convex) and $\psi(0^+) < \infty$, $\psi(1^-) < \infty$ then $I_1(r, F)$ for $V_n(\psi)$, $D_n^-(\psi)$, $D_n^-(\psi)$, $D_n^+(\psi)$ (or $D_n^+(\psi)$) are all equal.

Note. In testing $F = F_0$ against $F \in \omega_1$, arbitrary,

$$e_1 \text{ for } D_n^+(1) \leq e_1 \text{ for } D_n^-(1) \leq e_1 \text{ for } V_n(1).$$

This follows from the fact that $I_0(r, F_0)$ for $D_n^+(1)$, $D_n^-(1)$, $V_n(1)$ are equal whereas $D_n^+(1) \leq D_n^-(1) \leq V_n(1)$ so that

$$I_1(r, \omega_1) \text{ for } D_n^+(1) \leq I_1(r, \omega_1) \text{ for } D_n^-(1) \leq I_1(r, \omega_1) \text{ for } V_n(1).$$

Using the results of this section and §5, $e_1$ was found for the following examples. Note that for these examples $e_1$ for $D_n^+(1)$ and for $V_n(1)$ is the same as for $D_n^+(1)$. See §10 for details.
\( \text{TABLE 5} \)

\( e_I \) for \( D_n(1) \)

(a) \( F_\theta(x) = \Phi(x-\theta); \quad J = \frac{\theta^2}{8} \)

(b) \( F_\theta(x) = (1 + e^{-x+\theta})^{-1}; \quad J = -\ln \inf \int_0^1 \left[ xe^{-\theta/2} + (1-x) e^{\theta/2} \right]^{-2t} \, dx \)

\[ = \ln \left[ \frac{e^{\theta/2} - e^{-\theta/2}}{\theta} \right] \]

(c) \( F_\theta(x) = F_0(x-\theta), \quad \dot{F}_0(x) = \frac{1}{2} e^{-|x|}; \quad J = \frac{\theta}{2} - \ln(\frac{\theta}{2} + 1) \)

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TABLE 5

\( e_I \) for \( D_n(1) \)

(a) \( F_0 = F_0^{\theta+1}; \ J = -\ln(-\ln a) + \ln(1-a) - 1 - \frac{\ln a}{1-a} \) where \( a = \frac{1}{\theta+1} \)

(e) \( F_\theta = \frac{(e^{\theta F_0} - 1)}{(e^{\theta} - 1)}; \ J = \sup_{0 < t < 1} \left[ -t \ln(\theta/(e^t-1)) - \ln[(e^{\theta t}-1)/\theta t] \right] \)

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§9. EFFICIENCY OF SOME ONE-SIDED gof TESTS

Here we consider testing \( F \in \omega_0^- \) against \( F \in \omega_1^- \) where

\[
\omega_0^- = \{ \text{continuous } F : \sup(F_0^- - F) \psi_A(F_0^+) = a_0, F_0 \geq F \}
\]

\[
\omega_1^- = \{ \text{continuous } F : \sup(F_0^- - F) \psi_A(F_0^+) = a_1, F_0 \geq F \}
\]

where \( \psi_A \) is a non-negative function such that

\[
(1) \quad (x - u_{a,i}) \psi_A(x)
\]

is maximized over \( 0 \leq x \leq a \) at \( x = a \) for \( a \in A_1 = \{ 0 \leq a \leq 1, a \psi_A(a) \geq a_1 \} \) and \( u_{a,i} = a - a_i / \psi_A(a) \) is non-decreasing in \( \{ a : 0 \leq u_{a,i}, a < 1 \}, i = 0, 1, \) and \( 0 \leq a_0 < a_1 < \sup_{(0,1)} a \psi_A(a) \).

Let

\[
\tau_n = 2 \int \ln F_0 \, dF_n, \quad \tau'_n = 2 \int \ln(1 - F_0) \, dF_n,
\]

the Fisher and Pearson tests. (If \( F = F_0' \), \( -n \tau_n \) and \( -n \tau'_n \) are chi-square with 2n degrees of freedom. \( H_0 \) is accepted for small values of \( \tau_n \) or \( \tau'_n \).)

Let \( T_n \psi = -T'_n \psi \).

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Theorem 1. If (a) \( i = 0, x\psi(x) \to 0 \) as \( x \to 0 \), \( \psi \) is bounded non-negative and piecewise-continuous in \((0,1]\) or (b) \( i = 1, \psi(0^+) < \infty \), \( x - r/\psi(x) \) and \((x-u_{a,1}) \psi(x)\) are both maximized over \([u_{a,1}, a]\) at \( x = a \) for all \( a \in A_1 \). Then for \( D_n^-(\psi) \),

\[
I_i(r, \omega_i^-) = \inf \left\{ \ln \frac{x-r/\psi(x)}{x-a_i/\psi_A(x)} \right\} \]

\[
+ (1-x+r/\psi(x)) \ln \frac{1-x+r/\psi(x)}{1-x+a_i/\psi_A(x)} \]

where the \( \inf \) is taken over \((l > x, x\psi(x) > r)\) if \( i = 0 \) and over \( A_1 \) if \( i = 1 \), and

\[
\mu_i = \begin{cases} 
\sup_{A_0} \psi/\psi_A & \text{if } i = 0 \\
\inf_{A_1} \psi/\psi_A & \text{if } i = 1.
\end{cases}
\]

(c) If \( \psi(0^+) < \infty \), then for \( D_n^-(\psi) \)

\[
I_i(r, \omega_i^-) = \inf_{a \in A_1} \lambda_a \ln(\lambda_a/u_{a,1}) + (1-\lambda_a) \ln((1-\lambda_a)/(1-u_{a,1}))
\]

where

\[
\lambda_a = \sup_{(u_{a,1}, a)} (x-r/\psi(x)) \quad \text{and} \quad \mu_i = \inf_{A_1} \sup_{a \in A_1} (x-u_{a,1}) \psi(x)
\]

Theorem 2. (a) If \( x\psi_A(x) \to 0 \) as \( x \to 0 \), \( \psi_A \) is bounded and piecewise continuous in \((0,1]\) then

\[
I(\omega_i^-, \omega_0) = I_0(\mu_i, \omega_0^-) \quad \text{for } D_n^-(\psi_A).
\]

(b) If \( \psi_A(0^+) < \infty \), then

\[
I(\omega_0^-, \omega_i^-) = I_1(\mu_0, \omega_i^-) \quad \text{for } D_n^-(\psi_A).
\]
(c) If $\psi_A$ is bounded and piecewise continuous in $[0,1]$ then

$$J(F_0, \omega_1^-) \text{ (defined in §3) equals}$$

$$\sup_{0 < t < 1} \inf_{p \in A_1} - \ln f(p, t, a_1)$$

where

$$f(p, t, a_1) = p(l - \frac{a_1}{p\psi(p)})^t + (1-p) (1 + \frac{a_1}{(1-p)\psi(p)})^t.$$

(d) If $\psi_A = 1$,

$$J(\omega_0^-, \omega_1^-) = \begin{cases} -\frac{1}{2} \ln(l-(a_1-a_0)^2), & a_0 + a_1 \leq 1 \\ \frac{u}{u+v} \ln \frac{u}{l-a_1} + \frac{v}{u+v} \ln \frac{v}{a_1} - \ln(u+v), & a_0 + a_1 > 1 \end{cases}$$

where $u = \ln a_1/a_0$, $v = \ln (l-a_0)/(l-a_1)$.

(e) Suppose that the conditions (a) of Theorem 1 are satisfied and $x\psi_A(x)$ is uniquely maximized over $[0,1]$ at $x = 1$. Then for $D_n^-(\psi)$, \( e_\beta \to 1 \) as $a_1 \to \psi_A(1)$. Suppose also that the conditions (b) of Theorem 1 are satisfied. Then for $D_n^-(\psi)$

$$e_\alpha \to \frac{l - \mu_0/\psi(1)}{l - a_0/\psi_A(1)} \text{ as } a_1 \to \psi_A(1)$$

$$= 1 \text{ if } \sup_{A_1} \psi/\psi_A = \psi(1)/\psi_A(1)$$

and if $\psi_A = 1$, as $a_1 \to 1$,

$$e_1 \to \begin{cases} 0 & a_0 = 0 \\ 1 & a_0 > 0 \end{cases}$$

If $\psi_A = 1$ and $a_0 > 0$ then for $D_n^-(1)$, there exists $K < 1$ such that for $a_1 > K$, $e_1 = 1$.

51
Theorem 3. Let $\psi_A = 1$.

(a) For $\pi^-_{n,1}(1), \mu_0 = a_0 - a_0^2/2, \mu_1 = a_1^2/2,

\begin{align*}
I_0(r, \omega^-_0) &= \sup_t \left[-t(r + \frac{1}{2}) - \ln \left(\frac{e^{-a_0 t}}{t} + a_0 e^{-t}\right)\right] \\
I_1(r, \omega^-_1) &= \sup_t \left[-t(r + \frac{1}{2}) - \ln \left(\frac{1 - e^{-t(1-a_1)}}{t} + a_1 e^{-t}\right)\right].
\end{align*}

(b) For $\pi_n$,\[\mu_0 = -2(a_0 \ln a_0 - a_0 + 1)\]
\[\mu_1 = 2(1-a_1) (\ln(1 - a_1) - 1)\]
\begin{align*}
I_0(r, \omega^-_0) &= \sup_t \left[\frac{tr}{2} - \ln \left(\frac{1 - a_0^{t+1}}{t+1} + a_0\right)\right] \\
&= -r/2 - 1 - \ln(-r/2), & \text{if} & a_0 = 0
\end{align*}
\begin{align*}
I_1(r, \omega^-_1) &= \sup_{-1 < t < 0} \left[\frac{tr}{2} - \ln \left(\frac{(1-a_1)^{t+1}}{t+1} + a_1\right)\right].
\end{align*}

(c) For $\pi_n^\prime, \mu_1 = -\infty$ so that $e_{\alpha} = e_\beta = e_I = 0$.

Note. §10 contains formulae for $\mu_i, I_i(r, \omega^-_i), i = 0, 1$ for $\pi^-_{n,1}(\psi), \pi_n$ for general $\psi, \psi_A$.

Local Efficiency.

Theorem 4. Let $a_0 = 0$, i.e., $\omega^-_0 = \{F_0\}$. Let $\psi, \psi_A$ be bounded.

(a) For $D^-_n(\psi)$, $e_{\alpha} = 1$

\begin{align*}
\lim_{a_1 \to 0} e_{\beta} &= \inf_{a_1} \frac{\psi^2}{\psi_A^2} \frac{\sup \psi^2_{\psi_0}}{\sup \psi^2_{\psi_0}} \\
\lim_{a_1 \to 0} e_{I} &= \left[\frac{c_0(\psi)}{c_0(\psi)} \cdot \frac{2}{c_0(\psi_A)}\right]^2.
\end{align*}

where
\[ c_0(\psi) = \inf \frac{\psi_0}{\psi}, \]
\[ c_1(\psi) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \inf_x \left( \frac{\psi_0}{\psi_A} - \frac{\psi_0}{\psi} \left( \inf \frac{\psi}{\psi_A} - \varepsilon \right) \right). \]

(b) \[ T_{n.1}(\psi), \text{ as } a_1 \to 0 \]
\[ e_\alpha, e_\beta, e_I = \frac{a_1^2 \inf \frac{\psi^2}{\psi_A} \psi_A^2}{4 \cdot \nu(\psi) \cdot \inf \frac{\psi_0^2}{\psi_A^2}} + o(a_1^2) \]
where
\[ \nu(\psi) = \int_0^1 \left( \int_0^1 \psi \right)^2 - \left( \int_0^1 x \psi \right)^2. \]

(c) For \[ T_n, \text{ as } a_1 \to 0, \]
\[ e_\alpha, e_\beta, e_I = \frac{a_1^2 \inf \frac{x^{-2}}{\psi_A} \psi_A^{-2}}{4 \cdot \inf \frac{\psi_0^2}{\psi_A^2}} + o(a_1^2). \]

In Table 6 the efficiency for \( \psi = x^{-1/2} \) is to be interpreted as the limit of the efficiency as \( \psi \) bounded tends to \( x^{-1/2} \).
TABLE 6a

Efficiency of Tests of

\( F = F_0 \) against \( \sup (F_0 - F) = a_1 \) when \( F \leq F_0 \).

<table>
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<tr>
<th>( a_1 )</th>
<th>0.125</th>
<th>0.250</th>
<th>0.375</th>
<th>0.500</th>
<th>0.625</th>
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<td>1</td>
<td>1</td>
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<td>1</td>
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<td>0.004</td>
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<td>0.017</td>
<td>0.026</td>
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<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( \pi_n' )</td>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
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<tr>
<td>( V_n(1) )</td>
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<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
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</tr>
</tbody>
</table>

max. index of \( e_\alpha \) 0 | .0314 | .127 | .291 | .532 | .869 | 1.34 | 2.08 | 3 |

| \( D_n(1) \) | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| \( D_n(\psi_1) \) | 1 | 0.533 | 0.389 | 0.311 | 0.258 | 0.227 | 0.222 | 0.256 |
| \( D_n(x^{-1/2}) \) | 0.250 | 0.318 | 0.369 | 0.421 | 0.476 | 0.537 | 0.607 | 0.698 |
| \( T_{n-1}(1) \) | 0 | 0.012 | 0.046 | 0.103 | 0.180 | 0.277 | 0.397 | 0.551 |
| \( \pi_n \) | 0 | 0.001 | 0.004 | 0.012 | 0.025 | 0.046 | 0.048 | 0.163 |
| \( T_{n-2}(1) \) | 0 | 0.183 | 0.268 | 0.350 | 0.432 | 0.516 | 0.627 |
| \( \pi_n' \) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| \( V_n(1) \) | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

max. index of \( e_\beta \) 0 | .0314 | .127 | .291 | .532 | .869 | 1.34 | 2.08 | 3 |

| \( D_n(1) \) | 1 | .993 | .972 | .935 | .882 | .807 | .730 | .648 |
| \( D_n(\psi_1) \) | 1 | .802 | .694 | .623 | .574 | .541 | .525 | .549 |
| \( D_n(x^{-1/2}) \) | 1 | .608 | .588 | .557 | .528 | .504 | .479 | .447 |
| \( T_{n-1}(1) \) | 0 | 0.012 | 0.040 | 0.077 | 0.118 | 0.159 | 0.200 | 0.239 |
| \( \pi_n \) | 0 | 0.001 | 0.004 | 0.010 | 0.019 | 0.029 | 0.042 | 0.058 |
| \( \pi_n' \) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

max. index of \( e_I \) 0 | .00787 | .0323 | .0758 | .144 | .248 | .413 | .725 |

\( \mu_2 \) for \( D_n(1) \) 0 | .0625 | .125 | .1875 | .25 | .3126 | .2818 | .4717 |

* By Lemma iv, §6 and Theorem 3, §5.
† Included since when \( e_1 = 1 \) (see Table 2), the test "reject \( (F=\bar{F}_0) \iff D_n(1) > \mu_2 \)" is asymptotically Bayes if a positive prior can be defined.
‡ \( \mu_2 \) was obtained from Lemma iii, §6.
TABLE 6b

Efficiency of Testing:

\[ \sup(F_0 - F) = a_0 \] against \[ \sup(F_0 - F) = a_1 \] when \( F \leq F_0 \).

Note. For \( D_n^-(\psi) \) with \( \psi \) arbitrarily close to \( \psi_0 \) but bounded, and for \( \pi_n \), \( e_\alpha = e_\beta = e_1 = 0 \).

\[ a_0 = \frac{1}{8} \]

<table>
<thead>
<tr>
<th>( a_1 )</th>
<th>( .125+ )</th>
<th>( .25 )</th>
<th>( .375 )</th>
<th>( .5 )</th>
<th>( .625 )</th>
<th>( .75 )</th>
<th>( .875 )</th>
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</tr>
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<td>1</td>
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<td>1</td>
<td>1</td>
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<td>.5323</td>
<td>.8722</td>
<td>1.459</td>
<td>( \infty )</td>
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</table>

| \( e_\beta \) | \( D_n^-(1) \) | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| \( \quad \) | \( D_n^-(\psi_1) \) | 0 | 0 | 0 | 0 | 0 | 0 | .011 | 1 |
| \( \quad \) | \( D_n^-(x^{-1/2}) \) | 0 | 0 | .099 | .381 | .520 | .634 | .759 | 1 |
| \( \quad \) | \( T_n^{-1}(1) \) | 0 | 0 | 0 | .001 | .076 | .215 | .407 | 1 |
| \( \quad \) | \( \pi_n \) | 0 | 0 | 0 | 0 | 0 | .0006 | .073 | 1 |
| max. index of \( e_\beta \) | 0 | .03136 | .1268 | .2908 | .5323 | .8689 | 1.340 | 2.079 |

| \( e_\iota \) | \( D_n^-(1) \) | 1 | .993 | .972 | .935 | .885 | .848 | .844 | 1 |
| \( \quad \) | \( D_n^-(\psi_1) \) | 0 | 0 | 0 | 0 | 0 | 0 | .035 | 1 |
| \( \quad \) | \( D_n^-(x^{-1/2}) \) | 0 | 0 | .026 | .308 | .444 | .513 | .575 | 1 |
| \( \quad \) | \( T_n^{-1}(1) \) | 0 | 0 | 0 | .001 | .046 | .120 | .218 | 1 |
| \( \quad \) | \( \pi_n \) | 0 | 0 | 0 | 0 | 0 | .0002 | .022 | 1 |
| max. index of \( e_\iota \) | 0 | .00784 | .03227 | .07577 | .1438 | .2477 | .4133 | \( \infty \) |

\[ \mu_2 \] for \( D_n^-(1) \) | .1250 | .1875 | .2500 | .3125 | .3754 | .4452 | .5344 | 1 |
\[ a_0 = \frac{1}{4} \]

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<th>.5</th>
<th>.625</th>
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<th>.875</th>
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max. index of \( e_\alpha \) 0 | .03136 | .1268 | .2908 | .5493 | 1.0306 | \( \infty \) | 0

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max. index of \( e_\beta \) 0 | .03136 | .1268 | .2908 | .5323 | .8689 | 1.386 |

<table>
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max. index of \( e_I \) 0 | .007874 | .03227 | .07577 | .1458 | .2568 | \( \infty \) |

\( \mu_2 \) for \( D_n^-(1) \) 0.2500 | 0.3125 | 0.3750 | 0.4383 | 0.5080 | 0.5956 | 1 |

56
\[ a_0 = \frac{3}{8} \]

<table>
<thead>
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<th>(.5)</th>
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max. index of \( e_\alpha \) | 0 | .03136 | .1277 | .3128 | .6882 | \( \infty \) |

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max. index of \( e_\beta \) | 0 | .03136 | .1268 | .2908 | .5402 | .982 |

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max. index of \( e_{\lambda} \) | 0 | .007874 | .03227 | .07732 | .1604 | \( \infty \) |

\( \mu_2 \) for \( D_n^{-}(1) \) | .3750 | .4375 | .5009 | .5700 | .6554 | 1   |

57
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Theorem 1(ii) uses the following two lemmas.

Lemma 1(a). Under the conditions of Theorem 1(ii), for all \( \varepsilon > 0 \) there exist \( \mu, \nu \in (0,1) \) such that for all \( n \)

\[
E \left( \int_0^1 \left| \tilde{Y}_m \right|^m \nu^m \right) < \varepsilon^2.
\]

Proof. Let

\[
Y_n(t) = n^{1/2} \left( F_n(H_n^{-1}(t)) - t \right),
\]

\[
\tilde{Y}_n(t) = n^{1/2} \left( F_n(F_0^{-1}(t)) - t \right),
\]

\[
h_n = H_n(F_0^{-1}(t)),
\]

\[
\tilde{Y}_n(t) = Y_n(h_n) + n^{1/2}(h_n - t)
\]

(1) \( \therefore \)

\[
EY_n(t)^{2k} = \sum_{i=0}^{2k-1} \binom{2k}{i} EY_n(h_n)^{2k-i} [n^{1/2}(h_n - t)]^i + [n^{1/2}(h_n - t)]^{2k}.
\]

By our assumption on \( H_n \) there is a finite \( N_1 \) such that
\[ [n^{1/2}(h_n - t)]^i < N_i, \quad i = 0, \ldots, 2k-1 \]

Also
\[ Y_n(t)^2 \leq Y_n(1-t)^2. \]

Therefore \( EY_n(t)^{2k} \) is a pod \( 2k \) (polynomial of degree \( 2k \)) in \( t \), symmetric about \( t = 1/2 \).

Hence \( EY_n(t)^{2k} \) is a pod \( k \) in \( T = t-t^2 \). Also
\[ E_n Y_n(t)^{2k} = E(nU_n(t) - nt)^{2k} \]
is a pod \( 2k \) in \( n \). Therefore \( EY_n(t)^{2k} = g_t(n) + h_t(n^{-1}) \) where \( g_t \) and \( h_t \) are pod \( k \). But \( EY_n(t)^{2k} \rightarrow Ey(t)^{2k} \)
as \( n \rightarrow \infty \). Therefore \( g_t(n) \) is a constant in \( n \). Hence \( EY_n(t)^{2k} \) is a pod \( k \) in \( (T, n^{-1}) \). Also \( Y_n(0) = 0 \). Hence there is a finite \( M_k \) such that \( EY_n(t)^{2k} \leq (t-t^2)^M_k \) for all \( n, t \).

Similarly \( EY_n(t)^{2k+1} = n^{-1/2}(2t-1) \) (pod \( k \) in \( (T, n^{-1}) \)), so there is a finite \( N_k \) such that
\[ EY_n(t)^{2k+1} < n^{-1/2}(t-t^2) \cdot N_k \]
for all \( n, t \).

Hence by (1) there is a finite \( M^1_k \) such that for all \( n, t \)
\[ EY_n(t)^{2k} < (h_n - h_n^2) M^1_k + [n^{1/2}(h_n - t)]^{2k} \]

Let \( k = K(m)/2 \) where \( K(m) \) is the smallest even integer \( \geq m \).
Therefore
\[
\left( n^{m/2} \text{ET}_{n,m}(\psi)^m \right)^{2k/m} \leq n^{k} \text{ET}_{n,m}(\psi)^{2k} \leq n^{k} \text{ET}_{n,2k}(\psi)^{2k}, \quad \text{by Jenson's inequality}
\]
\[
= \int \text{ET}_{n} (t)^{2k} \psi(t)^{2k} \psi(t)^{2k} \quad \text{by Fubini's theorem}
\]
\[
< M_k^1 \int (h_n - \psi_n)^n \psi(t)^{2k} dt + \int [n^{1/2}(h_n - t)]^{2k} \psi(t)^{2k} dt.
\]
\[
< M_k^1(aI_1 + bI_2) + (dI_2^{1/2k} + eI_1^{1/2k})^{2k}
\]
\[
< \infty \quad \text{by assumption}
\]

Replacing \( \psi \) by
\[
\psi \quad \text{in } [0,u] U [v,1]
\]
\[
o \quad \text{in } (u,v)
\]

we have
\[
E(f(t) \left| \tilde{Y}_n \right|^m \psi(t) < M_k^1(aJ_1 + bJ_2) + (dJ_2^{1/2k} + 2J_1^{1/2k})^{2k} < \epsilon^2
\]
where
\[
J_1 = \int_0^u + \int_v^1 \left( t - t^2 \right) \psi(t)^K(m), \quad J_2 = \int_0^u + \int_v^1 \left| r \right|^K(m) \psi(t)^K(m)
\]

and we choose \( u, v \) such that \( J_1, J_2 < \epsilon^2/N \) where
\[
N = M_k^1(a+b) + (d+e)^{2k}
\]

**Lemma 1(b).** If \( H_n(F_0^{-1}(t)) \) is a continuous càdf on \([0,1]\) and tends to \( t \) as \( n \to \infty \) uniformly in \( 0 \leq t \leq 1 \) then

(i) \( E(\tilde{Y}_n(t) - \tilde{Y}_n(s))^4 \leq e_n + 12(t-s)^2 \) where \( e_n \to 0 \) and does not depend on \( s, t \) and \( \tilde{Y}_n(t) = n^{1/2}(F_n(F_0^{-1}(t)) - t) \), and
\[
0 \leq s < t < 1;
\]
(ii) \(\lim_{\delta \to 0} \liminf_{n \to \infty} P(\max_{|t-s| < \delta} |\tilde{Y}_n(t) - \tilde{Y}_n(s)| < \varepsilon) = 1\) for all \(\varepsilon > 0\).

Proof. (i) Let

\[
Y_1 = F_0(X_1), \quad S(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}
\]

\[
Z_1 = S(t-Y_1) - S(s-Y_1) - t+s,
\]

and

\[
\Delta = H_n(F_0^{-1}(t)) - H_n(F_0^{-1}(s))
\]

Then

\[
\Delta = t-s + f_n \quad \text{where} \quad f_n \to 0 \quad \text{uniformly}
\]

for \(0 < s < t < 1\)

\[
EZ_1^2 = \Delta(1-t+s)^2 + (1-\Delta)(t-s)^2
\]

\[
\leq (t-s) - 2 + f_n
\]

\[
EZ_1^1 \leq 1 \quad \text{since} \quad |Z_1| \leq 1.
\]
\[ E(\bar{Y}_n(t) - \bar{Y}_n(s))^2 = n^{-1} E Z_1^4 + 3(1 - n^{-1}) (EZ_1^2)^2 \]

\[ \leq e_n + 12(t-s)^2 \]

where \( e_n \to 0 \) and does not depend on \( s, t \).

(ii) This follows from (i) in the way that Theorem 6 follows from Theorem a in §3.1 [20] (cf. p. 184 [20]). More specifically, letting

\[ f(j,k,n) = P\left\{ \max_{1 \leq i \leq 2g} |\bar{Y}_n(\frac{i-1}{2g}) - \bar{Y}_n(\frac{i}{2g})| < g^{-2}, k \leq g \leq j \right\}, \]

\[ M_k = 12 \cdot \sum_{g=k}^{\infty} g^8 \cdot 2^{-g} \]

\[ D_{jk}^n = e_n \sum_{g=k}^{j} g^8 \cdot 2^g \quad \text{for} \quad j \geq k, \]

Then (cf (24) p. 178 [20]), \( f(j,k,n) \geq 1 - M_k - D_{jk}^n \). Therefore

\[ \liminf_{n} \liminf_{j} f(j,k,n) \geq \liminf_{n} \liminf_{j} f(j,k,n) \geq 1 - M_k. \]

Let \( \tau_j = \{ i/2^j : i = 0, 1, \ldots, 2^j \} \) and

\[ h(j,k,n) = P\left\{ \max_{t-s \in \tau_j} |\bar{Y}_n(t) - \bar{Y}_n(s)| < 2 \sum_{g=k}^{\infty} g^{-2} \right\}. \]

Then by p. 179 [20],

\[ h(j,k,n) \geq f(j,k,n). \]

Therefore

\[ \liminf_{n} \liminf_{j} h(j,n,k) \geq 1 - M_k. \]

Hence for all \( \epsilon > 0 \), letting
\[ g(n, \delta) = \lim_{n} \lim_{\delta \to 0} \inf \left\{ \max_{t-s < \delta, t, s \in \tau_j} \left| \bar{Y}_n(t) - \bar{Y}_n(s) \right| < \epsilon \right\} \]

This result follows since

\[ g(n, \delta) \leq \max_{t-s < \delta} \left| \bar{Y}_n(t) - \bar{Y}_n(s) \right| < \epsilon \]

(cf. the top of page 178 [20]).

Proof of Theorem 1.

(i) \( \int_{a}^{b} x^m \psi^m \) is a continuous functional, \( m = 1, 3, 5, \ldots \). Using Lemma 1(b) and (D2) of Theorem 3.2, p. 180 of [20] (a generalization of Donsker's Theorem) \( \Rightarrow n^{1/2} \Rightarrow (r+y)_{m} \xrightarrow{L} ||(r+y)_{m}||_{m}, m = 1, 3, 5, \ldots \)

The other results of (i) follows similarly.

(ii) By Lemma 1(a), \( E \int |\bar{Y}_n|^m \psi^m < M_0, n = 1, 2, \ldots \). For \( l \leq m < \infty \),

\[ (E |y+r|^m)^{1/m} = \|u(t-t^2)^{1/2} + r(t)\|_{m, \varphi(u)} \]

\[ \leq \|u(t-t^2)^{1/2}\|_{m, \varphi(u)} + \|r(t)\|_{m, \varphi(u)} \]

\[ = c(t-t^2)^{1/2} + |r(t)| \]

where

\[ c^m = \int |u|^m \varphi(u) \, du < \infty \]
Let

\[ A = \int_0^1 |y_r|^m \psi^m, \quad A_n = \int_0^1 |\tilde{y}_n|^m \psi^m. \]

\[ \therefore \text{For } 1 \leq m < \infty, \]

\[ (EA)^{1/m} \leq c \left( \int (t-t^2)^{1/2} \psi(t) \right)_{\infty} + \| r \cdot \psi \|_m \]

\[ \leq c \left( \int (t-t^2) \psi(t)^{K(m)} \right)^{1/K(m)} + \| r \cdot \psi \|_{K(m)} \]

\[ < \infty \]

by the assumptions and Fatou's Lemma.

For \( 0 < m < 1, \) \( A^{1/m} \leq \int |y_r| \psi \)

\[ \therefore \quad EA \leq (E \int |y_r|^m \psi^m)^{1/m} \quad \text{by Jensen's inequality} \]

\[ < \infty \quad \text{by the argument above.} \]

Hence using Lemma 1(a) for all \( \varepsilon > 0 \) there exist \( u \) and \( v \) in \( (0,1) \)

independent of \( n \) such that

\[ \varepsilon^2 \]

\[ E(\int_0^1 \int_0^1 |\tilde{y}_n|^m \psi^m < \varepsilon^2, \quad E(\int_0^1 \int_0^1 |y_r|^m \psi^m < \varepsilon^2. \]

Let

\[ B_n = \int_u^v |\tilde{y}_n|^m \psi^m, \quad B = \int_u^v |y_r|^m \psi^m. \]

66
By Theorem 2.1, p. 168 [20], $B_n \xrightarrow{p} B$

$\therefore$ for large enough $n$, $|P(B_n \leq x) - P(B \leq x)| < \varepsilon$

$$P(A \leq x + \varepsilon) - P(B \leq x) \leq P(A \leq x + \varepsilon) - P(B \leq x, A \leq x + \varepsilon)$$

$$= P(x < B - A \leq x + \varepsilon) \leq P(x < A \leq x + \varepsilon)$$

$$P(B \leq x) - P(A \leq x + \varepsilon) \leq P(A > x + \varepsilon, B \leq x)$$

$$\leq P(A - B \geq \varepsilon)$$

$$\leq \frac{E(A - B)}{\varepsilon}$$

$$< \varepsilon \quad \text{by (2)}$$

$\therefore \quad |P(B \leq x) - P(A \leq x + \varepsilon)| < \varepsilon + P(x < A \leq x + \varepsilon)$

Similarly

$|P(A_n \leq x + \varepsilon) - P(B_n \leq x)| < \varepsilon + P(x < A_n \leq x + \varepsilon)$

(3), (4) and (5) imply, for $n$ large enough

$$|P(A_n \leq x + \varepsilon) - P(A \leq x + \varepsilon)| < 3\varepsilon + P(x < A \leq x + \varepsilon) + P(x < A_n \leq x + \varepsilon).$$

Hence

$$|P(A_n \leq x) - P(A \leq x)| \to 0 \quad \text{as} \quad n \to \infty,$$
since \( A, A_n \) have continuous distributions.

\[ A_n \xrightarrow{\mathbb{L}} A \]

\[ T_n \cdot m(\psi) \xrightarrow{\mathbb{L}} \| r + y \|_m, \quad 0 < m < \infty. \]  

The rest of (ii) is proved similarly.

Theorem 2 uses the following.

**Lemma 2.** Let \( g \) be defined as in Theorem 2.

(i) w.p. 1 there is a \( u_1 \in (0, 1) \) such that

\[ |y(u)| \leq g(u) \quad \text{for } u \in (0, u_0) \cup (1-u_1, 1). \]

(ii) For \( c < 1 \) w.p. 1, there is a \( u_0 \in (0, 1) \) such that

\[ y(u) > c \cdot g_1(u) \text{ i.o. (infinitely often) in } (u_0, 1) \]

where \( g_1(u) = \left[ 2(u-u^2) \ln \frac{u}{1-u} \right]^{1/2} \) in \( [\frac{e}{1+e}, 1) \).

(6) **Proof (i).** \( W(t) = (1+t) y(\frac{t}{1+t}) \) is the Wiener process (e.g. [3]).

\[ W(t)^2 \leq 2t \ln \ln t^{-1} \quad \text{in } (0, t_0) \quad (e.g., [3]) \]

\[ \therefore \text{w.p. 1 there is a } u_0 \text{ such that} \]

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\[ |y(u)| \leq [2(u-u^2) \ln \ln \frac{1-u}{u}]^{1/2} \quad \text{in } (0, u_0) \]

\[ \leq g(u) \]

(i) follows by symmetry about \( u = \frac{1}{2} \).

(ii) By Problem 19, p. 560 [33] for \( c < 1 \), w.p. 1 there is a finite \( T \) such that \( W(t) > c(2t \ln \ln t)^{1/2} \) i.o. for \( t > T \).

(ii) follows from (6).

Proof of Theorem 2. (i) For \( k \leq m \), \( \| (r+y)\psi \|_k \leq \| r\psi \|_k \leq \| r \psi \|_{m^1} \]

\[ + \| y \psi \|_{m^1} < \infty \] by Lemma 2(i) and the assumptions.

(ii) By Lemma 2(ii) for \( c < 1 \), w.p. 1 there is a \( u_1 \) such that

\[ \sup_{u > u_1} y \psi \geq c \sup_{u > u_1} g_1 \psi, \] since \( \psi \) is bounded in \((0,1)\).

As \( u \to 1 \), \( g_1(u)/g(u) \to 1 \). \[ \therefore \] there is a \( u_2 \) such that

\[ \sup_{u > u_2} g_1 \psi \geq \frac{1}{2} \sup_{u \geq u_2} g \psi \]

\[ \therefore \text{w.p. 1, } \sup_{u > u_2} y \psi \geq \frac{1}{2} \sup_{u \geq u_2} g \psi. \]

\[ \therefore \text{by symmetry of } y(u) \text{ about } u = 1/2, \text{w.p. 1 there is a } u_3 \] such that

\[ \sup_{(0,1-u_3) \cup (u_3,1)} y \psi \geq \frac{1}{2} \sup_{(0,1-u_3) \cup (u_3,1)} g \psi = \infty \]

by assumption. \( \therefore \) w.p. 1 \( \sup y \psi = \infty \). \( \therefore \) w.p. 1 \( \sup |y| \psi = \infty \).
Proof of Theorem 3(a). \( B = \sum b_j f_j, \ (z) \overset{\Phi}{=} \{ \sum Y_j \lambda_j^{-1/2} f_j \} \) imply

\[
W_2 \overset{\Phi}{=} \int [\Sigma f_j (b_j + Y_j \lambda_j^{-1/2})]^2 = \Sigma(b_j + Y_j \lambda_j^{-1/2})^2
\]

since \( \{f_j\} \) are orthonormal. .

\[
E e^{-tW_2} = \prod_{j=1}^{\infty} (1+2t/\lambda_j)^{-1/2} \cdot e^{-S(t)}
\]

where

\[
S(t) = t \sum_{j=1}^{\infty} b_j^2 (1+2t/\lambda_j)^{-1}
\]

\[
= t \sum_{k=0}^{\infty} (-2t)^k \sum_{j=1}^{\infty} b_j^2 \lambda_j^{-k}
\]

\[
= t \sum_{k=0}^{\infty} (-2t)^k \int_0^1 \int_0^1 K_k(x,y) B(x) B(y) \, dx \, dy
\]

(Defining \( K_0(x,y) = \delta(x-y) \), the Dirac delta function)

\[
= t \int B^2 - 2t^2 \int K(x,y : -2t) \, dx \, dy
\]

Theorem 3(b) follows from the definition of \( W_2 \).

Proof of Example 1(a). It is shown in [3] that

\[
\prod_{j=1}^{\infty} (1+2t/\lambda_j)^{-1/2} = \left( \frac{z}{\sinh z} \right)^{1/2}
\]

when \( \psi = 1 \).

Let

\[
H(x) = \begin{cases} 1, & x \leq a \\ 0, & x > a \end{cases}, \quad R(x) = \begin{cases} 1, & x \leq b \\ 0, & x > b \end{cases} \quad \text{where} \quad 0 < b < a < 1.
\]
Then \( H(x) = \sum_{j=1}^{\infty} h_j f_j(x) \) where \( h_j = \int f_j^2 \Pi \left( e^{1/2}/j\pi \right) [1 - \cos(j\pi x)] \) and
\[
R(x) = \sum_{j=1}^{\infty} r_j f_j(x) \quad \text{where} \quad r_j = \int f_j R \left( e^{1/2}/j\pi \right) [1 - \cos(j\pi b)]. \quad \text{For} \quad k = 1, 2, \ldots
\]
\[
a \int_{0}^{b} du \int_{0}^{b} dv \ K_k(u,v) = \int_{0}^{b} H(u) R(v) K_k(u,v) \ du \ dv
\]
\[
= \sum_{j=1}^{\infty} h_j r_j \lambda_j^{-k}
\]
\[
= (-4)^k \cdot \left[ \varphi_{2k+2}(0) - \varphi_{2k+2}(\frac{a}{2}) - 2 \varphi_{2k+2}(\frac{b}{2}) \right. \\
\left. + \frac{1}{2} \varphi_{2k+2}(\frac{a+b}{2}) + \frac{1}{2} \varphi_{2k+2}(\frac{a-b}{2}) \right]
\]
since if \( x \geq 0 \)

(7) \[
2 \sum_{j=1}^{\infty} \cos(2j\pi x) \cdot (2\pi j)^{-2k+2} = (-1)^k \varphi_{2k+2}(x)
\]

where \( \varphi_n(x) \) is the \( n \)th Bernoulli polynomial (e.g., [28]).

Differentiating both sides and noting that \( \varphi_n(x) = \varphi_{n-1}(x) \) yields

\[
K_k(a,b) = (-4)^k \frac{1}{2} \left[ \varphi_{2k}(\frac{a+b}{2}) - \varphi_{2k}(\frac{a-b}{2}) \right], \quad k = 1, 2, \ldots
\]

Now
\[
\sum_{n=0}^{\infty} \varphi_n(x) t^n = \frac{t}{e^t - 1} e^{tx} \quad \text{and} \quad (7) \Rightarrow \varphi_{2k}(1-x) = \varphi_{2k}(x).
\]

Hence

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\[ 2 \sum_{k=0}^{\infty} t^{2k} \varphi_{2k}(x) = \frac{t}{e^{t} - 1} (e^{tx} + e^{-tx}) . \]

Hence

\[ \sum_{k=1}^{\infty} (-2t)^k K_k(a, b) = \sum_{k=0}^{\infty} (2z)^{2k} \frac{1}{2} [\varphi_{2k}\left(\frac{a+b}{2}\right) - \varphi_{2k}\left(\frac{a-b}{2}\right)] \]

\[ = \frac{z}{\sinh z} \sinh(zb) \cdot \sinh(za-z) . \]

**Proof of Example 1(b).** For this example,

\[ r(x) = \begin{cases} -e^x, & \text{in } [0, \frac{1}{2}] \\ -e^{-(1-x)}, & \text{in } [\frac{1}{2}, 1] \end{cases} , \]

and \( S(t) \) can be shown by Theorems 3(a) and Example 1(a) to equal

\[ e^{2\left[\frac{1}{2} \frac{1}{z} + \frac{1-e^{-2z}}{1+e^{-2z}}\right]} \]

where \( z = (2t)^{1/2} \).

Hence

\[ E \exp[-z^{2}w_{2}/2] = z^{1/2}(\sinh z)^{-1/2} \exp[-\theta^{2}\left(\frac{1}{2} - \frac{\tanh(z/2)}{z}\right)] \]

\[ = 2^{1/2} \exp[-\theta^{2}/2] \sum_{n=0}^{\infty} \frac{\theta^{2n}}{n!} z^{1/2-n} \sum_{k=0}^{\infty} a_{kn} \exp[-(k + \frac{1}{2})z] \]

where \( a_{kn}, A_{k}, K_{a}(c) \) are defined in the theorem.

Using the notation

\[ \text{LG}(x) = \int_{0}^{\infty} G(x) e^{-xt} \, dx , \]

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\[
L^{-1} \int_0^\infty a(x) e^{-xt} \, dx = a(x)
\]

\[
a(x) \otimes b(x) = \int_0^x a(y) \cdot b(x-y) \, dy
\]

we have

\[
P(W_2 \leq x) = L^{-1} \cdot t^{-1} \cdot E \cdot e^{-tW_2}
\]

\[
= 2^{3/2} \cdot e^{-\theta^2/2} \sum_{n=0}^\infty \sum_{k=0}^\infty a_{kn} \cdot L^{-1} \left( t^{-n/2-3/4} \cdot e^{-Axt^{1/2}} \right).
\]

But

\[
L^{-1} e^{-\frac{At^{1/2}}{2}} \cdot t^{-b} = \frac{1}{2} A^{-1/2} x^{-3/2} e^{-\frac{A^2}{4x} \cdot x^{b-1}} \frac{1}{\Gamma(b)}
\]

\[
= \frac{A^{b-3/2}}{\Gamma(b)} \cdot \pi^{-1/2} \int_0^\infty \exp\left[-2c \cosh^2 \theta \right] \cdot \sinh^{2b-1} \theta
\]

\[
\cdot \cosh^{2-2b} \theta \, d\theta,
\]

(setting \( c = \frac{A^2}{8x} \))

\[
= \frac{A^{2-3/2}}{\Gamma(b)} \cdot x^{b-3/2} \cdot \pi^{-1/2} \left( -\frac{1}{2} \frac{d}{dc} \right)^{3/2-2b}
\]

\[
\cdot \left[ e^{-c} \int_0^\infty e^{-c(1+\cosh \psi)} \left( \frac{1}{2} \sinh \psi \right)^{2b-1} \, d\psi \right]
\]

(if \( \frac{3}{2} - 2b \) is an integer, setting \( \psi = 2\theta \))

\[
= A^{2-3/2} \pi^{-1} x^{b-3/2} \left( -\frac{d}{dc} \right)^{3/2-2b} \left[ e^{-c\left(\frac{c}{2}\right)^v} K_v(c) \right]
\]

if \( b > 0 \) where \( v = b - 1/2 \), since by p. 82 [9], if \( v > -1/2 \)

\[
\Gamma\left(\frac{1}{2} + v\right) K_v(c) = \pi^{1/2} \left(\frac{c}{2}\right)^v \int_0^\infty e^{-c \cosh \psi \left(\sinh \psi\right)^2} \, d\psi.
\]
Hence \( P(W_2 > x_\alpha) \) is as given in the theorem where

\[
\begin{align*}
G_n(A, x) &= L^{-1} \cdot t^{-n/2-3/4} \cdot e^{-At^{1/2}} \\
&= A^{2-3/2} \cdot \pi^{-\frac{1}{2}} \cdot x^{n/2-3/4} \cdot \left( -\frac{d}{dc} \right)^{-n} \left[ e^{-c(c/2)^{-v}} K_v(c) \right]
\end{align*}
\]

at \( c = A^2/8x, v = \frac{n}{2} + \frac{1}{4} \). By p. 82 [9],

\[
\begin{align*}
\left( \frac{d}{dc} \right)^{-n} \left[ e^{-c(c/2)^{-v}} K_v(c) \right] \\
&= \frac{\left( \frac{\pi}{2} \right)^{1/2}}{\Gamma(v + \frac{1}{2})} \cdot 2^v \cdot \frac{d}{dc} \cdot \int_0^\infty e^{-c(t+2)} \cdot t^{v-1/2}(1 + \frac{t}{2})^{v-1/2} \cdot dt \\
&= \frac{\left( \frac{\pi}{2} \right)^{1/2}}{\Gamma(v + \frac{1}{2})} \cdot 2^v \cdot e^{-2c} \cdot \int_0^\infty e^{-ct} \cdot t^{v-1/2-n}(1 + \frac{t}{2})^{v-1/2-n} \cdot dt \\
&= e^{-2c} \cdot 2^v \cdot \frac{\Gamma(v-n + \frac{1}{2})}{\Gamma(v + \frac{1}{2})} \cdot \left( \frac{d}{dc} \right)^n \left[ e^c P_{v-n}(c) \right]
\end{align*}
\]

where \( P_v(c) = c^{-v} K_v(c) \).

\[
\Rightarrow \quad \frac{d}{dc} P_v(c) = -c P_{v+1}(c)
\]

\[
\Rightarrow \quad \frac{d}{dc} P_v(c) = \sum_{k=0}^{[i/2]} b_{k,i}(-1)^{i+k} e^{-2k} P_{v+i-k}(c)
\]

where \( \{ b_{k,i+1}, k = 0, 1, \ldots, \lfloor (i+1)/2 \rfloor \} \) are given in terms of \( \{ b_{k,i}, k=0,1,\ldots,\lfloor i/2 \rfloor \} \) by

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\[ b_{k,i+1} = b_{k-1,i} \cdot (i-2k+2) + b_{k,i} \]

where \( b_{0,1} = 1, b_{1,1} = 0 \).

Hence

\[ b_{k,i} = \frac{i!}{(i-2k)! \cdot 2^k \cdot k!} \]

Using these results, one can show that \( G_n(A,x) \) is as given just below 2.7.

Note. \( \theta = 0 \) yields (4.34) of [3].

Theorem 4 is proved similarly -- but more easily, and uses the fact that \( K_v(c) = K_{-v}(c) \).

In the computations we used the result on p. 5 of [9] that

\[ K_v(c) = \frac{\pi}{2 \sin \nu \pi} (I_{-v} - I_v) \]

where

\[ I_v = \sum_{m=0}^{\infty} \frac{(c^2)^{2m+v}/m! \cdot \Gamma(m+v+1)}{m!} \]

For \( c > 2 \) we used the modified asymptotic expansion of \( K_v(c) \) given by Airy on p. 544 of the Philosophical Magazine, 24 (1937).

Theorem 5 is a fairly straightforward generalization of VI.4.5, p. 230 and VII.2.3, p. 272 of [20].

For Theorems 6 and 7 and Example 3 we need the following,

Lemma 3. Let \( Q \) and \( P \) denote the probability distributions corresponding to the processes \( z_\theta \) and \( z \) for general \( \psi \). Let

\[ 75 \]
\[ X = - \int_0^1 \frac{1}{\varphi(u)} y(u) \, du. \]

Then

(i) \[ \mathbb{E}z(t)X = b(t) \]

(ii) \[ \frac{d\varphi}{d\mathcal{P}} = \exp[\theta x - \theta^2/2 \int_0^1 \varphi^2] . \]

**Proof.** (i) is straightforward. (ii) follows from (i) and Problem 11, p. 240 [20].

**Proof of Theorem 6.** By problem 16, p. 199 [20],

\[ P(-\lambda \leq y(t) \leq \lambda \text{ for } 0 < t < 0 | y(0) = y) = K(\nu y \lambda), \]

defined in the theorem.

Since \( \{y(t), t \in (0, v)\} \) and \( \{y(t), t \in (v, 1)\} \) are conditionally independent given \( y(v) = y \),

\[ \int_A y(t) \, d\mathcal{P} = -\int_A y(t) \, d\mathcal{P} = -\int_{-\lambda}^{\lambda} y f_t(y, \lambda) \, dy = 0 \]

where

\[ A = \{f(\cdot) : \sup_{(0, 1)} |f(t)| > \lambda\} \]

and

\[ f_v(y, \lambda) = \varphi(x(v-v^2)^{-1/2}) (v-v^2)^{-1/2} K(\nu y \lambda) K(1-\nu 0 y \lambda) = f_v(-y, \lambda) \]
\[ P(\sup |y + \theta p| \geq \lambda) = Q(A) = \int_A \frac{d\varrho}{dP} \, dP, \]

which by Lemma 3 has coefficient of $\theta$ equal to $\int_A x \, dP = 0$ and

coefficient of $\theta^2/2$ equal to

\[ \int_A (x^2 - \int \bar{p}^2) \, dP. \]

Now

\[ \int_A x^2 \, dP = \int_A \bar{p}(u) \, du \int_A \bar{p}(v) \, dv \int_A y(u) \, y(v) \, dP \]

\[ \therefore \min(u,v) - uv - \int_A y(u) \, y(v) \, dP = \int_A y(u) \, y(v) \, dP \]

\[ = \int_{-\lambda}^{\lambda} x dx \int_{-\lambda}^{\lambda} z dz \, f_{uv}(xz\lambda) \cdot (\text{joint density of } x = y(u) \text{ and } z = y(v)), \]

where

\[ f_{uv}(xz\lambda) = P(-\lambda \leq y(t) \leq \lambda \text{ in } (0,1) | y(u) = x, y(v) = z) \]

which is as given in Theorem 6 using conditional independence and (7) p. 199 [20] with $\alpha = \beta = \lambda$, which implies that $K(v-u, x, z, \lambda)$

\[ = P(-\lambda \leq y(t) \leq \lambda \text{ in } (u,v) | y(u) = x, y(v) = z). \]

The theorem follows using $P(A) = \alpha$ and

\[ \int \int \bar{p}(u) \bar{p}(v) (u \wedge v - uv) \, du \, dv = \int \bar{p}^2. \]
Proof of Theorem 7. Let $A = \{ f(\cdot) : \sup f + \sup -f \geq \lambda \}$,

$$(\sup y + \sup -y)|y(t) = x = (\sup y + \sup -y)|y(t) = -x$$

$$.\Rightarrow .\frac{\partial}{\partial \theta} Q(A)_{|\theta=0} = \int_A y(t) \, dP - \int_A y(t) \, dP = 0 \ , \quad A^c$$

$$.\Rightarrow .\frac{\partial^2}{\partial \theta^2} Q(A)_{|\theta=0} = \int_{A^c} x^2 \, dP - \int_{A^c} \frac{\partial^2}{\partial \theta^2} \cdot P(A)$$

$$\int_A y(u) y(v) \, dP = \int_{A^c} x d\bar{x} \int_{A^c} zdz f_{uv}(xz\lambda) \cdot$$

(joint density of $x = y(u)$ and $z = y(v)$) where

$$f_{uv}(xy\lambda) = \begin{cases} f_{vu}(yx\lambda), & v < u \\ P[\sup y + \sup -y \leq \lambda | y(u) = x, y(v) = y], & v > u \end{cases}$$

Let

$$D_1^+ = \sup y, \quad D_2^+ = \sup y, \quad D_3^+ = \sup y \quad (0u) \quad (uv) \quad (vl)$$

$$D_1^- = \sup -y, \quad D_2^- = \sup -y, \quad D_3^- = \sup -y \quad (0u) \quad (uv) \quad (vl)$$

$$.\Rightarrow .\text{for } v > u,$

$$f_{uv}(xz\lambda) = P(D_1^+ + D_j^- \leq \lambda, i, j = 1, 2, 3 | y(u) = x, y(v) = z)$$

$$= \int_{x_1^{+2} \leq \lambda} \prod_{i=1}^3 \frac{\partial^2}{\partial x_i \partial z_i} \ P(D_1^+ \leq x_1, D_i^- \leq z_1 | y(u) = x, y(v) = z) \ dx_i dz_i$$
by conditional independence.

\[
P(D_2^+ \leq A, D_2^- \leq B | y(u) = x, y(v) = z) = \mathbb{P}(\sum_{m=1}^{\infty} \frac{A}{v-u} \exp[-2 \frac{A}{v-u}] + \frac{B}{v-u} \exp[-2 \frac{B}{v-u}] - \exp[-2 \frac{C}{v-u}] - \exp[-2 \frac{D}{v-u}])
\]

by (6), p. 199 [20], since

\[
(8) \quad T(s) = (v-u)^{-1/2} [y(u + (v-u)s) - (1-s)x - sz] | y(u) = x, y(v) = z
\]

is a Brownian Bridge on \([0,1]\).

\[
\frac{\partial^2}{\partial a \partial b} P(D_2^+ \leq A, D_2^- \leq B | y(u) = x, y(v) = z) = H(u-v, x, z, a, b)
\]

given in the theorem. Hence \(f_{uv}(x, z, \lambda)\) is as given in the theorem.

The theorem follows.

**Proof of Example 3.**

\[
p(x) = \begin{cases} 
1-x, & x \in [0, \frac{1}{2}] \\
1+x, & x \in [\frac{1}{2}, 1]
\end{cases}
\]

(1) \(1-P(\sup|y+\theta_p| > \lambda) = P(-\lambda < y+\theta_p < \lambda \text{ in } (0,1])
\]

\[
= \int P(-\lambda+\theta x < y(x) < \lambda+\theta x \text{ in } (0, \frac{1}{2}) | y(\frac{1}{2}) = r) 
\cdot P(-\lambda+\theta(1-x) < y(x) < \lambda+\theta(1-x) \text{ in } (\frac{1}{2}, 1) | y(\frac{1}{2}) = r) \, d\phi(2r)
\]

\[
= \int H(2^{1/2} \lambda, 2^{1/2}(\lambda+\theta - \frac{\theta}{2}), 2^{1/2} \lambda, 2^{1/2}(\lambda-\theta + \frac{\theta}{2}))^2 \, d\phi(2r)
\]

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by symmetry and by (8) where 

\[ H(abx) = P(-a(l-t) - bt < y(t) < a(l-t) + bt \text{ in } (0,1) \]

\[ = \begin{cases} 
1 - \sum_{m=1}^{\infty} (e^{-2A_m} e^{-2B_m} - e^{-2C_m} e^{-2D_m}), & \text{when } a, b, \alpha, \beta \text{ are non-negative} \\
0, & \text{otherwise}
\end{cases} \]

where \( A_m, B_m, C_m \) and \( D_m \) are given in problem 15, p. 199 [20]. (i) follows on simplification, since \( H(x,d) = H(d, d+x, d, d-x) \).

(ii) follows similarly since if 

\[ G(a,b) = P(\sup(y+\theta p) \leq a, \sup -(y+\theta p) \leq b) \]

then the asymptotic power is 

\[ 1 - \int_{a+b}^{\lambda} \frac{\partial^2}{\partial a \partial b} G(a,b) = 1 - \int_0^{\lambda} G_2(\lambda-b_0, b_0) \, db_0 \]

where \( G_2(a,b) = \frac{\partial}{\partial b} G(a,b) \) so that 

\[ G_2(\lambda-b_0, b_0) = 2^{3/2} \int (\frac{\partial H}{\partial a} + \frac{\partial H}{\partial b}) \, H d\phi(x) \]

where \( H = H(abx) \), evaluated at \( a = 2^{1/2} b_0, b = 2^{1/2} (b_0 - \theta/2) + x, \) \( \alpha = 2^{1/2} (\lambda-b_0), \beta = 2^{1/2} (\lambda-b_0 + \theta/2) - x, \) since by conditioning on \( y(1/2) = x/\sqrt{2} \), \( G(a,b) \) is found to be 

\[ \int H(2^{1/2} b, 2^{1/2} b+x-\theta/2^{1/2}, 2^{1/2} a, 2^{1/2} a-x+\theta/2^{1/2})^2 \, d\phi(2^{1/2} \cdot x). \]

The range of integration follows since \( H = 0 \) if \( a, b, \alpha \) or \( \beta \) are negative.
10.2. Proofs of §5.

For Theorem 1 we need the following.

**Lemma 4.** For

(i) $T^{+}_{n,m}(\psi)$, $\psi$ bounded

(ii) $T_{n,m}(\psi)$, $m = 2, 4, 6, \ldots$, $\psi$ bounded

(iii) $T_{n,m}(\psi)$, $0 < m < \infty$, $\psi = 1$

$$I_0(r, F_0) = \int_0^1 \hat{H} \land \hat{H} \land \text{where} \ H \land \text{is a cdf on } [0,1] \text{ such that}$$

$$\frac{\partial}{\partial r} r^m \hat{H} \land \lambda (H-x)^{m-1} \psi^m = 0$$

$$\int_0^1 (H-x)^m \psi^m \, dx = r^m$$

and for (iii), $H(x) \geq x$.

**Proof.** For (i) and (ii),

$$I_0(r, F_0) = \inf I(Q, F_0) \text{ taken over cdfs } Q \text{ such that}$$

$$\| (Q, F_0) - (Q, F_0) \|_{F_0, m} \geq r$$

$$= \inf \int_0^1 \hat{H} \land \hat{H} \land$$

where the inf is taken over cdfs on $[0,1]$ $H$ such that

$$\int_0^1 (H-x)^m \psi^m = r^m$$

since the closer $H(x)$ is to $x$, the smaller is $\int \hat{H} \land \hat{H}$ if $H$ exists. Let $V = V(H, \hat{H}, x) = \hat{H} \land \hat{H} - \lambda (H-x)^m \psi^m$, $\lambda$ a constant. By the method of Lagrange multipliers we seek a function $H_\perp$ on $[0,1]$ giving an extremal of $\int_0^1 V \, dx$ subject to $H(0) = 0$, $H(1) = 1$. By the calculus of variation (e.g. [15]), $H_\perp$ satisfies Euler's equation:
\[ \frac{\partial V}{\partial H} = \frac{d}{dx} \frac{\partial V}{\partial \dot{H}}. \]

For (iii) we wish to find an extremal of \( L(H) = \int_0^x V \, dx \) where \( V = \dot{H} \ln \dot{H} - \lambda \cdot |H-x|^m \). Suppose \( H_2 \) gives an extremal. Then \( H_2 \) is a.c. For all intervals such that \( H_2(x) - x < 0 \) in \( (a,b) \) and \( H_2(x) - x = 0 \) at \( a, b \) let \( H_1(x) = a+b - H_2(a+b-x) \) in \( (a,b) \) and let \( H_1(x) = H_2(x) \) when \( H_2(x) \geq x \). Then

\[
\frac{1}{H_2} \ln H_2 = \frac{1}{H_1} \ln H_1 \quad \text{and} \quad \int_0^1 |H_2-x|^m = \int_0^1 |H_1-x|^m.
\]

A completely rigorous proof would require showing that the solution of Euler's equation is the minimising cdf. We have not attempted this.

**Proof of Theorem 1.** By Lemma 4, \( I_0(r, F_0) = \int \dot{H}_1 \ln \dot{H}_1 \) where \( \dot{H}_1 = x+J, J \geq 0, J(0) = J(1) = 0, J/(J+1) + \lambda m J^{m-1} = 0, \int_0^1 J^m = r^m \). Let \( t = J+1 \),

\[
\therefore \quad t - \ln t = \mu - \lambda J^m, \quad \mu \text{ a constant}
\]

\[
\therefore \quad x = \int \frac{dJ}{t-1} = -\frac{1}{m \lambda} \left[ \frac{d \ln t}{\left[ \frac{\mu - t + \lambda nt}{\lambda} \right]^{1-1/m}} \right] \text{, since } \lambda > 0
\]

\( \lambda \leq 0 \) leads to a contradiction. Let \( T = \ln t \),

\[ g(x) = \int_y^x (\mu+y - e^y)^{1/m-1} \, dy \]

\( \gamma \) a constant such that \(-m\lambda^{1/m} x = g(T)\). Let \( G(x) = g^{-1}(x) \)

\[ \therefore \quad T = G(-m\lambda^{1/m} x) \]
\[ G(0) = \gamma, \ j(0) = 0 \quad \Rightarrow \quad \mu = e^\gamma - \gamma. \]

Let \( \epsilon = G(-m^{1/m}) \), \( j(0) = 0 \quad \Rightarrow \quad \mu = e^\epsilon - \epsilon \), \(-m^{1/m} = g(\epsilon)\). \quad \therefore \epsilon < \gamma.

The theorem follows.

**Proof of Theorem 2.** Hoadley's Theorem 2 can be used to show that Lemma 4 extends to \( T_{n-2}(\psi_0) \).

\[ I_0(r, F_0) = \int_0^1 \hat{H} \ln \hat{H} \]

where

\[ (x-x^2) \hat{H} + 2\lambda(\hat{HH} - \hat{H}x) = 0 \]

\[ H(0) = 0 \quad \Rightarrow (x-x^2) \hat{H} + \lambda H^2 - H + 2(1-\lambda) \int_0^x \hat{H} \quad = 0 \]

\[ H(1) = 0 \quad \Rightarrow \lambda = 1 \quad \therefore \int_0^1 x \hat{H} \neq \frac{1}{2} \]

\[ \therefore \quad H - x = \frac{d(x-x^2)}{1+d \hat{x}} \quad \text{where} \quad d+1 \quad \text{is a positive constant} \]

\[ I_0(r,F_0) = \int \hat{H} \ln \hat{H} = \ln(1+d) \left( 1 + \frac{2}{d} \right) + \frac{2}{d} - \omega(1 + \frac{1}{d}) \]

\[ = \int \frac{(H-x)^2}{x-x^2} = r^2 \]
Proof of Theorem 3. When \( \psi = 1 \) this is well-known -- see Theorem 1 of [42], Theorems 5.1, 5.2 of [21]. Under the different condition that  
\[ \int \exp \{ s \sup_{(y,1)} |\psi(t)| \} \, dy < \infty \text{ for all } s, \] 
the theorem follows from Theorem 1 [43]. This version follows from Theorems 1, 2 of [1] when corrected: for the continuity of \( \rho_\psi(x), (x-x^2) \psi(x) \to 0 \) as \( x \to 0 \), 1 is not strong enough; the fifth line from the bottom of p. 1481 of [1] is incorrect as RHS depends on \( n \).

Proof of Theorem 4.

(a) \( I(\omega_1, F_0) \geq I_0(\mu_1, F_0) \) for \( D_n(\psi_A) \) by §3

\[ = I(\omega_2, F_0) \text{ by Hoadley's Theorem 1} \]

where

\[ \omega_2 = \{ \text{cdfs } F : \sup |F - F_0| \psi_A(F_0) \geq a_1 \} \]

\[ I(\omega_2, F_0) = I(F_2, F_0) \text{ where } F_2 \in \omega_1 \]

\[ \geq I(\omega_1, F_0). \]

(b) \( \mu_1(\omega_1) = \min(\mu_1(\omega_1) \text{ for } D_n^-(\psi), \mu_1(\omega_1) \text{ for } D_n^+(\psi)) \)

But

\[ \mu_1(\omega_1) \text{ for } D_n^+(\psi) = \mu_1(\bar{\omega}_1) \text{ for } D_n^-(\bar{\psi}) \]

where \( \bar{\omega}_1 \) is \( \omega_1 \) with \( \psi_A(x) \) replaced by \( \psi_A(1-x) \) and where \( \bar{\psi}(x) = \psi(1-x) \), and \( \mu_1(\omega_1) \) for \( D_n^-(\psi) = \mu_1(\omega_1^-) \) for \( D_n^-(\psi) \) where \( \omega_1^- \) is defined in §9.
<table>
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<tr>
<th>$r$</th>
<th>$I_0(r,F_0)$</th>
<th>$r$</th>
<th>$I_0(r,F_0)$</th>
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TABLE 7(a)

$I_0(r,F_0)$ for $T_{n,2}(1)$
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<th>$I_0(r,F_0)$</th>
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<th>$I_0(r,F_0)$</th>
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**TABLE 7(c)**

Table of $I_0(r,F_0)$ for $D_n(\psi)$

when $\psi(x) = \begin{cases} \psi_0(x), & 0.005 < x < 0.995 \\ \psi_0(x), & \text{otherwise} \end{cases}$

<table>
<thead>
<tr>
<th>$r$</th>
<th>$I_0(r,F_0)$</th>
<th>$r$</th>
<th>$I_0(r,F_0)$</th>
<th>$r$</th>
<th>$I_0(r,F_0)$</th>
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### TABLE 7(d)

Table of $I_0(r, F_0)$ for $T_{n+1}(1), T_{n+1}^+(1)$

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<th>$I_0(r, F_0)$</th>
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The rest follows from the formula for $\mu_1(\omega_1)$ given in Theorem 1(b), §9. Note that the conditions on $\psi, \psi_A$ can be weakened.

[1] easily generalizes to give

Lemma 5. Let $a(x,r)$ be defined by Theorem 3, §5.

(i) Let $\psi$ be non-negative piecewise continuous and bounded in $[0,1]$. Let $(1-x)\psi(x) \to 0$ as $x \to 1$. Suppose $F/F_0$ bounded and $G^{-1}(x) = \sup\{y : G(y) = x\}$ and (I) $f = F_0(F^{-1})$ continuous and $f(0) = 0, f(1) = 1$ or (II) $g = F(F_0^{-1})$ continuous and $g(0) = 0, g(1) = 1$. Then for $D_0^+(\psi), I_0^+(r,F)$ is continuous provided $D_0^+(F) < r < \sup (1-x)\psi(x)$ and equals \[ \inf \{a(F, r/\psi(F_0) + F_0 - F)\} \]

(ii) Let $\psi$ be non-negative, piecewise-continuous and bounded in $[0,1]$. Let $x\psi(x) \to 0$ as $x \to 0$. Suppose $(1-F)/(1-F_0)$ is bounded and (I) or (II). Then for $D_0^-\psi, I_0^-(r,F)$ is continuous provided $D_0^-(F) < r < \sup x\psi(x)$ and equals \[ \inf \{a(1-F, r/\psi(F_0) - F_0 + F)\} \]

(ii) Under the assumptions of (i) and (ii), for $D_0^-(\psi), I_0^-(r,F)$ is continuous for $D_0^-(F) < r < \max(\sup x\psi(x), \sup (1-x)\psi(x))$ and equals the minimum of $I_0^-(r,F)$ for $\psi_1^+(\psi)$ and $D_0^-(\psi)$.

(iv) For $V_0^-(1), V_0^-(F) < r < 1, I_0^-(r,F)$ is continuous and equals

\[ \inf \min \{a(F(x) - F(y), r - F(x) + F(y) + F_0(x) - F_0(y)), x > y \} \]

\[ a(1-F(x) + F(y), r + F(x) - F(y) - F_0(x) + F_0(y)) \]
(v) Under the conditions of (iii), for \( V_n(\psi) \), \( I_0(r,F) \) is continuous for \( V_F(F) < r < \sup_{G \in \mathcal{G}} V_F(G) \), and equals \(-\ln \max[\rho_V(r), \rho_V(-r)]\) where \( \rho_V(r) = \sup_{x > y} G(T(x,y,r), x, y, r) \).

\[
G(t,x,y,r) = \exp[-t(r+\psi(F_0(x)) - \psi(F_0(y)))] \cdot \varphi(t,x,y)
\]

\( T(x,y,r) \) is the root of \( r = \frac{\partial}{\partial r} \ln \varphi(t,x,y) \) when \( x > y \) and when the root exists and \( \varphi(t,x,y) = E e^{tZ} \), \( x > y \) where

\[
Z = \begin{cases} 
\psi(F_0(x)) - \psi(F_0(y)) & \text{w.p. } F(y) \\
\psi(F_0(x)) & \text{w.p. } F(x) - F(y) \\
0 & \text{w.p. } 1 - F(x)
\end{cases}
\]
TABLE 8
Details of $e_8$, §5:

When explicit the formulae for $\mu_1$ are given.

(a) $F_{\theta}(x) = \Phi(x-\theta)$; $I(F_{\theta}, F_0) = \theta^2/2$

\[
T_{n\cdot1}(l): \mu_1 = \Phi(\theta/\sqrt{2}) - \frac{1}{2} \quad D_n(\psi_1): \text{let } a = \Phi^{-1}(0.995) = 2.576; \quad \text{for } 2 \frac{3}{4} < \theta \leq 2a \text{ the sup occurs at } \theta = a.
\]

Table of $\mu_1$

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(b) $F_\theta(x) = (1 + e^{-x+\theta})^{-1}$: 

$$I(F_\theta, F_0) = a - 2 + e^\theta \int_0^\infty \frac{\ln(1+y) dy}{(1+ye^\theta)^2}$$

$$= \frac{e^\theta}{1-e^{-\theta}} - 2.$$ 

Let $c = 1 - e^{-\theta}.$

$T_{n,1}(1): \mu_1 = \frac{1}{2} + e^{-\theta}(c^{-2} \ln(1-c) + c^{-1})$

$T_{n,2}(1): \mu_1 = \frac{1}{2} + e^{-2\theta}(c^{-2} - 2c^{-3}) + e^{-\theta}(3c^{-2} - 2c^{-3} + c^{-1})$

$D_n(1): \mu_1 = 2(1-e^{-\theta/2})^{-1} - 1$

$T_{n,2}(\psi_0): \mu_1 = \theta(1 + e^{-\theta})/(1-e^{-\theta}) - 2$

$D_n(\psi_0):$ The sup occurs at $\left\{ \begin{array}{ll} \theta, & 0 \leq \theta \leq a \text{ where } a = \ln 199 = 5.293 \\ \theta/2, & \theta > 2a \end{array} \right.$

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(c) $F_\theta(x) = F_0(x-\theta)$, $F_0(x) = \frac{1}{2} e^{-|x|}$; $I(F_\theta, F_0) = e^{-\theta} + \theta - 1$

$T_{n,1}(1): \mu_1 = \frac{1}{2} - \frac{1}{4} e^{-\theta}(2+\theta)$

$T_{n,2}(1): \mu_1 = \frac{2}{3} - \frac{\theta e^{-\theta}}{2} - e^{-\theta} / 6 - e^{-2\theta} / 6$

$D_n(1), V_n(1): \mu_1 = 1 - e^{-\theta/2}$, since the sup is at $x = \theta/2$

$T_{n,2}(\psi_0): \mu_1^2 = (1-e^{-\theta}) (\ln 2 - \frac{1}{2}) + e^{-2\theta} [\ln(2e\theta - 1) + 2e^\theta + 2e^{-2\theta} - 4] / 16$

$\quad - 3(1-e^{-\theta}) / 2 + \theta - (e^{-1})^2 [\ln(1 - \frac{1}{2} e^{-\theta}) + \frac{1}{2} e^{-\theta}]$

$D_n(\psi_1): \text{let } a = \ln 100 = 4.605 \text{ The sup occurs at } \left\{ \theta, \frac{\theta}{2}, \theta > 2a \right\}$

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(a) $F_\theta = F_0^{\theta+1} : I(F_\theta, F_0) = \ln(\theta+1) - \frac{\theta}{\theta+1}$

$T_{n.1}(1): \mu_1 = \frac{1}{2} - \frac{1}{2+\theta}$

$T_{n.2}(1): \mu_1 = \frac{2^2}{3(3+2\theta)} (3+\theta)$

$D_n(1), V_n(1): \mu_1 = \theta/(\theta+1)(1+\theta^{-1})$

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(e) \( F_\theta = \frac{e^{F_0\theta}}{e^{\theta}} : I(F_\theta, F_0) = ln(\frac{\theta}{e^{\theta}}) + \theta e^{\theta}/(e^{\theta} - 1) - 1 \)

\[
\begin{align*}
T_{n,1}(1): & \quad \mu_1 = \frac{1}{2} - \frac{1}{\theta} + 1/(e^{\theta} - 1) \\
T_{n,2}(1): & \quad \mu_2 = -\frac{3}{2\theta}(e^{\theta} + 1)/(e^{\theta} - 1) + (2 + \theta^2)/\theta^2(e^{\theta} - 1) + (e^{\theta} + 1)/3 \\
D_n(1), V_n(1): & \quad \mu_1 = \frac{1}{\theta} ln((e^{\theta} - 1)/\theta) - \frac{1}{\theta} + 1/(e^{\theta} - 1)
\end{align*}
\]

**Table of \( \mu_1 \)**

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As \( \theta \to 0, I(F_\theta, F_0) \approx \theta^2/24 \cdot (1 + \theta^4/20) \)

and for \( T_{n,2}(\psi_0) \), \( \mu_2 \approx \theta^2/24 \cdot (1 - \theta^2/36) \).

Proof of Lemma. (i) Choose $F \in \omega_1$ of the form $F_0 + a_1 \cdot c \cdot f(F_0)$.

(ii) This follows from the case $\psi = \psi_A = 1$ by choosing

$$
F \in \omega_1 \text{ of the form } \begin{cases} 
0 & \text{in } [0, c) \\
F_0 & \text{in } [c, 1] 
\end{cases}
$$

(iii) If $\hat{H}$ exists and $H(x) - x \geq 0$ in $A$ and $H(x) - x \leq 0$ in $B$, then if $y \in B$, there is a $y_0$ in $B$ such that $[y_0, y] \subset B$ and $H(y_0) = y_0$, so that

$$
\int_B |H - x|^m \geq \int_B (x - H)^m \geq \int_B (x - H)^m \, d(x - H) = \frac{(y - H(y))^m}{m+1}
$$

$$
\therefore \quad \int_B |H - x|^m \geq \sup_B |H(y) - y|^m \, / (m+1)
$$

Similarly for $y \in A$. \quad \therefore \quad m_1 \geq a_{m+1}/(m+1).

(iv) $\inf \psi/\psi_A \cdot a_1 = \inf \psi/\psi_A \cdot \inf_{F: \max(\lambda^+, \lambda^-) \geq a_1} (\lambda^+ + \lambda^-) \leq \mu_1$

where $\lambda^+ = D^+(F_0)$ at $\psi = \psi_A$, $\lambda^- = D^-(F_0)$ at $\psi = \psi_A$.

Proof of Theorem. (This was given in [30] for $k, m = 2, \infty, \psi = \psi_A = 1$.)

With $\beta_n$ bounded away from 0,

$$
\alpha_n = \exp[-n I_0(\mu_1, F_0) + o(n)]
$$

\therefore \quad \alpha_n \text{ bounded away from 0 implies } n I_0(\mu_1, F_0) = O(1). \text{ But } I_0(r, F_0) = O(r^2) \text{ as } r \downarrow 0 \text{ and by the lemma, } \mu_1 \in O(a_1^c) \text{ as } a_1 \downarrow 0.

The theorem follows.

For Theorem 1 we need the following.

**Lemma 6.** Let $\psi$ be continuous and non-negative. If for some $\lambda_1$, 

$$H + \lambda_1 H^{-1} \psi = 0$$

has a unique non-negative solution $H_1$ on $[0,1]$ which vanishes at 0, 1 such that $\int_0^1 H_1 \psi = 1$, then $f_1 = H_1 \psi$ maximizes if $m \geq 1$ (minimizes if $m < 1$) $L(f) = \int\int K(s,t) f(s)^{m-1} f(t)^{m-1} \, ds \, dt$ among $f$ such that $\int_0^1 f^m = 1$ where $K(s,t) = (\min(s,t) - st) \psi(s) \psi(t)$ and the maximum (minimum) equals $\lambda_1^{-1}$. Further $\lambda_1 = \int_0^1 H_1^2$.

**Proof.** Let $K$ be any positive continuous symmetric function on $[0,1]^2$.

Let $q = m/(m-1)$. For $0 < m < 1$ it follows from Hölder's inequality that

$$\int |f \cdot g| \geq \|f\|_m \|g\|_q$$

(i.e., the direction of the inequality is reversed from when $m \geq 1$).

Applying Hölder's inequality twice, one obtains

$$\int\int K(s,t) |a(s) b(t)| \, ds \, dt \leq (\geq) \|a\|_q \|b\|_m (\int\int K^m)^{1/m} \text{ for } m \geq (,) 1.$$ 

Hence for $m \geq (,) 1$, $L(f)$ has a finite positive maximum (minimum) among $f$ such that

$$\int f^m = 1.$$
By the calculus of variations, if \( f_1 \) is any non-negative function that maximizes (minimizes) \( L(f) \) subject to (9), it satisfies

\[
(10) \quad f(s) = \lambda \int K(s,t) f(t)^{m-1} \, dt \quad \text{for some } \lambda
\]

Now (9) and (10) \( \Rightarrow \lambda^{-1} = L(f) \). Hence for \( f_1, \lambda = \lambda_1 \), the minimum (maximum) positive \( \lambda \) such that (9) and (10) have a non-negative solution. For the particular \( K \) of the lemma, setting \( H = f/\psi \), (10) is equivalent to \( \ddot{H} + \lambda H^{m-1} \psi^m = 0 \), since \( H(0) = H(1) = 0 \) (differentiating twice). Hence if this differential equation has a unique solution \((H_1, \lambda_2)\) satisfying the boundary conditions, then \( \lambda_2 = \lambda_1 \) and \( f_1 = H_1 \cdot \psi \).

Finally \( 1 = \int H_1^m \psi^m = -\int H_1 \frac{H_1'}{\lambda_1} = \int \frac{\psi^2}{\lambda_1} \).

**Proof of Theorem 1.**

(i) We give the proof for \( T_{n,m}(\psi), T_{n,m}^+(\psi), m \neq 2 \) only. The other cases are analogous.

By Lemma 4 we seek \( H = H(x,\lambda) \) such that \( \ddot{H}/H + m\lambda(H-x)^{m-1}\psi^m = 0 \), \( H \) is continuous, \( H(0) = 0, H(1) = 1, \dot{H} \geq 0 \) and \( \lambda = \lambda(r) \) is given by

\[
r^m = \int (H-x)^m \psi^m \quad \text{where we suppose for the moment that } H > x \text{ in } (0,1).
\]

It is clear that \( H \to x \) as \( r \to 0 \), and is easily shown that \( H = x + \alpha \) as \( r \to 0 \) where \( \alpha = (2-m)^{-1} \) and \( \ddot{H} + ma^{m-1} \psi^m = 0 \),

98.
a(0) = a(1) = 0. Hence \( a = \beta H_1 \) where \( \lambda_1, H_1 \) are given by Lemma 6 and \( \beta = (\lambda_1/m)^{1/(m-2)} \). \( \because \int a^2 = \beta^2 \lambda_1 \), by Lemma 6.

If for \( T_n \cdot \psi \), we did not assume that \( H > x \) in \((0,1)\)
then we would have instead that \( a \) is a non-negative multiple of \( H_1 \)
such that \( \dot{H}_1 + \lambda |H_1|^{m-1} \psi = 0, H_1(0) = H_1(1) = 0, \int_0^1 |H_1|^m \psi = 1 \);
by an analog of Lemma 5 \( f_1 = H_1 \cdot \psi \) maximizes \( L(|f|) \) among \( f \) such
that \( \int |f|^m = 1 \); hence we may take \( f_1 > 0 \) in \((0,1)\) which shows
that we may assume \( H > x \) in \((0,1)\).

\[
I_0(r, F_0) = \frac{\lambda \alpha}{2} \int a^2 \cdot (1 + o(1))
\]

and

\[
r = \lambda \alpha \int a^m \psi^m \cdot (1 + o(1))
= \lambda \alpha \frac{1}{m} \int a_1^2 \cdot (1 + o(1)).
\]

(1) and (2) of (i) follows. To show (3) of (i), we note
that \( f_1^2 = 2\lambda_1/m (c - f_1) \) for some constant \( c \). Integration over
\([0,1]\) yields \( \lambda_1 = 2\lambda_1/m (c-1) \). \( \therefore \) \( c = 1 + m/2 \)

\[
\therefore \quad \pm x \cdot \left( \frac{2\lambda_1}{m} \right)^{1/2} f_1 \int_0^1 (c-y^m)^{-1/2} dy
\]

\[
\therefore \quad \pm (2\lambda_1 \cdot m)^{1/2} c^{1/2-1/m} x = B(r_1/m : \frac{1}{m}, \frac{1}{2})
\]
in terms of the incomplete Beta function.
Further \( f_1 \geq 0 \) and \( f_1(0) = f_1(1) = 0 \) so that \( \hat{f}_1 \geq 0 \) in \((0, \frac{1}{2})\), \( \hat{f}_1 \leq 0 \) in \([\frac{1}{2}, 1)\). This yields on simplification that \((m\lambda_1/2)^{1/2} c^{1/2-1/m} = B(\frac{1}{2}, \frac{1}{m})\) which proves (3) of (i).

Note that \( f_1 \) is given by

\[
B\left( \frac{1}{1 + \frac{m}{2}} : \frac{1}{m}, \frac{1}{2} \right) = 2B\left( \frac{1}{m}, \frac{1}{2} \right) \cdot \left\{ \begin{array}{ll} x & \text{in } [0, \frac{1}{2}] \\ 1-x & \text{in } [\frac{1}{2}, 1] \end{array} \right.
\]

Finally, (4) and (5) of (i) follow from (2) of (i) and Theorem 2 of §5.

(ii) This follows from Theorem 3 of §5 since \( a(x,r) \) defined there equals \( r^2/2(x-x^2) + o(r^2) \) as \( r \to 0 \).

(iii) In a similar way this follows from Lemma 5(v) with \( F = F_0 \).

Theorem 2 follows easily from Theorem 1.

10.5. Proofs of §8.

Proof of Theorem 1. By Hoadley's Theorem 1,

\[
I_1(r,F) = \inf_{H \text{ cdf on } [0,1]} \inf_{\|H-X\| \geq r} I(H,f)
\]

where

\[
f = \frac{e^{\theta x} - 1}{e^\theta - 1}.
\]
by the calculus of variation \( I_1(r, F) = I(H_1, f) \) where
\[ \hat{H}_1 \hat{H}_1 - \theta + m\lambda(\hat{H}_1 - x)^{m-1} = 0, \lambda \text{ a constant, } \hat{H}_1 \geq x, \hat{H}_1(0) = \hat{H}_1(1) - 1. \]
The rest of the proof is analogous to that of Theorem 1, §5.

Proof of Theorem 2. (2) follows from Hoadley's first theorem with
\( r = 0, T(Q) = \min[\inf(b-Q), \inf(Q-a)], \) or from the first part of Lemma 2 of [52]. To show (3) consider first the case when \( b = 1; \) if \( H_1 \) is the (concave) envelope condition of \( \{a(x) \leq g(x)\} \) then (3) follows either as in the proof of Theorem 5,4 of [21] (where Hoadley showed that (3) holds for \( b(x) = 1, a(x) = H_0(x) - r, H_0 \) a continuous cdf on \([0,1]\) such that \( H_0(x) \geq x), \) or by an application of the second part of Lemma 2 of [52]. Hence (3) holds if \( H_1 \leq b. \)

For the discrete case (when \( a(x) = a_i, b(x) = b_i = \frac{\frac{1}{N} + 1}{N}, i = 1, \ldots, N), \) (3) follows from a repeated application of
\[ \inf \int_{x_0}^{x_1} \hat{H} \ln \hat{H} \, dx = \int_{x_0}^{x_1} \hat{H}_1 \ln \hat{H}_1 \, dx \]
where
\[ \hat{H} \geq 0, H(x_0) = y_0, H(x_1) = y_1 \]
\[ H(x_2) \leq y_2 \]
where
\[ H_1 \text{ joins } (x_0, y_0), (x_1, y_1) \text{ if } \frac{y_2 - y_0}{x_2 - x_0} \geq \frac{y_1 - y_0}{x_1 - x_0} \]
and
\[ H_1 \text{ joins } (x_0, y_0), (x_2, y_2), (x_1, y_1) \text{ otherwise, } \]
for \( x_0 < x_2 < x_1, y_0 < y_2 < y_1, \) (cf Lemma 3,2 [22] with \( F_0 = U). \)

To prove the general case let \( p_n = P(a(x) \leq U_n(x) \leq b(x) \)
in \((0,1)). \ a(x) \leq U_n(x), \ U_n(x^+) = U_n(x) \Rightarrow a(x^+) \leq U_n(x) \)
\[ p_n \leq q_{nN} = P(a_n \frac{1}{n} \leq U_n \frac{1}{N} \leq b_n \frac{1}{N}, i = 1, \ldots, n), \]
\( N = 1, 2, \ldots, \)
\[(11) \text{ . . . } \limsup \lim_{n \to \infty} n^{-1} \ln p_n \leq \lim_{N \to \infty} \lim_{n \to \infty} n^{-1} \ln q_{nN} = -I(H,U)\]

from the discrete case above.

For \(N = 1, 2, \ldots\), \(p_n \geq r_{nN} = P(a_i \leq U_n \leq b_i, \ i = 0, \ldots, N)\) where

\[
a_i = \begin{cases} 
  a_i \left(\frac{i+1}{N}\right)^+, & i = 0, \ldots, N-1 \\
  1, & i = N
\end{cases}
\]

\[
b_i = \begin{cases} 
  b_i \left(\frac{i-1}{N}\right), & i = 1, \ldots, N \\
  0, & i = 0
\end{cases}
\]

Since \(a(x^+) < b(x)\) in \((0,1)\) we may take \(N > 8^{-1}\) large enough so that \(a_i < b_i, i = 1, \ldots, N\).

\[\therefore \quad r_{nN} = P(a_i \leq U_n \leq b_i, \ i = 1, \ldots, N-1)\]

\[(12) \quad . \quad \liminf \lim_{n \to \infty} n^{-1} \ln p_n \geq \lim_{N \to \infty} \lim_{n \to \infty} n^{-1} \ln r_{nN} = -I(H,U)\]

from the discrete case (11) and (12) complete the proof.

**Proof of Theorem 3.** This is a direct application of Theorem 2. The restrictions on \(\psi\) arise from (1) of Theorem 2.

(e) follows from (d) as follows:
$H_0$ concave implies that $H_{st}$ is the envelope cdf of
\[(H_0 - t/\psi(H_0) \leq G, H_0(s) - t/\psi(H_0(s)) = G(s))\] so that for a given $t$,
$I(H_{st}, U)$ is minimized over $s$ by the envelope cdf $H_t$ of
\[[H_0 - t/\psi(H_0) \leq G] \quad \text{and} \quad I(H_t, U)\]
is minimized by $t = r$.

In the five examples of §8, $\theta > 0 \implies H_0 = F_0(\frac{F^{-1}}{\theta})$ is concave
while $\theta < 0 \implies H_0$ is convex. We shall only consider $\theta > 0$.

Note. If $F_\theta(x) = F_0(x-\theta)$, then
\[\theta > 0 \implies H_0 \text{ concave } \iff \frac{d}{dx} \ln \hat{F}_0(x) \text{ is non-increasing} \]
\[\theta < 0 \implies H_0 \text{ convex } \iff \frac{d}{dx} \ln \hat{F}_0(x) \text{ is non-decreasing} \]
assuming $d/dx \ln \hat{F}_0(x)$ is defined a.e.

Assume $0 \leq r < \mu_1$,

$H_0$ concave $\implies I_1(r, F) = I(H_1, U)$ for $D_n^-(\psi), D_n(\psi), V_n(\psi)$

where $H_1$ is the envelope cdf of $\{G : H_0 - r/\psi(H_0) \leq G\}$ assuming
$\psi(0+) < \infty, \psi(1-) < \infty$; hence if $G_0 = H_0 - r/\psi(H_0)$ is concave when
$G_0 \geq U$ (as is the case for $\psi = 1$ and $\psi$ close to $\psi_0$ but bounded)
then

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\[
H_1(x) = \begin{cases} 
  x \cdot G_0(x_1)/x_1, & 0 \leq x \leq x_1 \\
  G_0(x), & x_1 < x < x_2 \\
  1 - \frac{1-x}{1-x_2} \cdot (1-G_0(x_2)), & x_2 \leq x \leq 1
\end{cases}
\]

where \( x_1, x_2 \) are the unique points for which

\[
\frac{G_0(x_1)}{x_1} = G_0(x_1), \quad \frac{1 - G_0(x_2)}{1 - x_2} = G_0(x_2)
\]

so that

\[
(13) \quad I_1(r, F) = G_0(x_1) \ln \frac{G_0(x_1)}{x_1} + \int_{x_1}^{x_2} G_0 \ln G_0 + (1-G_0(x_2)) \ln \frac{1-G_0(x_2)}{1-x_2}.
\]

Actually for \( H_0 \) concave (13) holds provided \( x_1, x_2 \) are unique and \( G_0 \) is concave in \( (x_1, x_2) \).

For convenience we set \( y_i \equiv F^{-1}_\theta(x_i) \), \( i = 1, 2 \).

(a) \( F_\theta(x) = \Phi(x-\theta) \):

\[
J(F_0, F_\theta) = -\ln \inf_{0 < t < 1} \exp\left[-\frac{\theta^2}{2} (t-t^2)\right] = \frac{\theta^2}{8}.
\]

(i) \( D_n(1), D_n^-(1), V_n(1) \):

\( y_1 \) is the root of \( \frac{\Phi(y) - r}{\Phi(y-\theta)} = \exp\left[\frac{\theta^2}{2} - y\theta\right] \)

\( y_2 \) is the root of \( \frac{1-\Phi(y)+r}{1-\Phi(y-\theta)} = \exp\left[\frac{\theta^2}{2} - y\theta\right] \)
\[ I_\perp(r, F_\theta) = \frac{\theta^2}{2} (\Phi(y_2) - \Phi(y_1)) + \theta (\varphi(y_2) - \varphi(y_1)) + (\Phi(y_1) - r) \cdot \left( \frac{\theta^2}{2} - y_1 \right) + (\Phi(-y_2) + r) \cdot \left( \frac{\theta^2}{2} - y_2 \right) \]

(b) \( F_\theta(x) = (1 + e^{-x+\theta})^{-1} \):

\[ J(F_0, F_\theta) = -\ln \inf_{0 < t < 1} g(t) \]

where

\[ g(t) = \int_0^1 \left[ xe^{-\theta/2} + (1-x) e^{\theta/2} \right]^{-2t} dx \]

Therefore

\[ J(F_0, F_\theta) = \ln \left( (e^{\theta/2} - e^{-\theta/2})/\theta \right) \]

(i) \( D_n^{-1}(l), D_n(1), V_n(1) \):

\[ H_0(x) = \frac{x e^{\theta}}{1 + cx} \quad \text{where} \quad c = e^{\theta} - 1 \]

\[ \therefore x_1 \text{ is a root of } x^2 c(e^{\theta} - rc) - 2x cr - r = 0 \]

\[ \therefore x_1 = \frac{r + e^{\theta/2} \sqrt{r/c}}{e^{\theta} - cr} \quad \therefore \frac{e^{\theta}}{c} > \mu_1 > r \]

Since

\[ 1 - H_0(1-y) = \frac{y e^{-\theta}}{1 + y(e^{-\theta} - 1)}, \quad 1 - x_2 = [x_1]_{r \to -r, \theta \to -\theta} \]

\[ x_2 = 1 - \frac{-r + \sqrt{r/c}}{e^{-\theta} + r(e^{-\theta} + 1)} = \frac{e^{-\theta}(1+r) - \sqrt{r/c}}{e^{-\theta}(1+r) - r} \]

\[ H_0(x) = e^{\theta}/(1 + cx)^2 \]

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\[ \int_{x_1}^{x_2} \ln H \, dx = e^\theta \cdot \frac{\theta}{2} \left( \frac{1}{1 + cx_1} - \frac{1}{1 + cx_2} \right) - \frac{2e^\theta}{c} \int_{z_1}^{z_2} \frac{\ln y}{y^2} \, dy \]

\[ = \frac{e^\theta}{c} \left( \theta + 2(1 + \ln y) \right) \frac{z_2}{z_1} \]

where \( z_1 = 1 + cx_1, i = 1, 2. \)

\[ I_1(r, F_0) = \frac{x_1 e^\theta}{z_1^2} (\theta - 2 \ln z_1) + (1 - x_2) \frac{e^\theta}{z_2^2} (\theta - 2 \ln z_2) \]

\[ + \frac{e^\theta}{cz_2} (-\theta + 2 + 2 \ln z_2) - \frac{e^\theta}{cz_1} (-\theta + 2 + \ln z_1) \]

\[ I_1(r, F_0) = \left( \frac{2 \ln z_1 - \theta}{z_1^2} - \frac{2}{z_1} (1 - 2 \ln z_1) + \frac{e^\theta}{z_2^2} (\theta - 2 \ln z_2) \right. \]

\[ \left. + \frac{2}{z_2} (1 - 2 \ln z_2) \right) (1 - e^{-\theta})^{-1} \]

where

\[ z_1 = e^{\theta/2} \cdot \frac{e^{\theta/2} + \sqrt{rc}}{e^{\theta} - rc} = 1/[1 - r^{1/2}(1 - e^{-\theta})^{1/2}] \]

and

\[ z_2 = e^{\theta} \cdot \frac{1 - \sqrt{rc}}{1 - rc} = e^{\theta}/[1 + r^{1/2}(e^{\theta} - 1)^{1/2}] \]
(c) \( F_\theta(x) = F_0(x-\theta), \; \hat{r}_0(x) = \frac{1}{2} e^{-|x|}; \)

\[
J(F_0, F_\theta) = -\ln \inf_{0 < t < 1} \int \left( \frac{\hat{r}_0}{F_\theta} \right)^{1-t} dF_\theta = -\ln \inf_{0 < s < 1} \int_0^1 H_0^s dx.
\]

Let

\[
g(s) = \frac{e^{\theta s} - \theta}{2} + \frac{e^{-\theta s}}{2} + (\frac{e^{-\theta}}{4}) \int e^{-\theta/2} x^{-2s} dx, \quad \text{if } s = \frac{1}{2}
\]

\[
= \frac{e^{-\theta t} + e^{-\theta s}}{2} + \frac{1}{4s-2} (e^{-s\theta} - e^{-\theta t}), \quad \text{where } t = 1-s.
\]

\[
\therefore g(1-s) = g(s). \quad \therefore
\]

\[
g(s) \text{ is minimized at } s = \frac{1}{2},
\]

\[
g(\frac{1}{2}) = e^{-\theta/2} + \frac{e^{-\theta/2}}{2} \ln (\frac{1}{2} \cdot \frac{2}{e^\theta}) = e^{-\theta/2} \cdot (1 + \frac{\theta}{2})
\]

\[
\therefore J(F_0, F_\theta) = \frac{\theta}{2} - \ln(1 + \frac{\theta}{2}).
\]

(i) \( D_n^-(1), D_n(1), V_n(1): \)

\[
H_0(x) = \begin{cases} 
 e^{\theta x}, & 0 \leq x \leq \frac{1}{2} e^{-\theta} \\
 1 - \frac{e^{-\theta}}{4x}, & \frac{1}{2} e^{-\theta} < x < \frac{1}{2} \\
 1 - (1-x) e^{-\theta}, & \frac{1}{2} \leq x \leq 1
\end{cases}
\]

\[
\therefore x_1 = \frac{e^{-\theta}}{2-2r} \quad x_2 = \frac{(1 + 4re^\theta)^{1/2} - 1}{4re^\theta}
\]

(14) \( I_1(r, F_\theta) = x_1 g(x_1) \ln g(x_1) + (1-x_2) g(x_2) \ln g(x_2) + \int_{x_1}^{x_2} g(x) \ln g(x) \, dx \)

with \( g(x) = e^{-\theta/4x^2} \).
\[ I_1(r, F_\theta) = \frac{e^{-\theta}}{4x_1} [-\theta + \ln(4x_1^2)] + (1-x_2) \frac{e^{-\theta}}{4x_2} [-\theta + \ln(4x_2^2)] + \frac{e^{-\theta}}{4} \left[ (\theta + \ln 4) \left( \frac{1}{x_2} - \frac{1}{x_1} \right) + 2 \left( \frac{\ln x_2 + 1}{x_2} - \frac{\ln x_1 + 1}{x_1} \right) \right] \]

\[ F_\theta = F_0^{\theta+1} : \]

\[ H_0 = x^a \quad \text{where} \quad a = 1/(\theta+1), \]

\[ J(F_0, F_\theta) = -\ln \inf_t \int_{H_0^t} = -\ln \inf_{0 < t < 1} \frac{a^t}{1+(a-1)t}. \]

The \( \inf \) occurs when

\[ \frac{d}{dt} \left( t \ln a - \ln(1 + a-1 \cdot t) \right) = 0 \]

i.e., when \( t = 1/\ln a - 1/(a-1) \).

\[ J(F_0', F_\theta) = -\ln \left[ \frac{\frac{1}{\ln a} - (a-1)}{a-1} \cdot \ln a \right] \]

\[ = \begin{cases} 
-\ln(-\ln a) + \ln(1-a) - 1 - \frac{\ln a}{1-a}, & \theta > 0 \\
-\ln \ln a + \ln(1-a) - 1 + \frac{\ln a}{a-1}, & -1 < \theta < 0.
\end{cases} \]

(i) \( D_n(1), D_n^{-1}(1), V_n(1) \):

\[ \dot{H}_0 = ax^{a-1}. \quad x_1 \text{ is a root of } (x^a - r)/x = a \cdot x^{a-1} \]

\[ \therefore x_1 = (r/(1-a))^{1/a}. \quad x_2 \text{ is a root of } (1-x^a+r)/(1-x) = ax^{a-1}, \text{ i.e., } \]

\[ x^a(1-a) + ax^{a-1} = r+1. \quad (14) \text{ gives } I_1(r, F_\theta) \text{ with } g(x) = ax^{a-1}. \]

\[ I_1(r, F_\theta) = ax_1^a (\ln a + a-1 \cdot \ln x_1) + (1-x_2) ax_2^a (\ln a + a-1 \cdot \ln x_2) \]

\[ + \ln a \cdot (x_2^a - x_1^a) + (a-1) \left[ x_2^a (\ln x_2 - \frac{1}{a}) - x_1^a (\ln x_1 - \frac{1}{a}) \right]. \]
(e) $F_\theta = (e^{\theta F_0} - 1)/(e^\theta - 1)$:

$$J(F_0, F_\theta) = - \ln \inf_{0 < t < 1} \int \left( \frac{e^{\theta t} x}{e^\theta - 1} \right) dx$$

$$= - t_0 \ln \left( \frac{\theta}{e^\theta - 1} \right) - \ln \left( \frac{e^{\theta t_0} - 1}{e^\theta} \right)$$

where $t_0$ is the root of $- \ln a - \theta/(e^\theta - 1) + \ln \theta + 1/t = 0$.

$H_0(x) = 1/\theta \ln(a + bx)$ where $b = e^\theta - 1$. Therefore $\dot{H}_0 = 1/(a(1+bx))$.

(i) $D_1(n), D_2(n), V_1(n)$:

$z_1 = H_0(x_1)$ is the root of

$$\frac{(z-r) \cdot b}{e^{\theta z} - 1} = \frac{1}{a} e^{-\theta z}$$

i.e., $(z-r) \theta = 1 - e^{-\theta z}$.

$z_2 = H_0(x_2)$ is the root of

$$\frac{1 - z + r}{e^{\theta z} - 1} = \frac{1}{b} e^{-\theta z}$$

i.e., $(1-z+r) \theta = e^{\theta(1-z)} - 1$.

(14) gives $I_1(r, F_\theta)$ with

$$g(x) = \left[ \frac{\theta}{e^\theta - 1} + \theta x \right]^{-1}$$

Let $p_i = g(x_i), i = 1, 2$. Therefore

$$I_1(r, F_\theta) = x_1 p_1 \ln p_1 + (1-x_2) p_2 \ln p_2 - ((\ln p_2)^2 - (\ln p_1)^2)/2$$

(15) $V_1(n)$.

In [12] Chapman considered testing $F = F_0$ against 

$[F \leq F_0, \sup(F_0 - F) = \Delta, F$ continuous cdf] = $\omega^-$. He defined a test 

$\varphi = \varphi(x_1, \ldots, x_n)$ to be monotone if $x^{(1)}_\geq x^{(2)}_\geq \Rightarrow \varphi(x^{(1)}) \geq \varphi(x^{(2)})$

and p.o. (partially ordered) if $G_1 \leq G_2 \Rightarrow E_{G_1} \varphi \geq E_{G_2} \varphi$. He remarks 

that if $\varphi$ is continuous except for a finite number of jumps and $\varphi$ is p.o. then $\varphi$ is unbiased. He shows that tests of structure (d) 

(i.e., tests that depend on $F_0$ and $X_1, \ldots, X_n$ only through $F_0(X_1), \ldots, F_0(X_n)$) that are monotone, are p.o. Let 

\[
\pi = \inf_{F \in \omega^-} E_F \varphi, \quad \bar{\pi} = \sup_{F \in \omega^-} E_F \varphi.
\]

He shows that if $\varphi$ is p.o. and of structure (d), then 

\[
\pi = \inf_{0 \leq a \leq 1 - \Delta} E^{G_a(F_0)} \varphi, \quad \bar{\pi} = E^{G_a(F_0)} \varphi
\]

where 

\[
G^m_a(x) = \begin{cases} 
x & \text{in } [0, a) \cup [a + \Delta, 1] \\
a & \text{in } [a, a + \Delta)
\end{cases}
\]
\[ g^n(x) = \begin{cases} 
 0 & \text{in } [0, \Delta) \\
 x - \Delta & \text{in } [\Delta, 1) \\
 1 & \text{at } 1 
\end{cases} \]

He attempts to compute these bounds for large \( n \) for several fixed-\( \alpha \) tests, but as the central limit theorem cannot be used in the tails most of his results are wrong. We now give a slight generalization of (16) and use it to compute \( I_0(r, \omega_0^-), I_1(r, \omega_1^-) \) where

\[ \omega_0^- = \{ \text{cdf } F \leq F_0, \sup (F_0 - F) \psi_A(F_0) = a_0, F \text{ continuous} \} \]

\[ \omega_1^- = \{ \text{cdf } F \leq F_0, \sup (F_0 - F) \psi_A(F_0) = a_1, F \text{ continuous} \} \]

Chapman's argument extends easily to show that if

\[ \underline{\pi}_i = \inf_{F \in \omega_i^-} E F \varphi, \quad \bar{\pi}_i = \sup_{F \in \omega_i^-} E F \varphi \]

and \( \varphi \) is p.o. of structure (d), then

(17) \[ \underline{\pi}_i = \inf_{a \in A_i} E_{G_A(F_0)} \varphi \]

(18) \[ \bar{\pi}_i = E_{G_i(F_0)} \varphi \]

where
\[ g^i_a(x) = \begin{cases} 
 x & \text{in } [0, u_{a,i}) \cup [a, 1] \\
 u_{a,i} & \text{in } [u_{a,i}, a) 
\end{cases} \]

\[ G_i(x) = \begin{cases} 0, & u_{x,i} < 0 \\
 u_{x,i} & 0 \leq u_{x,i}, x < 1 \\
 l, & x = 1 
\end{cases} \]

\[ A_i = \{0 \leq a \leq 1, a \psi_A(a) \geq u_{a,i} \} \]

\[ u_{a,i} = a - a_i / \psi_A(a) \]

(19) and we assume that \( A_i \neq \emptyset \), i.e., \( a_i \leq \sup a \psi_A(a) \),

and

(20) \( (x - u_{a,i}) \psi_A(x) \) is maximized over \( 0 \leq x \leq a \) at \( x = a \)

for (3) to hold,

and

(21) \( u_{x,i} \) is non-decreasing in \( \{x : 0 \leq u_{x,i}, x < 1\} \) for (18) to hold, \( i = 0, 1 \).

Now if (17) holds, for \( i = 1 \), then

\[ \beta_n(r) = 1 - \inf_{F \in \mathcal{F}_1} \inf_{a \in A_1} E_{\psi} = 1 - \inf_{a \in A_1} E_{G^{\psi}(F)_0} \]

\[ = \sup_{a \in A_1} \exp\{-nI_1(r, G^{\psi}_a(F)_0) + o(n)\} \]
if \( \phi \) is regular in the sense of Hoadley's Theorem 1 or 2, when 
\[ F = G'_a(F_0), \]
and hence

\[
I'_{1}(r, \omega_{1}^{-}) = - \lim_{n \to \infty} n^{-1} \lim \beta_n(r_n) = \inf_{a \in A_1} I_1(r, G'_a(F_0)),
\]

provided

\[ 1 - E^\phi \left( G'_a(F_0) \right) \] \[ = \exp \left[ -nI_1(r, G'_a(F_0)) + o(n) \right] \] uniformly for \( a \in A_1 \).

Similarly, (13) implies that \[ \sup_{F \in \omega_1} I_1(r, F) = I_1(r, G_1(F_0)) \] if \( \phi \) is regular in the sense of Hoadley's Theorem 1 or 2 when \( F = G_1(F_0) \).

This can be used in measuring how good \( \phi \) can be. Similar (17), (16) and \( \phi \) regular imply that

\[
I_0(r, \omega_0^{-}) = I_0(r, G_0(F_0))
\]

and

\[
\sup_{F \in \omega_0} I_0(r, F) = \sup_{a \in A_0} I_0(r, G'_a(F_0)).
\]

Similarly if the test statistic is the form \( K(F_n) \) where \( K(\cdot) \) is regular in the sense above, then

\[
\mu_1(\omega_1^{-}) = \inf_{F \in \omega_1} K(F) = \inf_{a \in A_1} K(G'_a(F_0))
\]

\[
\sup_{F \in \omega_1} K(F) = K(G_1(F_0))
\]
\[ \mu_0(\omega_0) = \sup_{\omega_0} K(F) = K(G_0(F_0)) \]

\[ \inf_{\omega_0} K(F) = \inf_{a \in A_0} K(G_a^c(F_0)) \, . \]

In particular \( T_{n,1}^-(\psi) = -T_{n,1}^+(\psi), \) \( D_n(\psi), \) \( \pi_n \) and \( \pi'_n \) defined below are monotone and hence p.o. so that (17) and (18) hold; hence for regular \( \psi \) (e.g. \( \psi \) bounded) (22)-(25) hold, \( \pi_n, \pi'_n \) are the Fisher and Pearson tests considered by Chapman,

\[ \pi_n = 2 \int \ln F_0 \, dF_n \]

\[ \pi'_n = 2 \int \ln(1-F_0) \, dF_n \, . \]

if \( F = F_0, \) \(-n\pi_n\) and \(-n\pi'_n\) are chi-square with \( 2n \) degrees of freedom; \( H_0 \) is accepted for small values of \( \pi_n \) or \( \pi'_n \).

**Proof of Theorem 1.**

(a) follows from (9) and Lemma 5(ii) § 10.2

(b) follows from (c).

By an analog of Theorem 3(b) §8 (not applicable directly since \( G_a' \) is not continuous)

\[ I_1(r, G_a'(F_0)) = I(H, U) \]

if \( \psi(0+) < \infty, \) where \( H \) joins \((0,0), (u_{a,1}, \lambda_a)\) and \((1,1)\) linearly, since
\[ r < \sup_{(u_{a,1}^{a,1})} (x-u_{a,1}^{a,1}) \psi(x). \]

If (19), (20) hold, (c) follows from (22).

**Proof of Theorem 2.**

(a) \( I(\omega_1^-, \omega_0^-) \geq I_0(\mu_1^-, \omega_0^-) = I(\omega_2^-, \omega_0^-) \) where \( \omega_2 = \{ F : D_p^-(F) \geq a_1 \} \)

\[ I(\omega_2^-, \omega_0^-) = I(F_2, \omega_0^-) \]

for some \( F_2 \in \omega_2^-. \)

Let

\[ F_1 = \begin{cases} F_2, & F_2 \leq F_0 \\ 0, & F_2 > F_0 \end{cases} \]

Then \( I(F_2, \omega_0^-) \geq I(F_1, \omega_0^-) \geq I(\omega_1^-, \omega_0^-) \) since \( F_1 \in \omega_1^- \).

The proof of (b) is similar. For (c) we need

**Lemma 7.** Let

\[ \Delta = \{ F : \text{cdf on } [0,1], F(a_i) = v_i, i = 1, \ldots, M \} \]

where

\[ 0 < a_1 < a_2 < \cdots < a_M < 1, \quad 0 < v_1 < v_2 < \cdots < v_M < 1. \]

Then for \( 0 < t < 1 \),

\[ \sup_{H \in \Delta} \int_0^1 H^t \, dx = \int_0^1 H^t_0 \]

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where $H_0$ is the piecewise-linear cdf on $[0,1]$ through $(a_1,v_1)$, $i = 1, \ldots, M$.

**Note:** Lemma 3.2 [22] is a corollary of Lemma 7.

**Proof.** It is sufficient to prove that

$$\sup_{F \exists F(0)=0, F(a)=b, F \geq 0} \int_a^b \frac{t}{a} \cdot a = 0 \quad \text{for } a, b > 0.$$ 

This is done by a simple calculus of variations argument.

**Proof of (c).**

$$\omega^-_l = \bigcup_{p \in A_1} S_p$$

where we set $F(x) \equiv x$

and

$$S_p = \{F \text{ cdf on } [0,1] : \sup(x-F) \cdot \psi_A = a_1$$

$$= (F-F(p)) \psi_A(p), F \leq x\}.$$ 

Let $H_p$ be the cdf joining $(0,0), (p, p-a_1/\psi_A(p)), (1,1)$ linearly.

Then $H_p \in S_p$ so that

$$\int H_p^t \leq \sup_{H \in S_p} \int H^t \leq \sup_{H(p)=p \cdot a_1/\psi_A(p)} \int H^t = \int H_p^t, \text{ by Lemma } 7$$

Also

$$\int H_p^t = f(p, t, a_1).$$

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Hence

\[ J(F_0, \omega^-) = - \inf_t \sup_{p \in A_1} \sup_{H \in S_p} \ln \int H^t \]

\[ = \sup_t \inf_{p \in A_1} \ln f(p, t; \omega^-) \]

(a) \[ J(\omega^-, \omega^-) = - \inf_{0 < t < 1} \sup_{G_0 \subset \omega^-} \sup_{G_1 \subset \omega^-} \ln \int G_0^t G_1^{1-t} \]

\[ = - \inf_{0 < t < 1} \sup_{H \in C} \ln \int H^t \]

where \( H = G_0^{-1}(G_1^{-1}) \), \( G = G_1^{-1} \), \( F_0(x) = x \),

\[ C = \{ H : \sup(G-H) = a_0, H \leq G \text{ for some } G \text{ such that} \]

\[ \sup(G-x) = a_1, G \geq x \} \]

If \( \sup(G-x) = G(p)-p \), for some \( p \in (0, 1-a_1) \)

\[ G'(x) = \begin{cases} 
\frac{x}{p-a_1} \\
1 - (1-x) \frac{1-p-a_1}{1-p} 
\end{cases} \]

then \( \sup(G'-x) = a_1 \) and if

\[ D = \{ H : \sup(G-H) = a_0, H \leq G \text{ for some } G \sup(G-x) = G(p)-p = a_1, G \geq x \} \]

then

\[ \sup_{H \in D} \int H^t = \sup_{H \in D} \int H^t \text{ taken over } H \text{ such that} \]

\[ \sup(G'-H) = a_0, H \leq G' \]

\[ = \int \hat{A}_p^t \text{ applying Lemma 7 twice,} \]

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where $A_p$ is the cdf joining $(0,0), (p, p+a_1 - a_0), (1,1)$ linearly.

Now $\int \frac{A}{p} = f(1-p, t, a_1 - a_0)$. Hence

$$J(\omega_0, a_1) = \inf_{0 < t < 1} \sup_{a_1 < q < 1} \ln f(q, t, a_1 - a_0).$$

By convexity, given $t$ there is a unique $p \in (a,1)$ maximizing $f(p,t,a)$ and given $p$ there is a unique $t \in (0,1)$ minimizing $f(p,t,a)$.

Hence the $(p_0, t_0)$ such that

$$f(p_0, t_0, a) = \inf_{0 < t < 1} \sup_{a < q < 1} f(q, t, a)$$

is the unique solution of $f_1(p, t, a) = f_2(p, t, a) = 0$. Hence $p_c = (1+a)/2$ and $t_0 = 1/2$. Hence

$$\left\{ \begin{array}{ll}
\inf_{0 < t < 1} \sup_{a_1 < q < 1} f(q, t, a_1 - a_0) \\
= \left\{ \begin{array}{ll}
f\left(\frac{1+a_1 - a_0}{2}, \frac{1}{2}, a_1 - a_0\right), & \text{if } 1 + a_1 - a_0 > 2a_1 \\
\inf_{0 < t < 1} f(a_1, t, a_1 - a_0), & \text{if } 1 + a_1 - a_0 < 2a_1
\end{array}\right.
\end{array}\right.$$  

where

$$t_1 = \frac{\ln(a_0)}{\ln(\frac{1-a_0}{1-a_1})}$$

and $u, v$ are defined in the Theorem. (d) follows upon simplification.

(e) By Theorem 1(a), $\mu_{1} \to \psi(1)$. Hence

$$I_0(\mu_{1}, \omega_0) \left\{ \begin{array}{ll}
\to - \ln(a_0/\psi_{A}(1)) & \text{if } a_0 > 0 \\
= \ln(\psi(1)-\mu_{1}) + O(1) & \text{if } a_0 = 0
\end{array}\right.$$  

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Also \( \psi(1) - \mu_1 = O(\psi_A(1) - a_1) \). Hence by (a) \( e_\beta \to 1 \). By Theorem 1(b) as \( a_1 \to \psi_A(1) \),

\[
I_1(r, \omega^-) = -\left(1 - r/\psi(1)\right) \ln(1 - a_1/\psi_A(1)) + O(1).
\]

Hence by (b),

\[
e_\alpha \to \frac{1 - \mu_0/\psi(1)}{1 - a_0/\psi_A(1)}, \quad \mu_2 \to \psi(1)
\]

and

\[
I_0(\mu_2, \omega^-) \to -\ln(a_0/\psi_A(1)).
\]

Let \( \psi_A = 1 \) and \( a_0 = 0 \). Let \( \delta = 1 - a_1, \Delta = 1 - \mu_2 \). As \( a_1 \to 1 \),

\[
I_0(\mu_2, \omega^-) = -\ln \Delta \cdot (1 + o(1))
\]

\[
I_1(\mu_2, \omega^-) = \Delta \ln \Delta/\delta \cdot (1 + o(1)).
\]

Hence

\[
\Delta = \ln \ln \delta^{-1}/\ln \delta^{-1} \cdot (1 + o(1)).
\]

Hence

\[
I_0(\mu_2, \omega^-) = \ln \ln \delta^{-1} \cdot (1 + o(1)).
\]

But by (d), \( J(F_0, \omega^-) = 1/2 \ln \delta^{-1} \cdot (1 + o(1)) \). Hence \( e_I = o(1) \).

Let \( \psi_A = 1 \) and \( a_0 > 0 \). From the above, as \( a_1 \to 1 \),

\[
I_0(\mu_2, \omega^-) \to -\ln a_0. \quad \text{But by (d), } J(\omega_0, \omega^-) \to -\ln a_0. \quad \text{Hence } e_I \to 1 \]

as \( a_1 \to 1 \).
Let \( \psi_A = \psi = 1, a_0 > 0 \) and let \( a(\cdot, \cdot) \) be as defined in Theorem 3, §5. For \( a_1 \) close enough to 1, by Theorem 1(b), \( I_1(r, \omega^-_1) = a(a_1 - r, 1 - a_1) \) can be arbitrarily large for \( r < a_1 \) so that

\[ I_0(\mu_2, \omega^-_0) = a(\mu_2 - a_0, a_0); \]

hence \( \mu_2 = v/(u+v) \) where \( u \) and \( v \) are defined in (d), and on simplification \( I_0(\mu_2, \omega^-_0) = J(\omega^-_0, \omega^-_1) \) given in (d). Hence \( e_1 = 1. \)

Proof of Theorem 3. (a) By [13],

\[ I_1(r, G'(F_0)) = - \ln \inf \left\{ \exp(t(r+b(x))) \right\} dG'(x) \]

where

\[ b(x) = \int \frac{1}{\psi} - \int \frac{x\psi}{0} \quad \text{if} \quad bdG_a < \infty . \]

Taking \( \psi_A = \psi = 1, \) by (22)
\begin{align*}
I_1(r, \omega^-) &= \sup_{t} \inf_{a_1 \leq a \leq 1} \left[ -t(r + \frac{1}{2}) + \ln t \right. \\
& \hspace{1cm} \left. - \ln \left( 1 - e^{-ta_1} + e^{-ta} - e^{-t} + a_1 e^{-ta} \right) \right] \\
\text{and by (24), } \mu_1 &= a_1^2/2.
\end{align*}

However for each \( t \) the \( \inf \) over \( a \) occurs at \( a = 1 \).

\( I_0(r, \omega^-) \) and \( \mu_0 \) follow similarly from (23), (25).

(b) By [13] for \( r < 2 \)
\[
I_1(r, G_a'(F_0)) = -\ln \inf_{t} e^{-tr/2} \int e^{t/2} \, dG_a'
\]

where \( b(x) = 2 \ln x \), if \( \int bG_a' > -\infty \). For \( \Psi_A = 1 \),

\[
I_1(r, \omega^-) = \inf_{a_1 \leq a \leq 1} \sup_{t > -1} \left[ \frac{tr}{2} - \ln \left( \frac{(a-a_1)^{t+1} + 1-a^{t+1}}{t+1} + a_1 a^t \right) \right].
\]

Further for each \( t \) in \((-1,0)\) the \( \inf \) over \( a \) is attained at \( a = 1 \). But the \( t \) giving the \( \sup \) is unique and lies in \((a_1,0)\) so the result for \( I_1(r, \omega^-) \) follows.

\[
\mu_1 = 2 \inf_{a_1 \leq a \leq 1} \left( (a-a_1) \ln(1 - \frac{a_1}{a}) + a_1 - 1 \right)
\]

\[
I_0(r, \omega^-) = -\ln \inf_{t} e^{-tr/2} \int e^{tb/2} \, dG_0
\]

\[
\mu_0 = 2 \int_0^1 \ln x \cdot dG_0(x).
\]
(c) The formulae are as in (b) but with \( b(x) = 2 \ln(1-x) \),

\[
\int b \mathcal{D}_a = 2\left[a_1 - 1 - (1-a+a_1) \ln(1 + \frac{a_1}{1-a})\right] \to -\infty \quad \text{as} \quad a \to 1
\]

\[
\therefore \quad \mu_1 = -\infty.
\]

Proof of Theorem 4. (a) By Theorem 1, as \( a \downarrow 0 \), if \( \inf \frac{\psi}{\psi_A} > 0 \),

\[
I_0(r, \omega) = r^2 \cdot c_0(\psi)^2/2 (1 + o(1))
\]

\[
\therefore \quad I_0(\mu_1, \omega) = a_1^2 \lambda^2/2 + o(a_1^2) \quad \text{where} \quad \lambda = \frac{\inf \psi/\psi_A}{\inf \psi/\psi_0}.
\]

(1) follows using Theorem 2(a),

\[
I_1(r, \omega) = \inf_x (K \ln \frac{K}{z} + (1-K) \ln \frac{1-K}{1-z})
\]

where

\[
K = z + \frac{a_1}{\psi_A(x)} - \frac{r}{\psi(x)}, \quad z = x - a_1/\psi_A(x)
\]

\[
\therefore \quad \text{as} \quad a_1 \downarrow 0,
\]

\[
I_1(r) = \inf \frac{(\frac{a_1}{\psi_A} - \frac{r}{\psi})^2}{2(z-z^2)} (1 + o(1))
\]

\[
= \frac{c_1^2}{2} (\mu-r)^2 (1 + o(1))
\]

where
\[ c_1 = c_1(\psi) = \lim_{r \uparrow \mu} \inf \frac{a_1 \psi_0/\psi_A - r \psi_0/\psi}{\mu - r} \]

\[ \mu = a_1 \inf \psi/\psi_A, \quad 0 < r < \mu \]

\[ \therefore \quad \mu_2 \approx \frac{c_1 \mu}{c_0 + c_1} \quad \text{where} \quad c_0 = c_0(\psi) \]

and so

\[ I(\mu_2) \sim \frac{c_0^2 c_1^2 \mu^2}{2(c_0 + c_1)^2} \cdot \]

It follows from Theorem 2(c) that

\[ J(F_0, \omega_1) = \frac{a_1^2}{8} \sup \frac{\psi_2/\psi_0^2}{\psi_A/\psi_0^2} (1 + o(1)) \cdot \]

(2) follows.

(b) and (c) follow likewise from Theorem 3.
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