ASYMPTOTIC SUFFICIENCY AND EXACT ESTIMABILITY
IN BAYESIAN EXPERIMENTS

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TECHNICAL REPORT NO. 175
SEPTEMBER 1981

PREPARED UNDER THE AUSPICES
OF
NATIONAL SCIENCE FOUNDATION GRANT MCS 80-24649

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Abstract

Conditional independence is used to examine the connections between sufficiency in an asymptotic experiment and in finite sample size experiments, to build a "strictly Bayesian" definition of estimability and to exhibit the relation between asymptotic sufficiency and exact estimability in particular classes of experiments.

Key words and phrases: Bayesian experiment, conditional independence, asymptotic sufficiency, estimability and consistency, zero-one σ-fields, identification, stationary and IID experiments.

AMS/MOS Numbers: Primary 62A15; Secondary 62B05.
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1. Introduction

This paper has two objectives. First, we want to take another step in the direction started in our previous work on conditional independence and theory of reductions (sufficiency, ancillarity, etc.) of Bayesian experiments. We had analyzed the reductions in the "one shot" case (Florens-Mouchart (1977), Mouchart-Rolin (1978)) and in the sequential case (Florens-Mouchart (1980), Florens-Mouchart-Rolin (1980)). In this last case, we were only interested by the relation between sequential reductions and initial reductions. We now want to examine, in the case of sufficiency, the connection between a reduction admissible in every finite sample size model and a reduction admissible in the asymptotic experiment. This analysis will be done in Section 5.

Secondly, we want to define a Bayesian concept of consistency which satisfies two conditions. The first condition is that this concept does not require any reference to a "true value" of the parameter. Such a reference leads to the sampling analysis of Bayesian procedures and brings out the Bayesian fundamental structure which is, in our approach, a unique probability measure on the product of the parameter space and of the sample space. The second property needed is a good relation between consistency and identification. We have defined (Florens-Mouchart (1977)) the identification in Bayesian analysis as a property of minimum sufficiency of the $\sigma$-field in the
parameter space instead of the injectivity of the mapping which associates a sampling probability to a parameter and we want the consistency to imply the identification in our sense. This implication holds in a classical approach. These two requirements are satisfied by the definition of exact estimability given in Section 4. This definition can be viewed as a Bayesian concept of consistency.

These two purposes of the paper seem different, but we shall exhibit their connections in studying the exact estimability of some particular classes of experiments (stationary or I.I.D. experiments). All the topics presented in this paper belong to the foundations of statistics and have been treated by a great number of authors. Some references will be given in the paper, but we will not try to construct a complete bibliography. The results we will present are not surprising and are mainly a restatement of known results in sampling statistics. However, we think that we will prove the (mathematical) simplicity of Bayesian statistics, even in asymptotic theory.

As in our previous work, the main tool of our presentation is the conditional independence, and the definition and main results in this field will be presented in the second part of the introduction. Section 2 will be devoted to main definitions of Bayesian experiments. Sufficiency in asymptotic experiments and exact estimability will be presented in Sections 3 and 4.

Notations and conditional independence.

Let \((M, \mathfrak{m}, P)\) be a probability space and \(\mathfrak{n}\) be a sub \(\sigma\)-field of \(\mathfrak{m}\). We denote \(\overline{\mathfrak{n}}\) as the sub \(\sigma\)-field of \(\mathfrak{m}\) generating by \(\mathfrak{n}\) and all the null sets of \(\mathfrak{m}\). If \(\xi\) is a random variable, \(\xi \in \mathfrak{n}\) means that \(\xi\) is an
integrable \( \mathfrak{m} \)-measurable function. If \( \xi \in \mathfrak{m} \), \( \mathfrak{m} \) denotes the conditional expectation of \( \xi \) given \( \mathfrak{m} \) (see Hunt (1966)). \( \mathfrak{m}_1 \) and \( \mathfrak{m}_2 \) we denote as being sub-\( \sigma \)-fields of \( \mathfrak{m} \) and we call the projection of \( \mathfrak{m}_2 \) on \( \mathfrak{m}_1 \) (and we note \( \mathfrak{m}_1 \mathfrak{m}_2 \)) the sub-\( \sigma \)-field of \( \mathfrak{m}_1 \) generated by every version of the conditional expectation given \( \mathfrak{m}_1 \) of every integrable \( \mathfrak{m}_2 \)-measurable random variable. This definition crucially depends on \( P \) (see Mouchart-Rolin (1978)).

If \( \mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_3 \) are sub-\( \sigma \)-fields of \( \mathfrak{m} \), we say that \( \mathfrak{m}_1 \) and \( \mathfrak{m}_2 \) are conditionally independent given \( \mathfrak{m}_3 \) and we note \( \mathfrak{m}_1 \parallel \mathfrak{m}_2 |\mathfrak{m}_3 \) if one of the following equivalent properties is verified:

\[
\begin{align*}
(\text{i}) \quad & \forall \xi_1 \in \mathfrak{m}_1 \quad (i=1,2) \quad \mathfrak{m}_3 (\xi_1, \xi_2) \overset{a.s.}{=} \mathfrak{m}_3 (\xi_1, \mathfrak{m}_3 \xi_2), \\
(\text{ii}) \quad & \forall \xi \in \mathfrak{m}_2 \quad (\mathfrak{m}_1 \mathfrak{m}_3 \xi) \overset{a.s.}{=} \mathfrak{m}_3 \xi, \\
(\text{iii}) \quad & \forall \xi \in \mathfrak{m}_1 \mathfrak{m}_3 \quad \mathfrak{m}_2 (\mathfrak{m}_3 \xi) \overset{a.s.}{=} \mathfrak{m}_2 \xi,
\end{align*}
\]

(see Dellacherie-Meyer (1975) and Mouchart-Rolin (1978); all the following results are proved in this last paper).

Note that in the relation \( \mathfrak{m}_1 \parallel \mathfrak{m}_2 |\mathfrak{m}_3 \), any \( \mathfrak{m}_i \) \( (i=1,2,3) \) can be replaced by \( \overline{\mathfrak{m}_i} \). We recall the main results about conditional independence that we shall use in this paper.

**Lemma 1.1.** Let \( \mathfrak{m}_i \) \( (i=1,2,3,4) \) be sub-\( \sigma \)-fields of \( \mathfrak{m} \). The following properties are equivalent:

\[
\begin{align*}
(\text{i}) \quad & \mathfrak{m}_1 \parallel \mathfrak{m}_2 |\mathfrak{m}_3 \quad \text{and} \quad \mathfrak{m}_1 \parallel \mathfrak{m}_4 |\mathfrak{m}_2 \mathfrak{m}_3, \\
(\text{ii}) \quad & \mathfrak{m}_1 \parallel (\mathfrak{m}_2 \mathfrak{m}_4) |\mathfrak{m}_3, \\
(\text{iii}) \quad & \mathfrak{m}_1 \parallel \mathfrak{m}_4 |\mathfrak{m}_3 \quad \text{and} \quad \mathfrak{m}_1 \parallel \mathfrak{m}_2 |\mathfrak{m}_4 \mathfrak{m}_3.
\end{align*}
\]
Corollary 1.2. If \( m_1 \parallel m_2 | m_3 \) and

\[
\begin{align*}
    m_4 &\subset m_1, \\
    m_5 &\subset m_1 \lor m_3, \\
    m_6 &\subset m_2 \lor m_3,
\end{align*}
\]

we have:

(i) \( m_4 \parallel m_2 | m_3 \),

(ii) \( (m_1 \lor m_5) \parallel (m_2 \lor m_6) | m_3 \),

(iii) \( m_1 \parallel m_2 | m_3 \lor m_5 \lor m_6 \),

Lemma 1.3. For every \( m_1 \) and \( m_2 \) we have:

(i) \( m_1 \parallel m_2 | m_1 m_2 \)

(ii) \( \forall m_3 \subset m_1 \) such that \( m_1 \parallel m_2 | m_3 \) then \( m_1 m_2 \subset m_3 \)

(iii) If \( m_1 \parallel m_2 | m_3 \), then \( m_1 \parallel m_2 | m_2 m_3 \) for every \( m_3 \).

2. Bayesian Experiments

A Bayesian experiment is defined by a probability measure \( \pi \) on a measurable product space \((A \times S, G \otimes S)\). \((A, G)\) is the parameter space and \((S, G)\) is the sampling space. The marginal probabilities on \((A, G)\) and \((S, G)\) are respectively the prior probability and the predictive probability. If \( X \) is an event of \( G \), the conditional probability of \( X \) given \( G \) is the sampling probability of \( X \), and, if \( E \) is an event of \( G \), the conditional probability of \( E \) given \( G \) is the posterior probability of \( E \).

In this paper the existence of regular conditional probability is never assumed, except in some comments devoted to the comparison between Bayesian and classical viewpoints.
A sequential Bayesian experiment is defined by a Bayesian experiment 
\((A \times S, G \otimes \mathcal{S}, \pi)\) and by an increasing sequence of \(\sigma\)-fields \((\mathcal{S}_n)_n\) \((n=0,1,\ldots)\) included in \(\mathcal{S}\) such that \(\bigcup_{n=0}^\infty \mathcal{S}_n = \mathcal{S}\). We denote \(\pi_n\) the restriction of \(\pi\) to \(G \otimes \mathcal{S}_n\). Generally, \(\pi\) is constructed as the projective limit of a sequence of \(\pi_n\) by using Prokhorov's theorem, or by the combination of a probability \(\pi_0\) on the parameter space and the initial conditions \((\mathcal{S}_0)\) of the process and a sequence of conditional probabilities on \(\mathcal{S}_n\) given \(G\) and \(\mathcal{S}_{n-1}\) (Ionescu-Tulcea theorem – see Neveu (1964)). However, the construction of an asymptotic Bayesian experiment is not the object of this paper and we shall concentrate our analysis on the properties of the mathematical structure precedingly defined.

Let us remark that all the \(\pi_n\) and \(\pi\) have the same marginal probability on \((A,G)\) and that the marginal of \(\pi_n\) on \((S,\mathcal{S}_n)\) is the restriction of the marginal of \(\pi\) on \((S,\mathcal{S})\) to \(\mathcal{S}_n\). If \(\pi \in \mathcal{S}_n\), its conditional expectation \(G\) is the same for every \(\pi_{n'}\) such that \(n' \geq n\). (The sampling probability defined by \(\pi_n\) is the restriction of the sampling probability defined by \(\pi_{n'}\), \(n' \geq n\) or by \(\pi\).) It is fundamental to note that for every \(\xi \in G\) the sequence \(\mathcal{S}_n \xi\) of the conditional expectations of \(\xi\) given \(\mathcal{S}_n\) is a martingale closed on the right by \(\xi\). This fact implies that the sequence \((\mathcal{S}_n \xi)_n\) is uniformly integrable and converges almost surely to \(\mathcal{S} \xi\) (Dellacherie-Meyer (1980), Chap. V).

A sub-\(\sigma\)-field \(\mathcal{J}\) of \(\mathcal{S}\) is said to be sufficient if \(G\) and \(\mathcal{S}\) are independent conditionally to \(\mathcal{J}\), or, with our notation, if \(G \parallel \mathcal{S} | J\) (the letters \(G\) and \(\mathcal{S}\) are used to represent both the \(\sigma\)-fields on \(A\) and \(\mathcal{S}\) and the \(\sigma\)-fields of cylinders in \(G \otimes \mathcal{S}\)). The projection of \(G\) on \(\mathcal{S}\), noted \(G \mathcal{S}\) is the minimum sufficient \(\sigma\)-field: every \(J\) sufficient is such that \(G \mathcal{S} \subset J\). A sub-\(\sigma\)-field \(\mathcal{B}\) of \(G\) is sufficient if \(G \parallel \mathcal{S} | \mathcal{B}\)
and $G_\mathcal{S}$ is the (almost sure) minimal sufficient $\sigma$-field of it. A sub-$\sigma$-field $\mathcal{S}$ of $G$ is identified if $\mathcal{S} a.s. \subseteq G_\mathcal{S}$. The experiment is said to be identified if $G$ is identified.

Details on the preceding definitions can be found in our previous work, in particular in Florens-Mouchart (1977). The following lemma is not in our previous papers and will be used in Section 4.

**Lemma 2.1.** If $\mathcal{S}$ is a sufficient sub-$\sigma$-field of $G$ in $(\mathcal{A} \times \mathcal{S}, G \otimes \mathcal{S}, \pi)$ one has $G_\mathcal{S} a.s. \subseteq G_\mathcal{S}$.

Proof. In general, $G_\mathcal{S} a.s. \subseteq G_\mathcal{S}$. The converse inclusion follows from the fact that $G_\mathcal{S}$ is a sufficient sub-$\sigma$-field of $G$ in $(\mathcal{A} \times \mathcal{S}, G \otimes \mathcal{S}, \pi)$, or equivalently from the relation $G_\mathcal{S} \perp \mathcal{S} | G_\mathcal{S}$. This independence is implied (Lemma 1.1) by the two independences $G_\mathcal{S} \perp \mathcal{S} | \mathcal{S}$ (sufficiency of $\mathcal{S}$) and $G_\mathcal{S} \perp \mathcal{S} | G_\mathcal{S}$ (definition of $G_\mathcal{S}$). 

3. **Asymptotic Sufficiency**

Let us consider an asymptotic Bayesian experiment $(\mathcal{A} \times \mathcal{S}, G \otimes \mathcal{S}, \pi, (\mathcal{G}_n)_{n \geq 0})$ and a sequence of sub-$\sigma$-fields of $G$, denoted $(\mathcal{G}_n)_{n \geq 0}$ satisfying the following hypotheses:

- $H_1: \forall n \geq 0 \quad \mathcal{G}_n \subseteq \mathcal{S}_n$,
- $H_2: \forall n \geq 0 \quad G \perp \mathcal{S}_n | \mathcal{G}_n$.

In other words, $\mathcal{G}_n$ is a sufficient $\sigma$-field of the experiment "stopped at time $n". The sequence $(\mathcal{G}_n)_{n}$ is increasing, but the sequence $(\mathcal{G}_n)_{n}$ is not, in general, an increasing one.

We denote $\mathcal{G}_\infty$ as the tail $\sigma$-field of the sequence $(\mathcal{G}_n)_{n \geq 0}$ defined by

$$\mathcal{G}_\infty = \bigcap_{n \geq 0} \bigvee_{p > 0} \mathcal{G}_{n+p}.$$ 

(see Mouchart-Rolin (1978), Section 3)
represents the $\sigma$-field generated by $\bigvee_{p>0} \mathcal{I}_{n+p}$ and all the null sets of $\mathcal{G}$ (for the restriction of $\pi$ to $\mathcal{G}$). If $(\mathcal{I}_n)_n$ is an increasing sequence $\mathcal{I}_\infty = \bigvee_{n>0} \mathcal{I}_n$. $\mathcal{I}_\infty$ is also called the asymptotic $\sigma$-field of the sequence $(\mathcal{I}_n)_n$.

**Theorem 3.1.** If for any $n$, $\mathcal{I}_n$ is a sufficient $\sigma$-field in $(A\times S, G \otimes \mathcal{G}_n, \pi_n)$ (hypothesis H1 and H2), $\mathcal{I}_\infty$ is a sufficient $\sigma$-field in $(A\times S, G \otimes \mathcal{G}, \pi)$. Then $\mathcal{I}_\infty$ is said to be asymptotically sufficient.

**Proof.** We have to prove $G \bigparallel \mathcal{G} \big|_{\mathcal{I}_\infty}$.

(i) We first remark that $\forall n \geq 0 \forall m \geq 0 \ G \bigparallel \mathcal{G}_n \big|_{p=0} \mathcal{I}_{n+p}$. This conditional independence follows from $G \bigparallel \mathcal{G}_{n+m} \big|_{p=0} \mathcal{I}_{n+m}$. As $\bigwedge_{p=0}^{m-1} \mathcal{I}_{n+p} \subseteq \mathcal{G}_{n+m}$ $G \bigparallel \mathcal{G}_{n+m} \big|_{p=0} \mathcal{I}_{n+m}$ implies $G \bigparallel \mathcal{G}_n \big|_{p=0} \mathcal{I}_{n+m}$ (Corollary 1.3) which implies $G \bigparallel \mathcal{G}_n \big|_{p=0} \mathcal{I}_{n+m}$ as $\mathcal{G}_n \subseteq \mathcal{G}_{n+m}$.

(ii) Let us prove that $G \bigparallel \mathcal{G}_n \big|_{p=0}^{\infty} \mathcal{I}_{n+p}$. This relation follows from a convergence theorem of martingale (see, e.g., Doob (1953) or Dellacherie-Meyer (1980)). Let $\eta \in \mathcal{G}_n$. The sequence $(G \bigvee_{p=0}^{m} \mathcal{I}_{n+p})^\eta$ is a martingale adapted to the increasing sequence of $\sigma$-fields $(G \bigvee_{p=0}^{m} \mathcal{I}_{n+p})^m$ and converges a.s. to $(G \bigvee_{p=0}^{\infty} \mathcal{I}_{n+p})^\eta$. For the same reason $(\bigvee_{p=0}^{m} \mathcal{I}_{n+p})^\eta$ converges a.s. to $(\bigvee_{p=0}^{\infty} \mathcal{I}_{n+p})^\eta$. From (i) we have the equality $(G \bigvee_{p=0}^{m} \mathcal{I}_{n+p})^\eta = (G \bigvee_{p=0}^{\infty} \mathcal{I}_{n+p})^\eta$, then we have the a.s. equality of the limits and the conditional independence stated verified.

(iii) We now prove that $\forall n \ G \bigparallel \mathcal{G} \big| \bigvee_{p=0}^{\infty} \mathcal{I}_{n+p}$. Note that $\forall m, n$

$$
\bigvee_{p=0}^{\infty} \mathcal{I}_{m+p} \subseteq \mathcal{G}_n \bigvee_{p=0}^{\infty} \mathcal{I}_{n+p}.
$$

Then the conditional independence

$$
G \bigparallel \bigvee_{p=0}^{\infty} \mathcal{I}_{m+p} \big|_{p=0} \mathcal{G}_n \bigvee_{p=0}^{\infty} \mathcal{I}_{n+p}
$$

is trivially true. We have also
proven that \( G \supseteq s_n \mid \nu \succsim \nu_{n+p} \). These two conditional independences imply (Lemma 1.1) \( G \supseteq s_n \mid \nu \succsim \nu_{m+p} \) for any \( n, m \). If \( \xi \in G, \left( s_n \mid \nu \succsim \nu_{m+p} \right) \xi = a.s. \left( \nu \succsim \nu_{m+p} \right) \xi. \) By a convergence theorem of martingale \( \lim_{n \to \infty} \left( s_n \mid \nu \succsim \nu_{m+p} \right) \xi = a.s. \left( \nu \succsim \nu_{m+p} \right) \xi = a.s. \left( \nu \succsim \nu_{m+p} \right) \xi. \) The last equality implies \( G \supseteq \nu \succsim \nu_{n+p} \).

(iv) Finally, let \( \xi \in G \). We have proven that \( s \xi \in \nu \succsim \nu_{n+p} \) for any \( n \). Then \( s \xi \in \bigcap_{n \geq 0} \nu \succsim \nu_{n+p} = \nu \infty \) and \( G \supseteq \nu \infty. \)

Corollary 3.2. If \( \nu_n \) is sufficient for \( \beta \) in \((A \times S, G \otimes s_n, \pi_n)\) (i.e., \( \forall n \geq 0 \nu_n \subseteq s_n \) and \( G \supseteq s_n \mid \nu_n \)), \( \nu \infty \) is asymptotically sufficient for \( \beta \) (i.e., \( G \supseteq \nu \nu \infty \)).

The proof is identical to the proof of the preceding theorem.

Theorem 3.1 can be viewed as a Bayesian version of a family of very similar results. For example, Theorem 3.1 (and Section 4) of Dinkin (1978) or the main theorem given by Diaconis-Freedman (1978) (in which more references can be found) analyze the same kind of situations. (See also Lauritzen (1980).)

In these papers the authors consider the following problem: let a family \( (Q_n)_{n \geq 0} \) of conditional probabilities on \( s_n \) given \( \nu_n \) (satisfying some comparability assumptions); can we characterize the (convex) family of all probabilities \( P \) on \( s \) such that, for any \( P \), the conditional probability of \( P \) on \( s_n \) given \( \nu_n \) is \( Q_n \)? Our problem is quite similar, but instead of considering all probabilities satisfying this condition we consider a parameterized family and we assume that the parameter space is provided with a \( \sigma \)-field such that the considered family is a transition probability and that
there exists a (prior) probability measure on this $\sigma$-field. This regularity assumption allows us to use the powerful tool of conditional independence and simplifies the proof. To make the connection easier we rewrite our result in the case of the existence of regular conditional probabilities. Let

$$P^G_\mathcal{G}$$

be the conditional probabilities on $\mathcal{G}$ given $G$ (the parametrized family of probabilities) and $\mu$ the prior probability. $\mu$ and $P^G_\mathcal{G}$ define a probability $\pi$ in $G \otimes \mathcal{G}$ by extending $\pi(E \times X) = \int_E P^G(X) d\mu$. Let $P_\mathcal{G}$ be the marginal probability of $\pi$ on $\mathcal{G}$. Our assumption ($G \|_{\mathcal{S}_n} \mathcal{J}_n \forall n > 0$)

can be written as:

$$P^G_\mathcal{S}_n (X) = P^G_\mathcal{J}_n (X) \ a.s. \ \forall X \in \mathcal{S}_n$$

($P^G_\mathcal{J}_n$ and $P^G_{\mathcal{J}_n}$ are the conditional probabilities on $\mathcal{S}_n$ given $G \otimes \mathcal{J}_n$ and $G \otimes \mathcal{J}_n$, respectively).

The family $(P^G_\mathcal{J}_n)_n$ plays the role of the family $(Q^G_\mathcal{J}_n)_n$. Then we construct $P^G_\mathcal{J}_\infty$, the conditional probability on $\mathcal{S}$ given $\mathcal{J}_\infty$ and we have proven that

$$P^G_\mathcal{G} (X) = P^G_\mathcal{J}_\infty (X) \ a.s. \ \forall X \in \mathcal{S}$$

implies $P^G_\mathcal{J}_\infty (X) = \int P^G_\mathcal{J}_\infty (X) \ dP^G_{\mathcal{J}_\infty} \ a.s. \ \text{then}$

the family of $P^G_\mathcal{G}$ is (a.s.) characterized by the family of $P^G_{\mathcal{J}_\infty}$ (the family of conditional probabilities on $\mathcal{J}_\infty$ given $G$).

Theorem 3.1 gives a way to construct an asymptotic sufficient $\sigma$-field given a sequence of $\sigma$-fields sufficient for any sample size. We are now interested by the converse problem which is related to the minimum sufficiency in the asymptotic experiment.

**Theorem 3.3.** Let $\mathcal{J}$ be an asymptotic sufficient $\sigma$-field. Then there exists a sequence $\mathcal{J}_n$ sufficient for $\mathcal{S}_n$ such that $\mathcal{J}$ is the asymptotic $\sigma$-field of this sequence.

**Proof.** Let $\mathcal{J}_n = \mathcal{S}_n \mathcal{J}$ be the projection of $\mathcal{J}$ on $\mathcal{S}_n$. By Lemma 1.3 (iii) we have $G \|_{\mathcal{S}_n} \mathcal{S}_n \mathcal{J}$ as a consequence of $G \|_{\mathcal{S}_n} \mathcal{J}$ (implied by the assumption
We just have to verify the relation \( \mathcal{J} = \bigcap_{n>0} \bigvee_{p>0} \mathcal{G}_{n+p} \mathcal{J} \). Let \( \mathcal{J}_m = \mathcal{J} \cap \mathcal{G}_m \). We easily verify that \( \forall \mathcal{J}_m = \bigcap_{n>0} \bigvee_{p>0} \mathcal{G}_{n+p} \mathcal{J}_m \). As \( \mathcal{J} = \bigvee_{m \geq 0} m \), we have to prove \( \bigvee_{m \geq 0} m \bigvee_{n \geq 0} \bigvee_{p \geq 0} \mathcal{G}_{n+p} \mathcal{J}_m = \bigvee_{n \geq 0} \bigvee_{p \geq 0} \mathcal{G}_{n+p} \mathcal{J}_m \). This equality follows from the two following properties, which can be verified easily:

- If \( m_{m,n} \) is a family of \( \sigma \)-fields such that \( \forall m \bigvee_{n \geq m} m_{m,n} \), and if \( n' \geq n \), then \( \bigvee_{m \geq n} m_{m,n} = \bigvee_{n \geq n'} m_{m,n} \).

- If \( m_n \) is an increasing family of \( \sigma \)-field and \( \nu \) another \( \sigma \)-field we have \( \bigvee_{n \geq n} m_n = \nu \bigvee_{n \geq n} m_n \).

\[ \square \]

Theorem 3.4. A sufficient sub-\( \sigma \)-field of the asymptotic experiment is minimum if and only if it is the asymptotic \( \sigma \)-field of a sequence of minimum sufficient sub-\( \sigma \)-fields of \( \mathcal{G}_n \).

Proof. Let \( \mathcal{J} \) be a minimum sufficient \( \sigma \)-field of the asymptotic experiment. Then \( \mathcal{J} = \mathcal{G} \) a.s. and \( \mathcal{G}_n \mathcal{J} = \mathcal{G}_n (\mathcal{G}) = \mathcal{G}_n \) a.s. Then \( \mathcal{G}_n \mathcal{J} \) is a minimum sufficient sub-\( \sigma \)-field of \( \mathcal{G}_n \).

- Reciprocally, let \( \mathcal{J}_n \) be a sequence of minimum sufficient \( \sigma \)-fields of \( \mathcal{G}_n \) and \( \mathcal{J}_\infty \) be the asymptotic \( \sigma \)-field of \( (\mathcal{J}_n)_n \). Let \( \mathcal{J} \) be another sufficient sub-\( \sigma \)-field of \( \mathcal{G} \). \( \mathcal{G}_n \mathcal{J} \) is sufficient in \( \mathcal{G}_n \) and we have \( \mathcal{G}_n \mathcal{J} \subseteq \mathcal{G}_n \mathcal{J} \). Then \( \bigvee_{n \geq 0} \bigvee_{p \geq 0} \mathcal{G}_{n+p} \mathcal{J} \subseteq \mathcal{G}_n \mathcal{J} \) and \( \mathcal{J}_\infty \mathcal{J} \subseteq \mathcal{G}_n \mathcal{J} \).

We shall now analyze the asymptotic sufficiency in particular classes of experiments and we shall give a result about sufficiency in a stationary experiment which will be helpful in the analysis of estimability.

We now assume that there exists a measurable space \( (X, Y) \) such that \( S = \prod_{n \geq 0} X_n \) and \( s = \omega \ Y_n \) with \( X_n = X \) and \( Y_n = Y \) for any \( n \). Let us consider an asymptotic Bayesian experiment with such a sampling space. This
experiment is defined by a probability \( \pi \) on \( A \times \left( \prod_{n \geq 0} X_n \right), G \otimes \left( \otimes_{n \geq 0} Y_n \right) \)

and the sequence of \( S_n \) is now the sequence of \( \otimes_{p=0}^n Y_p \). Let us denote \( \pi_I \) the restriction of \( \pi \) to \( G \otimes \left( \otimes_{n \in I} Y_n \right) \) for any finite subset of \( \mathbb{N} \). If \( I^+p = \{ i+p | i \in I \} \) and if \( |I| \) is the number of elements in \( I \), let us note that \( \pi_I \) and \( \pi_{I^+p} \) are both probabilities in \( (A \times \chi^{|I|}, G \otimes \mathcal{G}^{|I|}) \). This experiment is said to be a stationary Bayesian experiment if, for any \( I \) and \( p \), \( \pi_I = \pi_{I^+p} \). Let us remark that this definition is a property of the joint process generating the parameter and the sample. An experiment constructed by a prior probability and a stationary sampling process is not necessarily a stationary Bayesian experiment. In particular, experiments defined by a stationary sampling process indexed by incidental parameters (in the case of Kiefer and Wolfowitz (1956)) do not satisfy the stationary condition. Conversely, if a Bayesian experiment is stationary, the sampling process is (a priori almost surely) stationary. We now use the following notation: \( \chi^m_n \) denotes \( \otimes_{p=n}^m Y_p (m \geq n) \) and \( \chi^\infty_n = \otimes_{p=n}^\infty Y_p \). \( \pi^m_n \) denotes the restriction of \( \pi \) to \( G \otimes \chi^m_n \).

We consider the sequence of \( (\chi_n)_{n \geq 0} \) and its asymptotic \( \sigma \)-field \( \chi_\infty = \bigcap_{n \geq 0} \chi^\infty_n \). \( \chi_n \) is not, in general, a sufficient \( \sigma \)-field of \( A \times \prod_{p=0}^n X_p, G \otimes \chi^n, \pi_n \), but the stationary hypothesis implies the asymptotic sufficiency of \( \chi_\infty \).

**Theorem 3.5.** In a stationary Bayesian experiment the asymptotic \( \sigma \)-field \( \chi_\infty \) of the sequence of \( \sigma \)-fields of coordinates is asymptotically sufficient.

**Proof.** The proof is similar to the proof of Theorem 3.1. The stationary hypothesis implies:
\[ \forall \xi \in \mathcal{G} \quad \chi_0^{n+\xi} \text{ a.s. } \chi_n^{m+\xi} \]

because \( \pi_0^n = \pi_m^{n+m} \). The equality of the limits when \( n \to \infty \) is then satisfied and we get \( \mathcal{G} \parallel_0 \chi_0^\infty \parallel_\chi_n^\infty \). We use the same argument as in part (iv) of the proof of Theorem 3.1 and we get \( \mathcal{G} \parallel_0 \chi_0^\infty \bigcap_{n \geq 0} \chi_n^\infty \).

One can remark that the stationary assumption of the preceding theorem is too strong of a hypothesis. The result given by this theorem could be obtained if \( \chi_0^{n,\xi} = \chi_m^{n+m,\xi} \) a.s. for every \( \xi \in \mathcal{G} \) and every \( m \) and \( n \). This property means that the posterior probability given a finite sample size does not depend on the beginning time of the sampling. Without assumptions on the predictive process, this does not imply stationarity. However, the assumption of Theorem 3.4 is more easily checked because it can be verified without computing the posterior probability.

4. Exact Estimability

Let a Bayesian experiment \((\mathcal{A} \times \mathcal{S}, \mathcal{G} \circ \mathcal{S}, \pi)\) and \( \mathcal{B} \) be a sub-\( \sigma \)-field of \( \mathcal{G} \). \( \mathcal{B} \) will be said to be \textit{exactly estimable} if \( \mathcal{B} \parallel_\mathcal{G} | \mathcal{S} \). This conditional independence means that the conditional probability of an event of \( \mathcal{B} \) is a.s. 0 or 1: If \( E \in \mathcal{B} \) and if \( 1_E \) is the characteristic function of \( E \), the exact estimability of \( \mathcal{B} \) implies \( (1_E)^2 \text{ a.s. } 1_E \) by using the first characteristic of the conditional independence. \( \mathcal{B} \parallel_\mathcal{G} | \mathcal{S} \) formalizes the fact that \( \mathcal{B} \) is "perfectly known" after the observation of the sample. Note that \( \mathcal{B} \parallel_\mathcal{G} | \mathcal{S} \) is equivalent to \( \mathcal{B} \parallel \mathcal{B} | \mathcal{S} \). (\( \mathcal{B} \subseteq \mathcal{G} \) implies that \( \mathcal{B} \parallel_\mathcal{G} | \mathcal{S} \Rightarrow \mathcal{B} \parallel_\mathcal{B} | \mathcal{S} \). Reciprocally, \( \mathcal{B} \parallel_\mathcal{B} | \mathcal{S} \) and \( \mathcal{B} \parallel_\mathcal{G} | \mathcal{B} \circ \mathcal{S} \), which is trivially true, implying \( \mathcal{B} \parallel_\mathcal{G} | \mathcal{S} \)). Then \( (1_E)^2 \text{ a.s. } 1_E \) for any \( E \in \mathcal{B} \) is a necessary and sufficient condition for the exact estimability. The
analysis of "0-1 sets" has a long history in probability theory and in statistics and we do not give a complete list of references. The statistical paper closer than our work is probably the paper by Breiman, LeCam and Schwartz (1964). The connection between 0-1 sets and the analysis of the relation between sufficiency and ancillarity was presented in Mouchart-Rolin (1978), in which more references can be found. The presentation of exact estimability in terms of conditional independence will clarify the connection with sufficiency and allows us to build a homogenous presentation of many problems in Bayesian statistics.

Exact estimability can be viewed as a Bayesian definition of consistency without any reference to a concept of a "true parameter". Let us consider a sequential Bayesian experiment \((A \times S, G \otimes S, \pi, (G_n)_{n \geq 0})\) and a sub-\(\sigma\)-field \(\mathcal{G}\) of \(G\) asymptotically exactly estimable, i.e., such that \(\mathcal{G} \perp G|S\). This condition is equivalent to \(G \xi = \xi\) a.s. for any \(\xi \in \mathcal{G}\) (because \((G \otimes S) \xi = \xi\) and is also equivalent to \(\lim\limits_{n \to \infty} G_n \xi = \xi\) (because \(G_n \xi \to G \xi\) a.s.). In other words, \(\mathcal{G}\) is exactly estimable if and only if the sequence of conditional expectations of any \(\mathcal{G}\)-measurable integrable function \(\xi\) is a consistent estimator of \(\xi\) for the almost sure convergence.

Note that the probability used in this almost sure convergence is the probability \(\pi\) on the product \(G \otimes S\): there exists a set \(U\) in \(G \otimes S\) such that \(\pi(U) = 1\) and that for any \((a, s) \in U\) the limit of \(G_n\) (a real function of \(s\)) is equal to \(\xi\) (a real function of \(a\)).

In most of the previous works (see, e.g., Martin and Vagueisy (1969), Doob (1949), Friedman (1963), Berk (1970), Jones and Rothenberg (1980), etc.) asymptotic sampling properties of Bayes estimators or decisions were analyzed. Our presentation is based on the asymptotic feature of the joint probability
on the parameter space and on the sampling space and is different. In this paper we do not examine the connection between the two approaches.

Two definitions are necessary. An experiment is said to be exactly estimable if $G$ is exactly estimable. A sub-$\sigma$-field $\mathcal{J}$ of $\mathcal{G}$ is said to be 0-1 in the sampling probability if $\mathcal{J} \parallel \mathcal{G} | G$ (or equivalently $\mathcal{J} \parallel \mathcal{J}^* | G$). This definition is equivalent to the definition of the exact estimability (which is a 0-1 property in the posterior probability). The following result relates these two definitions.

Theorem 4.1. Let $A \times S, G \otimes S, \pi$ be a Bayesian experiment and $\mathcal{F}$ a sub-$\sigma$-field of $G$. Then $\mathcal{F}$ is exactly estimable if and only if there exists a sub-$\sigma$-field $\mathcal{J}$ of $\mathcal{F}$, 0-1 in the sampling probability such that $\mathcal{F}$ is a.s. the projection of $\mathcal{J}$ on $G$.

Proof.

(i) Let us assume that $G \mathcal{J} \subset \mathcal{F}$ such that $\mathcal{J} \parallel \mathcal{G} | G$ and $\mathcal{F} = G \mathcal{J}$ a.s.

We have to prove $G \mathcal{J} \parallel G | \mathcal{F}$ which is equivalent to $\mathcal{F} \parallel G | \mathcal{F}$ because $\mathcal{F} = G \mathcal{J}$ a.s.

$G \mathcal{J} \parallel G | \mathcal{F} \iff \forall \xi \in G \mathcal{J} \ (G \otimes S) \xi = a.s. \Rightarrow \mathcal{F} \xi = a.s. \Rightarrow \mathcal{F} \xi$.

As $G \mathcal{J}$ is generated by the family of the $\sigma$-fields generated by the conditional expectations of any $\eta \in \mathcal{F}$ given $G$, it is sufficient to verify the preceding property for $\xi = G \eta \ \eta \in \mathcal{F}$.

Then $\xi = a.s. \Rightarrow \mathcal{F} \xi = a.s. \Rightarrow G \mathcal{F} \eta$, which is implied by characterization (iii) of $\mathcal{J} \parallel \mathcal{J} | G$ (see Section 1).

(ii) Reciprocally, if $\mathcal{F}$ is exactly estimable, let us define $\mathcal{J} = G \mathcal{F}$, the projection of $\mathcal{F}$ in $\mathcal{G}$. We can reproduce the same proof as in the direct proof and we get $\mathcal{J} \parallel \mathcal{J} | G$. So $\mathcal{J}$ is a 0-1 $\sigma$-field
in the sampling probability. Finally, we must prove that $\mathcal{B} \text{ a.s. } G\mathcal{J}$, or equivalently $G(\mathcal{B}) \text{ a.s. } \mathcal{B}$. From $\mathcal{G} \perp \mid \mathcal{S} \mid \mathcal{G}$ we get $G(\mathcal{B}) \text{ a.s. } \mathcal{B}$. From $\mathcal{B} \perp \mid \mathcal{G} \mid \mathcal{S}$ we get $\mathcal{G} \text{ a.s. } \mathcal{B}$ and the proposition is then verified.

Theorem 4.1 would be repeated in exchanging $\mathcal{B}$ and $\mathcal{J}$. In fact, we have proven that 0.1 σ-fields in the sampling probability and in the posterior probability are "in duality" by the projection operator.

In sampling theory, identification is defined by the injectivity of the mapping which associates a sampling probability to any parameter and is often presented as a necessary condition for the existence of a consistent estimator (see, e.g., LeCam and Schwartz (1960), Rothenberg (1971), Schönfeld (1975), Deistler and Seifert (1978), etc.). The Bayesian definition of identification has been introduced (Florens (1974), Florens-Mouchart (1977), and Picci (1977)) in connection with sufficiency on the parameter space (see Barantkin (1961)). A Bayesian experiment is identified if the "all" priori probability is revised by the sample in the following sense: there does not exist a sub-σ-field on the parameter space such that, conditionally to this σ-field, the prior and the posterior probabilities are the same. However, connection between consistency and identification exists in Bayesian experiments and is given by the following corollary of Theorem 4.1.

**Corollary 4.2.** Any sub-σ-field $\mathcal{B}$ of $\mathcal{G}$ exactly estimable is identified.

**Proof.** $\mathcal{B}$ is identified if $\mathcal{B} \subset \mathcal{G}$. By Theorem 4.1 $\mathcal{B}$ exactly estimable implies $\mathcal{B} \text{ a.s. } G\mathcal{J}$ with $\mathcal{J} \subset \mathcal{S}$. From the definition of the projection it is clear that $\mathcal{J} \subset \mathcal{S}$ implies $G\mathcal{J} \subset G\mathcal{S}$ and the result follows.

The identification is not in general a sufficient condition for exact estimability. However, in some particular sampling processes this condition
becomes sufficient. It is the case of I.I.D. (identically independently distributed) sampling process that we shall examine now. An IID Bayesian experiment is a stationary Bayesian experiment such that (with the notation of Section 3) \( \otimes \chi_n \mid \otimes \chi_n \mid G \) for any finite subsets \( I \) and \( J \) of \( \mathbb{N} \) satisfying \( I \cap J = \emptyset \). We have the following result.

**Theorem 4.3.** An identified IID Bayesian experiment is exactly estimable.

**Proof.** We use the notation of Section 3. In an IID Bayesian experiment \( \chi_{\infty} \) is a sufficient sub-\( \sigma \)-field of \( \mathcal{G} \) (Theorem 3.5) and then (Lemma 2.1) the projections of \( \mathcal{G} \) and \( \chi_{\infty} \) on \( G \) are a.s. equal. The identification of the experiment implies that \( G_{a.s.} = \chi_{\infty} \). We have to verify that \( \chi_{\infty} \) is 0-1 in the sampling probability and the theorem will be proved by using Theorem 3.1.

The fact that \( \chi_{\infty} \) is 0-1 in the sampling probability is a version of the well known 0-1 law of Kolmogorov (see, e.g., Neveu (1964)), but we shall sketch a proof of this result because our definitions are not exactly the classical ones (in particular, the definition of \( \chi_{\infty} \) involves \( \otimes \chi_p \) instead of \( \otimes \chi_p \)).

By the independence assumption we have

\[
\chi_0^n \mid \chi_{n+p}^{n+q} \mid G \quad \forall 1 \leq p \leq q.
\]

By a convergence theorem if martingale, we get \( \chi_0^n \mid \chi_{n+p}^{\infty} \mid G \), which is equivalent to \( \chi_0^n \mid \chi_{n+p}^{\infty} \mid G \) and implies \( \chi_0^n \mid \chi_{\infty} \mid G \). A last application of a convergence theorem if martingale gives the result \( \chi_0^n \mid \chi_{\infty} \mid G \).

This theorem may be completed by making two remarks.

(i) An IID Bayesian experiment is identified if and only if the experiment in which only one observation is generated is identified.
More precisely, we can easily prove that any \( \mathcal{G} \) sub-\( \sigma \)-field of \( G \) sufficient for \( Y_n \) is sufficient for \( Y_0^n \) for any \( n \) and then is sufficient for \( Y_0^\infty \). This result is obtained by recurrence: if \( \mathcal{G} \) is sufficient for a any \( Y_n \), we have \( G \upharpoonright Y_n \mid \mathcal{G} \) \( \forall n \). In particular, \( G \upharpoonright Y_0 \mid \mathcal{G} \). Let us assume \( G \upharpoonright \bigotimes_{p=0}^{n} Y_n \mid \mathcal{G} \) and \( Y_0^n \mid G \) imply \( G \upharpoonright Y_{n+1} \mid Y_0^n \mid \mathcal{G} \) (Lemma 1.1). This last independence and \( G \upharpoonright Y_0^n \mid \mathcal{G} \) imply \( G \upharpoonright Y_0^{n+1} \mid \mathcal{G} \) (Lemma 1.1). The limit independence follows from the usual convergence theorem of martingales.

(ii) Theorem 4.3 may be easily extended in the following way: the minimum sufficient sub-\( \sigma \)-field of the parameter space of an IID Bayesian experiment is asymptotically exactly estimable. In Theorem 4.3 this minimum sufficient \( \sigma \)-field was \( G \) itself, but the proof may be reproduced in the case of a sub-\( \sigma \)-field.

The result proved in Theorem 4.3 shows the most natural way to analyze the exact estimability of an experiment: an experiment is exactly estimable if there exists a sufficient sub-\( \sigma \)-field in the sampling space which is 0-1 in the sampling probability (0-1 \( \sigma \)-fields greater than \( Y_\infty \) was characterized for some classes of processes, see, e.g., Sendler (1978)). For example, a sufficient condition for the exact estimability of an identified stationary experiment is that the asymptotic \( \sigma \)-field \( Y_\infty \) was 0-1 in the sampling probability. We know that the IID processes are not the only processes for which this property was verified. Moving average processes of finite order satisfy the 0-1 law and define, in general, stationary experiments.

The last case we want to examine is the exact estimability of the parameter \( \sigma \)-field in the presence of exogenous variables. The main example of
such a model is the regression model, but one can find many other examples, in particular, in econometric literature. For the simplicity of our exposition we shall restrict our analysis to the case of ancillarity of a sub-vector of the sampling vector. Extensions can be easily done with different definitions of exogeneity (see Florens-Mouchart (1977 and 1980)).

Let \((A, \mathcal{C})\) be the parameter space and \((Y, \mathcal{U})\) and \((Z, \mathcal{Z})\) the two measurable spaces. The experiment we shall consider is defined by a probability \(\pi\) on \(A \times \bigotimes_{n=0}^{\infty} X_n \times \bigotimes_{n=0}^{\infty} Z_n, \mathcal{G} \otimes \mathcal{U}_0 \otimes \mathcal{Z}_0^\infty\) with \(Y_n = Y, \mathcal{U}_n = \mathcal{U}, \mathcal{Z}_n = Z, \mathcal{Z}_n = Z\) for every \(n\). We made the following assumptions:

- **H1**: the asymptotic model is identified.

- **H2**: \(m \geq 0, \ \mathcal{U}_m \mathcal{Z}_m, G \otimes Z_m\) (the sampling distribution of \(\mathcal{U}_m\) given the sequence of \(Z_m\) "only depends" on \(Z_m\))

- **H3**: \(Z_0^\infty\) is ancillary: \(G \mathcal{Z}_0^\infty\) (which is equivalent to \(G \mathcal{Z}_n\) for any subset \(I\) of integers)

- **H4**: The experiment is IID conditionally to the sequence \((Z_n)_{n \geq 0}\).

More precisely, the experiment is conditionally stationary: for any \(I\) and any \(p\), the conditional probability on \(G \otimes \mathcal{U}_n\) given \(\otimes Z_n\) is almost surely equal to the conditional probability on \(G \otimes \mathcal{U}_n\) given \(\otimes Z_n\). We assume that the probabilities on \(\otimes Z_n\) and \(\otimes Z_n\) are equivalent. The experiment is conditionally independent: for every \(I\) and \(J \subset \mathbb{N}(I \cap J = \emptyset)\), \(\mathcal{U}_n \bigotimes \mathcal{U}_J \mathcal{Z}_n\).

Hypotheses H1-H4 extend to conditional models the hypotheses of Theorem 4.3. Let us point out that we cannot apply Theorem 4.3 to prove the exact
estimability of \$G\$ because we do not make assumptions on the process generating the exogenous variables. This process can be non-stationary and non-independent. The following result shows that a weaker condition of this process is sufficient.

**Theorem 4.4.** If the asymptotic \$\sigma\$-field \(Z_\infty = \bigcap_{n>0} \overline{Z_n}\) is 0.1, then \$G\$ is exactly estimable.

**Proof.** We only give a short proof because the arguments are mainly the same as in the previous theorem of this paper.

(i) The first step is to verify the asymptotic sufficiency of the asymptotic \$\sigma\$-field of the two processes. This \$\sigma\$-field can equivalently be written \(\bigcap_{p>0} (\overline{\psi_p} \otimes \overline{Z_p}) = \bigcap_{p>0} (\overline{\psi_p} \otimes Z_p) = \bigcap_{p>0} \overline{\psi_p} \otimes \bigcap_{p>0} \overline{Z_p}\) (see Mouchard-Rolin (1978), Section 3 and Theorem 3.3). From H3 and H4 we get \((Z_0^n \otimes \psi_0^n) \xi = (Z_{n+p}^p \otimes \psi_{n+p}^p) \xi\ \forall \xi \in G, \ V_n, p\). The same proof as in Theorem 3.5 can be repeated and we obtain the result.

(ii) The second step is to extend the zero-one law of Theorem 4.3. First we notice that H4 implies \(\psi_0^n \perp \psi_{n+1}^{n+p} | G \otimes \overline{Z_0}\) which gives

(a) \(\psi_0 \perp \bigcap_{p>0} \overline{\psi_p} | G \otimes \overline{Z_0}\).

Secondly, let us note that H2 implies \(Z_0^\infty \perp G\) and \(Z_0^\infty \perp G | \bigcap_{p>0} \overline{Z_p}\).

It was assumed that \(Z_0^\infty \perp \bigcap_{p>0} \overline{Z_p}\). Then Lemma 1.1 gives

(b) \(Z_0^\infty \perp \bigcap_{p>0} \overline{Z_p} | G\).

Thirdly, H2 and H4 give \(Z_0^n \perp \psi_{n+1}^{n+p} | G \lor \overline{Z_{n+1}^{n+p}}\). We get \(Z_0^n \perp \psi_{n+1}^{n+1} | G \lor \overline{Z_{n+1}^{n+1}}\) and then
(c) \( Z_0^\infty \bigcap_{p \geq 0} \overline{\psi_p} | G \vee \bigcap_{p \geq 0} \overline{Z_p} \),

(b) and (c) give (Lemma 1.1)

(d) \( Z_0^\infty \bigcap_{p \geq 0} (\overline{\psi_p} \otimes \overline{Z_p}) | G \)

and (a) and (d) imply \( \psi_0^\infty \otimes Z_0^\infty \bigcap_{p \geq 0} (\overline{\psi_p} \otimes \overline{Z_p}) | G \).

(iii) We can conclude the proof: \( \bigcap_{p \geq 0} (\overline{\psi_p} \otimes \overline{Z_p}) \) is sufficient so its projection on \( G \) is exactly \( G \) because the experiment is identified and \( \bigcap_{p \geq 0} (\overline{\psi_p} \otimes \overline{Z_p}) \) is 0-1 in the sampling process. Its projection is then exactly estimable.

One may remark that, as in Theorem 3.5, the hypotheses of Theorem 4.4 are too strong, but they can be easily verified (in particular, without computation of posterior probabilities). The assumption \( Z_0^\infty \bigcap_{p \geq 0} \overline{Z_p} \) replaces in our theorem the usual assumptions about the asymptotic behavior of exogenous variables (see, e.g., Malinvaud (1970)). This hypothesis is obviously satisfied if the process generating the exogenous variables is independent, but we know (see, e.g., Cohn (1965) and Iosifescu and Theodorescu (1969)) that there exist many non-independent and non-stationary processes satisfying this zero-one law.

Acknowledgements

This work was done during a summer visit at the Department of Statistics at Stanford University and has greatly benefited from helpful discussions in this department, in particular with P. Diaconis. I am also grateful to J. M. Rolin for previous discussions on this topic. The usual disclaimer applies.
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