REMARKS ON ASYMPTOTIC PROPERTIES OF
GENERALIZED BAYES ESTIMATORS IN NON-REGULAR CASES

BY

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Abstract

It is shown that the generalized Bayes estimator (GBE) with respect
to the quadratic loss and the Lebesgue measure, i.e. the Pitman estimator,
is two-sided asymptotically efficient in a non-regular case. In another
non-regular case the two-sided asymptotic efficiency is also discussed.
Further, the asymptotic expansion of the GBE and its asymptotic density
are given up to the order $n^{-1}$ in a truncated normal distribution case.

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1. Introduction

Recently the asymptotic efficiency including higher order in regular cases has been extensively studied by Akahira and Takeuchi (1978, 1981), Pfanzagl and Wefelmeyer (1978), Ghosh, Sinha and Wieand (1980), and others. The asymptotic sufficiency in non-regular cases has been discussed by Akahira (1976a) and recently extended by Weiss (1979). The asymptotic efficiency in non-regular cases has been discussed by Takeuchi (1974), Akahira (1975a, 1975b, 1976b), and Akahira and Takeuchi (1979b) in special situations and recently by Akahira (1982) in more general cases.

In this paper it is shown that the generalized Bayes estimator (GBE) with respect to the quadratic loss and the Lebesgue measure, i.e., the Pitman estimator, is two-sided asymptotically efficient in a non-regular case. In another non-regular case the two-sided asymptotic efficiency is also discussed. Further the asymptotic expansion of the GBE and its density are given up to the order \( n^{-1} \) in a truncated normal distribution case.

2. Definitions and Assumptions

Let \( X \) be an abstract sample space whose generic point is denoted by \( x \), \( \mathcal{B} \) a \( \sigma \)-field of subsets of \( X \) and \( \{ P_\theta : \theta \in \Theta \} \) a set of probability measures on \( \mathcal{B} \), where \( \Theta \) is called a parameter space. We assume that \( \Theta \) is an open set of \( \mathbb{R}^1 \). Consider \( n \)-fold direct products \( (X^n, \mathcal{B}^n) \) of \( (X, \mathcal{B}) \) and the corresponding product measure \( P^n_\theta \) of \( P_\theta \). An estimator of \( \theta \) is defined to be a sequence \( \{ \hat{\theta}_n \} \) of \( \mathcal{B}^n \)-measurable functions \( \hat{\theta}_n \) on \( X^n \) into \( \Theta \). For simplicity we denote \( \{ \hat{\theta}_n \} \) by \( \hat{\theta}_n \).

For an increasing sequence of positive numbers \( \{ c_n \} \) (\( c_n \) tending to infinity) an estimator \( \hat{\theta}_n \) is called consistent with order \( \{ c_n \} \)
(or \(\{c_n\}\)-consistent for short) if for every \(\varepsilon > 0\) and every \(\eta \in \Theta\) there exist a sufficiently small positive number \(\delta\) and a sufficiently large positive number \(L\) satisfying the following:

\[
\lim_{n \to \infty} \sup_{\Theta : |\Theta - \eta| < \delta} P_{\Theta}^n[|c_n^{\hat{\Theta}} - \theta| \geq L] < \varepsilon
\]

(Akahira, 1975a).

A \(\{c_n\}\)-consistent estimator is said to be asymptotically median unbiased if for every \(\eta \in \Theta\) there exists a positive number \(\delta\) such that

\[
\lim_{n \to \infty} \sup_{\Theta : |\Theta - \eta| < \delta} |P_{\Theta}^n[\hat{\Theta}_n = \Theta] - \frac{1}{2}| = 0;
\]

\[
\lim_{n \to \infty} \sup_{\Theta : |\Theta - \eta| < \delta} |P_{\Theta}^n[\hat{\Theta}_n \geq \Theta] - \frac{1}{2}| = 0.
\]

For an AMU estimator \(\hat{\Theta}_n^*\) it is called two-sided asymptotically efficient if for any AMU estimator \(\hat{\Theta}_n\) and any \(t > 0\)

\[
\lim_{n \to \infty} [P_{\Theta}^n[c_n^{\hat{\Theta}_n^* - \theta} < t] - P_{\Theta}^n[c_n^{\hat{\Theta}_n - \theta} < t] \geq 0.
\]

We assume that for each \(\Theta \in \Theta\) \(P_{\Theta}\) is absolutely continuous with respect to a \(\sigma\)-finite measure. We denote a density \(dP_{\Theta}/d\mu\) by \(f(x, \Theta)\). Let \(X_1, X_2, \ldots, X_n, \ldots\) be a sequence of i.i.d. random variables with the density \(f(x, \Theta)\). Suppose that \(X = \Theta = R^1\). Let \(L_n(u)\) be a bounded non-negative and monotone increasing function of \(|u|\) and \(\pi(\Theta)\) be a non-negative function. Define a posterior density \(p_n(\Theta|\bar{x}_n)\) and posterior risk \(r_n(d|\bar{x}_n)\) by

\[
p_n(\Theta|\bar{x}_n) = \prod_{i=1}^{n} f(x_i, \Theta) \pi(\Theta) \left[\int_{\Theta} \prod_{i=1}^{n} f(x_i, \Theta) \pi(\Theta) d\Theta\right]^{-1}
\]
and

\[ r_n(d|\tilde{x}_n) = \int_{\Theta} L_n(d-\theta) \ p_n(\theta|\tilde{x}_n) d\theta , \]

respectively, where \( \tilde{x}_n = (x_1, \ldots, x_n) \).

Now suppose that \( \lim_{n \to \infty} L_n(u/c_n) = L^*(u) \) for all \( u \in \mathbb{R}^1 \). We define

\[ r_n^*(d|\tilde{x}_n) = \int_{\Theta} L^*(c_n(d-\theta)) \ p_n(\theta|\tilde{x}_n) d\theta . \]

An estimator \( \hat{\theta}_n \) is called a generalized Bayes estimator with respect to a loss function \( L^* \) and a prior density \( \pi \) if

\[ r_n^*(\hat{\theta}_n|\tilde{x}_n) = \inf_{d \in \Theta} r_n^*(d|\tilde{x}_n) . \]

Since

\[ \lim_{n \to \infty} \inf_{d \in \Theta} \int_{\Theta} L_n(d-\theta) \ \bar{p}(\theta) d\theta = \inf_{d \in \Theta} \int_{\Theta} L^*(c_n(d-\theta)) \ \bar{p}(\theta) d\theta | = 0 \]

uniformly in every posterior density \( \bar{p}(\theta) \), it follows that for a generalized Bayes estimator \( \hat{\theta}_n \)

\[ \lim_{n \to \infty} \inf_{d \in \Theta} r_n(d|\tilde{x}_n) - r_n^*(\hat{\theta}_n|\tilde{x}_n) = 0 . \]

In the subsequent discussion we shall deal with the case when \( \pi(\theta) \equiv 1 \).

Suppose that \( \theta \) is a location parameter, i.e. \( f(x, \theta) = f(x-\theta) \).

Further we assume the following:

(A.1) \( f(x) > 0 \) for \( a < x < b \);

\[ f(x) = 0 \] for \( x \leq a, \ x \geq b \).

(A.2) \( f(x) \) is continuously differentiable in the interval \( (a,b) \) and

\[ 0 < \lim_{x \to a+0} f(x) = \lim_{x \to b-0} f(x) < \infty . \]
Under the assumptions (A.1) and (A.2) it is shown by Akahira (1975a) that the maximum order of consistency is \( n \). In the subsequent discussions we shall deal with the case when \( c_n = n \).

3. Two-Sided Asymptotic Efficiency of the Generalized Bayes Estimator

We consider the generalized Bayes estimator (GBE) with respect to the quadratic loss function, i.e., the Pitman estimator. Since

\[
\prod_{i=1}^{n} f(x_i - \theta) > 0 \quad \text{for} \quad \max x_i - b < \theta < \min x_i - a ,
\]

\[
\prod_{i=1}^{n} f(x_i - \theta) = 0 \quad \text{otherwise} ,
\]

it is easily seen that the GBE \( \hat{\theta}_{GB} \) is given by

\[
\hat{\theta}_{GB} = \frac{\int_{\theta}^{\bar{\theta}} \prod_{i=1}^{n} f(x_i - \theta) d\theta}{\int_{\theta}^{\bar{\theta}} \prod_{i=1}^{n} f(x_i - \theta) d\theta} ,
\]

where \( \theta = \max x_i - b \) and \( \bar{\theta} = \min x_i - a \). Let \( \theta_0 \) be the true parameter. Put \( \theta = \theta_0 + (A/n) \) and \( \bar{\theta} = \theta_0 + (B/n) \). Then we have

\[
\hat{\theta}_{GB} - \theta_0 = \frac{\int_{\theta_0}^{\theta_0 + \frac{A}{n}} (\theta - \theta_0) \prod_{i=1}^{n} f(x_i - \theta) d\theta}{\int_{\theta_0}^{\theta_0 + \frac{B}{n}} \prod_{i=1}^{n} f(x_i - \theta) d\theta} .
\]

Since putting \( t = n(\theta - \theta_0) \) we have
\begin{equation}
(3.1) \quad n(\hat{\theta}_{GB} - \theta) = \frac{\int_{B}^{A} t \prod_{i=1}^{n} f(X_i - \theta_0 - \frac{t}{n}) dt}{\int_{B}^{A} \prod_{i=1}^{n} f(X_i - \theta_0 - \frac{t}{n}) dt}.
\end{equation}

For sufficiently large \( n \) we have

\[
\prod_{i=1}^{n} f(X_i - \theta_0 - \frac{t}{n}) = \exp \left[ - \sum_{i=1}^{n} \left\{ \log f(X_i - \theta_0) - \log f(X_i - \theta_0 - \frac{t}{n}) \right\} \right] 
\cdot \exp \left\{ \sum_{i=1}^{n} \log f(X_i - \theta) \right\} 
\sim \exp \left[ \frac{t}{n} \sum_{i=1}^{n} \frac{\partial \log f(X_i - \theta)}{\partial \theta} \right] \cdot \exp \left\{ \sum_{i=1}^{n} \log f(X_i - \theta_0) \right\}.
\]

Since \( \sum_{i=1}^{n} \{ \partial \log f(X_i - \theta_0)/\partial \theta \}/n \) converges in probability to 0 as \( n \) tends to infinity, it follows from (3.1) that for sufficiently large \( n \)

\begin{equation}
(3.2) \quad n(\hat{\theta}_{GB} - \theta_0) \sim \frac{\int_{B}^{A} t dt}{\int_{B}^{A} dt} = \frac{1}{2}(A + B) = n \left( \frac{\theta + \overline{\theta}}{2} - \theta_0 \right).
\end{equation}

It is seen by the similar discussion to Example 3 in Akahira (1982) that the case when the density \( f(x) \) satisfies the assumptions (A.1) and (A.2) is essentially reduced to the uniform distribution on \((a, b)\) (see also Example 1 in Akahira, 1982). It follows from Akahira (1982) that the estimator \((\hat{\theta} + \overline{\theta})/2\) is two-sided asymptotically efficient. Hence it is seen from (3.2) that under assumptions (A.1) and (A.2) the generalized Bayes estimator \( \hat{\theta}_{GB} \) with respect to the quadratic loss function is two-sided asymptotically efficient.
In (A.2) we assume that \( \lim_{x \to a^+} f(x) = \lim_{x \to b^-} f(x) \). Here we consider a typical example of the cases when \( \lim_{x \to a^+} f(x) \neq \lim_{x \to b^-} f(x) \). Suppose that

\[
\begin{align*}
f(x) &= \begin{cases} 
  c e^{-x} \text{ for } 0 < x < 1; \\
  0 \quad \text{for } x \leq 0, x \geq 1, 
\end{cases}
\end{align*}
\]

where \( c \) is some constant.

Let \( \theta_0 \) be the true parameter. If the loss function \( L^*(u) \) is quadratic, i.e. \( L^*(u) = u^2 \), then the GBE \( \hat{\theta}_{GB}^* \) has

\[
n(\hat{\theta}_{GB}^* - \theta_0) = \frac{T(e^{2T} + 1)}{e^{2T} - 1} - 1,
\]

where \( T = n\{1 - (\max X_i - \min X_i)\} \). If the loss function \( L^*(u) \) is the absolute value of \( u \), i.e. \( L^*(u) = |u| \), then the GBE \( \hat{\theta}_{GB}^* \) has

\[
n(\hat{\theta}_{GB}^{**} - \theta_0) = \log \frac{e^{-T} + e^{T}}{2}.
\]

Here we modify the GBE's \( \hat{\theta}_{GB}^* \) and \( \hat{\theta}_{GB}^{**} \) to be AMU and let them be \( \tilde{\theta}_{GB}^* \) and \( \tilde{\theta}_{GB}^{**} \), respectively. However these modified GBE's \( \tilde{\theta}_{GB}^* \) and \( \tilde{\theta}_{GB}^{**} \) may not be two-sided asymptotically efficient.

4. Asymptotic Expansion of the GBE in the Truncated Normal Density Case

In this section we shall obtain the asymptotic expansion of the GBE up to the order \( n^{-1} \) and its asymptotic density in the truncated normal density case.

We consider the following truncated normal density \( f(x) \) given by
\[
    f(x) = \begin{cases} 
    ce^{-x^2/2} & \text{for } |x| < 1; \\
    0 & \text{for } |x| \geq 1 , 
    \end{cases}
\]

where \( c \) is some constant. Then we have

\[
    \prod_{i=1}^{n} f(x_i - \theta) = \begin{cases} 
    c^n \exp \left( -\frac{1}{2} \sum_{i=1}^{n} x_i^2 + n\theta \bar{x} - \frac{n}{2} \Theta^2 \right) & \text{for } \theta < \theta < \Theta; \\
    0 & \text{otherwise}, 
    \end{cases}
\]

where \( \bar{x} = \frac{\sum_{i=1}^{n} x_i}{n} \), \( \theta = \max x_i - 1 \) and \( \Theta = \min x_i + 1 \). Without loss of generality we may assume that the true parameter \( \theta_0 \) is equal to zero. It is easily seen that the GBE \( \hat{\theta}_{GB} \) is given by

\[
    \hat{\theta}_{GB} = \frac{\int_{\frac{\theta}{\Theta}}^{\frac{\theta}{\Theta}} f(x_i - \theta) \, d\theta}{\int_{\frac{\theta}{\Theta}}^{\frac{\theta}{\Theta}} f(x_i - \theta) \, d\theta}.
\]

Putting \( \theta = t/\sqrt{n} \) and \( Z = \sqrt{n} \bar{x} \), we have

\[
    \hat{\theta}_{GB} = \frac{1}{\sqrt{n}} \int_{\frac{\theta}{\Theta}}^{\frac{\theta}{\Theta}} e^\frac{-t^2}{2} + Zt \, dt.
\]

Since

\[
    \int_{\frac{\theta}{\Theta}}^{\frac{\theta}{\Theta}} e^\frac{-t^2}{2} + Zt \, dt = \frac{1}{2} n\sqrt{n}(\theta^2 - \theta_0^2) + \frac{1}{3} zn^2(\theta^3 - \theta_0^3) + \frac{1}{4}(z^2 - 1) n^2 \sqrt{n}(\theta^4 - \theta_0^4) + o_p \left( \frac{1}{\sqrt{n}} \right); \]

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\[
\int_{\sqrt{n} \theta}^{\sqrt{n} \theta - \frac{t^2}{2} + Zt} e^{\frac{Z^2}{2}} \, dt
\]

\[
= \sqrt{n} (\theta - \bar{\theta}) \{ 1 + \frac{1}{2} Z \sqrt{n} (\theta + \bar{\theta}) + \frac{1}{6} (Z^2 - 1) n (\theta^2 + \theta \bar{\theta} + \bar{\theta}^2) + o_p \left( \frac{1}{n} \right) \},
\]

it follows that

(4.1) \quad \hat{n}_{GB} = \frac{\frac{1}{2} n(\theta + \bar{\theta}) + \frac{1}{3\sqrt{n}} Z n^2 (\theta^2 + \theta \bar{\theta} + \bar{\theta}^2) + \frac{1}{8n} (Z^2 - 1) n^3 (\theta + \bar{\theta}) (\theta^2 + \bar{\theta}^2) + o_p \left( \frac{1}{n} \right)}{1 + \frac{1}{2\sqrt{n}} Z n (\theta + \bar{\theta}) + \frac{1}{6n} (Z^2 - 1) n^2 (\theta^2 + \theta \bar{\theta} + \bar{\theta}^2) + o_p \left( \frac{1}{n} \right)}

\[
= \frac{1}{2} n(\theta + \bar{\theta}) + \frac{1}{12\sqrt{n}} Z n^2 (\theta - \bar{\theta})^2 - \frac{1}{16n} (Z^2 - 1) n^3 (\theta + \bar{\theta}) (\theta^2 + \bar{\theta}^2 - (\theta - \bar{\theta})^2)
\]

\[- \frac{1}{24n} n^3 (\theta + \bar{\theta})^3 + o_p \left( \frac{1}{n} \right).
\]

Put \( S = n(\theta + \bar{\theta})/2 \) and \( T = n(\theta - \bar{\theta})/2 \). Then it follows from (4.1) that the stochastic expansion of \( \hat{\theta}_{GB} \) is given by

(4.2) \quad \hat{n}_{GB} = S + \frac{1}{3\sqrt{n}} Z T^2 - \frac{1}{2n} Z^2 S (S^2 - T^2) + \frac{1}{6n} S (S^2 - 3T^2) + o_p \left( \frac{1}{n} \right).

Next we shall obtain the asymptotic expansion of the density of \( \hat{\theta}_{GB} \).

It is seen from (4.2) that the asymptotic expansion of the characteristic function \( \phi_n(t) \) of \( \hat{\theta}_{GB} \) is given by

\[
\phi_n(t) = E \left[ e^{itS} \left\{ 1 + \frac{it}{3\sqrt{n}} Z S^2 - \frac{it}{2n} Z^2 S (S^2 - T^2) + \frac{it}{6n} S (S^2 - 3T^2)
\right.
\]

\[- \frac{t^2}{18n} Z^2 T^4 + o_p \left( \frac{1}{n} \right) \right\} \right].
\]

Since the conditional expectations are given by

\[
E(Z|S,T) = O\left( \frac{1}{n} \right); \quad E(Z^2|S,T) = \sigma^2 + O\left( \frac{1}{n} \right),
\]

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it follows that

\[ E(ZT^2|S) = E[E(Z|S,T)T^2|S] = O\left(\frac{1}{n}\right) \]
\[ E(Z^2S^3|S) = S^3E(Z^2|S) = \sigma^2S^3 + O\left(\frac{1}{n}\right) \]
\[ E(Z^2T^2|S) = SE(Z^2T^2|S) = \sigma^2S + O\left(\frac{1}{n}\right) \]
\[ E(Z^2T^4|S) = \sigma^2 + O\left(\frac{1}{n}\right) \]

where \( \sigma^2 = \int_{-1}^{1} x^2 f(x) dx \).

Since the conditional density \( g(T|S) \) of \( T \) given \( S \) is given by

\[
g(T|S) = \begin{cases} 
ke^{-|T-S|} & \text{for } T > |S| \\
0 & \text{otherwise}
\end{cases}
\]

where \( k \) is some constant, the conditional expectations are given by

\[ E(T|S) = k(|S| + 1) \]
\[ E(T^2|S) = k(S^2 + 2|S| + 2) \]

Hence it follows that

\[
\phi_n(t) = E[e^{itS}] \left[ 1 - \frac{it}{2n} S^3E(Z^2|S) + \frac{it}{2n} SE(Z^2T^2|S) \\
+ \frac{it}{6n} S^3 - \frac{it}{2n} SE(T^2|S) - \frac{t^2}{18n} E(Z^2T^4|S) \right]
\]

\[
= E(e^{itS}) - \frac{it}{2n} \sigma^2E(S^3 e^{itS}) + \frac{it}{2n} \sigma^2E(Se^{itS})
\]
\[
+ \frac{it}{6n} E(S^3 e^{itS}) - \frac{it}{2n} kE[(S^3 + 2S|S| + S)e^{its}]
\]
\[- \frac{t^2}{18n} \sigma^2E(e^{itS}) + o\left(\frac{1}{n}\right)
\]
\[ = E(e^{itS}) - \frac{\sigma^2}{18n} t^2 E(e^{itS}) + \frac{1}{2n} (\sigma^2 - k)it E(Se^{itS}) \]

\[ - ikt E(S|S| e^{itS}) + \frac{1}{6n} (1 - 3\sigma^2 - 3k)it E(S^3 e^{itS}) + o\left(\frac{1}{n}\right). \]

Further, it is seen that the characteristic function \( \phi_n(t) \) has the following form:

\[ \phi_n(t) = \frac{4c^2}{4c^2 + t^2} + \frac{v_1 t^2}{n(4c^2 + t^2)} + \frac{v_2 t^2}{n(4c^2 + t^2)^2} + \frac{v_3 t^4}{n(4c^2 + t^2)^3} + \frac{v_4 t^2}{n(4c^2 + t^2)^4} + o\left(\frac{1}{n}\right), \]

where \( c = ce^{-1/2} \) and \( v_i (i=1, 2, \ldots, 6) \) are certain constants. Hence it is seen by a similar argument to Takeuchi (1975) that the asymptotic density \( g_n(u) \) of the GBE \( \hat{\theta}_{GB} \) up to the order \( n^{-1} \) has the following form:

\[ g_n(u) = c e^{-2c'} |u| \left\{ 1 + \frac{c_0}{n} + \frac{c_1}{n} |u| + \frac{c_2}{n} u^2 + \frac{c_3}{n} u^3 + o\left(\frac{1}{n}\right) \right\}, \]

where \( c_i (i=0, 1, 2, 3) \) are certain constants.

In the truncated normal density case we may construct the AMU estimator \( \hat{\theta}_n^* \) attaining at an arbitrary point \( t \) the bound of the distribution of \( n|\hat{\theta}_n - \theta| \) for AMU estimators \( \hat{\theta}_n \) using the discretized likelihood method (see Akahira and Takeuchi, 1979a). Then the AMU estimator \( \hat{\theta}_n^* \) is given by
\[
\theta_n^* = \begin{cases} 
\bar{x} & \text{for } \theta + \frac{t}{n} \leq \bar{x} \leq \bar{\theta} - \frac{t}{n}, \quad \bar{\theta} - \theta \leq \frac{2t}{n}; \\
\bar{\theta} - \frac{t}{n} & \text{for } \bar{x} > \bar{\theta} - \frac{t}{n}, \quad \bar{\theta} - \theta \leq \frac{2t}{n}; \\
\theta + \frac{t}{n} & \text{for } \bar{x} < \theta + \frac{t}{n}, \quad \bar{\theta} - \theta \leq \frac{2t}{n}; \\
\frac{1}{2}(\theta + \bar{\theta}) & \text{for } \bar{\theta} - \theta > \frac{2t}{n},
\end{cases}
\]

where \( \bar{x} = \sum_{i=1}^{n} x_i / n \).

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References


