ON ASYMPTOTIC DEFICIENCY OF ESTIMATORS
IN THE MULTIPARAMETER EXPONENTIAL FAMILY CASE

BY

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ON ASYMPTOTIC DEFICIENCY OF ESTIMATORS
IN THE MULTIPARAMETER EXPONENTIAL FAMILY CASE\(^1\)

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ABSTRACT

It was shown by Akahira (1981) that the asymptotic deficiency of any
two estimators in the restricted class \(D\) of the third order asymptoti-
cally median unbiased and best asymptotically normal (BAN) estimators is
given by the difference between the coefficients of order \(n^{-1}\) of the
variances of the estimators. In this paper it is shown that the asymptotic
deficiency of any two estimators in the class \(D\) is obtained from the
coefficients of order \(n^{-1}\) of their covariances for the multiparameter
exponential family case.

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1. INTRODUCTION

Recently higher order asymptotic efficiency of estimators based on independently and identically distributed samples has been studied (e.g. Pfanzagl and Wefelmeyer, 1978; Takeuchi and Akahira, 1978; Ghosh, Sinha and Wieand, 1980; Akahira and Takeuchi, 1981). Third order asymptotic efficiency of the maximum likelihood estimator in the multiparameter exponential case was discussed by Takeuchi and Akahira (1978).

Let $\mathcal{D}$ be the class of the estimators whose element $\hat{\theta}_n$ is BAN and third order asymptotically median unbiased and asymptotically expanded as

$$\sqrt{n} (\hat{\theta}_n - \theta) = U + \frac{1}{\sqrt{n}} Q + o_p \left( \frac{1}{\sqrt{n}} \right)$$

and $Q = Q_p (1), E[U_{\alpha, \beta}^k] = o(1), (k=1,2)$ for all $\alpha, \beta = 1, \ldots, p$, where $E$ denotes asymptotic expectation and $U = (U_1, \ldots, U_p)'$ and $Q = (Q_1, \ldots, Q_p)'$ and the joint distribution of $\sqrt{n} (\hat{\theta}_n - \theta)$ admits Edgeworth expansion. It is clear that $\mathcal{D} = \mathcal{A}_3 = \mathcal{A}_2$.

It was shown by Akahira (1981) that the asymptotic deficiency of any two estimators in the class $\mathcal{D}$ is given by the difference between the coefficients of order $n^{-1}$ of their variances. Let $\hat{\theta}_n$ and $\hat{\theta}_{k_n}$ be estimators in the class $\mathcal{D}$ based on samples sizes $n$ and $k_n$ with $k_n > n$. To compare $\hat{\theta}_n$ and $\hat{\theta}_{k_n}$ we consider the asymptotic deficiency defined by Hodges and Lehmann (1970). Assume that the asymptotic deficiency of $\hat{\theta}_n$ w.r.t. $\hat{\theta}_{k_n}$ $d = \lim_{n \to \infty} (k_n - n)$ exists. Then it is shown that $d$ is obtained from the coefficients of order $n^{-1}$ of their covariances.

2. RESULTS

Let $(\mathcal{X}, \mathcal{B})$ be a sample space. We consider a family of probability measures on $\mathcal{B}$, $P = \{P_\theta: \theta \in \Theta\}$, where the index set $\Theta$ is called a
parameter space. We assume that $\Theta$ is an open set in a Euclidean $p$-space $\mathbb{R}^p$ with a norm denoted by $\| \cdot \|$. Then an element $\theta$ of $\Theta$ may be denoted by $(\theta_1, \ldots, \theta_p)$. Consider $n$-fold direct products $(X^{(n)}, \mathcal{B}^{(n)})$ of $(X, \mathcal{B})$ and the corresponding product measures $P^{(n)}_\Theta$ of $P_\Theta$. An estimator of $\Theta$ is defined to be a sequence $\{\hat{\theta}^{(n)}_n\}$ of $\mathcal{B}^{(n)}$-measurable functions $\hat{\theta}^{(n)}_n$ on $X^{(n)}$ into $\Theta$ ($n=1, 2, \ldots$). For simplicity we denote an estimator as $\hat{\theta}^{(n)}_n$ instead of $\{\hat{\theta}^{(n)}_n\}$. Then $\hat{\theta}^{(n)}_n$ may be denoted by $(\hat{\theta}^{(n)}_{n1}, \ldots, \hat{\theta}^{(n)}_{np})$. For an increasing sequence of positive numbers $\{c_n\}$ ($c_n$ tending to infinity) an estimator is called consistent with order $\{c_n\}$ (or $\{c_n\}$-consistent for short) if for every $\varepsilon > 0$ and every $\Theta \in \Theta$ there exist a sufficiently small positive number $\delta$ and a sufficiently large number $L$ satisfying the following:

$$\lim_{n \to \infty} \sup_{\|\Theta - \delta\| < \delta} P^{(n)}_\Theta \{c_n \|\hat{\theta}^{(n)}_n - \Theta\| \geq L\} < \varepsilon \quad \text{(Akahira, 1975).}$$

For each $k=1, 2, \ldots$, a $\{c_n\}$-consistent estimator $\hat{\theta}^{(n)}_n$ is $k$-th order asymptotically median unbiased (or $k$-th order AMU) estimator if for each $\Theta \in \Theta$ and each $\alpha=1, \ldots, p$, there exists a positive number $\delta$ such that

$$\lim_{n \to \infty} \sup_{\|\Theta - \delta\| < \delta} c_k^{-1} \left| P^{(n)}_\Theta \{\hat{\theta}^{(n)}_n \leq \Theta_\alpha \} - \frac{1}{2} \right| = 0 ;$$

$$\lim_{n \to \infty} \sup_{\|\Theta - \delta\| < \delta} c_k^{-1} \left| P^{(n)}_\Theta \{\hat{\theta}^{(n)}_n \geq \Theta_\alpha \} - \frac{1}{2} \right| = 0 .$$

For $\hat{\theta}$ $k$-th order AMU, $G_0(t, \Theta_\alpha) + c_k^{-1} G_1(t, \Theta_\alpha) + \ldots + c_k^{-(k-1)} G_{k-1}(t, \Theta_\alpha)$ ($\alpha=1, \ldots, p$) is called the $k$-th order asymptotic marginal distribution of $c_n(\hat{\theta}^{(n)}_n - \Theta)$ (or $\hat{\theta}^{(n)}_n$ for short) if
\[
\lim_{n \to \infty} c_n^{-k} \left| p_{\theta}^{(n)} \{ c_n (\hat{\theta}_{n\alpha} - \theta_\alpha) < t \} - G_0(t, \theta_\alpha) - c_n^{-1} G_1(t, \theta_\alpha) - \ldots - c_n^{-(k-1)} G_{k-1}(t, \theta_\alpha) \right| = 0.
\]

We note that \( G_i(t, \theta_\alpha) \) \((i=1, \ldots, k-1; \alpha=1, \ldots, p)\) may be generally absolutely continuous functions, hence the asymptotic marginal distributions for any fixed \( n \) may not be a distribution function.

Suppose that \( \hat{\theta}_n \) is \( k \)-th order AMU and has the \( k \)-th order asymptotic marginal distribution \( G_0(t, \theta_\alpha) + c_n^{-1} G_1(t, \theta_\alpha) + \ldots + c_n^{-(k-1)} G_{k-1}(t, \theta_\alpha) \) \((\alpha=1, \ldots, p)\) and the joint distribution of \( \hat{\theta}_n \) admits asymptotic expansion up to \( k \)-th order, i.e., the order of \( c_n^{-(k-1)} \). Letting \( \theta_0 \) \((\epsilon \Theta)\) be arbitrary but fixed. Denote \( \theta_0 \) by \((\theta_{01}, \ldots, \theta_{0p})\). Let \( \alpha \) be arbitrary but fixed in \( 1, \ldots, p \). We consider the problem of testing the hypothesis \( H^+: \theta_\alpha = \theta_{0\alpha} + tc_n^{-1} (t > 0) \) against \( K: \theta_\alpha = \theta_{0\alpha} \). Put \( \Phi_{\frac{1}{2}} = \{ \{ \phi_n \} : E_n(\theta_\alpha) + tc_n^{-1} (\phi_n) = \frac{1}{2} + o(c_n^{-(k-1)}) \} \), \( 0 \leq \phi_n (\hat{\theta}_n) \leq 1 \) for all \( \hat{\theta}_n \in X^{(n)} \) \((n=1, 2, \ldots)\).

Putting \( A_{\hat{\theta}_{n\alpha}}^{\theta_0\alpha} = \{ c_n (\hat{\theta}_{n\alpha} - \theta_{0\alpha}) \leq t \} \), we have

\[
\lim_{n \to \infty} p_{\theta_0\alpha + tc_n^{-1} (A_{\hat{\theta}_{n\alpha}}^{\theta_0\alpha}, \theta_\alpha)}^{(n)} = \lim_{n \to \infty} p_{\theta_0\alpha + tc_n^{-1}}^{(n)} \{ \hat{\theta}_{n\alpha} \leq \theta_{0\alpha} + tc_n^{-1} \} = \frac{1}{2}.
\]

Hence it is seen that a sequence \( \{ X_{A_{\hat{\theta}_{n\alpha}}^{\theta_0\alpha}} \} \) of the indicators of \( A_{\hat{\theta}_{n\alpha}}^{\theta_0\alpha} \) belongs to \( \Phi_{\frac{1}{2}} \). If

\[
\sup_{\{ \phi \} \in \Phi_{\frac{1}{2}}} \lim_{n \to \infty} c_n^{k-1} (E_n(\theta_\alpha) (\phi_n) - H_0^+(t, \theta_0\alpha) - c_n^{-1} H_1^+(t, \theta_0\alpha) - \ldots - c_n^{-(k-1)} H_{k-1}^+(t, \theta_0\alpha)) = 0,
\]

then we have
\[ G_0(t, \theta_{0\alpha}) \leq H_0^+(t, \theta_{0\alpha}) ; \]

and for any positive integer \( j \leq k \) if \( G_i(t, \theta_{0\alpha}) = H_i^+(t, \theta_{0\alpha}) \) (i=1, \ldots, j-1) then

\[ G_j(t, \theta_{0\alpha}) \leq H_j^+(t, \theta_{0\alpha}) . \]

Consider next the problem of testing the hypothesis \( H^-: \theta_\alpha = \theta_{0\alpha} + t\sigma_n^{-1} \)
\( (t < 0) \) against \( K: \theta_\alpha = \theta_{0\alpha} \). If

\[
\sup_{\{\phi_n\} \in \Phi_{n2}} \lim_{n \to \infty} c_n^{k-1} \left\{ F_n(n) \left( \frac{\phi_n}{\phi_{n2}} \right) - H_0^-(t, \theta_{0\alpha}) - c_n \right\}^{-1} H_1^-(t, \theta_{0\alpha}) - \ldots - c_n^{-(k-1)} H_k^-(t, \theta_{0\alpha}) \right\} = 0 ,
\]

then we have

\[ G_0(t, \theta_{0\alpha}) \geq H_0^-(t, \theta_{0\alpha}) ; \]

and for any positive integer \( j \leq k \) if \( G_i(t, \theta_{0\alpha}) = H_i^-(t, \theta_{0\alpha}) \) (i=1, \ldots, j-1), then \( G_j(t, \theta_{0\alpha}) \geq H_j^-(t, \theta_{0\alpha}) . \)

\( \hat{\theta}_n \) is called the k-th order asymptotically efficient in the class \( A_k \) of all k-th order AMU estimators if its k-th order asymptotic marginal distribution attains uniformly the bound of the k-th order asymptotic marginal distributions of k-th order AMU estimators, that is, for each \( \alpha_1, \ldots, p \)

\[ G_i(t, \theta_\alpha) = \begin{cases} H_i^+(t, \theta_\alpha) & \text{for } t > 0 , \\ H_i^-(t, \theta_\alpha) & \text{for } t < 0 , \end{cases} \]

i=0, \ldots, k-1 (Akahira and Takeuchi, 1976, 1981). (Note that for \( t=0 \) and each \( \alpha_1, \ldots, p \) we have \( G_i(0, \theta_\alpha) = H_i^+(0, \theta_\alpha) = H_i^-(0, \theta_\alpha) \) (i=0, \ldots, k-1) from the condition of k-th order asymptotically median unbiasedness.)
In the subsequent discussions we shall essentially deal with the case when \( c_n = \sqrt{n} \). We suppose that for each \( \theta \in \Theta \), \( P_{\theta} \) is absolutely continuous with respect to a \( \sigma \)-finite measure \( \mu \). We denote the density \( dP_{\theta}/d\mu \) by \( f(x, \theta) \). Suppose that \( X_1, X_2, \ldots, X_n \) are independently and identically distributed according to an exponential type distribution, i.e. with the density

\[
f(x, \theta) = c(\theta) h(x) \exp\left\{ \sum_{i=1}^{n} s_i(\theta) t_i(x) \right\},
\]

where \( \theta \) is a \( p \)-dimensional real vector-valued parameter which belongs to an open set \( \Theta \subset \mathbb{R}^p \), and \( s_i(\theta) \)'s are continuous real valued functions and \( t_i(\theta) \)'s are real valued measurable functions. It is known that the maximum likelihood estimator, the generalized Bayes estimator and the best invariant estimator are third order asymptotically efficient estimators in the class \( D \) (e.g. see Akahira and Takeuchi, 1981).

Let \( k_n \) be integers with \( k_n > n \). We assume that \( d = \lim_{n \to \infty} (k_n - n) \) exists. Let \( \hat{\theta}_n \) and \( \hat{\theta}_n^{*} \) be any two estimators in the class \( D \) based on sample size \( n \) and \( k_n \), respectively. Then \( d \) is called the asymptotic deficiency of \( \hat{\theta}_n^{*} \) w.r.t. \( \hat{\theta}_n \) (Hodges and Lehmann, 1970). Put

\[
T_n = \sqrt{n} (\hat{\theta}_n - \theta); \quad T_n^{*} = \sqrt{k_n} (\hat{\theta}_n^{*} - \theta).
\]

Since \( \hat{\theta}_n \) belongs to the class \( D \), it follows by Theorem 5.1.4 in Akahira and Takeuchi (1981, page 153) that for each \( \alpha \) the cumulants of \( T_n = \sqrt{n} (\hat{\theta}_n - \theta) \) are expressed as the following form:

\[
E(T_{n\alpha}) = \frac{1}{\sqrt{n}} \mu_{\alpha} + o(\frac{1}{n});
\]

\[
\text{Cov}(T_{n\alpha}, T_{n\beta}) = \tau_{\alpha\beta} + \frac{1}{n} \left\{ \eta(\mu_{\alpha}, \mu_{\beta}) + \tau_{\alpha\beta} \right\} + o(\frac{1}{n});
\]
\[ \kappa_3(T_{n\alpha}, T_{n\beta}, T_{n\gamma}) = \frac{1}{\sqrt{n}} \beta_{\alpha\beta\gamma} + o\left(\frac{1}{n}\right) ; \]
\[ \kappa_4(T_{n\alpha}, T_{n\beta}, T_{n\gamma}, T_{n\delta}) = \frac{1}{n} \beta_{\alpha\beta\gamma\delta} + o\left(\frac{1}{n}\right) , \]

where \( I_{\alpha\beta} \) denotes \((\alpha, \beta)\)-element of the inverse matrix of the Fisher information matrix \( I = (I_{\alpha\beta}) \).

Since \( \hat{\theta}_{k_n}^* \) is also an estimator in the class \( D \), it follows by Theorem 5.1.4 in Akahira and Takeuchi (1981, page 153) that for each \( \hat{\theta}_{k_n}^* \) has the cumulants of \( T_{n\alpha}^* = \sqrt{k_n} \left( \hat{\theta}_{k_n}^* - \theta_{\alpha} \right) \) of the following type:

\[ E(T_{n\alpha}^*) = \frac{1}{\sqrt{n}} \mu_{\alpha} + o\left(\frac{1}{n}\right) ; \]
\[ \text{Cov}(T_{n\alpha}^*, T_{n\beta}^*) = I_{\alpha\beta} + \frac{1}{n} \{ (\mu_{\alpha}, \mu_{\beta}) + \tau_{\alpha\beta}^* \} + o\left(\frac{1}{n}\right) ; \]
\[ \kappa_3(T_{n\alpha}^*, T_{n\beta}^*, T_{n\gamma}^*) = \frac{1}{\sqrt{n}} \beta_{\alpha\beta\gamma} + o\left(\frac{1}{n}\right) ; \]
\[ \kappa_4(T_{n\alpha}^*, T_{n\beta}^*, T_{n\gamma}^*, T_{n\delta}^*) = \frac{1}{n} \beta_{\alpha\beta\gamma\delta} + o\left(\frac{1}{n}\right) . \]

Note that the only term of order \( n^{-1} \) are different in the cumulants between any two estimators in the class \( D \) (Akahira and Takeuchi, 1981). Then it is shown by Theorem 5.1.5 in Akahira and Takeuchi (1981, page 153) that the density of the joint distribution of \( \sqrt{n} (\hat{\theta}_{k_n} - \theta) \) is given by

\[ f_n(y_1, \ldots, y_p) = \frac{|I|^{1/2}}{(\sqrt{2\pi})^p} \left[ \exp\left\{ \sum_{\alpha} \sum_{\beta} I_{\alpha\beta} y_\alpha y_\beta \right\} \right] \]
\[ \cdot \left\{ 1 + \frac{1}{6\sqrt{n}} \sum_{\alpha} \sum_{\beta} \sum_{\gamma} \beta_{\alpha\beta\gamma} H_{\alpha\beta\gamma} + \frac{1}{\sqrt{n}} \sum_{\alpha} \mu_{\alpha} H_{\alpha} \right. \]
\[ + \frac{1}{72n} \sum_{\alpha} \sum_{\beta} \sum_{\gamma} \sum_{\alpha'} \sum_{\gamma'} \sum_{\beta'} \beta_{\alpha\beta\gamma} \beta_{\alpha'\beta'\gamma'} H_{\alpha\beta\gamma\alpha'\beta'\gamma'} \]
\[ + \frac{1}{24n} \sum_{\alpha} \sum_{\beta} \sum_{\gamma} \sum_{\delta} (\beta_{\alpha\beta\gamma\delta} + 4\mu_{\alpha} \beta_{\beta\gamma\delta}) H_{\alpha\beta\gamma\delta} \right\} \]
\[
+ \frac{1}{2n} \sum_{\alpha} \sum_{\beta} \{ \mu_{\alpha} \nu_{\beta} + \eta(\mu_{\alpha}, \nu_{\beta}) \} H_{\alpha \beta} + \frac{1}{2n} \sum_{\alpha} \sum_{\beta} \tau_{\alpha \beta} H_{\alpha \beta} \}
\]
\[ + o\left(\frac{1}{n}\right),\]

where \( H_{\alpha \beta} \ldots \) are (multivariate) Hermite polynomials defined by

\[
H_{\alpha} = \sum_{\beta} I_{\alpha \beta} y_{\beta};
\]
\[
H_{\alpha \beta} = H_{\alpha} H_{\beta} - I_{\alpha \beta};
\]
\[
H_{\alpha \beta \gamma} = H_{\alpha} H_{\beta} H_{\gamma} - I_{\alpha \beta} H_{\gamma} - I_{\alpha \gamma} H_{\beta} - I_{\beta \gamma} H_{\alpha},
\]

etc.

Next we shall obtain the distributions of \( \sqrt{n} \left( \hat{\theta}_{n} - \theta \right) \) and \( \sqrt{n} \left( \hat{\theta}_{k} - \theta \right) \)
up to order \( n^{-1} \). Since \( \sqrt{n} \left( \hat{\theta}_{n} - \theta_{n} \right) \leq y_{\alpha} \) \( (\alpha = 1, \ldots, p) \) if and only if

\[
\sqrt{n} \left( \hat{\theta}_{n} - \theta_{n} \right) \leq \sqrt{n} \left( \hat{\theta}_{n1} - \theta_{n1} \right) \leq (1 - \frac{d}{2n}) y_{\alpha}, \ldots, \sqrt{n} \left( \hat{\theta}_{np} - \theta_{np} \right) \leq (1 - \frac{d}{2n}) y_{p},
\]

it follows that for sufficiently large \( n \)

\[
\text{(2)} \quad P_{(n)}^{(n)} \{ \sqrt{n} \left( \hat{\theta}_{n1} - \theta_{1} \right) \leq y_{1}, \ldots, \sqrt{n} \left( \hat{\theta}_{np} - \theta_{p} \right) \leq y_{p} \}
\]

\[
= P_{(n)}^{(n)} \{ \sqrt{n} \left( \hat{\theta}_{n1} - \theta_{n1} \right) \leq (1 - \frac{d}{2n}) y_{1}, \ldots, \sqrt{n} \left( \hat{\theta}_{np} - \theta_{np} \right) \leq (1 - \frac{d}{2n}) y_{p} \}.
\]

From (1) and (2) it follows that the density of the joint distribution of

\[
\sqrt{n} \left( \hat{\theta}_{n} - \theta \right)
\]

is given by

\[
\text{(3)} \quad f_{(n)} (y_{1}, \ldots, y_{p}) = \frac{1}{(2\pi)^{p/2}} \left[ \exp \left( - \frac{1}{2} \left( \sum_{\alpha} I_{\alpha \beta} y_{\alpha} y_{\beta} \right) \right) \right] \cdot \left\{ 1 + \ldots + \frac{1}{2n} \sum_{\alpha} \sum_{\beta} \tau_{\alpha \beta} H_{\alpha \beta} + \frac{d}{2n} \left( \sum_{\alpha} \sum_{\beta} I_{\alpha \beta} y_{\alpha} y_{\beta} - p \right) + o\left(\frac{1}{n}\right) \right\},
\]

where the dots "..." in \{ \} mean that they are the same as that of (1). On
the other hand, it follows by (1) that the density of the joint distribution
of \( \sqrt{n} \left( \hat{\theta}_{k} - \theta \right) \) under \( P_{(n)}^{(n)} \) is given by

\[ \]
(4) \[ f_{\hat{\theta}^*_k/n} (y_1, \ldots, y_p) = \frac{|I|^{1/2}}{(\sqrt{2\pi})^p} \left[ \exp\left\{ -\frac{1}{2} \sum_{\alpha} \sum_{\beta} I_{\alpha\beta} y_{\alpha} y_{\beta} \right\} \right. \]
\[ \times \left( 1 + \ldots + \frac{1}{2n} \sum_{\alpha} \sum_{\beta} \tau_{\alpha\beta} H_{\alpha\beta} \right) + o\left( \frac{1}{n} \right), \]

where the dots "..." in \{ \} means that they are the same as that of (1).

From (3) and (4) it follows that the density of the joint distribution of \( \sqrt{n} (\hat{\theta}^*_{k/n} - \theta) \) under \( P_{\theta} \) up to order \( n^{-1} \) as \( n \to \infty \) if and only if

\[ d = \frac{\sum_{\alpha} \sum_{\beta} (\tau_{\alpha\beta}^* - \tau_{\alpha\beta}) H_{\alpha\beta}}{\sum_{\alpha} \sum_{\beta} I_{\alpha\beta} y_{\alpha} y_{\beta} - p}. \]

Since

\[ \sum_{\alpha} \sum_{\beta} I_{\alpha\beta} y_{\alpha} y_{\beta} - p = \sum_{\alpha} y_{\alpha} H_{\alpha} - \sum_{\alpha} \sum_{\beta} I_{\alpha\beta}^\alpha I_{\alpha\beta} \]
\[ = \sum_{\alpha} \sum_{\beta} I_{\alpha\beta} H_{\alpha} H_{\beta} - \sum_{\alpha} \sum_{\beta} I_{\alpha\beta}^\alpha I_{\alpha\beta} \]
\[ = \sum_{\alpha} \sum_{\beta} I_{\alpha\beta}^\alpha H_{\alpha\beta}, \]

it follows that

\[ d = \frac{\sum_{\alpha} \sum_{\beta} (\tau_{\alpha\beta}^* - \tau_{\alpha\beta}) H_{\alpha\beta}}{\sum_{\alpha} \sum_{\beta} I_{\alpha\beta}^\alpha H_{\alpha\beta}}. \]

Hence we have established the following:

**Theorem.** The densities of the joint distributions of \( \hat{\theta}_n \) and \( \hat{\theta}^*_{k/n} \) in the class \( \mathcal{D} \) are equal to up to order \( n^{-1} \) if and only if the asymptotic deficiency of \( \hat{\theta}^*_{k/n} \) w.r.t. \( \hat{\theta}_n \) is given by

\[ d = \frac{\sum_{\alpha} \sum_{\beta} (\tau_{\alpha\beta}^* - \tau_{\alpha\beta}) H_{\alpha\beta}}{\sum_{\alpha} \sum_{\beta} I_{\alpha\beta}^\alpha H_{\alpha\beta}}. \]
Remark: In the one parameter case it is easily seen by the above theorem that the asymptotic deficiency is given by $d = I(\tau^* - \tau)$ independent of $y_1, \ldots, y_p$. (See Akahira, 1981.) In the multiparameter case, as is immediately seen from the Theorem, the asymptotic deficiency depends on $(y_1, \ldots, y_p)$. This essentially differs from the one parameter case.

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