"EFFICIENCY IN SUBSAMPLING METHODS"

BY

LOUIS GORDON

TECHNICAL REPORT NO. 21
MARCH 22, 1971

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"Efficiency in Subsampling Methods"

Louis Gordon*

Stanford University

Abstract

Hartigan (JASA, 1969) constructs an exact confidence interval for the common median of a set of continuous symmetric and independent random variables; the technique involves computing a number of subsample means from the given sample. In this paper, sufficient conditions are given for the related test of location for i.i.d. random variables having variance to have asymptotic efficiency 1 relative to one and two-sided t-tests. A related problem involving the asymptotic variance of the sample distribution function of subsample means is also discussed.

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1. Introduction

J. A. Hartigan (JASA, 1969) introduced a scheme for generating exact confidence intervals for the common median of a collection of independent, but not necessarily identically distributed, symmetric random variables. Forsythe and Hartigan (Biometrika, 1970) examined the lengths of Hartigan's intervals in the case of identically distributed normal variates when the subsampling scheme consisted of all half-samples.

This paper considers two questions related to Hartigan's ideas. First, the confidence interval formulation yields a test of location for i.i.d. symmetric random variables. Under certain sufficient conditions, the asymptotic efficiency of the Hartigan test relative to the t-test is shown to be 1. The proof is accomplished by an extension of Hartigan's technique of examining the random distribution function of the subsample means.

Second, a closer look at the fine structure of the subsample c.d.f. is achieved by evaluating its asymptotic variance at fixed points for two specific families of subsampling schemes. This quantity is seen to differ markedly for the two closely related families considered.
2. Preliminaries

The following notation and conventions will be used throughout the discussion. $F$ will denote a continuous distribution function symmetric about 0 with variance $\sigma^2$. $Y_1, Y_2, \ldots$ are a sequence of independent random variables distributed as $F$. $S_n$ will stand for a random variable distributed as the n-th partial sum of the $Y$'s. If two independent sums are needed, they will be represented as $S^{(1)}_n$ and $S^{(2)}_n$. The normalized n-fold convolution of $F$ will be written $F_n$ and is the distribution function of $S_n/n^{1/2}$. $Z$ will denote a standard normal random variable having distribution function $\Phi$.

Let $A$ and $B$ be subsets of the indices. The symmetric difference of $A$ and $B$, written $A\Delta B$, is $(A \setminus B) \cup (B \setminus A)$. We shall write $\nu(A)$ for the cardinality of the set $A$. $S_A$ will represent the sum $\sum Y_i | i \in A$. $\mathcal{O}_n$ will denote a collection of non-empty subsets of indices such that (1) each subset in $\mathcal{O}_n$ is a subset of $\{1, \ldots, n\}$, and (2) $\mathcal{O}_n \cup \{\emptyset\}$ is a group under the operation symmetric difference. For convenience, we shall occasionally call $\mathcal{O}_n$ a group. We shall write $m_n$ for $\nu(\mathcal{O}_n)$. Finally, $P_{\epsilon,n}$ will denote the proportion of subsets in $\mathcal{O}_n$ having cardinality far from $n/2$; that is,

$$P_{\epsilon,n} = \nu(A \in \mathcal{O}_n \mid \nu(A) - \frac{n}{2} > \epsilon n)^{-1} m_n.$$
3. The Basic Result

We first prove a lemma that is a modest extension of Hartigan's Theorem 4. This lemma is then subject to the usual extension to uniform convergence a la the Glivenko-Cantelli theorem. The extension can then be used to approximate the behavior of the test statistic based on Hartigan's confidence intervals for testing, in a translation parameter family, the hypothesis of symmetry about 0.

**Lemma 1.** For fixed \( t \), there exists a sequence \( g_n \) decreasing to 0 and a positive constant \( B_0 \) such that, for any \( \epsilon < 1/10 \) and any integers \( p, q \) for which \( p + q = n \) and \( |p - \frac{1}{2} n| < n \epsilon \) we have

\[
|E[I_{\{S_p^{(1)} + S_q^{(2)} \leq t \sqrt{n}\}} - \Phi(t/\sigma)] I_{\{S_p^{(1)} - S_q^{(2)} \leq t \sqrt{n}\}} - \Phi(t/\sigma)]| < g_n + B_0 \epsilon.
\]

**Proof.** By the central limit theorem, there exists a sequence \( \delta_n \), decreasing to 0, for which \( \sup_x |F_n(x) - \Phi(x)| < \delta_n \).

We now write

\[
I(t) = |E[I_{\{S_p^{(1)} + S_q^{(2)} \leq t \sqrt{n}\}} - \Phi(t/\sigma)] I_{\{S_p^{(1)} - S_q^{(2)} \leq t \sqrt{n}\}} - \Phi(t/\sigma)]| \leq |E\Phi(t \sigma^{-1}(n/p)^{-1/2} - |S_q^{(2)}/\sigma \sqrt{q}| (q/p)^{1/2}) - \Phi(t/\sigma)|
\]

\[
- 2\Phi(t/\sigma) [F_n(t/\sigma) - \Phi(t/\sigma)]| \leq |E\Phi(t \sigma^{-1}(n/p)^{-1/2} - |S_q^{(2)}/\sigma \sqrt{q}| (q/p)^{1/2}) - \Phi(t/\sigma)| + \delta_p + 2\delta_n
\]

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Now \( (n/p)^{-1/2} - \sqrt{2} \leq 5\epsilon \) and \( (q/p)^{1/2} - 1 \leq 5\epsilon \) so that by a Taylor expansion with remainder,

\[
I(t) \leq \left| E(\Phi(t\sigma^{-1}\sqrt{2} - |S_{q}^{(2)}/\sigma \sqrt{q}|)) - \Phi^2(t/\sigma) \right| + 3d_{n}/4 + 5\epsilon|t|\sigma^{-1} + E|S_{q}^{(2)}/\sigma \sqrt{q}|]/\sqrt{2\pi}
\]

Since \( \Phi^2(t\sigma^{-1}) = E\Phi(t\sigma^{-1}\sqrt{2} - |Z|) \), and \( q > (1/2 - \epsilon)n \), we may use weak convergence to choose \( \epsilon_n \) decreasing to 0 such that

\[
I(t) \leq \epsilon_n + 5\epsilon|t|\sigma^{-1} + 1]/\sqrt{2\pi}
\]

Hence the assertion.

Lemma 2. Let \( \mathcal{G}_{n} \) be a sequence of groups on the first \( n \) indices for which \( P_{\epsilon,n} \rightarrow 0 \) for any positive \( \epsilon \). Then for any fixed \( t \)

\[
m_{n}^{-1} \sum_{A \in \mathcal{G}_{n}} I[S_{A} - S_{A_{c}} \leq t \sqrt{n}] \xrightarrow{P} \Phi(t/\sigma).
\]

Complementation is taken relative to the set \( \{1, \ldots, n\} \).

Proof. The proof turns on Hartigan's idea of showing that the difference converges to 0 in \( L^2 \). We shall suppress "\( n \)" in the notation for the sake of simplicity. Choose \( \epsilon \) positive with \( \epsilon < 1/10 \).
Consider the subset of $\mathcal{G}$ given by

$$\mathcal{G}_\epsilon = \{ A \in \mathcal{G} \mid v(A) - \frac{1}{2} n \leq n \epsilon \}. \quad \text{Let } \mathcal{I}_\epsilon = \{ (A, B) \in \mathcal{G}_\epsilon \times \mathcal{G}_\epsilon \mid A \cap B \in \mathcal{G}_\epsilon \}.$$ 

Note that $v(\mathcal{G}_\epsilon \times \mathcal{G}_\epsilon \setminus \mathcal{I}_\epsilon) \leq (2m^2 + m) P_\epsilon$.

We shall write $AB$ for $A \cap B$ when $A$ and $B$ are subsets of indices.

If $(A, B) \in \mathcal{I}_\epsilon$, we may write

$$S_A - S_A^C = S_{AB} + S_{AB^C} - S_{A^B} - S_{A^B^C},$$

and

$$S_B - S_B^C = S_{AB} - S_{A^B} + S_{A^B^C} - S_{A^B^C}.$$

Hence we have the simultaneous representation:

$$S_A - S_A^C = S^{(1)}_p + S^{(2)}_q,$$

$$S_B - S_B^C = S^{(1)}_p - S^{(2)}_q,$$

where $p + q = n$, and $p = v(A \cap B)$ so that $|p - \frac{1}{2} n| \leq n \epsilon$ since $A \cap B \in \mathcal{G}_\epsilon$. Note that the hypothesis of Lemma 1 is satisfied.

Now,

$$E(m^{-1} \sum_{A \in \mathcal{G}_\epsilon} [I[S_A - S_A^C \leq t \sqrt{n}] - \Phi(t/\sigma)]^2)$$

$$= m^{-2} \sum_{(A, B) \in \mathcal{G}_\epsilon \times \mathcal{G}_\epsilon \setminus \mathcal{I}_\epsilon} + \sum_{(A, B) \in \mathcal{I}_\epsilon} [I[S_A - S_A^C \leq t \sqrt{n}] - \Phi(t/\sigma)]$$

$$\cdot [I[S_B - S_B^C \leq t \sqrt{n}] - \Phi(t/\sigma)]$$

$$\leq m^{-2} (4(2m^2 + m) P_\epsilon + (g_n + B_0 \epsilon) m^2).$$

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Hence, since $P_\epsilon$ and $g_n$ converge to 0, the difference converges in $L^2$ to 0.

Remark. Note that contained in the hypothesis of Lemma 2 is the implicit assumption that $m_n \to \infty$. This follows since the average size of a non-empty set in the group is $(m+1)n/2m$.

Lemma 3. Let $\bar{Y}_n$ denote the average of the first $n$ Y's. Let $t$ be fixed and let $P_{\epsilon,n} \to 0$. Then

$$m_n^{-1} \sum_{A \in \mathcal{S}_n} I_{\{S_A^{1/2} \leq (t \sqrt{n})^{-1} \leq \sqrt{n} \bar{Y}_n \leq t\}} - I_{\{S_A - S_{Ac} \leq t \sqrt{n}\}} \to 0.$$ 

Proof. We show the quantity in question converges in $L^1$. Where convenient and unambiguous, the $n$ will be suppressed in the notation.

Let $d_n$ be the same decreasing sequence as in the last proof. Choose $\epsilon$ positive and $\epsilon < 1/10$. Let $\mathcal{S}_\epsilon$, $p$, and $q$ also be as before. Then, for $A \in \mathcal{S}_\epsilon$, write

$$J(A) = \left| E\left[ I_{\{S_A^{1/2} \leq (t \sqrt{n})^{-1} \leq \sqrt{n} \bar{Y}_n \leq t\}} - I_{\{S_A - S_{Ac} \leq t \sqrt{n}\}} \right] \right|$$

$$= E\left[ I_{\{S_A^{1/2} \leq (t \sqrt{n})^{-1} \leq (S_A + S_{Ac}) \leq (S_A - S_{Ac})^{-1/2}\}} \right]$$

$$+ E\left[ I_{\{S_A^{1/2} \leq (t \sqrt{n})^{-1} > (S_A + S_{Ac}) \geq (S_A - S_{Ac})^{-1/2}\}} \right]$$

$$\leq 2[qd_n + E[\Phi((n/q)^{1/2}[-\frac{t}{\sigma} - (1-n/p) S_A^{-1/2}])]
- \Phi((n/q)^{1/2}[-\frac{t}{\sigma} + S_A^{-1/2}])] \leq \frac{1}{4}d_n/4 + B_1 \epsilon.$$
where \( B_1 \) is a constant independent of the choice of \( \epsilon \).

Hence,

\[
E[m^{-1} \sum_{A \in G} \text{I} \left\{ S_{A_n}^{1/2} \nu(A)^{-1} - \sqrt{n} \frac{1}{n} \bar{Y}_n \leq t \right\} \{ S_{A} - S_{A^c} \leq t \sqrt{n} \}] \leq m^{-1} \left( \sum_{A \in G \setminus \Omega_\epsilon} 2 + \sum_{A \in \partial \Omega_\epsilon} J(A) \right) \leq m^{-1} (2mP_\epsilon + m[L_0 n^{-1/4} + B_1 \epsilon])
\]

We therefore have \( L^1 \) convergence to 0.

**Notation.** We define the sample distribution function of subsample means relative to \( G_n \) to be \( G_n(t) = \sum_{A \in G_n} \text{I} \{ S_{A} \nu(A)^{-1} \leq t \} \).

**Theorem 4:** Let \( G_n \) be a sequence of groups on \( n \) indices. Let \( Y_1, Y_2, \ldots \) be i.i.d. random variables distributed as \( F \), symmetric about 0, with variance \( \sigma^2 \). Let \( G_n \) be a sequence of groups with \( P_{\epsilon,n} \to 0 \) for each \( \epsilon > 0 \). Then

\[
\sup_t |G_n(tn^{-1/2} + \bar{Y}_n) - \phi(t \sigma^{-1})| \xrightarrow{P} 0.
\]

**Proof.** Combination of Lemmas 2 and 3 yields \( G_n(tn^{-1/2} + \bar{Y}_n) \xrightarrow{P} \phi(t \sigma^{-1}) \) for each fixed \( t \). Since \( \phi(t \sigma^{-1}) \) and the \( G_n(tn^{-1/2} + \bar{Y}_n) \) are increasing in \( t \), the standard uniform convergence argument of the Glivenko-Cantelli theorem yields the result.
4. Efficiency

**Notation.** Let $Y_1^*, Y_2^*, \ldots$ be a sequence of continuous i.i.d. random variables with variance $\sigma^2$. The quantities $\bar{Y}_n^*$, $S_n^*$, and $G_n^*(t)$ are the analogues to the quantities $\bar{Y}_n$, $S_A$, and $G_n(t)$ discussed in the previous section, computed using $Y_i^*$ instead of $Y_i$.

**Definition.** Let $T_n^*$ be a consistent estimator for $\sigma^2$. We shall call the statistic $\sqrt{n} \frac{\bar{Y}_n^*}{T_n^*}$ a $t$-statistic and will denote by $\tau_n^\alpha$ the value for which $P(\sqrt{n} \frac{\bar{Y}_n^*}{T_n^*} > \tau_n^\alpha) = \alpha$.

**Notation.** Analogously, $\xi^\alpha$ will denote the real number for which $P[Z > \xi^\alpha] = \alpha$.

The Hartigan scheme for constructing confidence intervals yields tests for the Hypothesis $H_0 : \beta_0 = 0$ when $Y_1^*, Y_2^*, \ldots$ are i.i.d. random variables distributed as $F(x - \beta_0)$. A test of size $\gamma + \delta$ relative to a group $\mathcal{G}_n$ is given by: Reject $H_0$ if $G_n^*(0) \notin [\gamma, 1-\delta]$. We compare the power of this test against the power of the $t$-test: Reject $H_0$ if $\sqrt{n} \frac{\bar{Y}_n^*}{T_n^*} \notin [\tau_n^{1-\delta}, \tau_n^\gamma]$, when the true common distribution of the $Y_1^*, \ldots, Y_n^*$ is $F(x - \beta\sqrt{n})$.

**Theorem 1.** Let $F$ be a continuous c.d.f., symmetric about 0, with variance $\sigma^2$. Let $\mathcal{G}_n$ be a sequence of groups on $n$ indices for which $P_{\epsilon, n} \rightarrow 0$ for each fixed positive $\epsilon$. Let $Y_1^*, \ldots, Y_n^*$ be distributed independently as $F(x - \beta\sqrt{n})$. Then
\[ \lim_{n \to \infty} P(\sqrt{n} \, \bar{Y}_n^* / T_n^* \notin [y_n^{1-\delta}, y_n^\delta]) = \lim_{n \to \infty} P((G_n^*(0) \notin [\gamma, 1-\delta]) \]
\[ = P(Z + \beta \sigma^{-1} \notin [\xi^{1-\delta}, \xi^\delta]). \]

**Proof.** It is clear that for \( i \leq n \) and \( Y_i^* = Y_i + \beta_n^{-1/2} \) that
\[ \lim_{n \to \infty} P(\sqrt{n} \, \bar{Y} + \beta / T_n^* \notin [y_n^{1-\delta}, y_n^\delta]) = P(Z + \beta \sigma^{-1} \notin [\xi^{1-\delta}, \xi^\delta]). \]

Now \( G_n^*(0) = \Sigma_{i=1}^{m^{-1}} \left( (S_n \, v(A)^{-1} - \bar{Y}_n) / \sqrt{n} \right) \geq - \sqrt{n} \, \bar{Y}_n - \beta \)

Hence, from Theorem 2.4, \( G_n^*(0) - \phi(- \sqrt{n} \, \bar{Y}_n \sigma^{-1} - \beta \sigma^{-1}) \to 0 \).

Hence \( \lim P(G_n^*(0) \notin [\gamma, 1-\delta]) = P(\phi(Z - \beta \sigma^{-1}) \notin [\gamma, 1-\delta]) \). But the latter probability is just
\[ P(Z - \beta \sigma^{-1} \notin [\xi^{1-\delta}, \xi^\delta]) = P(Z + \beta \sigma^{-1} \notin [\xi^{1-\delta}, \xi^\delta]). \]

**Corollary.** For near translation alternatives, with a family of continuous distributions symmetric about the mean, possessing a variance, the t-test (both one and two-sided) has asymptotic relative efficiency 1 with respect to the test indicated in Theorem 1 based on Hartigan confidence intervals when, for the groups \( G_n^*, P_{e,n} \to 0 \) for all positive \( e \).

For a discussion of asymptotic efficiency, see D. Fraser's book on nonparametric statistics.
Remark. The condition $P_{\varepsilon,n} \to 0$ is not so strong as may appear at first glance. The following are two equivalent formulations of a sufficient condition for the above to hold for all positive $\varepsilon$:

(1) Let $\mathcal{F}_n$ be a group on the indices $\{1, \ldots, n\}$. Then if $1 \leq i < j \leq n$ implies there exist sets $A, B$ in $\mathcal{F}_n$ for which $i \in A$, $j \notin A$, $j \in B$, and $i \notin B$, we have that $P_{\varepsilon,n} \to 0$ for all $\varepsilon > 0$.

(2) Let $R_n$ be the $m_n \times n$ incidence matrix of 0's and 1's where

$$(R_n)_{ij} = I\{j \in A_i\}.$$  

$P_{\varepsilon,n} \to 0$ for each $\varepsilon > 0$ if $R_n$ is of rank $n$.

Note in particular that (1) and (2) are indeed satisfied for $\mathcal{F}_n$ the power set of $\{1, \ldots, n\}$.

Remark. A more delicate handling of the approximations yields, under the same conditions on $P_{\varepsilon,n}$, that efficiency is also 1 for subsample means $2S_A/n$, $A \in \mathcal{F}_n$. 

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5. A Related Problem

The normalized and centered subsample c.d.f.

\[ H_n(t) = m_n^{-1} \sum_{\{S^{-1} \frac{\hat{A}}{n} \leq \tilde{y}_n \leq t_n^{-1/2}\}} \]

plays a crucial role in the preceding discussion. The asymptotic variance, \( m_n \text{E}[H_n(t) - \Phi(t/\sigma)]^2 \), of this c.d.f. was therefore of interest. Surprisingly, this variance is strongly dependent on the structure of the group in question. This assertion will be supported by computing the asymptotic variances associated with two very similar groups. One group will consist entirely of half samples; in the second, all but one set will be half-samples.

Notation. \( F \) will now denote a c.d.f. symmetric about 0, with variance \( \sigma^2 \), and finite fourth moment \( \mu_4 \). \( F \) will be assumed to have a continuous bounded density function \( f \), and \( F_n \) will then have density \( f_n \). We will write \( \kappa = (\mu_4 - 3\sigma^4)/24\sigma^4 \).

Corresponding to a group \( G \) is its 0-1 incidence matrix \( R \). The rows of \( R \) correspond to non-empty sets in \( G \); a 1 appears in column \( j \) if the index \( j \) appears in the set corresponding to the row in question. We shall need three sequences of incidence matrices.

Define \( R^*_1 = (1) \). We define inductively the \( 2^\ell - 1 \times 2^\ell - 1 \) matrices \( R^*_\ell \), for \( \ell \geq 2 \). In particular

\[
R^*_\ell = \begin{pmatrix}
R^*_{\ell-1} & R^*_{\ell-1} & 0 \\
2^{\ell-1} & 2^{\ell-1} & 0 & e^T \\
0 & e^T & 1 \\
R^*_{\ell-1} & J-R^*_{\ell-1} & e
\end{pmatrix}
\]
where $J$ is the matrix of all 1's and $e$ is a vector of all 1's.

We now define $\hat{R}_2^\ell$ and $\hat{R}_2^\ell$ by $\hat{R}_2^\ell = (R^*_2, 0)$ and

$$
\hat{R}_2^\ell = \begin{pmatrix}
R^*_2 & 0 \\
2^{\ell-1} & 1 \\
e^T & 1 \\
J-R^*_2 & e
\end{pmatrix}
$$

Note that $P_{\epsilon,2^\ell} \to 0$ because of condition (1) in the remark following Theorem 3.1 as does $P_{\epsilon,2^{\ell-1}}^*$, the proportion corresponding to $R_{2^{\ell-1}}^*$, for the same reason. Hence $P_{\epsilon,2^{\ell-1}} \to 0$ for each positive $\epsilon$ as well.

Finally observe that all rows of $\hat{R}_2^\ell$ have $2^{\ell-1}$ 1's and $2^{\ell-1}$ 0's. Also, all but one row of $R_2^\ell$ have $2^{\ell-1}$ 1's and $2^{\ell-1}$ 0's; the remaining row possesses all 1's. Observe that $\hat{R}_2^\ell$ has dimensions $2^{\ell-1} \times 2^\ell$ and that $R_2^\ell$ has dimensions $2^{\ell+1-1} \times 2^\ell$.

We shall now compute the asymptotic variances of $\hat{G}_2^\ell(t 2^{-\ell/2} + \tilde{Y})$ and $G_2^\ell(t 2^{-\ell/2} + \tilde{Y})$. We start by stating some results related to the Edgeworth expansion for densities (see Feller, Chapter XV).

**Definition.** We define $\delta_n(x) = n(f_n(x) - \varphi(x))$ and $\Delta_n(x) = n(F_n(x) - \Phi(x))$, where $\varphi(x)$ is the standard normal density. We shall also write $\delta(x) = \kappa \varphi^{iv}(x)$ and $\Delta(x) = \kappa \varphi^{iii}(x)$. 

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Lemma 1. There exist a sequence of functions $a_n(x)$ and a sequence of constants $b_n$ such that:

\[(1) \quad |a_n(x)| \leq b_n \quad \text{and} \quad b_n \rightarrow 0\]

\[(2) \quad \Delta_n(x) = \Delta(x) + \int_0^x a_n(t) \, dt .\]

**Proof.** The result follows directly after noting that the hypothesis of symmetry yields $F_n(0) = \frac{1}{2}$ and $\int x^2 F(dx) = 0$.

Then integrate the Edgeworth expansion for $F_n$. 

Lemma 2:

\[
\lim_{n \to \infty} nE[I \left( \frac{1}{n} \sum_{i=1}^{n} X_i \leq t \sqrt{2n} \right) \left( \frac{1}{n} \sum_{i=1}^{n} X_i \geq t \sqrt{2n} \right) - \phi^2(t \sigma^{-1})] \\
= 2E \Delta(t \sqrt{2} \sigma^{-1} - |Z|) - 2\kappa(\phi(t \sqrt{2} \sigma^{-1})\phi'(0)) + \phi^2(t \sqrt{2} \sigma^{-1}) \phi(0)) .
\]

**Proof.** Note that $\phi^2(t \sigma^{-1}) = E \phi(t \sqrt{2} \sigma^{-1} - |Z|)$. Let

\[
D_n = nE[I \left( \frac{1}{n} \sum_{i=1}^{n} X_i \leq t \sqrt{2n} \right) \left( \frac{1}{n} \sum_{i=1}^{n} X_i \geq t \sqrt{2n} \right) - \phi^2(t \sigma^{-1})] \\
= nE[F_n(t \sqrt{2} \sigma^{-1} - |S_n|) - \phi(t \sqrt{2} \sigma^{-1} - |Z|)] .
\]

We may apply Lemma 1 to the remainder term followed by the central limit theorem to then obtain
\[ D_n = nE \left[ \phi(t \sqrt{2} \sigma^{-1} - |S_n/\sigma \sqrt{n}|) - \phi(t \sqrt{2} \sigma^{-1} - |Z|) \right] \\
+ E \Delta(t \sqrt{2} \sigma^{-1} - |Z|) + o(1) \]  
(1)

The first term of the sum in (1) may be written, using the symmetry of \( F_n \), as \( D_n^* = 2 \int_0^\infty \phi(t \sqrt{2} \sigma^{-1} - |\xi|) \delta_n(\xi) \, d\xi \). Now let \( s = t \sqrt{2} \sigma^{-1} \) and note that \( \phi(s - |\xi|) \) is Lebesgue integrable in \([0, \infty)\) and \( |\delta_n(x) - \delta(x)| \leq b_n \) so \( D_n^* = 2 \int_0^\infty \phi(s-\xi) \delta(\xi) \, d\xi + o(1) \).

Now integrate by parts four times:

\[
D_n^* = 2k \int_0^\infty \phi(s-\xi) \varphi_{111}(\xi) \, d\xi + o(1)
\]

\[
= k \int_0^\infty \varphi_{111}(s-\xi) 2\varphi(\xi) \, d\xi - 2k(\varphi_{111}(0) \varphi(s) + \varphi(0) \varphi_{111}(s)) + o(1)
\]  
(2)

Combining equations (1) and (2) yields the result.

\[ \square \]

**Lemma 3.**

\[ EA(s-|Z|) = \frac{1}{2} \Delta(s/\sqrt{2}) \varphi(s/\sqrt{2}) + 2k \varphi_{111}(s/\sqrt{2}) \varphi(s/\sqrt{2}) + 3k[\varphi^1(s/\sqrt{2})]'^2/2. \]

**Proof.** We wish to evaluate \( I(s) = 2k \int_0^\infty \varphi_{111}(s-\xi) \varphi(\xi) \, d\xi \). Note first that \( \varphi^{(k)}(s-\xi) = \frac{d^k}{ds^k} \varphi(s-\xi) \). Hence

\[
I(s) = 2k \frac{d^3}{ds^3} \int_0^\infty \varphi(s-\xi) \varphi(\xi) \, d\xi = k \sqrt{2} \frac{d^3}{ds^3} \varphi(s/\sqrt{2}) \varphi(s/\sqrt{2}).
\]

The assertion follows.

\[ \square \]
Lemma 4. \[ \lim n E \left[ I \left( \frac{S_n(1) + S_n(2)}{\sqrt{n}} \leq t \right) \frac{I \left( \frac{S_n(1) - S_n(2)}{\sqrt{n}} \leq t \right)}{\sqrt{n}} - \Phi^2(t \sigma^{-1}) \right] \]

\[ = \Delta(t \sigma^{-1}) \Phi(t \sigma^{-1}) + 3\kappa [\Phi^4(t \sigma^{-1})]^{1/2}. \]

Proof. Let \( D_n \) be the expectation on the left in the statement of the lemma. Then, from Lemma 2,

\[ D_n = 2E \Delta(s - \|Z\|) - 2\kappa \varphi(s) \Phi_{ii}(0) + \Phi_{ii}(s) \varphi(0) + o(1), \]

where \( s = t \sqrt{\sigma^{-1}} \). From Lemma 3,

\[ D_n = \left[ \Delta(s/\sqrt{2}) \varphi(s/\sqrt{2}) + 4\kappa \Phi_{ii}(s/\sqrt{2}) \varphi(s/\sqrt{2}) + 3\kappa [\Phi^4(s/\sqrt{2})]^{1/2} \right] \]

\[ - 2\kappa \left( \frac{1}{\sqrt{2\pi}} \varphi(s) + \frac{1}{\sqrt{2\pi}} (s^2 - 1) \varphi(s) \right) + o(1). \]

The lemma follows.

Theorem 5. Let \( n = 2^\ell \) then for \( \hat{G}_{2^n} \), the group having incidence matrix \( \hat{R}_{2^n} \),

\[ \lim \hat{n}_{2^n} E \left[ \hat{G}_{2^n}(t/\sqrt{2^n} + \hat{Y}_{2^n}) - \Phi(t \sigma^{-1}) \right]^2 \]

\[ = \Phi(t \sigma^{-1}) (1 - \Phi(t \sigma^{-1})) + 6\kappa [\Phi^4(t \sigma^{-1})]^{1/2}. \]

Proof. Let the expectation on the left of the equality in the statement of the theorem be denoted \( D_n \). Then
\[ D_n = (2n-1)^{-1} \sum_{\sigma} E[I_{\{S_A - S_A^c \leq t \sqrt{2n}\}} - \phi(t\sigma^{-1})][I_{\{S_B - S_B^c \leq t \sqrt{2n}\}} - \phi(t\sigma^{-1})] \]

where set complementation is relative to \{1, \ldots, 2n\}.

If \( A \neq B \) then \( S_A - S_A^c = S_{AB} + S_{AB^c} - S_{AB}^{c} - S_{AB^c}^{c} \)

and \( S_B - S_B^c \) may be similarly decomposed. We thereby obtain the simultaneous representation \( S_A - S_A^c = S_n^{(1)} + S_n^{(2)} \) and \( S_B - S_B^c = S_n^{(1)} - S_n^{(2)} \). Therefore

\[ D_n = \phi(t\sigma^{-1})(1 - \phi(t\sigma^{-1})) - 2\Delta(t\sigma^{-1}) \phi(t\sigma^{-1}) + 2n E[I_{\{S_n^{(1)} + S_n^{(2)} \leq t \sqrt{2n}\}} - \phi(t\sigma^{-1})][I_{\{S_n^{(1)} - S_n^{(2)} \leq t \sqrt{2n}\}} - \phi(t\sigma^{-1})] + o(1). \]

From Lemma 4,

\[ D_n = \phi(t\sigma^{-1})(1 - \phi(t\sigma^{-1})) - 2\Delta(t\sigma^{-1}) \phi(t\sigma^{-1}) + 2\Delta(t\sigma^{-1}) \phi(t\sigma^{-1}) + 6\kappa[\phi^4(t\sigma^{-1})]^2 + o(1). \]

Theorem 6. Let \( n = 2^\ell \) and let \( \varphi_{2n} \) be the group corresponding to the incidence matrix \( R_{2n} \). Then

\[ \lim_{m \to \infty} E[G_{2n}(t/\sqrt{2n} + \bar{Y}_{2n}) - \phi(t\sigma^{-1})]^2 \]

\[ = \phi(t\sigma^{-1})(1 - 2\phi(t\sigma^{-1})) + [\phi(t\sigma^{-1}) - \phi(-t\sigma^{-1})]^+ + 12\kappa[\phi^4(t\sigma^{-1})]^2. \]
Proof. Let $D_n$ again correspond to the quantity whose limit is to be taken, found on the left side of the equality in the statement. Let $N = \{1, 2, \ldots, 2n\}$ and observe that $N \in \mathcal{G}_{2n}$. As before, we may write

$$D_n = (4n-1)^{-1} \sum \sum \left[ I; \left( S_A \sqrt{A} - Z_{2n} \leq t / \sqrt{2n} \right) - \Phi(t\sigma^{-1}) \right] \times \left[ I; \left( S_B \sqrt{B} - Z_{2n} \leq t / \sqrt{2n} \right) - \Phi(t\sigma^{-1}) \right]$$

There are four types of pairs $(A, B)$ possible:

1. $(A, A)$, $4n-1$ pairs
2. $(N, A)$ or $(A, N)$, $A \neq N$, $2(4n-2)$ pairs
3. $(A, B)$, $A \neq N \neq B \neq A \neq B^c$, $(4n-2)(4n-4)$ pairs
4. $(A, A^c)$, $A \neq N$, $4n-2$ pairs.

These classes make the following contributions:

1. $\Phi(t\sigma^{-1})(1 - \Phi(t\sigma^{-1})) + o(1)$
2. $o(1)$
3. This analysis is identical to that in Theorem 5 and so the contribution is

$$\lim E[I; \left( g_n(1) + g_n(2) \leq t \sqrt{2n} \right) - \Phi(t\sigma^{-1})][I; \left( g_n(1) - g_n(2) \leq t \sqrt{2n} \right) - \Phi(t\sigma^{-1})] + o(1) = 12\kappa[\Phi(t\sigma^{-1})]^2 + o(1).$$

4. The contribution is

$$E[I; \left( S_A - S_{A^c} \leq t \sqrt{2n} \right) - \Phi(t\sigma^{-1})][I; \left( S_{A^c} - S_A \leq t \sqrt{2n} \right) - \Phi(t\sigma^{-1})] + o(1).$$
This quantity may be evaluated in the usual manner to yield
\((\phi(t\sigma^{-1}) - \phi(-t\sigma^{-1}))^+ - \phi^2(t\sigma^{-1})\). Adding the contributions yields
the result.

Remark. Although it may not be evident at first glance, the asymptotic
error is indeed symmetric about 0.
References


