ACCURATE BOOTSTRAP CONFIDENCE LIMITS
IN PARAMETRIC MODELS

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THOMAS J. DICICCIO AND JOSEPH P. ROMANO

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The problem of constructing confidence limits for a scalar parameter is considered. Under weak conditions, Efron's (1987) accelerated bias-corrected bootstrap confidence limits are correct to second order in parametric families. In this article, a new method for setting approximate confidence limits is proposed as an attempt to alleviate two problems inherent in Efron's method. The accelerated bias-corrected method is not fully automatic since it requires the calculation of an analytical adjustment; furthermore, it is typically not exact, though for many situations, particularly scalar-parameter families, exact answers are available. In a broader amount of generality, the proposed method is exact when exact answers exist, and it is second-order accurate otherwise. The method is automatic and for scalar parameter models it can be iterated to achieve higher accuracy, with the number of computations being linear in the number of iterations. However, when nuisance parameters are present, only second-order accuracy seems obtainable.

Some key words: Accelerated bias-corrected method; Bootstrap; Confidence limits; Higher-order accuracy; Least favourable family; Orthogonal parameters; Pivots.
1. INTRODUCTION

Consider the problem of constructing confidence limits for a parameter \( \theta \) indexing a parametric family of distributions \( \{ P_\theta, \theta \in \Theta \} \), where \( \Theta \) is some interval on the real line. Approximate confidence limits for \( \theta \) based on a bootstrap estimate of the sampling distribution of an estimator \( \hat{\theta} \) were introduced by Efron (1981). Further refinements and generalizations are discussed in Efron (1984, 1987), with the most recent method being the so-called accelerated bias-corrected percentile (BC\(_a\)) confidence limits.

A standard approximate upper 1\( - \alpha \) confidence limit of the form \( \hat{\theta} + \hat{\sigma} z^{(1-\alpha)} \), where \( \hat{\sigma} \) is an estimate of the standard error of \( \hat{\theta} \) and \( z^{(\alpha)} \) is the \( \alpha \) quantile of the standard normal distribution, is based on the asymptotic Gaussian approximation \( (\hat{\theta} - \theta)/\hat{\sigma} \sim N(0, 1) \). Because this limit is typically accurate only as a first-order approximation, several alternative confidence procedures, based on analytical corrections and resampling methods, designed to improve coverage accuracy, have been proposed. Specifically, Efron (1987) develops the accelerated bias-corrected method of constructing confidence limits by assuming a rather weak approximation. In particular, he assumes that for some monotone transformation \( g \),

\[
\frac{g(\hat{\theta}) - g(\theta)}{1 + a g(\theta)} + z_0 \sim N(0, 1),
\]

where the acceleration constant \( a \) and the bias constant \( z_0 \) are independent of \( \theta \). When (1.1) holds exactly, the resulting BC\(_a\) limits are exact, and quite generally the BC\(_a\) limits are second-order correct in the sense described by Efron (1987). One advantage of this method is that the BC\(_a\) limits can be constructed without knowledge of \( g \).

In this article, a new procedure is proposed as an attempt to alleviate two problems inherent in the BC\(_a\) method. The first problem concerns the difficulty in computing the acceleration constant \( a \) by Efron's (1987) formula (4.4). Indeed, for some cases,
as discussed in Loh and Wu (1987), this formula is not applicable. The computation of $a$ may be a delicate matter and the $BC_a$ procedure does not seem to be generally automatic. The second problem is that the $BC_a$ method does not typically produce exact limits in some situations, though clear exact limits are available. For example, the $BC_a$ method fails to be exact for constructing a confidence limit for a scale parameter in a parametric scale family. For a broad range of situations, the limits proposed here are exact, and they are otherwise highly accurate in a sense to be described.

In Section 2, Efron's $BC_a$ method is reviewed, and the new procedure is derived under the assumption that Efron's pivotal approximation (1.1) is exact. In Section 3, it is seen that this procedure is also exact under even much weaker assumptions. In Section 4, we discuss how to iterate the procedure to achieve arbitrary accuracy in those cases where the conditions for exactness given Section 3 are not met. Each round of iteration is shown to improve the accuracy of the procedure and the number of computations required is linear in the number of iterations. In Section 5, we discuss how the method is extended to situations involving nuisance parameters. Some simulation results are presented in Section 6 to substantiate the theory.
2. REVIEW OF THE \( BC_\alpha \) AND A NEW METHOD

In this section, we provide a simple construction of Efron's (1987) \( BC_\alpha \) method under assumption (1.1). A new method is also derived which is exact under (1.1) and avoids having to compute \( z_0 \) or \( a \). The argument for the latter method will be considered more generally in the next section.

We begin by assuming that, for some monotone transformation \( g \), the estimate \( \hat{\theta} \) of \( \theta \) satisfies

\[
U = \frac{g(\hat{\theta}) - g(\theta)}{1 + ag(\theta)} + z_0 \sim H,
\]

where \( U \) is a random variable with distribution function \( H \). Of course, (1.1) implies (2.1). For simplicity, assume \( H^{-1} \) exists and let \( u^{(\alpha)}_{\alpha} = H^{-1}(\alpha) \). Without loss of generality it may be assumed that \( g \) is monotonically increasing. Let \( G_\theta(t) = pr_\theta(\theta \leq t) \).

Also, let

\[
\overline{G}_\phi(s) = pr_\phi(\phi \leq s),
\]

where \( \phi = g(\theta) \) and \( \hat{\phi} = g(\hat{\theta}) \), so that

\[
G_\theta(t) = \overline{G}_\phi\left\{ g(t) \right\}
\]

and

\[
G_{\theta^{-1}}(\alpha) = g^{-1}\left\{ \overline{G}_{\phi^{-1}}(\alpha) \right\}.
\]

By inverting (2.1), it follows that, for all \( \phi \),

\[
pr_\phi(\phi \leq \hat{\phi}_{U,\alpha}) = 1 - \alpha,
\]

where

\[
\hat{\phi}_{U,\alpha} = \hat{\phi} - \frac{(1 + a \hat{\phi})(u^{(\alpha)}_{\alpha} - z_0)}{1 + a(u^{(\alpha)}_{\alpha} - z_0)}.
\]
Thus, an exact upper 1–α confidence limit for \( \phi \) is \( \hat{\phi}_{U,\alpha} \). Correspondingly, an exact upper confidence limit for \( \theta \) is

\[
\hat{\theta}_{U,\alpha} = g^{-1}(\hat{\phi}_{U,\alpha}).
\]  

(2.7)

Thus far, in order to construct \( \hat{\theta}_{U,\alpha} \), one needs to know \( H, g, a, \) and \( z_0 \).

Note that

\[
\bar{G}_{\hat{\phi}}(s) = H\left( \frac{s - \hat{\phi}}{1 + a \hat{\phi}} + z_0 \right)
\]

and

\[
\bar{G}_{\hat{\phi}}^{-1}(\alpha) = \hat{\phi} + (1 + a \hat{\phi})(u^{(\alpha)} - z_0)
\]

In particular, the bootstrap distribution \( \bar{G}_{\hat{\phi}} \) of \( \hat{\phi} \) satisfies

\[
\bar{G}_{\hat{\phi}}(s) = *H\left( \frac{s - \hat{\phi}}{1 + a \hat{\phi}} + z_0 \right)
\]

and

\[
\bar{G}_{\hat{\phi}}^{-1}(\alpha) = \hat{\phi} + (1 + a \hat{\phi})(u^{(\alpha)} - z_0)
\]

The upper 1–α confidence limit \( \hat{\phi}_{U,\alpha} \) of \( \phi \) is obtained from the bootstrap distribution of \( \hat{\phi} \) by the relationship

\[
\hat{\phi}_{U,\alpha} = \bar{G}_{\hat{\phi}}^{-1}\left\{ H\left( z_0 - \frac{u^{(\alpha)} - z_0}{1 + a(u^{(\alpha)} - z_0)} \right) \right\}.
\]

Similarly, the upper 1–α limit \( \hat{\theta}_{U,\alpha} \) for \( \theta \) can be obtained from the bootstrap distribution \( G_{\hat{\theta}} \) of \( \hat{\theta} \) by the relationship

\[
\hat{\theta}_{U,\alpha} = G_{\hat{\theta}}^{-1}\left\{ H\left( z_0 - \frac{u^{(\alpha)} - z_0}{1 + a(u^{(\alpha)} - z_0)} \right) \right\}.
\]  

(2.8)

Notice that \( \hat{\theta}_{U,\alpha} \) can be computed from (2.8) without knowledge of \( g \), but one still needs to know \( H, z_0 \) and \( a \). Efron assumes \( H \) is standard normal and gives formulae for \( a \) and \( z_0 \).
Thus far, we have two expressions (2.7) and (2.8) for \( \hat{\theta}_{U,\alpha} \). How can the upper limit \( \hat{\theta}_{U,\alpha} \) be constructed without explicit knowledge of \( H, a \) and \( z_0 \)? The following theorem gives a formula.

**Theorem 2.1.** Assume (2.1). Let \( \theta_0 \) be any value of \( \theta \), let \( \theta_0' = G_{\theta_0}^{-1}(\alpha) \), and let

\[
\theta_1 = G_{\hat{\theta}}^{-1}\left\{G_{\theta_0'}(\theta_0)\right\}.
\]

Then, \( \theta_1 = \hat{\theta}_{U,\alpha} \).

For any value \( \theta \), \( G_{\theta} \) may be simulated. Formula (2.9) requires that \( G_{\theta} \) be simulated at the three values \( \hat{\theta}, \theta_0, \) and \( \theta_0' \). In order to reduce the calculations, one may choose \( \theta_0 \) to be \( \hat{\theta} \). In this case, one needs to simulate two bootstrap distributions, \( G_{\hat{\theta}} \) and \( G_{\theta_0'} \) to calculate the upper confidence limit. Formula (2.9) is invariant under a transformation of \( \theta \), as may be seen by use of (2.2)-(2.4).

In the next two sections, we investigate the behavior of \( G_{\hat{\theta}}^{-1}\left\{G_{\theta_0'}(\theta_0)\right\} \) as an upper limit when (2.1) does not hold. In Section 3, it is nevertheless seen to be exact under a weaker assumption than (2.1). In Section 4, we will see that \( \theta_0 \) may be interpreted as a contemplated value of the upper confidence limit. Indeed, when (2.1) is not exact, one may start with a contemplated value \( \theta_0 \) and produce \( \theta_1 \) by (2.9). The closer the value of \( \theta_0 \) to the correct upper limit, the closer \( \theta_1 \) will be to the right upper limit. Moreover, this procedure can be iterated. Under expansion assumptions, each round of iteration reduces the error of the proposed limit by a factor \( n^{-1/2} \).

**Proof of Theorem 2.1.** First, we derive a formula for \( \hat{\theta}_{U,\alpha} \). Let \( \phi_0 \) be any value of \( \phi \) and let
\[ \phi_0' = \overline{G}_{\phi_0^{-1}}(\alpha) = \phi_0 + (1 + a \phi_0)(u^{(\alpha)} - z_0) . \]

Then, it is easy to verify that

\[ \overline{G}_{\phi_0'}(\phi_0) = H \left\{ z_0 - \frac{u^{(\alpha)} - z_0}{1 + a u^{(\alpha)} - z_0} \right\} , \]

and therefore the true upper limit for \( \phi \) defined by (2.6) is also given by

\[ \hat{\phi}_{U,\alpha} = \overline{G}_\phi^{-1}\left\{ \overline{G}_{\phi_0'}(\phi_0) \right\} . \]

Letting \( \theta_0 = g^{-1}(\phi_0) \) and using (2.3)-(2.4), it follows that \( \hat{\theta}_{U,\alpha} = g^{-1}(\hat{\phi}_{U,\alpha}) \) is given by formula (2.9).
3. CONDITIONS FOR EXACTNESS

In this section, we develop conditions under which an upper endpoint $\theta_4$ given by (2.9) is exact. In particular, a necessary and sufficient condition is obtained for the proposed intervals to be exact.

Before proceeding, recall that it is always possible in principle to construct an exact upper $1-\alpha$ confidence bound in the parametric setting with no nuisance parameters. To do this, one simply tests, for each fixed $\theta$, the null hypotheses that $\theta$ is true based on the distribution $G_\theta$ of $\hat{\theta}$. This idea is an old one and it is discussed in Schenker (1987). Specifically, suppose that, for fixed $\alpha$, the quantile graph of $G_\theta^{-1}(\alpha)$ versus $\theta$ is continuous and strictly monotone increasing. Then, the value of $\theta$ satisfying

$$\hat{\theta} = G_\theta^{-1}(\alpha)$$

(3.1)

is an exact $1-\alpha$ upper confidence limit for $\theta$. This value, as in Section 2, will be denoted $\hat{\theta}_{U,\alpha}$ and is the same exact limit that Efron (1987, Section 5) calls $\theta_{EX} \left[ 1-\alpha \right]$.

Given that an exact method exists in this setting, what are the reasons to consider other methods? First of all, the equation (3.1) is difficult to solve because the distributions $G_\theta$ often do not have analytical expressions. Schenker (1987) discusses a simulation method to solve (3.1) where $G_\theta$ is simulated at many values of $\theta$, but this method suffers from the substantial amount of computation required. The method proposed here selectively chooses based on the data two or three values of $\theta$ at which to compute $G_\theta$ and then obtains a confidence limit based on a formula involving these distributions.

The first result in this section imposes a condition on the family of distributions $G_\theta$ of $\hat{\theta}$, stated as Condition A.

**Condition A.** Suppose the family $G_\theta$ satisfy the following: If, for any choice of $t_1$, $t_2$, 

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\[ \theta_1 \text{ and } \theta_2, \ G_{\theta_1}(t_1) = G_{\theta_2}(t_2), \text{ then } G_{t_1}(\theta_1) = G_{t_2}(\theta_2). \]

**Example 1.** If \( G_{\theta}(t) = H(t - \theta) \) is a location family, and \( H^{-1} \) exists, then Condition A is satisfied.

**Example 2.** If \( G_{\theta}(t) = H(t / \theta) \) is a scale family, where \( H^{-1} \) exists, then Condition A is satisfied.

**Example 3.** If

\[
G_{\theta}(t) = H \left\{ \frac{g(t) - g(\theta)}{1 + ag(\theta)} + z_0 \right\},
\]

where \( g \) is monotone increasing, \( a \) and \( z_0 \) are constants and \( H^{-1} \) exists, then Condition A is satisfied. Note that this is the family derived from assumption (2.1). The fact that Condition A holds may not be apparent but can be checked by simple algebra.

In these examples, if \( H^{-1} \) does not exist, the exact upper limit \( \hat{\theta}_{U, \alpha} \) is not well-defined; however, an appropriate upper limit, based on the graph of \( G_{\theta}^{-1}(\alpha) \), can be appropriately defined.

The following result generalizes Theorem 2.1.

**Theorem 3.1.** Suppose \( G_{\theta}^{-1} \) exists. Given a value \( \theta_0 \) of \( \theta \), let \( \theta_0' = G_{\theta_0}^{-1}(\alpha) \), and let

\[
\theta_1 = G_{\theta}^{-1} \left\{ G_{\theta_0'}(\theta_0) \right\}. \tag{3.2}
\]

Then, a necessary and sufficient condition for \( \hat{\theta}_{U, \alpha} = \theta_1 \) for all choices \( \theta_0 \) of \( \theta \) is that Condition A holds.

**Proof.** By assumption, \( \alpha = G_{\theta_0}(\theta_0') \), and hence by (3.1) it follows that

\[
G_{\theta_0}(\theta_0') = G_{\hat{\theta}_{U, \alpha}}(\hat{\theta}).
\]
By Condition A,

$$G_{\theta_0'}(\theta_0) = G_{\theta}(\hat{\theta}_{U\alpha}).$$

Now, to prove sufficiency, apply $G_{\theta}^{-1}$ to both sides. To prove necessity, work backwards.

Theorem 3.1 imposes a condition on the family of distributions $G_{\theta}$ of $\hat{\theta}$ for the approximate limit $\theta_1$ given by (3.2) to be exact. Next, we consider conditions concerning the existence of pivotal quantities which are necessary and sufficient for exactness.

Specifically, suppose there exists a quantity $R(\hat{\theta}, \theta)$ which is pivotal in the sense that the distribution of $R(\hat{\theta}, \theta)$ when $\theta$ is true is independent of $\theta$. Theorem 3.1 already contains some special cases. For example, if $R(\hat{\theta}, \theta) = \hat{\theta} - \theta$, $R(\hat{\theta}, \theta) = \hat{\theta}/\theta$, or

$$R(\hat{\theta}, \theta) = \frac{g(\hat{\theta}) - g(\theta)}{1 + ag(\theta)} + z_0$$

is pivotal, then (3.2) is true. In general, what are the necessary and sufficient conditions on $R$ such that (3.2) remains true? To begin, assume that $R(\hat{\theta}, \theta)$ is pivotal with distribution function $H$ and that $u^{(\alpha)} = H^{-1}(\alpha)$ exists. Also, assume that $R(s, t)$ is monotone increasing in $s$ for each fixed $t$. Note that

$$G_{\theta}(s) = H\{R(s, \theta)\}.$$ 

Assume that for each fixed value of $s$, $R(s, t)$ is monotone decreasing in $t$. Define an inverse $k_s$ as follows. For fixed $s$, if $R(s, t) = u$, then $k_s(u) = t$, so that $R\{s, k_s(u)\} = u$ and $k_s\{R(s, t)\} = t$. Thus, $k_s(u)$ is monotone decreasing in $u$, and

$$pr_{\theta}\{R(\hat{\theta}, \theta) \leq u^{(\alpha)}\} = pr_{\theta}\{\theta \leq k_{\theta}(u^{(\alpha)})\} = 1 - \alpha.$$
It follows that the exact 1-\( \alpha \) confidence limit for \( \theta \) is

\[
\hat{\theta}_{U, \alpha} = k_{\hat{\theta}}(u^{(\alpha)}) .
\]

To obtain an expression for \( G_{\hat{\theta}}^{-1}(\alpha) \) in terms of \( R \), define an inverse as follows. For fixed \( t \), if \( R(s, t) = u \), then \( h_t(u) = s \) so that \( R\{h_t(u), t\} = u \) and \( h_t\{R(s, t)\} = s \). Thus, \( h_t(u) \) is monotone increasing in \( u \), and

\[
G_{\hat{\theta}}^{-1}(\alpha) = h_{\hat{\theta}}(u^{(\alpha)}) .
\]

Consider the following condition concerning \( R \).

**Condition B.** For all \( s, t, \) and \( u \),

\[
R\{t, h_t(u)\} = R\{k_t(u), s\} .
\]

The existence of a pivot is always true in the parametric setting because \( R(\hat{\theta}, \theta) = G_{\hat{\theta}}(\hat{\theta}) \) has the uniform distribution on \((0, 1)\).

**Theorem 3.2.** A necessary and sufficient condition for \( \hat{\theta}_{U, \alpha} = \theta_1 \), where \( \theta_1 \) is given by (3.2), is that there exists a pivot \( R \) satisfying Condition B.

**Proof of Theorem 3.2.** The exact upper 1-\( \alpha \) confidence limit for \( \theta \) is

\[
k_{\hat{\theta}}(u^{(\alpha)}) = h_{\hat{\theta}} \left[ R\{k_{\hat{\theta}}(u^{(\alpha)}), \hat{\theta}\} \right] = G_{\hat{\theta}}^{-1} \left[ H \left[ R\{k_{\hat{\theta}}(u^{(\alpha)}), \hat{\theta}\} \right] \right] . \quad (3.3)
\]

For given \( \theta_0, \theta_0' = h_{\theta_0}(u^{(\alpha)}) \) and

\[
G_{\theta_0'}(\theta_0) = H \left[ R\{\theta_0, h_{\theta_0}(u^{(\alpha)})\} \right] .
\]
Thus,

\[ G_{\hat{\theta}}^{-1} \left\{ G_{\theta_0} (\theta_0) \right\} = G_{\hat{\theta}}^{-1} \left[ H \left\{ R \left\{ \theta_0, h_{\theta_0} (u^{(a)}) \right\} \right\} \right]. \] (3.4)

Comparing (3.3) and (3.4), a necessary and sufficient condition for the exactness of the formula is

\[ R \left\{ \theta_0, h_{\theta_0} (u^{(a)}) \right\} = R \left\{ k_{\theta} (u^{(a)}), \hat{\theta} \right\}, \]

which is equivalent to Condition B. Conversely, there always exists a pivot \( R (\hat{\theta}, \theta) = G_{\theta} (\hat{\theta}). \)

Notice that the form of \( R \) is not required to compute \( \theta_1 \). We now investigate an equivalent condition on \( R \) which may be easier to verify in practice, stated as Condition C.

**Condition C.** For some function \( f, R (s, t) = f \left\{ R (t, s) \right\} \) for all \( s, t \).

Note that if \( R \) satisfies Condition C, then \( f = f^{-1} \) and \( f \) is monotone decreasing. We now have the following.

**Proposition 3.1.** Condition B and Condition C are equivalent.

**Proof of Proposition 3.1.** Assume Condition C, so that

\[ R (s, t) = f \left\{ R (t, s) \right\} = u, \] say. Now, \( R (s, t) = u \) implies \( h_t (u) = s \). Also,

\[ f \left\{ R (t, s) \right\} = u \] implies \( R (t, s) = f (u) \), so that \( k_t \left\{ f (u) \right\} = s \) and

\[ h_s \left\{ f (u) \right\} = k_s (u). \] Then, \( R \left\{ t, h_t (u) \right\} = R (t, s) = f (u) \) and
\[ R\{h_s(u), s\} = R \left[ h_s \left\{ f(u) \right\}, s \right] = f(u). \]

Hence, Condition B holds. Conversely, if Condition B holds, then \( R\{t, h_t(u)\} \) is a function of \( u \) alone, say \( R\{t, h_t(u)\} = f^{-1}(u) \). Then, \( R(s, t) = R\{h_t(u), t\} = u \) implies

\[ R\{t, h_t(u)\} = R(t, s) = f^{-1}(u). \]

Thus, \( R(t, s) = f^{-1}\{R(s, t)\} \).

We now investigate a further equivalent condition.

**Condition D.** If for any values of \( s, s', t, \) and \( t' \), \( R(s, t) = R(s', t') \), then \( R(t, s) = R(t', s') \).

**Proposition 3.2.** Condition C and Condition D are equivalent.

**Proof of Proposition 3.2.** Assume Condition C and suppose that \( R(s, t) = R(s', t') \). Then,

\[ R(t, s) = f^{-1}\{R(s, t)\} = f^{-1}\{R(s', t')\} = R(t', s') \]

Conversely, assume Condition D. Define a function \( f \) by \( R(s, t) = f\{R(t, s)\} \); the assumption assures that \( f \) is well defined. Then, \( R(s, t) = f\{R(t, s)\} \).
The main results of this section may be summarized as follows. Condition A is necessary and sufficient for the exactness of formula (3.2). Moreover, Condition A is equivalent to the existence of a pivot satisfying any of Conditions B, C, or D. Notice that Condition D, for the special choice of pivot \( R(\hat{\theta}, \theta) = G_{\hat{\theta}}(\theta) \), is exactly Condition A.

Rather than insist on the formula \( G_{\hat{\theta}}^{-1} \left\{ G_{\theta_0}(\theta_0) \right\} \) to be exact for arbitrary \( \theta_0 \), one can obtain weaker conditions than those already obtained by considering the choice of \( \theta_0 = \hat{\theta} \). Then, Conditions A, B, and D can be weakened to the following equivalent necessary and sufficient conditions for the exactness of the formula taking \( \theta_0 = \hat{\theta} \). The proofs are similar. Condition C does not appear to have a natural weaker condition.

Condition \( A' \). If \( G_s(r) = G_t(s) \), then \( G_r(s) = G_s(t) \).

Condition \( B' \). For all \( t, u \):

\[
R\left\{ t, h_t(u) \right\} = R\left\{ k_t(u), t \right\}.
\]

Condition \( D' \). \( R(r, s) = R(s, t) \) implies \( R(s, r) = R(t, s) \).
4. ITERATIVE CORRECTIONS AND ASYMPTOTICS

Thus far, we have seen that the formula

\[ \theta_1 = G_{\hat{\theta}}^{-1} \left\{ G_{\hat{\theta}}, (\theta_0) \right\} \]

is an exact upper 1-\(\alpha\) confidence bound for a wide class of problems. In this section, we study the behavior of this bound when it is not exact; in particular, it is not assumed Conditions A-D hold. However, since

\[ R(\hat{\theta}, \theta) = \frac{g(\hat{\theta}) - g(\theta)}{1 + ag(\theta)} + z_0 \]

is an approximate pivot quite generally (see DiCiccio and Tibshirani, 1987) for suitable choice of \(g\), \(a\) and \(z_0\), and since Condition B is true for this choice of \(R\), the resulting formula should remain approximately valid. Indeed, this is true under weak conditions.

First, notice that formula (4.1) may be iterated. That is, starting with a contemplated endpoint \(\theta_0\), (4.1) produces a new endpoint \(\theta_1\). Letting \(\theta_1' = G_{\hat{\theta}}^{-1}(\alpha)\), one may form a new endpoint \(\theta_2\) by setting

\[ \theta_2 = G_{\hat{\theta}}^{-1} \left\{ G_{\hat{\theta}}, (\theta_1') \right\} \]

and this process can be continued. As will be seen, each round of iteration reduces the level error of the resulting interval. Unlike other iterative methods proposed, the amount of computation for this method is linear in the number of iterations. Other iterative procedures, as in Hall (1986) and Beran (1987), often involve nested bootstrap calculations, so that the number of calculations is exponential in the number of iterations.

The following fact says that if one starts with an upper bound \(\theta_0\) which is correct, the resulting upper limit \(\theta_1\) will remain correct. This suggests that further iteration will
converge to the correct upper limit \( \hat{\theta}_{U, \alpha} \) when starting with an inexact \( \theta_0 \).

**Proposition 4.1.** If \( \theta_0 = \hat{\theta}_{U, \alpha} \), then \( \theta_0 = \theta_1 \).

**Proof.** By definition, \( \hat{\theta}_{U, \alpha} \) satisfies \( G_{\hat{\theta}_{U, \alpha}}(\hat{\theta}) = \alpha \). Since \( \theta_0 = \hat{\theta}_{U, \alpha} \), it follows that \( \theta_0' = \hat{\theta} \). Then,

\[
\theta_1 = G_{\hat{\theta}}^{-1}\left\{ G_{\theta_0'}(\theta_0) \right\} = G_{\hat{\theta}}^{-1}\left\{ G_{\theta_0}(\theta_0) \right\} = \theta_0 .
\]

**Remark 4.1.** It is useful to remember that at any stage of the iterations, one can determine if any contemplated endpoint \( \theta_j \) is correct without calculating \( \theta_{i+1} \) by checking \( G_{\theta_j}(\hat{\theta}) = \alpha \).

The main result of this section is now developed. Assume, as in Hall (1987), that the distribution of \( \hat{\theta} \) can be approximated by an Edgeworth expansion. In anticipation of asymptotic results, place a subscript \( n \) on \( \hat{\theta} \) so that \( \hat{\theta}_n = \hat{\theta} \) depends on \( n \), as does its distribution \( G_{\theta, \theta} \) under \( \theta \). Let \( \sigma_n(\theta) \) denote the standard deviation of \( \hat{\theta}_n \) under \( \theta \).

Let

\[
J_{n, \theta}(x) = \Pr \left\{ \frac{n^{1/2}(\hat{\theta}_n - \theta) / \sigma_n(\theta)}{x} \leq z \right\} = G_{\theta, \theta}\left\{ \sigma_n(\theta) x / n^{1/2} + \theta \right\} .
\]

Assume \( J_{n, \theta}(x) \) satisfies

\[
J_{n, \theta}(x) = \Phi(x) + \sum_{i=1}^{\nu} n^{-i/2} p_i(z ; \theta) \phi(x) + O_P(n^{-(\nu+1)/2}) ,
\]

where \( \Phi \) and \( \phi \) are the standard Gaussian distribution function and density function, respectively. Let \( z_{n, \theta}(\alpha) = J_{n, \theta}^{-1}(\alpha) \) so that \( z_{n, \theta}(\alpha) \) satisfies

\[
z_{n, \theta}(\alpha) = \alpha + \sum_{i=1}^{\nu} n^{-i/2} p_{i, 1}(z^{(\alpha)} ; \theta) + O_P(n^{-(\nu+1)/2}) ,
\]

(4.2)
where $z^{(\alpha)} = \Phi^{-1}(\alpha)$. The functions $p_i(z, \theta)$ and $p_{i,1}(z, \theta)$ are polynomials in $z$ which depend continuously on $\theta$.

**Theorem 4.1.** Assume expansions (4.2) and (4.3) are valid. Suppose $\theta_{0,n}$ satisfies

$$|\theta_{U,\alpha} - \theta_{0,n}| = O_P(n^{-\nu/2}).$$

Let $\theta_{0,n}' = G_{n,\theta_n}'(\alpha)$ and

$$\theta_{1,n} = G_{n,\theta_n'}^{-1}\left\{G_{n,\theta_{0,n}}'(\theta_{0,n})\right\}.$$

Then,

$$|\theta_{U,\alpha} - \theta_{1,n}| = O_P(n^{-(\nu+1)/2}).$$

**Proof of Theorem 4.1.** For ease of notation, assume $\sigma_n(\theta) = 1$. This assumption can be achieved by adopting the variance stabilizing parametrization that one is in the variance stabilized case. All approximations, denoted $\approx$, are understood to be valid to order $n^{-(\nu+1)/2}$ in $P_\theta$ probability. Since $\hat{\theta}_{U,\alpha}$ satisfies $G_{n,\hat{\theta}_{U,\alpha}}^{-1}(\alpha) = \hat{\theta}_n$, and $G_{n,\hat{\theta}_{U,\alpha}}^{-1}(\alpha)$ is by (4.3) approximately

$$G_{\hat{\theta}_{U,\alpha}}^{-1}(\alpha) \approx \hat{\theta}_{U,\alpha} + \left\{ \frac{z_{\alpha} + \sum_{i=1}^{\nu} n^{-i/2} p_{i,1}(\hat{\theta}_{U,\alpha}, z_{\alpha})}{n^{1/2}} \right\},$$

it follows that

$$\hat{\theta}_{U,\alpha} \approx \hat{\theta}_n - \left\{ \frac{z_{\alpha} + \sum_{i=1}^{\nu} n^{-i/2} p_{i,1}(\hat{\theta}_{U,\alpha}, z_{\alpha})}{n^{1/2}} \right\}.$$  \hspace{1cm} (4.4)

Similarly,

$$\theta_{0,n}' \approx \theta_{0,n} + \left\{ \frac{z_{\alpha} + \sum_{i=1}^{\nu} n^{-i/2} p_{i,1}(\theta_{0,n}, z_{\alpha})}{n^{1/2}} \right\}.$$  \hspace{1cm} (4.5)
Combining (4.4), (4.5) and the fact that \( |\theta_{0,n} - \hat{\theta}_{U,\alpha}| = O_P\left(n^{-\nu/2}\right) \) yields

\[
\theta_{0,n}' \approx \theta_{0,n} + \hat{\theta}_n - \hat{\theta}_{U,\alpha}.
\]  

(4.6)

Let \( z_{n,\theta} = n^{1/2}(x - \theta) \). By (4.2), we have in general

\[
G_{n,\theta}(x) \approx \Phi(z_{n,\theta}) + \sum_{i=1}^{\nu} n^{-i/2} p_i(\theta, z_{n,\theta}) \phi(z_{n,\theta}).
\]

(4.7)

Combining (4.6) and (4.7) yields

\[
G_{n,\theta_0,\alpha}'(\theta_{0,n}) \approx \Phi\left\{ n^{1/2}(\hat{\theta}_{U,\alpha} - \hat{\theta}_n) \right\} + \sum_{i=1}^{\nu} n^{-i/2} p_i\left\{ \theta_{0,n}', n^{1/2}(\hat{\theta}_{U,\alpha} - \hat{\theta}_n) \right\} \phi\left\{ n^{1/2}(\hat{\theta}_{U,\alpha} - \hat{\theta}_n) \right\}.
\]

(4.8)

Now, by (4.4) and (4.7)

\[
G_{n,\theta_0}(\hat{\theta}_{U,\alpha}) \approx \Phi\left\{ n^{1/2}(\hat{\theta}_{U,\alpha} - \hat{\theta}_n) \right\} + \sum_{i=1}^{\nu} n^{-i/2} p_i\left\{ \theta_n, n^{1/2}(\hat{\theta}_{U,\alpha} - \hat{\theta}_n) \right\} \phi\left\{ n^{1/2}(\hat{\theta}_{U,\alpha} - \hat{\theta}_n) \right\}.
\]

But, \( \theta_{0,n}' \approx \hat{\theta}_n \), so that (4.7) and (4.8) are equivalent to order \( O_P\left(n^{-(\nu+1)/2}\right) \). Thus,

\[
G_{n,\theta_0}(\hat{\theta}_{U,\alpha}) \approx G_{n,\theta_0,\alpha}'(\theta_{0,n}).
\]

Apply \( G_{n,\hat{\theta}_n}^{-1} \) to both sides to get the result.
5. NUISANCE PARAMETERS

In this section, we indicate how the procedure given by (2.9) can be extended to set approximate confidence limits for a scalar parameter in multiparameter families and non-parametric situations.

For the multiparameter case, consider a family of distributions indexed by the vector parameter \( \eta = (\eta^1, \ldots, \eta^p) \), and suppose that the scalar parameter \( \theta = t(\eta) \) is of interest. Let \( \hat{\eta} \) be an estimator of \( \eta \), and let \( \hat{\theta} = t(\hat{\eta}) \) be the corresponding estimator of \( \theta \). The distribution function of \( \hat{\theta} \) is given by \( G_\eta(s) = pr_\eta(\hat{\theta} \leq s) \), and the bootstrap distribution for \( \hat{\theta} \) is \( G_{\hat{\eta}} \).

Of course, when \( G_\eta \) depends on \( \eta \) only through \( \theta = t(\eta) \), the results of the previous sections apply since the problem is effectively a parametric one with no nuisance parameters. Unfortunately, \( G_\eta \) usually does not depend only on \( \theta \). To handle this more complicated situation, suppose the model has been parametrized so that \( \eta^1 = \theta \) and \( \eta^2, \ldots, \eta^p \) is orthogonal to \( \theta \), i.e.,

\[
\text{cov}(n^{1/2}\hat{\theta}, n^{1/2}\hat{\eta}^i) = O(n^{-1}) \quad (i = 2, \ldots, p).
\]

Let \( \lambda = (\eta^2, \ldots, \eta^p) \). Then, (2.9) can be adapted to find an approximate upper 1-\( \alpha \)-confidence limit for \( \theta \) by proceeding as follows. Start with an initial value \( \theta_0 \) of \( \theta \), perhaps the percentile limit \( G^{-1}_\eta(1-\alpha) \), set \( \theta'_0 = G^{-1}_{(\theta_0, \lambda)}(\alpha) \), and finally take

\[
\theta_1 = G^{-1}_{(\theta'_0, \lambda)}\left\{ G_{(\theta'_0, \lambda)}(\theta_0) \right\}.
\]

In many situations, it may be inconvenient to transform a given model to achieve an orthogonal parametrization. For such cases, the least favourable family construction used by Efron (1987) allows the procedure to be implemented in terms of the original parameters. Suppose that

\[
\text{cov}(n^{1/2}\hat{\eta}^i, n^{1/2}\hat{\eta}^j) = \kappa^i,j \quad (i, j = 1, \ldots, p),
\]

\[18\]
and let $t_i = \partial \eta^i / \partial \eta$ and $\mu^i = \sum \kappa^i t_i$. Consider the line in the parameter space, parametrized by $\eta$, given by $\eta(\sigma) = \eta + \sigma \hat{\mu}$, where $\hat{\mu} = (\hat{\mu}^1, \ldots, \hat{\mu}^p)$ and $\hat{\mu}_i = \mu_i(\eta)$. Then, $\hat{\eta} = \eta(0)$, and the bootstrap distribution for $\hat{\theta}$ is $G_{\eta(0)}$. To implement the procedure, commence with an initial value $\theta_0$ for $\theta$, choose $\sigma_0$ to satisfy $\theta_0 = t \{ \eta(\sigma_0) \}$ and let $\theta_0' = G_{\eta(\sigma_0)}^{-1}(\alpha)$. The approximate limit for $\theta$ is

$$\theta_1 = G_{\hat{\eta}}^{-1} \left\{ G_{\eta(\sigma_0')} \theta_0 \right\},$$

where $\sigma_0'$ satisfies $\theta_0' = t \{ \eta(\sigma_0') \}$. It can be shown that the error in the coverage level of $\theta_1$ is of order $O(n^{-1})$; however, because of the presence of nuisance parameters, the order of this error cannot be reduced by further iteration.

Efron (1987) discusses the reduction of nonparametric problems to multiparameter ones and he presents a construction of the appropriate least favourable family. By using his construction, the procedure outlined above can be extended for use in nonparametric situations. The properties of these approximate limits in multiparameter families and nonparametric cases are currently being investigated, and further results concerning second-order accuracy will appear elsewhere.
6. SIMULATIONS

In this section, two examples are considered to study the behavior of the iterative method when the conditions for exactness do not hold. First, consider a sample of size $n=5$ from a bivariate Gaussian distribution with unknown means and variances and unknown correlation coefficient $\theta$, and let $\hat{\theta}$ be the usual sample correlation estimate. For this example, an exact formula for $G_\theta$ exists; see Garwood (1933). Approximate and exact upper and lower 97.5% confidence limits for $\theta$ having observed $\hat{\theta} = 0.0, 0.3$, and 0.8 are given in Table 1. When $\hat{\theta} = 0.3$, for example, the exact upper 97.5% confidence limit is $\hat{\theta}_{U,0.025} = 0.8851$. The percentile limit is 0.9386, and the accelerated bias-corrected percentile limit is 0.9194. Note that the acceleration constant is 0 in this case. Starting with the percentile method limit $\theta_0$, the procedure described in Section 4 is iterated five times, with $\theta_i$ denoting the result of the $i$th iteration. By the fourth iteration, the limits agree with $\hat{\theta}_{U,0.025}$ to three decimals. Note that any contemplated upper limit $\theta_U$ is exact if it satisfies $G_{\theta_U}(\hat{\theta}) = 0.025$. In this example, $G_{\theta_U}(\hat{\theta})$ equals 0.025 to three decimals when $i = 4$ and $i = 5$. Further results on upper and lower limits for this problem, based on $\hat{\theta} = 0.0, 0.3$, and 0.8 are summarized in Table 1.

Next, consider a sample of size $n=10$ observations from the $Beta(\theta, 2)$ family of distributions with densities

$$f_{\theta}(x) = \theta(\theta+1)x^{\theta-1}(1-x) \quad 0 < x < 1,$$

where $\theta > 0$. The problem is to construct upper and lower 95% confidence limits for $\theta$ based on the maximum likelihood estimate $\hat{\theta}$. In this case, $G_{\theta}$ does not have a known explicit form, so that simulation is required to approximate $G_{\theta}$. Thus, whenever necessary, $G_{\theta}$ is computed by drawing $B$ samples, each of size 10, from $f_{\theta}$, and computing $\hat{\theta}$ for each sample. The empirical distribution of these $B$ values, $G_{\theta}$, serves as a stochastic approximation to $G_{\theta}$. Given $\hat{\theta} = 1.4518$, Table 2 shows the $BC_{\alpha}$ limits and the results
of iterating the new method 10 times, starting with the percentile limit \( \theta_0 \). For any upper endpoint \( \theta_U \), \( G_{\theta_U}(\hat{\theta}) \) is approximated as well. In this case, even if \( \theta_U \) were exact, the estimated value \( \hat{G}_{\theta_U}(\hat{\theta}) \) would not be exactly 0.05 due to simulation error. Indeed, given \( \hat{\theta} \) and a fixed choice of \( \theta_U \), \( \hat{G}_{\theta_U}(\hat{\theta}) \) has a standard error of

\[
G_{\theta_U}(\hat{\theta}) \left\{ 1 - G_{\theta_U}(\hat{\theta}) \right\} \frac{1}{B},
\]

which for \( \theta_U \) near the exact limit, can be estimated by 0.05-0.95/\( B \). Table 2 was constructed using \( B = 10,000 \), so this estimated standard error is 0.0044. Although the results in Table 2 do not seem to monotonically improve with the number of iterations, all of the limits after the first iteration are quite satisfactory and the lack of convergence can be explained by the simulation error. In Table 3, the results are repeated for three iterations, but with \( B \) increased to 30,000, and the results are better. It is clear that if one wants extremely high accuracy by iteration, \( B \) must be large enough to make (6.1) small.

To study the overall coverage probabilities of the proposed limits, a larger simulation was done. Table 4 summarizes the results of drawing 500 samples of size 10 from \( f_\theta \) when \( \theta = 1 \), and recording the proportion of times the proposed limits cover \( \theta \). Here, \( B = 500 \) is used. For example, based on two iterations, 96.8% of the 500 nominal 95% upper limits exceeded \( \theta \), while 94.2% of the nominal 95% lower limits were less than \( \theta \). Table 4 also shows the results obtained for \( \theta = 2 \) and \( \theta = 10 \). The coverage accuracy of the approximate limits is quite satisfactory in view of the inherent simulation error.

Acknowledgement: We would like to thank Christian Leger for doing the computer work to generate Tables 2-4.
### TABLE 1

Upper and Lower 97.5% Confidence Limits for the Correlation Coefficient (n = 5),
Upper Limit = $\theta_U$, Lower Limit = $\theta_L$

<table>
<thead>
<tr>
<th>Method</th>
<th>Exact</th>
<th>Perc.</th>
<th>BC$_a$</th>
<th>$\theta_1$</th>
<th>$\theta_2$</th>
<th>$\theta_3$</th>
<th>$\theta_4$</th>
<th>$\theta_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta_U$</td>
<td>0.8114</td>
<td>0.8783</td>
<td>0.8783</td>
<td>0.7817</td>
<td>0.8223</td>
<td>0.8071</td>
<td>0.8131</td>
<td>0.8108</td>
</tr>
<tr>
<td>$G_{\theta_U}(\hat{\theta})$</td>
<td>0.0250</td>
<td>0.0107</td>
<td>0.0107</td>
<td>0.0331</td>
<td>0.0223</td>
<td>0.0261</td>
<td>0.0246</td>
<td>0.0252</td>
</tr>
<tr>
<td>$\theta_L$</td>
<td>-0.8114</td>
<td>-0.8783</td>
<td>-0.8783</td>
<td>-0.7817</td>
<td>-0.8223</td>
<td>-0.8071</td>
<td>-0.8131</td>
<td>-0.8108</td>
</tr>
<tr>
<td>$1-G_{\theta_L}(\hat{\theta})$</td>
<td>0.0250</td>
<td>0.0107</td>
<td>0.0107</td>
<td>0.0331</td>
<td>0.0223</td>
<td>0.0261</td>
<td>0.0246</td>
<td>0.0252</td>
</tr>
</tbody>
</table>

- $\hat{\theta} = 0.0$

<table>
<thead>
<tr>
<th>Method</th>
<th>Exact</th>
<th>Perc.</th>
<th>BC$_a$</th>
<th>$\theta_1$</th>
<th>$\theta_2$</th>
<th>$\theta_3$</th>
<th>$\theta_4$</th>
<th>$\theta_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta_U$</td>
<td>0.8851</td>
<td>0.9386</td>
<td>0.9194</td>
<td>0.8678</td>
<td>0.8899</td>
<td>0.8837</td>
<td>0.8855</td>
<td>0.8850</td>
</tr>
<tr>
<td>$G_{\theta_U}(\hat{\theta})$</td>
<td>0.0250</td>
<td>0.0077</td>
<td>0.0128</td>
<td>0.0324</td>
<td>0.0231</td>
<td>0.0256</td>
<td>0.0248</td>
<td>0.0250</td>
</tr>
<tr>
<td>$\theta_L$</td>
<td>-0.7026</td>
<td>-0.7606</td>
<td>-0.8159</td>
<td>-0.6728</td>
<td>-0.7160</td>
<td>-0.6962</td>
<td>-0.7056</td>
<td>-0.7012</td>
</tr>
<tr>
<td>$1-G_{\theta_L}(\hat{\theta})$</td>
<td>0.0250</td>
<td>0.0160</td>
<td>0.0094</td>
<td>0.0304</td>
<td>0.0227</td>
<td>0.0261</td>
<td>0.0245</td>
<td>0.0252</td>
</tr>
</tbody>
</table>

- $\hat{\theta} = 0.3$

<table>
<thead>
<tr>
<th>Method</th>
<th>Exact</th>
<th>Perc.</th>
<th>BC$_a$</th>
<th>$\theta_1$</th>
<th>$\theta_2$</th>
<th>$\theta_3$</th>
<th>$\theta_4$</th>
<th>$\theta_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta_U$</td>
<td>0.9720</td>
<td>0.9885</td>
<td>0.9761</td>
<td>0.9705</td>
<td>0.9722</td>
<td>0.9720</td>
<td>0.9720</td>
<td>0.9720</td>
</tr>
<tr>
<td>$G_{\theta_U}(\hat{\theta})$</td>
<td>0.0250</td>
<td>0.0049</td>
<td>0.0189</td>
<td>0.0274</td>
<td>0.0245</td>
<td>0.0250</td>
<td>0.0250</td>
<td>0.0250</td>
</tr>
<tr>
<td>$\theta_L$</td>
<td>-0.2224</td>
<td>-0.0384</td>
<td>-0.4129</td>
<td>-0.2986</td>
<td>-0.1860</td>
<td>-0.2386</td>
<td>-0.2149</td>
<td>-0.0257</td>
</tr>
<tr>
<td>$1-G_{\theta_L}(\hat{\theta})$</td>
<td>0.0250</td>
<td>0.0461</td>
<td>0.0120</td>
<td>0.0189</td>
<td>0.0284</td>
<td>0.0236</td>
<td>0.2256</td>
<td>0.0247</td>
</tr>
</tbody>
</table>

- $\hat{\theta} = 0.8$
### TABLE 2

Upper and Lower 95% Confidence Limits for a Beta Parameter (n = 10)

\( \hat{\theta} = 1.4518, \ B = 10,000 \). Upper Limit = \( \theta_U \), Lower Limit = \( \theta_L \)

<table>
<thead>
<tr>
<th>Method</th>
<th>( \theta_U )</th>
<th>( G_{\theta U} (\hat{\theta}) )</th>
<th>( \theta_L )</th>
<th>( 1-G_{\theta L} (\hat{\theta}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Perc.</td>
<td>2.3653</td>
<td>0.0218</td>
<td>0.9711</td>
<td>0.0993</td>
</tr>
<tr>
<td>( \theta_1 )</td>
<td>2.1531</td>
<td>0.0463</td>
<td>0.8755</td>
<td>0.0570</td>
</tr>
<tr>
<td>( \theta_2 )</td>
<td>2.1313</td>
<td>0.0562</td>
<td>0.8573</td>
<td>0.0516</td>
</tr>
<tr>
<td>( \theta_3 )</td>
<td>2.1497</td>
<td>0.0528</td>
<td>0.8638</td>
<td>0.0496</td>
</tr>
<tr>
<td>( \theta_4 )</td>
<td>2.1475</td>
<td>0.0495</td>
<td>0.8627</td>
<td>0.0501</td>
</tr>
<tr>
<td>( \theta_5 )</td>
<td>2.1333</td>
<td>0.0541</td>
<td>0.8669</td>
<td>0.0500</td>
</tr>
<tr>
<td>( \theta_6 )</td>
<td>2.1302</td>
<td>0.0547</td>
<td>0.8723</td>
<td>0.0500</td>
</tr>
<tr>
<td>( \theta_7 )</td>
<td>2.1378</td>
<td>0.0547</td>
<td>0.8752</td>
<td>0.0559</td>
</tr>
<tr>
<td>( \theta_8 )</td>
<td>2.1377</td>
<td>0.0567</td>
<td>0.8670</td>
<td>0.0519</td>
</tr>
<tr>
<td>( \theta_9 )</td>
<td>2.1530</td>
<td>0.0504</td>
<td>0.8648</td>
<td>0.0513</td>
</tr>
<tr>
<td>( \theta_{10} )</td>
<td>2.1404</td>
<td>0.0556</td>
<td>0.8688</td>
<td>0.0508</td>
</tr>
<tr>
<td>( \theta_{10} )</td>
<td>2.1548</td>
<td>0.0513</td>
<td>0.8734</td>
<td>0.0562</td>
</tr>
</tbody>
</table>

### TABLE 3

Upper and Lower 95% Confidence Limits for a Beta Parameter (n = 10)

\( \hat{\theta} = 1.4518, \ B = 30,000 \). Upper Limit = \( \theta_U \), Lower Limit = \( \theta_L \)

<table>
<thead>
<tr>
<th>Method</th>
<th>( \theta_U )</th>
<th>( G_{\theta U} (\hat{\theta}) )</th>
<th>( \theta_L )</th>
<th>( 1-G_{\theta L} (\hat{\theta}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Perc.</td>
<td>2.3801</td>
<td>0.0170</td>
<td>0.9693</td>
<td>0.0970</td>
</tr>
<tr>
<td>( \theta_1 )</td>
<td>2.1514</td>
<td>0.0499</td>
<td>0.8698</td>
<td>0.0511</td>
</tr>
<tr>
<td>( \theta_2 )</td>
<td>2.1515</td>
<td>0.0499</td>
<td>0.8653</td>
<td>0.0496</td>
</tr>
<tr>
<td>( \theta_3 )</td>
<td>2.1461</td>
<td>0.0512</td>
<td>0.8669</td>
<td>0.0512</td>
</tr>
<tr>
<td>( \theta_{10} )</td>
<td>2.1452</td>
<td>0.0505</td>
<td>0.8708</td>
<td>0.0565</td>
</tr>
<tr>
<td>Method</td>
<td>$\theta = 1$</td>
<td>$\theta = 2$</td>
<td>$\theta = 3$</td>
<td></td>
</tr>
<tr>
<td>--------</td>
<td>-------------</td>
<td>-------------</td>
<td>-------------</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Upper</td>
<td>Lower</td>
<td>Upper</td>
<td>Lower</td>
</tr>
<tr>
<td>Perc.</td>
<td>0.996</td>
<td>0.908</td>
<td>0.996</td>
<td>0.930</td>
</tr>
<tr>
<td>BC$_a$</td>
<td>0.962</td>
<td>0.940</td>
<td>0.976</td>
<td>0.976</td>
</tr>
<tr>
<td>$\theta_1$</td>
<td>0.964</td>
<td>0.946</td>
<td>0.976</td>
<td>0.962</td>
</tr>
<tr>
<td>$\theta_2$</td>
<td>0.966</td>
<td>0.942</td>
<td>0.976</td>
<td>0.964</td>
</tr>
</tbody>
</table>
REFERENCES


