BARTLETT ADJUSTMENT FOR EMPIRICAL LIKELIHOOD

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THOMAS DIICCCIO\textsuperscript{2}, PETER HALL\textsuperscript{3}, JOSEPH ROMANO\textsuperscript{2}

STANFORD UNIVERSITY AND AUSTRALIAN NATIONAL UNIVERSITY

Summary

The chi-squared approximation to the distribution of the empirical likelihood ratio statistic is considered in cases where inference is required for a smooth function of the mean of the distribution from which the sample is drawn. It is shown for such cases that the error in the chi-squared approximation is of order $O(n^{-1})$, where $n$ is the sample size, and that, by use of a Bartlett adjustment factor, the error can be reduced to order $O(n^{-2})$. An explicit formula for the adjustment is derived. The increase in coverage accuracy of approximate confidence intervals obtained by using the adjustment is illustrated in a simulation study.

\textbf{Short Title.} Nonparametric Bartlett Adjustment.

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\textbf{Key words and phrases.} Bartlett adjustment, chi-squared approximation, empirical likelihood ratio statistic, nonparametric confidence region, signed root empirical likelihood ratio statistic.

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1. Introduction

The use of empirical likelihood to construct approximate confidence regions in non-parametric settings has been developed by Owen (1987, 1988) using a least favorable family discussed by Efron (1981). In particular, Owen has shown for various functions of the underlying distribution that asymptotically, as the sample size $n$ increases, the empirical likelihood ratio statistic has a chi-squared distribution. Empirical likelihood and parametric likelihood surfaces are compared in DiCiccio, Hall, and Romano (1988). The coverage accuracy of empirical likelihood is compared with other methods that make use of least favorable families in DiCiccio and Romano (1988). The present article concerns the chi-squared approximation to the distribution of the empirical likelihood ratio statistic when interest centers on inference about a smooth function of the mean of the distribution underlying the sample. Examples in this context include nonparametric estimation of means, ratios or products of means, variances, ratios or products of variances, correlation coefficients, etc.; Bhattacharya and Ghosh (1978) have developed rigorous Edgeworth expansion theory for this type of situation. It is shown here that the error in the chi-squared approximation is of order $O(n^{-1})$, and furthermore, that this error can be reduced to the very small order $O(n^{-2})$ by the use of a Bartlett adjustment. Various authors, including Barndorff-Nielsen and Cox (1984), Lawley (1956), and McCullagh (1987, p. 212), have discussed Bartlett adjustments for parametric models.

To review empirical likelihood, suppose that $X_1, \ldots, X_n$ is a sample from an unknown $r$-variate distribution $F_0$ having mean $\mu_0$ and nonsingular covariance matrix $\Sigma_0$. Let $\theta^T = (\theta_1, \ldots, \theta_q)$ be a $q$-dimensional parameter ($q \leq r$) that can be expressed as a function of the mean of the underlying distribution, and put $\theta_0 = \theta(\mu_0)$. The empirical likelihood function $L$ for this parameter is defined by considering distributions $F_p, p = (p_1, \ldots, p^n)$, supported on the sample, where $X_i$ is assigned mass $p^i$. For a specified value $\theta_1$ of the parameter of interest, the empirical likelihood $L(\theta_1)$ is defined to be the maximum value of $\prod p^i$ over all such distributions that satisfy $\theta(\sum X_i p^i) = \theta_1$. If no distribution $F_p$ satisfying the constraint exists, then by definition $L(\theta_1) = 0$. Since $\prod p^i$ attains its overall maximum when $p^i = n^{-1}, i = 1, \ldots, n$, it follows that the empirical likelihood function is maximized
at $\hat{\theta} = \theta(\bar{X})$, where $\bar{X} = n^{-1} \sum X_i$ is the sample mean. The empirical likelihood ratio statistic is

$$W_0 = -2 \log \{L(\theta_0)/L(\hat{\theta})\} = -2 \log \{n^n L(\theta_0)\}.$$ 

It is shown in Section 3 that

$$P(W_0 \leq z) = P(\chi^2_q \leq z) + O(n^{-1}),$$

and thus the error in coverage level of confidence regions obtained by using the chi–squared approximation to the distribution of the empirical likelihood ratio statistic is of order $O(n^{-1})$. It is also shown that the chi–squared approximation can be improved by using a Bartlett adjustment; that is,

$$P(\lambda_0 \{E(nR^T R)/q\}^{-1} \leq z) = P(\chi^2_q \leq z) + O(n^{-2}),$$

where $R$ is a $q$–dimensional vector defined in Section 3 such that $W_0 = nR^T R + O_p(n^{-3/2})$. The error in coverage level of confidence regions obtained using the Bartlett adjustment is of order $O(n^{-2})$. These expansions follow from the usual assumptions to guarantee the existence of certain Edgeworth expansions; namely, the existence of moments of $F_0$, that $F_0$ satisfies Cramer’s condition, and that $\theta(\cdot)$ is smooth.

An expansion for $\{E(nR^T R)/q\}$ of the form $1 - an^{-1}$ is available from expression (3.12), and this expansion can be used in place of $E(nR^T R)$ in (1.1) without affecting the order of the error term. The $n^{-1}$ term in this expansion involves population moments which are usually unknown. In practice, these population moments can be replaced by $\sqrt{n}$–consistent estimates with the order $O(n^{-2})$ for the error in the chi–squared approximation still being achieved. For example, sample counterparts of the population moments could be used.

It should be noted that an asymptotic expansion for the expectation of $W_0$ does not exist. For example, suppose $q = 1$ and the functional of interest if the mean. Then, the likelihood ratio statistic $W_0$ is infinity when the true mean $\theta_0$ falls outside the range of the data. If $F_0$ is not degenerate, this happens with positive probability, and so $E(W_0) = \infty$. 

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Therefore, the problem of adjusting $W_0$ by a factor depending on its mean is a subtle one. Nevertheless, (1.1) is true. The reason is that $n^{-1}W_0$ can be approximated in a distributional sense by $R^TR$ (to order $n^{-5/2}$), and an adjustment determined by the mean of $R^TR$ is effective in improving the accuracy of a chi-squared approximation.

In the case where $q = 1$ and the functional of interest is the population mean, expression (3.10) yields

$$\left\{ E(nR^TR)^{-1} = 1 - n^{-1} \left( -\frac{1}{3} \frac{\mu_2^3}{\mu_2^3} + \frac{1}{2} \frac{\mu_4}{\mu_2^2} \right) + O(n^{-2}) \right\}, \tag{1.2}$$

where $\mu_2, \mu_3,$ and $\mu_4$ are central moments of the population. If the underlying distribution is the chi-squared with $\nu$ degrees of freedom, then the adjustment from (1.2) is $1 - (9 + 20\nu^{-1})(6n)^{-1}$. Owen (1988) considers the case $\nu = 1$ and $n = 20$, for which the adjustment is 0.7583. Hence, the corresponding critical values obtained from the chi-squared distribution with one degree of freedom are multiplied by a factor $0.7583^{-1} = 1.3187$.

This adjustment indicates that the true coverage level of nominal 90% confidence intervals obtained by the chi-squared approximation for $W_0$ is 84.80%; in a simulation study consisting of 1,000 trials, Owen observed a coverage rate of 87.2%. Owen also reported results for the mean based on Stigler's (1977) Data Set 9, which has $n = 20$. Using sample moments in (1.2), a Bartlett adjustment of 1.1665 is obtained in this case, and this adjustment suggests that the nominal 95% confidence intervals obtained by Owen's method have true coverage rate of 93.04%.

Further simulation results are presented in Section 2.
2. Simulations

In order to assess the increase in coverage accuracy achieved by utilizing a Bartlett adjustment, a modest simulation experiment was performed. The functional of interest is the mean of a univariate population. For various distributions, sample sizes, and nominal coverage levels, 5000 simulated data sets were generated and two-sided confidence intervals were constructed by using three different methods. The first method is the unadjusted empirical likelihood method which employs a simple chi-squared critical value \( c_{1-\alpha} \). In the second method which utilizes the theoretical Bartlett adjustment \( 1-an^{-1} \), where \( a = \mu_4/2\mu_2^2 - \mu_3^2/3\mu_2^3 \), the chi-squared critical value, \( c_{1-\alpha} \) is replaced by \( c_{1-\alpha}/(1-an^{-1}) \). This method, of course, assumes knowledge of the actual population moments. The final method uses an estimated Bartlett correction, in which the population moments in \( a \) are replaced by estimated sample moments. Here, we have employed unbiased estimates of these moments.

Table 1 reports the results based on simulated standard normal data at samples sizes 10 and 20. Table 2 reports the results for chi-squared data with one degree of freedom, for sample sizes of 20 and 40. Finally, Table 3 reports the results for data sampled from the \( t \)-distribution with 5 degrees of freedom, at sample sizes 15 and 30. The actual population values of \( a \) for these three situations are 3/2, 29/6, and 9/2.

Overall, the results are quite satisfactory. Even for normal data, the unadjusted empirical likelihood intervals have coverage levels significantly different from the nominal level. In fact, the observed coverage of empirical likelihood is always below the nominal level. In all cases, the theoretical and estimated Bartlett adjustments substantially improve coverage accuracy. The most difficult situation is the chi-squared distribution. In this case, the theoretical adjustment performs quite well; however, the estimated adjustment is only a modest improvement over the unadjusted empirical likelihood method. The difficulty in this case arises because the sample estimate of \( a \) is quite skewed and biased, with the result being that the estimated adjustment does not have as big an effect as the theoretical adjustment. The success of the estimated Bartlett correction method is
especially surprising in the $t$-distribution situation as it involves a sample estimate of the fourth moment of the underlying population.
3. Derivation of the Bartlett adjustment

In this section, the Bartlett adjustment is derived in a sequence of ten steps.

**Step 1. Empirical likelihood for the mean.** Suppose that $X_1, \ldots, X_n$ is a random sample of size $n$ taken from an unknown $r$-variate distribution $F_0$ having mean $\mu_0$ and nonsingular covariance matrix $\Sigma_0$. The $j$th component of $X_i$ is denoted by $X_{ij}$, $j = 1, \ldots, r$, so that $X_i = (X_{i1}, \ldots, X_{ir})^T$. Let $\overline{L}$ be the empirical likelihood function for the mean; for a specified vector $\mu = (\mu_1, \ldots, \mu_r)^T$, $\overline{L}(\mu)$ is defined to be the maximum value of $\Pi p^i$ over all vectors $p = (p^1, \ldots, p^n)$ that satisfy both $\sum p^i = 1$ and $\sum X_{ip^i} = \mu$.

An explicit expression for $\overline{L}(\mu)$ can easily be derived. The maximum value of $\Pi p^i$ over all vectors $(p^1, \ldots, p^n)$ satisfying

$$\sum p^i = 1, \sum (X_i - \mu)p^i = 0 \quad (3.1)$$

is found by considering

$$\sum \log(p^i) - ns \left( \sum p^i - 1 \right) - nt^T \sum (X_i - \mu)p^i,$$

where the scalar $s$ and the vector $t = (t^1, \ldots, t^r)^T$ are Lagrange multipliers. Straightforward differentiation shows that the maximum is attained when $p^i = n^{-1} \{s + t^T(X_i - \mu)\}^{-1}$, and the constraints (3.1) imply that $s = 1$. Thus,

$$p^i = p^i(\mu) = n^{-1} \{1 + t^T(X_i - \mu)\}^{-1},$$

where $t = t(\mu)$ is given by the equation

$$\sum \{1 + t^T(X_i - \mu)\}^{-1}(X_i - \mu) = 0. \quad (3.2)$$

The empirical likelihood function for the mean is given at the point $\mu$ by $\overline{L}(\mu) = \Pi p^i(\mu)$.

Since $\Pi p^i$ attains its largest value over all vectors $(p^1, \ldots, p^n)$ satisfying $\sum p^i = 1$ when $p^i = n^{-1}$, $i = 1, \ldots, n$, it follows that the empirical likelihood function $\overline{L}$ is maximized at $\hat{\mu} = \bar{X} = n^{-1} \sum X_i$, and $\overline{L}(\hat{\mu}) = n^{-n}$. The empirical likelihood ratio at the point $\mu$ is

$$\frac{\overline{L}(\mu)}{\overline{L}(\hat{\mu})} = \Pi \{1 + t^T(X_i - \mu)\}^{-1},$$
and minus twice the logarithm of this ratio is
\[ \overline{W}(\mu) = 2 \sum \log \{1 + t^T(X_i - \mu)\}, \] (3.3)
where \( t \) is defined by (3.2). The empirical likelihood ratio statistic for the mean is \( \overline{W}_0 = \overline{W}(\mu_0) \).

In terms of variables \( Y_1, \ldots, Y_n \) defined by \( Y_i = \sum_{0}^{-1/2}(X_i - \mu_0) \), expression (3.3) becomes
\[ \overline{W}(\mu) = 2 \sum \log \{1 + u^T(Y_i - \nu)\}, \] (3.4)
where \( u = \sum_{0}^{1/2} t \) and \( \nu = \sum_{0}^{-1/2}(\mu - \mu_0) \). Note that \( u \) satisfies
\[ \sum \{1 + u^T(Y_i - \nu)\}^{-1}(Y_i - \nu) = 0. \] (3.5)
Comparison of (3.4) and (3.5) with (3.2) and (3.3) shows that there is no loss of generality in making the assumption \( \mu_0 = 0 \) and \( \sum_{0} = I \).

**Step 2. Expansion of \( \overline{W}_0 \).** The first stage of this step is to obtain an expansion for \( t \) that solves (3.2) in the case \( \mu = 0 \). Let
\[ T = n^{-1} \sum (1 + t^T X_i)^{-1} X_i = n^{-1} \sum \{1 - t^T X_i + (t^T X_i)^2 - (t^T X_i)^3 + \ldots \} X_i. \]
It is convenient for the expansions that follow to introduce the notation
\[ \alpha^{i_1 \ldots i_k} = E(X_{i_1}^{j_1} \ldots X_{i_k}^{j_k}), \quad A_{i_1 \ldots i_k} = n^{-1} \sum (X_{i_1}^{j_1} \ldots X_{i_k}^{j_k} - \alpha^{i_1 \ldots i_k}); \]
in particular, \( \alpha^j = 0 \), \( A^j = n^{-1} \sum X_i^j \), and \( \alpha^{jk} = \delta^{jk} \), \( (j, k = 1, \ldots, r) \), where \( \delta^{jk} \) is Kronecker’s delta.

In terms of this notation, provided that \( t \) is \( O_p(n^{-1/2}) \),
\[ T^j = \alpha^{j} - A^{j} - A^{jk} A_{k}^{\ell} + \alpha^{jkl} A_{k}^{\ell} + A^{jkl} A_{k}^{\ell} - \alpha^{jkm} A_{k}^{\ell} \alpha^{km} + O_p(n^{-2}). \]
Use is made here of the usual convention in which summation over repeated indices is understood. Solving \( T = 0 \) gives
\[ t^j = A^{j} - A^{jk} A_{k}^{\ell} + \alpha^{jkl} A_{k}^{\ell} + A^{ij} A_{k}^{\ell} A_{k}^{\ell} + A^{ijkl} A_{k}^{\ell} A_{k}^{\ell} - A^{jm} \alpha^{jkm} A_{k}^{\ell} A_{m}^{\ell} + O_p(n^{-2}). \]
\[ -2\alpha^{jkm} A_{k}^{\ell} A_{m}^{\ell} + 2\alpha^{jkn} \alpha^{km} A_{k}^{\ell} A_{m}^{\ell} \]
Now, \( \sum (1 + t^T X_i)^{-1}(t^T X_i) = 0 \) implies
\[
\sum t^T X_i = \sum \left\{ (t^T X_i)^2 - (t^T X_i)^3 + (t^T X_i)^4 \right\} + O_p(n^{-3/2}).
\]
Hence,
\[
W_0 = 2 \sum \log(1 + t^T X_i) = 2 \sum \left\{ t^T X_i - \frac{1}{2}(t^T X_i)^2 + \frac{1}{3}(t^T X_i)^3 - \frac{1}{4}(t^T X_i)^4 \right\} + O_p(n^{-3/2}),
\]
and
\[
n^{-1}W_0 = n^{-1} \sum \left\{ (t^T X_i)^2 - \frac{4}{3}(t^T X_i)^3 + \frac{3}{2}(t^T X_i)^4 \right\} + O_p(n^{-5/2})
\]
\[
= t^j j^j + A^j k^i t^k - \frac{4}{3} \alpha^j k^l t^k t^l - \frac{3}{2} \alpha^j k^m t^k t^m + O_p(n^{-5/2})
\]
\[
= A^j A^i - A^j k^i A^k A^l + \frac{2}{3} \alpha^j k^l A^j A^k A^l + \alpha^j k^m A^i A^k A^l A^m + O_p(n^{-1}).
\]

(3.6)

**Step 3. Expansion of \( W(\mu) \).** From expressions (3.2) and (3.3), it is evident that the preceding expansion for \( W_0 \) can be used to develop a similar expansion for \( W(\mu) \) by replacing \( X_i \) with \( X_i - \mu \) throughout (3.6). Assuming that \( \mu \) is \( O(n^{-1/2}) \), this process involves replacing \( A^j \) by \( A^j - \mu^j \), replacing \( A^j k^i \) by \( A^j k^i - (\mu^j A^k + \mu^k A^j + \mu^j \mu^k = A^j k + O_p(n^{-1}) \), and replacing \( A^j k^l \) by \( A^j k^l - (\mu^j \alpha^k l + \mu^k \alpha^j l + \mu^l \alpha^j k) + O_p(n^{-1}). \) Let \( A = X = (A^j) \). Then
\[
n^{-1}W(\mu) = (A - \mu)^i (A - \mu)^j
\]
\[
- \left\{ A^j k^i (\mu^j A^k + \mu^k A^j) + \mu^i \mu^k \right\} (A - \mu)^j (A - \mu)^k
\]
\[
+ \frac{2}{3} \alpha^j k^l (A - \mu)^j (A - \mu)^k (A - \mu)^l + A^j k^i A^k A^l (A - \mu)^j (A - \mu)^k
\]
\[
+ \frac{2}{3} \left\{ A^j k^l - (\mu^j \alpha^k l + \mu^k \alpha^j l + \mu^l \alpha^j k) \right\} (A - \mu)^j (A - \mu)^k (A - \mu)^l
\]
\[
- 2\alpha^j k^m A^j k^l (A - \mu)^j (A - \mu)^k (A - \mu)^l
\]
\[
+ \alpha^j k^m \alpha^l m (A - \mu)^j (A - \mu)^k (A - \mu)^l (A - \mu)^m
\]
\[
- \frac{1}{2} \alpha^j k^m A^j k^l (A - \mu)^j (A - \mu)^k (A - \mu)^l (A - \mu)^m + O_p(n^{-5/2}).
\]

(3.7)
Step 4. Empirical likelihood for a function of the mean. Suppose that \( \theta = (\theta^1, \ldots, \theta^q)^T \) is a \( q \)-dimensional parameter (\( q \leq r \)) that depends on the underlying distribution only through its mean, and let \( \theta_0 = \theta(\mu_0) = \theta(0) \). The empirical likelihood ratio statistic for this parameter, denoted by \( W_0 \), can be obtained as the minimum of \( \bar{W}(\mu) \) over all vectors \( \mu \) that satisfy \( \theta(\mu) = \theta_0 \). An expansion of \( W_0 \) can thus be derived from expression (3.7).

Let \( \bar{\mu} \) be the value of \( \mu \) that minimizes \( \bar{W}(\mu) \) subject to the constraint \( \theta(\mu) = \theta_0 \). Consider the expansion \( \bar{\mu} = A - (\mu_1 + \mu_2 + \mu_3) + O_p(n^{-2}) \) where \( \mu_1, \mu_2, \) and \( \mu_3 \) are \( O_p(n^{-1/2}), O_p(n^{-1}), \) and \( O_p(n^{-3/2}) \), respectively. In the following arguments, \( \mu_1, \mu_2, \) and \( \mu_3 \) are determined successively. For these derivations, put

\[
\theta^u_{j_1 \ldots j_k} = \partial^k \theta(\mu)/\partial \mu^{j_1} \ldots \partial \mu^{j_k} \bigg|_{\mu = 0}, \ u = 1, \ldots, q,
\]

and let \( \Theta = (\theta^u_j) \), so that \( \Theta \) is a \( q \times r \) matrix. It is assumed that \( \Theta \) is of full rank \( q \).

If \( \mu \) is \( O_p(n^{-1/2}) \), then \( n^{-1}\bar{W}(\mu) = (A - \mu)^j(A - \mu)^j + O_p(n^{-3/2}) \), and the constraint \( \theta(\mu) = \theta_0 \) is equivalent to \( \theta^u_j \mu^j = O_p(n^{-1}) \), \( u = 1, \ldots, q \). Solving for \( \mu_1 \) requires minimizing \( \mu_1^j \mu_1^j \) subject to \( \Theta(A - \mu_1) = 0 \). A Lagrange multiplier argument shows that \( \mu_1 = MA \), where \( M = \Theta^T(\Theta \Theta^T)^{-1} \Theta \).

Again, if \( \mu \) is \( O_p(n^{-1/2}) \), then

\[
n^{-1}\bar{W}(\mu) = (A - \mu)^j(A - \mu)^j - A^k(A - \mu)^j(A - \mu)^k + \frac{2}{3} \alpha^{jkl} (A - \mu)^j(A - \mu)^k(A - \mu)^l + O_p(n^{-2}),
\]

and the constraint \( \theta(\mu) = \theta_0 \) is equivalent to \( \theta^u_j \mu^j + \frac{1}{2} \theta^u_{jk} \mu^j \mu^k = O_p(n^{-3/2}) \), \( u = 1, \ldots, q \). Now having fixed \( \mu_1 = MA \), solving for \( \mu_2 \) requires minimizing \( \mu_2^j \mu_2^j + 2 \mu_2^l (\bar{\mu}_1^l + B^j) \) subject to \( \Theta \mu_2 = U \), where \( B = (B^1, \ldots, B^r)^T \) and \( U = (U^1, \ldots, U^q)^T \), with \( B^j = \alpha^{jkl} \mu_1^k \mu_1^l - A^k \mu_1^l \) and \( U^u = \frac{1}{2} \theta^u_{jk} (A - \mu_1)^j(A - \mu_1)^k \). A Lagrange multiplier argument shows that \( \mu_2 = NU - (I - M)B \), where \( N = \Theta^T(\Theta \Theta^T)^{-1} \).

Finally, if \( \mu \) is \( O_p(n^{-1/2}) \), then expansion (3.7) for \( n^{-1}\bar{W}(\mu) \) holds with error of order \( O_p(n^{-5/2}) \), and the constraint \( \theta(\mu) = \theta_0 \) is equivalent to \( \theta^u_j \mu^j + \frac{1}{2} \theta^u_{jk} \mu^j \mu^k + \frac{1}{6} \theta^u_{jklt} \mu^j \mu^k \mu^l = O_p(n^{-2}) \), \( u = 1, \ldots, q \). Now having fixed \( \mu_1 = MA \) and \( \mu_2 = NU - (I - M)B \), solving for \( \mu_3 \) requires minimizing \( \mu_3^j \mu_3^j + 2 \mu_3^l (\mu_1^l + \mu_2^l + B^j + C^j) \) subject to \( \Theta \mu_3 = V \), where
\[ C = (C^1, \ldots, C^r)^T \text{ and } V = (V^1, \ldots, V^q)^T, \text{ with} \]
\[
C^j = 2\alpha^{jkl} \mu^k_1 \mu^l_1 - A^{jk} \mu^k_2 + A^{jk} \mu^k_1 \mu^l_1 + A^{j\ell} A^{k\ell} \mu^k_1 \mu^l_1 - A^{j\ell} A^{k\ell} \mu^k_1 \mu^l_1 \\
-2\alpha^{jkm} A^{\ell m} \mu^k_1 \mu^l_1 + 2\alpha^{jkn} \alpha^{\ell mn} \mu^k_1 \mu^l_1 \mu^m_1 - \alpha^{jkm} \mu^k_1 \mu^l_1 \\
\text{and} \]
\[
V^u = -\theta^{u}_{jk} \mu^j_1 (A - \mu_1)^k + \frac{1}{6} \theta^{u}_{jkl} (A - \mu_1)^j (A - \mu_1)^k (A - \mu_1)^l.
\]

A Lagrange multiplier argument shows that \( \mu_3 = NV - (I - M)C \).

To summarize these calculations,
\[
\tilde{\mu} = (I - M)(A + B + C) - N(U + V) + O_p(n^{-2}),
\]
and
\[
n^{-1} W_0 = n^{-1} W(\tilde{\mu})
\]
\[
= \mu^i_1 \mu^i_1 - A^{jk} \mu^j_1 \mu^k_1 + 2\alpha^{jkl} \mu^j_1 \mu^k_1 \mu^l_1 + 2\mu^j_2 \mu^j_2 + A^{j\ell} A^{k\ell} \mu^j_1 \mu^k_1 \\
+ \frac{2}{3} A^{jkl} \mu^j_1 \mu^k_1 \mu^l_1 - 2\alpha^{jkm} A^{\ell m} \mu^j_1 \mu^k_1 \mu^l_1 + \alpha^{jkn} \alpha^{\ell mn} \mu^j_1 \mu^k_1 \mu^m_1 \mu^l_1 \\
+ \frac{1}{2} \alpha^{jkm} \mu^j_1 \mu^k_1 \mu^m_1 + \mu^j_2 \mu^j_2 + 2\alpha^{jkl} \mu^j_1 \mu^k_1 \mu^l_2 - 2A^{jk} \mu^j_1 \mu^k_2 \\
+ 2\mu^j_2 \mu^j_2 + O_p(n^{-5/2}).
\]

(3.8)

**Step 5. Signed root of \( n^{-1} W_0 \).** The signed root of \( n^{-1} W_0 \) is a \( q \)-dimensional vector \( R_0 = (R^1_0, \ldots, R^q_0)^T \) such that \( R_0^T R_0 = n^{-1} W_0 \). Consider the expansion \( R_0 = R_1 + R_2 + R_3 + O_p(n^{-2}) \), where \( R_1, R_2, \) and \( R_3 \) are \( O_p(n^{-1/2}) \), \( O_p(n^{-1}) \), and \( O_p(n^{-3/2}) \), respectively. Let \( O = \Theta^T (\Theta \Theta^T)^{-1/2} \), so that \( O \) is an \( r \times q \) matrix having the properties \( O O^T = M \) and \( O^T O = I \). Furthermore, let \( R = R_1 + R_2 + R_3 \).

It is evident from expansion (3.8) that \( R_1 = O^T A = O^T \mu_1 \). Then \( \mu_1 = OR_1 \), so that \( \mu^i_1 = O^{ju} R^u_1 \). Using this identity in (3.8), it can be shown that
\[
R^u_2 = -\frac{1}{2} A^{jku} O^{ju} O^{kv} R^v_1 + \frac{1}{3} \alpha^{jkl} O^{ju} O^{kv} O^{\ell w} R^v_1 R^w_1 + O^{ju} D^j
\]
\[
= -\frac{1}{2} O^{ju} M^{k\ell} A^{jk} A^\ell + \frac{1}{3} \alpha^{jkl} O^{ju} M^{km} M^{\ell n} A^m A^n \\
+ \frac{1}{2} D^{uv} \theta^{v}_{jk} (I - M)^j (I - M)^k A^\ell A^m,
\]
where $D = NU$ and $P = (\Theta \Theta^T)^{-1/2}$, and

$$
R_3^u = \left( -A^j \mu_2^k + \mu_3^j + \frac{1}{2} A^j D^k \right) Oj^u \\
+ \left( A^j k^t \mu_2^t - \frac{5}{8} A^j l^t A^{km} M^{lm} - \frac{1}{3} \alpha^{jkl} D^t \right) Oj^u O^{kv} R_1^v \\
+ \left( \frac{1}{3} A^j k^t - \alpha^{jkm} A^{lm} - A^j m^k A^{ln} \right) Oj^u O^{kv} O^t w R_1^v R_1^w \\
+ \left( \alpha^{jkn} \alpha^{ln} - \frac{5}{9} \alpha^{jkn} \alpha^{lmo} M^{no} - \frac{1}{4} \alpha^{jkm} \right) Oj^u O^{kv} O^t w O^{mz} R_1^v R_1^w R_1^x \\
= \frac{1}{4} Oj^u N^{kv} \theta_{\ell m} (I - M)^{\ell n} (I - M)^{\ell m} A^{nk} A^{no} + \frac{3}{8} Oj^u M^{lm} M^{kn} A^{jk} A^{km} A^{ln} \\
+ \frac{1}{3} Oj^u \alpha^{jkl} M^{km} N^{oq} A^{lp} A^q A^r + \frac{1}{3} Oj^u M^{km} M^{ln} A^{jk} A^{km} A^{ln} \\
- \frac{5}{12} Oj^u \alpha^{jkm} M^{ln} M^{ko} M^{lp} A^{m} A^{lp} A^q - \frac{5}{12} Oj^u \alpha^{jkn} M^{ln} M^{ko} M^{lp} A^{m} A^{lp} A^q \\
+ \frac{4}{9} Oj^u \alpha^{jkn} \alpha^{lmo} M^{lp} M^{m} M^{lp} A^q A^r A^t - \frac{1}{4} Oj^u \alpha^{jkm} M^{ln} M^{ko} M^{lp} A^q A^r A^t \\
- \frac{1}{2} P^{uv} \theta_{jk} (I - M)^{jk} (I - M)^{jk} A^{nk} A^{no} A^{lp} A^q \\
- P^{uv} \theta_{jk} (I - M)^{jk} (I - M)^{jk} A^{nk} A^{no} A^{lp} A^q \\
+ P^{uv} \theta_{jk} (I - M)^{jk} (I - M)^{jk} \alpha^{mn} M^{np} M^{pq} A^{lp} A^q.
$$

By defining $R, R_1, R_2,$ and $R_3$ as above,

$$
n^{-1} W_0 = R^T R + O_p(n^{-5/2}) \\
= R_3^u R_1^u + 2R_1^u R_2^u + 2R_1^u R_3^u + R_2^u R_2^u + O_p(n^{-5/2}).
$$

*Step 6. Formulae for moments.* This step contains some formulae that are required for calculating the joint cumulants of $R$.

Suppose that $X$ is a random vector having the same distribution as $X_i$, and suppose that $h^1, h^2, \ldots$ are real-valued functions such that $E\{h^j(X)\} = 0$, $j = 1, 2, \ldots$. Let
\[ H^j = h^j(X) \text{ and } Z^j = n^{-1} \sum h^j(X_i). \] Then
\[
E(Z^j Z^k) = n^{-1} E(H^j H^k), \quad E(Z^j Z^k Z^l) = n^{-2} E(H^j H^k H^l),
\]
\[
E(Z^j Z^k Z^l Z^m) = n^{-3}(n - 1)\{ E(H^j H^k)E(H^m H^m)[3] \} + n^{-3} E(H^j H^k H^l H^m),
\]
\[
E(Z^j Z^k Z^l Z^m Z^n) = n^{-3}\{ E(H^j H^k)E(H^m H^n H^n)[10] \},
\]
\[
E(Z^j Z^k Z^l Z^m Z^n Z^o) = n^{-3}\{ E(H^j H^k)E(H^m H^n)E(H^n H^o)[15] \} + O(n^{-4}),
\]
where
\[
E(H^j H^k)E(H^l H^m)[3] = E(H^j H^k)E(H^l H^m) + E(H^j H^l)E(H^k H^m) + E(H^j H^m)E(H^k H^l),
\]
and similarly for \( E(H^j H^k)E(H^l H^m H^n)[10] \) and \( E(H^j H^k)E(H^l H^m)E(H^n H^o)[15] \).

**Step 7. Joint third-order cumulants of \( R \).** The joint third-order cumulants of \( R \) are given (McCullagh, 1987, p. 31) by
\[
\text{cum}(R^u, R^v, R^w) = E(R^u R^v R^w) - E(R^u)E(R^v)E(R^w) + 2E(R^u)E(R^v)E(R^w)
\]
\[
= E(R^u_1 R^v_1 R^w_1) + E(R^u_2 R^v_1 R^w_1)[3] - E(R^u_2)E(R^v_1 R^w_1)[3] + O(n^{-3}).
\]
It follows from the formulae of Steps 5 and 6 that
\[
E(R^u_1 R^v_1) = n^{-1} \delta^{uv}, \quad E(R^u_1 R^v_1 R^w_1) = n^{-2} \alpha^{ijkl} O^{ju} O^{kv} O^{lw},
\]
\[
E(R^u_2) = n^{-2} \left\{ -\frac{1}{6} \alpha^{ijkl} O^{ju} M^{kl} + \frac{1}{2} P^{uv} \theta_{jk}(I - M)^{jk} \right\},
\]
\[
E(R^u_2 R^v_1 R^w_1) = n^{-2} \left\{ -\frac{1}{6} \alpha^{ijkl} O^{ju} M^{kl} \delta^{vw} - \frac{1}{3} \alpha^{ijkl} O^{ju} O^{kv} O^{lw}
\]
\[+ \frac{1}{2} P^{ux} \theta_{jk}(I - M)^{jk} \delta^{uv} \right\} + O(n^{-3}).
\]

Therefore,
\[
E(R^u_2 R^v_1 R^w_1) = E(R^u_2)E(R^v_1 R^w_1) - \frac{1}{3} E(R^u_1 R^u_1 R^u_1) + O(n^{-3}), \tag{3.9}
\]
and hence \( \text{cum}(R^u, R^v, R^w) = O(n^{-3}) \).
Step 8. Joint fourth-order cumulants of R. The joint fourth-order cumulants of \( R \) are given (McCullagh, 1987, p. 31) by

\[
\begin{align*}
\text{cum}(R^u, R^v, R^w, R^z) &= E(R^u R^v R^w R^z) - E(R^u R^v)E(R^w R^z)[3] - E(R^u)E(R^v R^w R^z)[4] \\
&+ 2E(R^u)E(R^v)E(R^w R^z)[6] - 6E(R^u)E(R^v)E(R^w)E(R^z) \\
&= E(R^u_1 R^w_1 R^x_1 R^z_1) + E(R^u_2 R^w_2 R^x_2 R^z_2)[4] + E(R^u_3 R^w_3 R^x_3 R^z_3)[4] \\
&+ E(R^u_2 R^w_2 R^x_2 R^z_2)[6] - E(R^u_1 R^w_1 R^z_1)[3] \\
&- E(R^u_2 R^w_2)E(R^w_1 R^z_1)[12] - E(R^u_3 R^w_3)E(R^w_1 R^z_1)[12] \\
&- E(R^u_2 R^w_2)E(R^w_1 R^z_1)[6] + E(R^u_1 R^w_1 R^z_1)[6] \\
&- E(R^u_2)E(R^w_1 R^z_1)[12] + 2E(R^u_2)E(R^w_1)E(R^w_1 R^z_1)[6] \\
&+ O(n^{-4}). \tag{3.10}
\end{align*}
\]

Let

\[
\begin{align*}
s_1 &= \alpha^{jklm} O^{iu} O^{kv} O^{lw} O^{mx}, \quad s_2 = \delta^{uv} \delta^{wx} + \delta^{uw} \delta^{vx} + \delta^{ux} \delta^{vw}, \\
s_3 &= \alpha^{jkl} \alpha^{mno} M^{no} (O^{ju} O^{kv} O^{lw} O^{mx} + O^{ju} O^{kv} O^{lx} O^{mw} + O^{ju} O^{kw} O^{lx} O^{mv} \\
&+ O^{iu} O^{kw} O^{lx} O^{mu}), \\
s_4 &= \alpha^{jkn} \alpha^{lmo} M^{no} (O^{ju} O^{kv} O^{lw} O^{mx} + O^{ju} O^{kw} O^{lx} O^{mv} + O^{ju} O^{kv} O^{lw} O^{mv} \\
&+ O^{ju} O^{kw} O^{lx} O^{mv}), \\
s_5 &= \theta_{mn}^{y} (I - M)^{mn} \alpha^{jkl} (P^{yu} O^{iu} O^{kw} O^{lx} + P^{yu} O^{iu} O^{kw} O^{lx} + P^{yu} O^{iu} O^{kv} O^{lx} + P^{yu} O^{iu} O^{kv} O^{lw}),
\end{align*}
\]

Ignoring terms of order \( O(n^{-4}) \), it follows from the formulae of Steps 5 and 6 that

\[
\begin{align*}
E(R^u_1 R^w_1 R^x_1 R^z_1) - E(R^u_1 R^w_1)E(R^x_1 R^z_1)[3] &= n^{-3} (s_1 - s_2), \\
E(R^u_2 R^w_2 R^x_2 R^z_2)[4] - E(R^u_2 R^w_2)E(R^x_1 R^z_1)[12] &= n^{-3} (-6s_1 + 2s_2 - \frac{1}{6}s_3 + \frac{2}{3}s_4 + \frac{1}{2}s_5), \\
E(R^u_2 R^w_2 R^x_2 R^z_2)[6] - E(R^u_2 R^w_2)E(R^x_1 R^z_1)[6] &= n^{-3} (3s_1 - s_2 + \frac{1}{6}s_3 - \frac{5}{9}s_4 - \frac{1}{2}s_5), \\
E(R^u_2 R^w_2 R^x_2 R^z_2)[4] - E(R^u_2 R^w_2)E(R^x_1 R^z_1)[12] &= n^{-3} (2s_1 - \frac{1}{9}s_4),
\end{align*}
\]

and from expression (3.9) that

\[
E(R^u_2)E(R^x_1 R^z_1)[4] - E(R^u_2)E(R^x_1 R^z_1)[12] + 2E(R^u_2)E(R^x_1)E(R^x_1 R^z_1)[6] = 0.
\]
Hence, by expression (3.10), \( \text{cum}(R^u, R^v, R^w, R^z) = O(n^{-4}) \).

**Step 9. Approximation to the distribution of \( W_0 \).** Let the mean vector and the covariance matrix of \( n^{1/2}R \) be denoted by \( \delta = (\delta^1, \ldots, \delta^q)^T \) and \( \Delta \), respectively. Then \( \delta \) is \( O(n^{-1/2}) \), and \( \Delta = \mathbf{I} + \Delta_1 + O(n^{-2}) \), where \( \Delta_1 \) is \( O(n^{-1}) \). It has been shown in Steps 7 and 8 that the joint third- and fourth-order cumulants of \( n^{1/2}R \) are \( O(n^{-3/2}) \) and \( O(n^{-2}) \), respectively; furthermore, the higher-order joint cumulants are \( O(n^{-3/2}) \) or smaller. Thus, to error of order \( O(n^{-3/2}) \), \( n^{1/2}R \) has a multivariate normal distribution.

It follows then that the \( s \)th cumulant of \( nR^T R \) (Johnson and Kotz, 1970, p. 153) is

\[
\kappa_s = 2^{s-1}(s-1)! \sum_{u=1}^{q} (1 + s\beta^u)(\lambda^u)^s + O(n^{-3/2}),
\]

where \( \lambda^1, \ldots, \lambda^q \) are the eigenvalues of \( \Delta, \beta^1, \ldots, \beta^q \) are \( O(n^{-1}) \), and \( \sum \beta^u\lambda^u = \delta^T\delta \). Note that \( \sum(\lambda^u)^s = \text{tr}(\Delta^s) \) and that \( \lambda^u = 1 + \lambda^u_i \), where \( \lambda^u_i \) is \( O(n^{-1}) \). In particular, \( E(nR^T R) = \sum \lambda^u + \delta^T\delta = q + \sum \lambda^u_i + \delta^T\delta \). Therefore,

\[
\kappa_s = 2^{s-1}(s-1)! \sum_{u=1}^{q} \{ 1 + s(\lambda^u_i + \beta^u\lambda^u) \} + O(n^{-3/2})
\]

\[
= 2^{s-1}(s-1)! \left( q + s \left( \sum \lambda^u_i + \delta^T\delta \right) \right) + O(n^{-3/2})
\]

\[
= 2^{s-1}(s-1)!q \left( E(nR^T R)/q \right)^s + O(n^{-3/2}).
\]

Ignoring terms of order \( O(n^{-3/2}) \), the \( s \)th cumulant of \( (nR^T R) \{ E(nR^T R)/q \}^{-1} \) is \( 2^{s-1}(s-1)!q \), which is the \( s \)th cumulant of \( \chi_q^2 \).

By the validity of Edgeworth expansions for \( n^{1/2}R \) in this situation (Bhattacharya and Rao, 1976, p. 51 ff) and since \( W_0 = nR^T R + O_p(n^{-3/2}) \), it follows that

\[
P \left( W_0 \{ E(nR^T R)/q \}^{-1} \leq z \right) = P \left( (nR^T R) \{ E(nR^T R)/q \}^{-1} \leq z \right)
\]

\[
= P(\chi_q^2 \leq z) + O(n^{-3/2}). \tag{3.11}
\]

Moreover, by an argument based on the oddness and evenness of polynomials in Edgeworth expansions that is given, for example, by Barndorff-Nielsen and Hall (1988), the \( O(n^{-3/2}) \) term in (3.11) is actually \( O(n^{-2}) \). Therefore,

\[
P \left( W_0 \{ E(nR^T R)/q \}^{-1} \leq z \right) = P(\chi_q^2 \leq z) + O(n^{-2}).
\]
Step 10. Expansion of $E(nR^TR)$. Recall from Step 5 that

$$R_u^u R_u^u = R_1^u R_1^u + 2R_2^u R_2^u + 2R_3^u R_3^u + R_4^u R_4^u + O_p(n^{-5/2}).$$

Thus,

$$E(R_u^u R_u^u) = E(R_1^u R_1^u) + 2E(R_2^u R_2^u) + 2E(R_3^u R_3^u) + E(R_4^u R_4^u) + O(n^{-3}).$$

Now define

$$t_1 = \alpha^{jkl} \alpha^{mn0} M_{jm} M^b M^d, \quad t_2 = \alpha^{jkl} \alpha^{mn0} M_{jk} M^b M^d,$$

$$t_3 = \alpha^{jkl} M_{jk} M_{tm}, \quad t_4 = \alpha^{jkl} N_{jm} \theta^u_{mn} (I - M)^{mk}(I - M)^{nl},$$

$$t_5 = Q^{uv} \theta^u_{jk} \theta^v_{lm} \{(I - M)^{jk}(I - M)^{km} + 2(I - M)^{jt}(I - M)^{km}\},$$

$$t_6 = \alpha^{jkl} N_{jm} \theta^u_{mn} (I - M)^{mn} M^{kt},$$

where $Q = (\Theta \Theta^T)^{-1}$. Recall that $M = \Theta^T (\Theta \Theta^T)^{-1} \Theta$ and $N = \Theta^T (\Theta \Theta^T)^{-1}$ are defined in Step 4. In terms of this notation,

$$E(R_1^u R_1^u) = n^{-1} q, E(R_2^u R_2^u) = n^{-2} \left( \frac{1}{3} t_1 - \frac{1}{2} t_3 + \frac{1}{2} t_4 + \frac{1}{2} q \right) + O(n^{-3}),$$

$$E(R_3^u R_3^u) = n^{-2} \left( \frac{43}{72} t_1 - \frac{73}{72} t_2 + \frac{5}{8} t_3 - t_4 + \frac{1}{12} t_5 - \frac{3}{8} q \right) + O(n^{-3}),$$

$$E(R_4^u R_4^u) = n^{-2} \left( -\frac{7}{36} t_1 + \frac{1}{36} t_2 + \frac{1}{4} t_3 + \frac{1}{4} t_5 - \frac{1}{6} t_6 - \frac{1}{4} q \right) + O(n^{-3}).$$

Therefore,

$$E(nR^TR) = nE(R_u^u R_u^u)$$

$$= q + n^{-1} \left( \frac{5}{3} t_1 - 2t_2 + \frac{1}{2} t_3 - \frac{1}{4} t_4 + \frac{1}{4} t_5 \right) + O(n^{-2}),$$

and

$$\{E(nR^TR)/q\}^{-1} = 1 - (qn^{-1}) \left( \frac{5}{3} t_1 - 2t_2 + \frac{1}{2} t_3 - \frac{1}{4} t_4 + \frac{1}{4} t_5 \right) + O(n^{-2}).$$  \hspace{1cm} (3.12)

15
References


TABLE 1
Estimated Coverage
Normal Data
5000 Simulations

$n = 10$

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**TABLE 2**
Estimated Coverage
Chi-squared Distribution with Mean 1
5000 Simulations

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\[ n = 40 \]

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TABLE 3  
Estimated Coverage  
$t$-distribution with 5 Degrees of Freedom  
5000 Simulations

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