RESIDUAL LIFE TIME AT GREAT AGE

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TECHNICAL REPORT NO. 36
MARCH 15, 1972

PREPARED UNDER THE AUSPICES
OF
NATIONAL SCIENCE FOUNDATION GRANT GP-30711X

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Residual life time at great age

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Summary. The asymptotic behaviour of the residual life time at time $t$ is investigated (for $t \to 0$). We derive weak limit laws and their domains of attraction and treat rates of convergence and moment convergence. The presentation exploits the close similarity with extreme value theory.

1970 AMS subject classification. Primary 60 F 05, secondary 62 N 05.

Key words and phrases. Residual life time, regular variation, distribution tail, extreme value theory.

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Introduction

Consider a light bulb. It has a certain life time $X$, which is a random variable with probability distribution function $F$. Suppose $F(x)<1$ for all $x$. After having burned $t$ hours, there remains a residual life time, with a distribution function $F_t$ defined by

$$1 - F_t(x) = P(X-t > x | X > t).$$

The residual life time is a random variable $X_t$ which is defined on the conditional probability space $\{X > t\}$.

If $X$ has an exponential distribution $F(x) = 1 - e^{-\lambda x}$, then so has $X_t$. Indeed

$$1 - F_t(x) = P(X > x+t | X > t) = \frac{e^{-\lambda(x+t)}}{e^{-\lambda t}} = e^{-\lambda x}.$$

It is well known that this characterizes the exponential distribution:

$F_t = F_0$ for all $t > 0$ implies that $F_0(x) = 1 - e^{-\lambda x}$ for some $\lambda > 0$.

If $F_t$ does depend on $t$, then it is possible that the family $\{F_t\}_{t>0}$ has a weak limit $G$ as $t \to \infty$. It is easy to see that $G$ must be of the exponential type. It is also easy to characterize the class of distribution functions for which $F_t$ tends to $G$. 

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Suppose now we allow a scale transformation. Suppose there exists a positive function $a$ such that $F_t(a(t)x)$ has a weak limit $G$ as $t \to \infty$. It will be shown that the possible limit types in this situation are

$$E_\alpha(x) = \begin{cases} 0 & \text{for } x < 0 \\ 1-(1+x)^{-\alpha} & \text{for } x \geq 0 \end{cases}$$

and

$$\Pi(x) = \begin{cases} 0 & \text{for } x < 0 \\ 1-e^{-x} & \text{for } x \geq 0 \end{cases},$$

where $\alpha$ is a positive constant. We remark that these limit distributions bear some resemblance to the limit distributions $\phi_{\alpha}$ and $\Lambda$ in extreme value theory (see section 2). This resemblance is due to the fact that both theories have a common basis, Karamata’s theory of regular variation. In section 5, we shall show how Karamata’s theorems translate into a remarkable moment theorem: convergence of a positive moment as $t \to \infty$ is equivalent to weak convergence of the (properly normed) residual life time distributions $F_t$.

If we allow as norming functions for $F_t$ both a scale transformation and a shift, then in addition to the limit distributions above discrete limit distributions appear. The new limit types are:

$$E_{\gamma,\alpha}(x) = \begin{cases} 0 & \text{for } x < 0 \\ 1-\exp^{-\gamma[1+\alpha \log(1+x)]]} & \text{for } x \geq 0 \end{cases}$$

and

$$\Pi_\gamma(x) = \begin{cases} 0 & \text{for } x < 0 \\ 1-\exp^{-\gamma[1+x]} & \text{for } x \geq 0, \end{cases}$$
with $\alpha, \gamma > 0$ and $[a]$ the integral part of $a$.

The limit types constitute a two parameter family. Indeed, set $p = \gamma > 0$, $pc = a^{-1} > 0$ and:

$$\Pi_{p,c}(x) = \sum_{\gamma, p}^{-1} \Pi_{c^{-1}}(cx) \left[ = 1 - \exp\left(-p\left(1 + \frac{\log(1+cx)}{cp}\right)\right) \text{ for } x \geq 0 \right].$$

On letting $p$ or $c$, or $p$ and $c$ tend to zero we obtain the distributions

$$\Pi_{0,c}(x) = \sum_{c}^{-1}(cx)$$

$$\Pi_{p,0}(x) = \Pi\left(\frac{x}{p}\right)$$

$$\Pi_{0,0}(x) = \Pi(x).$$

In this paper, we consider the general situation (allowing both a scale transformation and a shift). We shall derive the possible limit distributions (section 1) and their domains of attraction (sections 2 and 3). The sections 4 and 5 refer to continuous limit distributions. In section 4, we give approximation theorems for finite values of $t$. In section 5, we shall prove that convergence of one moment entails weak convergence of the residual life times.
1. **The possible limit distributions.**

We say that the distribution function $F$ is in the domain of r.l.t. attraction of the distribution function $G$ and write $F \in D_r(G)$ if:

a) $G$ is non degenerate

b) $F(x) < 1$ for all $x$

c) there exist $a(t) > 0$ and $b(t)$ such that

\[
\lim_{t \to \infty} P\left(\frac{X-b(t)}{a(t)} < x \mid X > t\right) = G(x)
\]

for all continuity points $x$ of $G$. (Here $X$ is a random variable with distribution $F$).

Suppose $F \in D_r(G)$. Then (1) holds. In particular $G$ satisfies:

\[
\lim_{t \to \infty} \frac{1-F(b(t) + x(t))}{1-F(t)} = 1-G(x)
\]

in all continuity points $x$ of $G$ for which $G(x) > 0$. We shall use this relation to derive a functional equation for the limit distribution $G$.

**Lemma 1.** For each continuity point $y$ of $G$ for which $0 < G(y) < 1$ there exist constants $A(y) > 0$ and $B(y)$ such that

\[
1-G(B(y) + xA(y)) = (1-G(x))(1-G(y))
\]

for all $x$ for which $G(x)$ is positive.
**Proof.** Let $y$ be a continuity point of $G$ such that $0 < G(y) < 1$. Without loss of generality we may assume that $y = 0$. This may be achieved by a translation of the $x$-axis and it simplifies notation. Note that (2) then implies that $b(t) > t$ for $t$ sufficiently large.

We define

$$H_t(x) = \frac{1-F(b(t) + xa(t))}{1-F(t)}.$$

The functions $H_t$ converge weakly to $1-G$ on the interval $I = \{x \mid G(x) > 0\}$ as $t \to \infty$. Replacing $t$ by $b(t)$ in the numerator we obtain

$$(4) \quad H_t \left( \frac{b(b(t)) - b(t)}{a(t)} + x \frac{a(b(t))}{a(t)} \right) = H_b(t)(x) \cdot H_t(0)$$

which converges weakly to $(1-G(x))(1-G(0))$ on $I$ as $t \to \infty$.

We shall now prove that

$$(5) \quad \lim_{t \to \infty} \frac{a(b(t))}{a(t)} = A(0) \quad \text{and} \quad \lim_{t \to \infty} \frac{b(b(t)) - b(t)}{a(t)} = B(0)$$

exist and are finite.

Let $(A,B)$ be a limit point as $t \to \infty$. Then $A \in [0,\infty)$ and $B \in [0,\infty]$. Taking the limit of an appropriate sequence $t_n \to \infty$ in (4) we obtain

$$(6) \quad 1-G(B+xA) = (1-G(x))(1-G(0)).$$

In first instance we restrict ourselves to $x \geq 0$. The equality (6)
then is valid if both \( x \) and \( B + xA \) are continuity points of \( G \). It also holds in all continuity points \( x \) of \( G \) if \( A \) or \( B \) is infinite. Since the right-hand side is not identically zero on \((0, \infty)\) we see that \( A \) and \( B \) are finite.

Suppose \( A > 0 \). Then (6) holds for all \( x \in I \). (We assume \( G \) to be right continuous). Let \((A', B')\) also be a limit point with \( A' > 0 \). Then we have

\[
1 - G(B' + xA') = (1 - G(x))(1 - G(0)) = 1 - G(B + xA).
\]

Since \( G \) is continuous in \( x = 0 \), it is continuous in \( B + 0A = B \), and \( 1 - G(0) > (1 - G(0))^2 = 1 - G(B) > 0 = 1 - G(\infty) \). This implies that \( G \) has at least two positive points of increase, \( x_1 \) and \( x_2 \), and hence

\[
B' + x_i A' = B + x_i A \quad \text{for } i = 1, 2
\]

which proves that \( A' = A \) and \( B' = B \).

Now suppose \( A = 0 \). This is impossible because then the left-hand side of (6) is constant but the right-hand side is not.

**Lemma 2.** \( A(y) \geq 1 \).

**Proof.** Again we may assume that \( y = 0 \) since a translation of the \( x \)-axis in (3) does not affect the coefficient \( A(y) \).

Since \( G(0) \) is positive, relation (2) implies that there exist \( q < 1 \) and \( b_0 \) such that \( 1 - F(b(t)) \leq q(1 - F(t)) \) for all \( t > b_0 \).

We define sequences \((a_n)\) and \((b_n)\) by:
\[ b_{n+1} = b(b_n) \text{ and } a_{n+1} = a(b_n). \]

The sequence \( b_n \) is increasing. Suppose it has a finite limit \( b \).

Then \( F(b) < 1 \) and by (2)

\[
\frac{1-F(b-0)}{1-F(b-0)} = \lim_{n \to \infty} \frac{1-F(b_n)}{1-F(b_n)} = 1-G(0) < 1.
\]

This contradiction shows that \( b_n \to \infty \). Hence we obtain from (5) that

\[
\frac{a_{n+1}}{a_n} \to A(0) \text{ and } \frac{b_{n+1} - b_n}{a_n} \to B(0).
\]

The first relation shows that for any \( A > A(0) \) we have

\[ a_n \leq KA^n \text{ for all } n \text{ for some } K > 0, \]

the second that \( b_{n+1} - b_n \leq K_1 A^n \)

for all \( n \) for some \( K_1 > 0 \). Hence \( b_{n+1} - b_0 \leq K_1 (1 + A + \cdots + A^n) \)

and \( b_n \to \infty \) implies \( A > 1 \). Hence \( A(0) > 1 \).

**Lemma 3.** There exists a real number \( x_1 \) such that \( G(x) = 0 \) if \( x < x_1 \)

and \( 0 < G(x) < 1 \) if \( x > x_1 \).

**Proof.** Let \( y = 0 \) be a continuity point of \( G \) such that \( 0 < G(0) < 1 \).

Suppose \( G(x) > 0 \) for all \( x \). Letting \( x \) tend to \( -\infty \) in (3)

with \( y = 0 \) we obtain \( 1 = 1-G(0) \). Contradiction.

Define the sequence \( y_0, y_1, y_2, \ldots \) by

\[ y_{n+1} = B(0) + y_n A(0). \]

Then \( 1-G(y_n) = (1-G(0))^n \) by (3). Hence \( y_n \) increases, \( B(0) > 0 \) and
since \( A(0) \geq 1 \) the sequence \( (y_n) \) diverges and \( 1-G(x) \) is strictly positive.

**Lemma 4.** Let the distribution function \( G \) satisfy (1) and let \( H \) be an unbounded non-increasing function on an interval \( (x_0, \infty) \) which agrees with \( 1-G \) for all \( x \) for which \( G(x) > 0 \) and which satisfies

\[
H(B(y) + xA(y)) = H(x) \cdot H(y)
\]

for all \( x > x_0 \)

for some pair \((A(y), B(y))\) defined in lemma 1. Then

\[
1-G(x) = \min(1, H(x))
\]

for all \( x > x_0 \).

**Remark.** A result of this lemma is that for example the distribution function

\[
G(x) = \begin{cases} 
0 & \text{for } x < 1 \\
1-e^{-x} & \text{for } x \geq 1
\end{cases}
\]

cannot be a limit distribution in (1).

**Proof.** We may assume that \( y = 0 \) is a continuity point of \( G \) such that \( G(0) > 0 \) and such that (7) holds. Then \( x_1 = \inf\{x \mid G(x) > 0\} \) is negative and \( x_1 \) is finite by lemma 3. It suffices to prove that each left neighborhood \((x_1-\epsilon, x_1)\) contains a point \( x \) such that \( H(x) \geq 1 \).

Observe that \( H(B(0) + xA(0)) < H(0) \) for all \( x > x_1 \) and hence \( B(0) + xA(0) > 0 \) for all \( x > x_1 \). This implies that we can find \( x \in (x_1-\epsilon, x_1) \) such that \( B(0) + xA(0) > x_1 \). We may also assume that \( x \)}
and \( B(0) + xA(0) \) are continuity points of \( H \).

Using relation (5) we find that the left hand side of (4) converges to \( H(B(0) + xA(0)) \). By (1) we have

\[
\lim_{{t \to \infty}} \min(l, H_t(x)) = 1 - G(x) = 1
\]

and hence (4) gives: \( H(B(0) + xA(0)) \geq H(0) \). Together with (7) thisimplies that \( H(x) \geq 1 \).

In order to solve the functional equation (3), we need two preliminary lemmas.

**Lemma 5.** Let \( \phi: (x, x_1) \to \mathbb{R} \) be a function and let \( b > 0 \) and \( c \) be real numbers such that

\[
\phi(x + b) = \phi(x) + c \quad \text{ for all } x > x_1
\]

Then

\[
\phi(x) = b^{-1}cx + \phi_0(x)
\]

where \( \phi_0 \) is periodic modulo \( b \).

**Proof.** Direct computation shows that \( \phi(x) - b^{-1}cx \) is periodic modulo \( b \).

**Lemma 6.** Let \( \phi: (x, x_1) \to \mathbb{R} \) be a non-decreasing unbounded positive function and for \( i = 1, 2 \) let \( A_i \geq 1 \), \( B_i \) and \( C_i > 0 \) be real numbers such that
\( \phi(B_1 + xA_1) = \phi(x) + C_1 \quad \text{for } x > x_1 \)

Then

\( B_1 + B_0A_1 = B_0 + B_1A_0 \) \( (8) \)

**Proof.** For all \( x > x_1 \) we have

\[
\phi(B_1 + (B_0 + xA_0)A_1) = (\phi(x) + C_0) + C_1 \\
\phi(B_0 + (B_1 + xA_1)A_0) = (\phi(x) + C_1) + C_0
\]

Hence

\[
\phi(B_1 + B_0A_1 + xA_0A_1) = \phi(B_0 + B_1A_0 + xA_0A_1)
\]

and \( \phi \) is periodic modulo \((B_1 + B_0A_1) - (B_0 + B_1A_0)\) on \((B_1 + B_0A_1, \infty)\)

unless \((8)\) holds.

**Theorem 1.** The distribution functions with non-empty domain of
residual lifetime attraction are of the types \( \Pi, \Pi_\gamma, \Xi_\alpha \) or \( \Xi_{\alpha, \gamma} \)
defined in the introduction.

**Proof.** Suppose \( G \) has non-empty domain of r.l.t. attraction. Let

\( S \) be the set of continuity points \( y \) of \( G \) for which \( G(y) \) is
positive and set \( x_1 = \inf\{x | G(x) > 0\} \). From lemma 1, we know that
for each \( y \in S \)

\( l-G(B(y) + xA(y)) = (l-G(x))(l-G(y)) \) \( (3) \)

for all \( x > x_1 \).
We consider two cases.

**Case 1.** $A(y) = 1$ for all $y \in S$.

Define $\phi(x) = -\log(1-G(x))$ for $x > x_1$. Relation (3) becomes

\[ \phi(B(y) + x) = \phi(x) + \phi(y). \]

Fix $y \in S$. Then $\phi(y) > 0$ and hence $B(y) > 0$. By lemma 5:

\[ \phi(x) = c \cdot x + \phi_0(x) \]

with $c = \phi(y) \cdot (B(y))^{-1} > 0$ and with $\phi_0$ periodic modulo $B(y)$.

Hence $\lim_{x \to \infty} x^{-1} \cdot \phi(x) = c$ exists, $c$ is independent of $y$ and so is the function $\phi_0$. There are two possibilities: either $\phi_0$ is constant or $\phi_0$ is periodic modulo $p$ with $p > 0$ minimal.

a) $\phi_0$ is constant: $\phi(x) = c(x+d)$. Define $H(x) = e^{-c(x+d)}$ for all $x \in \mathbb{R}$. Then $H(x) = 1-G(x)$ for $x > x_1$ and $H$ satisfies the functional relation (7) for all $x \in \mathbb{R}$. Lemma 4 yields

\[ 1-G(x) = \min(1, e^{-c(x+d)}) \]

and $G$ is of type $\Pi$.

b) $\phi_0$ is not constant and has minimal period $p > 0$: The function $\phi$ has the properties: $\phi(x+p) - \phi(x) = cp$ by (10); $\phi(x) \in \{cp, 2cp, 3cp, \ldots\}$ since $\phi_0$ is periodic modulo $B(y) = c^{-1} \phi(y)$ for all $y \in S$.

It follows that $\phi(x) = cp \cdot \left[ x+d \right]_p$. As above we set $H(x) = \exp(-cp \left[ x+d \right]_p)$ for all $x \in \mathbb{R}$ and $G$ is of type $\Pi_y$ with $y = cp > 0$. 

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Case 2. $A(y) > 1$ for some $y \in S$.

If we apply lemma 6 to $\phi(x) = -\log(1-G(x))$ we find that $A(y) > 1$
for all $y \in S$. We write

$$B(y) + xA(y) = x_0 + (x-x_0)A(y)$$

where $x_0 = (1-A(y))^{-1}B(y)$ does not depend on $y$ by lemma 6. Hence
we may assume $x_0 = 0$. Then (3) becomes

$$1-G(xA(y)) = (1-G(x))(1-G(y)).$$

For fixed $y \in S$ we must have $xA(y) > y$ for all $x > x_\perp$. Taking
$y > 0$ we see that $x_\perp > 0$.

Now set $x = e^\xi$, $\psi(\xi) = -\log(1-G(x))$. Then

$$\psi(\xi + a(\eta)) = \psi(\xi) + \psi(\eta)$$

for $\xi > \log x_\perp$ and all $\eta \in \log S$. Note that $a(\eta) = \log A(e^\eta) > 0$.
Hence we in fact have relation (9) once more. There are two
possibilities:

a) $\psi(\xi) = c(\xi + d)$, $H(x) = e^{-c(\xi + d)} = (e^d x)^{-c}$ for all $x > 0$ and
$G$ is of type $\Xi_\alpha$ with $\alpha = c > 0$, or

b) $\psi(\xi) = cp[\xi + d]$, $H(x) = \exp - cp[\xi + d] = \exp - cp[\frac{1}{p} \log e^d x]$ for all
$x > 0$ and $G$ is of type $\Xi_{\gamma, \alpha}$ with $\gamma = cp$, $\alpha = p^{-1}$.

Finally observe that $[x + [y]] = [x + y]$ holds for all reals
$x$ and $y$. Hence any function $G$ of one of the four types satisfies
(3) for all $y > x_\perp$ if we take $A$ and $B$ to be right continuous.

Hence $G \in D_+(G)$.
2. The domain of r.i.t. attraction of $\Xi_\alpha$ and of $\Pi$.

The limit distributions in extreme value theory are usually denoted by

\[ \phi_\alpha(x) = e^{-x^{-\alpha}} \quad \text{for } x > 0 \]
\[ \Lambda(x) = e^{-e^{-x}} \]
\[ \psi_\alpha(x) = e^{(-x)^\alpha} \quad \text{for } x < 0 \]

with $\alpha > 0$. Let $G$ be one of these limit distributions. A distribution function $F$ lies in the domain of attraction of $G$ and we write $F \in D(G)$ if there exist $a_n > 0$ and $b_n$ such that

\[ \lim_{t \to \infty} P^n(b_n + x a_n) = G(x) \]

for all $x$. (The maxima of $n$ independent random variables each distributed according to $F$ converge, properly normed, in distribution to a random variable with distribution $G$). Gnedenko (1943) determined these domains of attraction and since then there has been a substantial list of publications on this subject.

In this paragraph we shall prove that the domain of residual lifetime attraction of a continuous limit distribution coincides with the domain of attraction of some extreme value limit distribution.

The proof depends on a continuation principle derived in lemma 8. It would however be much more interesting if theorem 2 could be proved by probabilistic arguments.
Lemma 7. If there exists functions \( a(t) > 0 \) and \( b(t) \) such that

\[
H_t(x) = \frac{1-F(b(t)+xa(t))}{1-F(t)}
\]

converges to \( 1-G(x) \) for all \( x \) such that \( G(x) > 0 \) as \( t \to \infty \)
and \( G \) is continuous, then

\[
\lim_{t \to \infty} \frac{1-F(t+0)}{1-F(t-0)} = 1.
\]

Proof. For any \( q < 1 \) there exists \( y \) such that \( q < 1-G(y) < 1 \).
For this \( y \) by (2) we have that \( b(t) + ya(t) > t \) and
\( l-F(b(t) + ya(t)) < q(l-F(t)) \) for \( t > t_0 \). Hence certainly
\( l-F(t+0) > q(l-F(t-0)) \) for \( t > t_0 \).

Lemma 8. If \( H_t(x) \) converges to \( 1-G(x) \) for all \( x \) such that
\( G(x) > 0 \) as \( t \to \infty \) and \( G \) is continuous, then \( H_t(x) \) converges
to a continuous unbounded non-increasing function \( H \) on an interval
\( (x_0, \infty) \) as \( t \to \infty \). If \( G \) is of type \( II \) then \( x_0 = -\infty \) and
\( H(x) = Ce^{-\lambda x} \) with \( C > 0 \) and \( \lambda > 0 \). If \( G \) is of type \( E_\alpha \) for
some \( \alpha > 0 \) then \( x_0 \) is finite and \( H(x) = C(x-x_0)^{-\alpha} \) with \( C > 0 \).

Proof. We assume that \( G(0) > 0 \).

Let \( I_0 = (c_0, \infty) \) be the maximal interval on which \( H = \lim_{t \to \infty} H_t \)
exists and on which either \( H(x) = C e^{-\lambda x} \) or \( H(x) = C(x-x_0)^{-\alpha} \).
Set \( I_1 = (c_1, \infty) = \{ x | B(0) + xA(0) \in I_0 \} \) with \( A(0) \) and \( B(0) \) as
in lemma 1.
Note that \( c_1 < c_0 \) unless \( c_0 = -\infty \) in case of \( C \cdot e^{-\lambda x} \) or \( c_0 = x_2 \) in case of \( C \cdot (x - x_0)^{-\alpha} \); then \( c_1 = c_0 \).

We shall prove that \( H_t(x) \) converges on \( I_\perp \) as \( t \to \infty \). Then \( I_\perp \subset I_0 \) and the proof of the lemma is complete.

Relation (4) holds for all \( x \). In particular for \( x \in I_\perp \) the left-hand side of (4) converges to \( H(B(0) + xA(0)) \) as \( t \to \infty \) and hence \( \lim_{t \to \infty} H_b(t)(x) \) exists on \( I_\perp \). This limit again satisfies relation (3) of lemma 1 on \( I_\perp \), hence the limit equals \( H \) on \( I_\perp \).

In order to prove that \( H_t(x) \) converges on \( I_\perp \) as \( t \to \infty \), we choose a sequence \( t_n \to \infty \) such that \( 1 - F(t_{n+1}) \sim 1 - F(t_n) \) as \( n \to \infty \). This is possible by lemma 7. Then also \( 1 - F(b(t_{n+1})) \sim 1 - F(b(t_n)) \) as \( n \to \infty \) by (2). We define

\[
\tilde{a}(t) = a(b(t_n)), \tilde{b}(t) = b(b(t_n)) \text{ for } F(b(t_n)) \leq F(t) < F(b(t_{n+1}))
\]

(We may and do suppose that \( b(t_n) \) is strictly increasing).

Then

\[
\frac{1 - F(b(t) + x\tilde{a}(t))}{1 - F(t)} = H_t \left[ \frac{\tilde{b}(t) - b(t)}{a(t)} + x \frac{\tilde{a}(t)}{a(t)} \right]
\]

converges to \( H(x) \) on \( I_\perp \). (The left-hand side we write as

\[
\frac{1 - F(b(b(t_n)) + xa(b(t_n)))}{1 - F(t)}
\]
with \(1 - F(b(t_{n+1})) \leq 1 - F(t) \leq 1 - F(b(t_{n}))\). Any limit point \((A, B)\) of \[
\left[ \frac{\theta(t)}{a(t)}, \frac{\theta(t) - b(t)}{a(t)} \right]
\]
satisfies:

\[
H(x) = H(B + xA)
\quad \text{for all } x \in I_0.
\]

Since \(H\) is continuous and increasing on \(I_0\), we have \(A = 1\) and \(B = 0\). Hence we may replace \(\theta(t)\) and \(\theta(t) - b(t)\) by \(a(t)\) and \(b(t)\) on the left-hand side of (11) without altering the limit.

**Theorem 2.**

\[
D_r(\hat{a}) = D(\phi_a)
\]

\[
D_r(\Pi) = D(\Lambda).
\]

**Proof.** The following statements are equivalent:

1) \(F \in D(\Lambda)\)

2) \(F^n(b_n + xa_n) \rightarrow \Lambda(x)\) for all \(x\)

3) \(n \log F(b_n + xa_n) \rightarrow e^{-x}\) for all \(x\)

4) \(n(1 - F(b_n + xa_n)) \rightarrow e^{-x}\) for all \(x\)

5) \(\lim_{r \to 0} \frac{1 - F(\theta(r) + xa(r))}{r} = e^{-x}\) for all \(x\)

6) \(\lim_{t \to \infty} \frac{1 - F(b(t) + xa(t))}{1 - F(t)} = e^{-x}\) for all \(x\)

7) \(\lim_{t \to \infty} \frac{1 - F(b(t) + xa(t))}{1 - F(t)} = e^{-x}\) for all \(x > 0\)

8) \(F \in D_r(\Pi)\).

1 \(\iff\) 2 \(\iff\) 3 \(\iff\) 4 \(\iff\) 5 are well known. See Gnedenko (1943).
5 \Rightarrow 6 is trivial; 6 \Rightarrow 5 is an easy consequence of lemma 7. 6 \Rightarrow 7 is trivial; 7 \Rightarrow 6 is proved in lemma 8. 8 \Rightarrow 7 is trivial; 6 \Rightarrow 8 is trivial.

The proof that \( D_r(\Xi_\alpha) = D(\phi_\alpha) \) is similar.

**Corollary 1.** If \( F \in D_r(\Xi_\alpha) \), then

\[
\lim_{t \to \infty} \frac{\frac{X-t}{\alpha(t)} \leq x}{X > t} = \Xi_\alpha(x) \quad \text{for all } x.
\]

**Proof.** \( F \in D(\phi_\alpha) \) if and only if (Gnedenko, 1943)

\[
\lim_{t \to \infty} \frac{1-F(tx)}{1-F(t)} = x^{-\alpha} \quad \text{for all } x > 0.
\]

**Corollary 2.** If \( F \in D_r(\Pi) \), then

\[
\lim_{t \to \infty} \frac{X-t}{\alpha(t)} \leq x | X > t} = \Pi(x) \quad \text{for all } x
\]

with

\[
a(t) = \frac{\int_0^\infty (1-F(s))ds}{1-F(t)}.
\]

**Proof.** \( F \in D_r(\Pi) \) if and only if (de Haan, 1970)

\[
\lim_{t \to \infty} \frac{1-F(t+xa(t))}{1-F(t)} = e^{-x}
\]

for all real \( x \) with the auxiliary function \( \alpha \) as in the statement of the corollary.
3. The domains of r.l.t. attraction of the discrete limit distributions.

Let $G$ be a discrete limit distribution. $G$ is of type $\Pi_{p,c}$ with $p > 0$, $c \geq 0$. For $c > 0$ we have

$$1 - \Pi_{p,c}(x) = \begin{cases} 1 & \text{for } x < 0 \\ \exp - p \left( 1 + \frac{\log(1 + cx)}{pc} \right) & \text{for } x \geq 0 \end{cases}$$

We define $\Pi_{p,0} = \lim_{c \to 0} \Pi_{p,c}$ as in the introduction.

$\Pi_{p,c}$ is a right continuous distribution with discontinuity point, $t_0, t_1, t_2, \ldots$ where

$$a') \quad t_n = \begin{cases} c^{-1}(e^{npc} - 1) & \text{if } c > 0 \\ n & \text{if } c = 0 \end{cases}$$

$$b') \quad (1 - \Pi_{p,c}(t_{n+1}))(1 - \Pi_{p,c}(t_n))^{-1} = e^{-p}.$$

The sequence of discontinuity points of the distribution function $G$ of type $\Pi_{p,c}$ will satisfy

$$a'') \quad \frac{t_{n+1} - t_n}{t_n - t_{n-1}} = e^{pc}$$

$$b'') \quad (1 - G(t_{n+1}))(1 - G(t_n))^{-1} = e^{-p}.$$

Conversely these relations exactly characterize the distribution functions of type $\Pi_{p,c}$.

**Lemma 2.** Let $F_0$ be a discrete distribution function whose discontinuity points form an unbounded increasing sequence
\[ t_0, t_1, t_2, \ldots \text{ such that} \]

\[
a) \lim_{n \to \infty} \frac{t_{n+1} - t_n}{t_n - t_{n-1}} = e^{pc} > 1
\]

\[
b) \lim_{n \to \infty} \frac{1 - F_0(t_{n+1})}{1 - F_0(t_n)} = e^{-p} < 1
\]

Then \( F_0 \in D_r(\Pi_{p,c}) \).

**Proof.** Set \( a_n = t_{n+1} - t_n, \ b_n = t_{n+1} \) for \( n = 0, 1, 2, \ldots \). Then

\[
H_n(x) = \frac{1 - F_0(b_n + xa_n)}{1 - F_0(t_n)}
\]

takes on the values

\[
(1 - F_0(t_{n+1}))(1 - F_0(t_n))^{-1}, \ (1 - F_0(t_{n+2}))(1 - F_0(t_n))^{-1}, \ldots
\]

between the successive discontinuity points

\[
0, \ a_n^{-1}a_{n+1}, \ a_n^{-1}(a_{n+1} + a_{n+2}), \ldots
\]

and hence \( H_n \) converges weakly to a function which has discontinuity points \( 0, e^{pc}, e^{pc} + e^{2pc}, \ldots \) and takes on the values \( e^{-p}, e^{-2p}, \ldots \) in between.

Thus \( \min\{1, H_n(x)\} \) converges weakly to \( 1 - \Pi_{p,c}(x) \) and since \( F_0 \) is discrete this proves that \( F_0 \in D_r(\Pi_{p,c}) \). (Relation (1) holds with the norming functions \( a(t) \) and \( b(t) \) defined by \( a(t) = a_n, b(t) = b_n \) for \( t_n \leq t < t_{n+1} \)).
Definition. Let $F_1$ and $F_2$ be distribution functions such that $F_i(x) < 1$ for $i = 1, 2$ and all $x$. $F_1$ and $F_2$ are tail equivalent if

$$\lim_{x \to \infty} \frac{1-F_1(x)}{1-F_2(x)} = 1.$$

Lemma 10. If $F_1 \in D_r(G)$ and $F_2$ is tail equivalent to $F_1$, then $F_2 \in D_r(G)$. Moreover the residual lifetimes of the two distributions converge with the same norming constants $a(t)$ and $b(t)$.

Proof. Suppose that

$$\frac{1-F_1(b(t) + xa(t))}{1-F_1(t)} \sim 1-G(x) \quad \text{as} \quad t \to \infty$$

for all $x$ for which $G(x)$ is positive. This implies in particular that $b(t) + xa(t)$ diverges for all such $x$ as $t \to \infty$. Hence the asymptotic relation remains valid if we replace $F_1$ by $F_2$.

Theorem 3. Suppose $p > 0$ and $c > 0$. The distribution function $F$ lies in the domain of residual lifetime attraction of $\Pi_{p,c}$ if and only if $F$ is tail equivalent to a discrete distribution function $F_0$ which satisfies a) and b) in lemma 9.

Proof. The if statement follows on combining lemmas 9 and 10. In order to prove that any distribution function $F \in D_r(\Pi_{p,c})$ is tail equivalent to a discrete distribution function $F_0$ which satisfies a) and b), we
shall construct a sequence \((b_n)\) such that \((1-F(b_{n+1}))(1-F(b_n))^{-1} \to e^{-p}\) and we shall show that \(F\) is tail asymptotic to a discrete distribution \(F_0\) which only takes on the values \(F(b_n)\).

Let \(G\) be a translate of \(H_{P,c}\) such that \(G(0) = 1-e^{-p}\) and such that 0 is a continuity point of \(G\). Suppose (1) holds. Then

\[
\lim_{t \to \infty} \frac{1-F(b(t))}{1-F(t)} = 1-G(0) = e^{-p} < 1
\]

and as in the proof of lemma 2 we may define sequences \((a_n)\) and \((b_n)\) such that

\[
a_{n+1} = a(b_n), \quad b_{n+1} = b(b_n) \quad \text{and} \quad b_n \to \infty.
\]

We shall now prove that \(F\) is tail equivalent to a distribution function \(F_0\) which only takes on the values \(F(b_n), \ n = 1, 2, \ldots\)

Suppose not. Then, since \((1-F(b_{n+1}))(1-F(b_n)) \to e^{-p}\) by (16), there exists a sequence \(s_k \to \infty\) and \(n(k)\) such that

\[
\lim_{k \to \infty} \frac{1-F(s_k)}{1-F(b_{n(k)})} = q_1 \quad \text{with} \quad e^{-p} < q_1 < 1.
\]

Define

\[
H_t(x) = \frac{1-F(b(t) + xa(t))}{1-F(t)}
\]

Then
\[
H_{b_n}(k) \left( \frac{b(s_k)-b(b_n(k))}{a(b_n(k))} + x \frac{a(s_k)}{a(b_n(k))} \right) = H_{s_k}(x) \frac{1-F(s_k)}{1-F(b_n(k))}
\]

and using the arguments of lemma 1 we find

\[
1-G(B_1 + xA_1) = (1-G(x)).q_1
\]

for some real numbers \( A_1 \) and \( B_1 \) for all \( x \) for which \( G(x) \) is positive. We remark that \( 1-G(x) \) only takes on the values \( 1, e^{-p}, e^{-2p}, \ldots \). Hence \( q_1 = e^{-kp} \) for some integer \( k \).

Contradiction.

\( F_0 \) is clearly a discrete distribution function. Let \( t_0, t_1, \ldots \) be its discontinuity points such that

\[
F_0(t) = F(b_n) \quad \text{for} \quad t_n \leq t < t_{n+1}.
\]

Then

\[
\frac{1-F_0(t_{n+1})}{1-F_0(t_n)} = \frac{1-F(b_{n+1})}{1-F(b_n)} \to e^{-p}
\]

by (16) which proves b).

The functions

\[
H_n(x) = \frac{1-F(t_n+xa_n)}{1-F(t_{n-1})}
\]

satisfy
\[ H_n \left( \frac{t_{n+1} - t_n}{a_n} + x \frac{a_{n+1}}{a_n} \right) = H_{n+1}(x)H_n(0). \]

Letting \( n \) tend to infinity we obtain as in the proof of lemma 1

\[ \frac{a_{n+1}}{a_n} \to A, \quad \frac{t_{n+1} - t_n}{a_n} \to B > 0 \quad \text{as } n \to \infty. \]

which implies

\[ \frac{t_{n+1} - t_n}{t_n - t_{n-1}} = \frac{t_{n+1} - t_n}{a_n} \cdot \frac{a_n}{a_{n-1}} \cdot \frac{a_{n-1}}{a_n} \to A. \]

This proves a) since \( A > 1 \) by lemma 2 and \( A = e^{pc} \) by lemma 9.

**Remark.** Let \( p_n = F_0(t_n) - F_0(t_n^-) \) be the height of the jump in \( t_n \). Then b) in lemma 9 is equivalent to

\[ b^*) \quad \lim_{n \to \infty} p_{n+1} p_n^{-1} = e^{-p} < 1. \]
4. Approximation for finite $t$.

In extreme value theory, there exist convenient sufficient conditions, due to R. von Mises (1936), for a distribution function to belong to the domain of attraction of some limit distribution:

\[ F \in D(\Phi_\alpha) \text{ if } \lim_{t \to \infty} \frac{tF'(t)}{1-F(t)} = \alpha \]

\[ F \in D(\Lambda) \text{ if } \lim_{t \to \infty} \frac{d}{dt} \frac{1-F(t)}{F'(t)} = 0. \]

By theorem 2 these conditions are also sufficient for a distribution function to belong to the domain of residual lifetime attraction of a continuous limit distribution. For residual lifetimes, one has in addition some simple inequalities which give upper and lower bounds for the normed residual lifetime distributions for larger values of $t$.

**Theorem 4.** Let $F$ be a distribution function which has a positive density $F'$ for $t \geq t_0$ and let $\alpha_1$ and $\alpha_2$ be positive real numbers such that

\[ \alpha_1 \leq \frac{tF'(t)}{1-F(t)} \leq \alpha_2 \quad \text{for } t \geq t_0. \]

If $X$ is a random variable with distribution $F$, then

\[ \Xi_{\alpha_1}(x) \leq P\left(\frac{X-t}{t} \leq x | X > t \right) \leq \Xi_{\alpha_2}(x) \]

for all $x$ for $t \geq t_0$.  

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Proof. Integration between $t$ and $(1+x)t$ with $x > 0$ gives

$$
\alpha_1 \int_t^{(1+x)t} \frac{du}{u} \leq \int_t^{(1+x)t} \frac{F'(u)}{1-F(u)} \, du \leq \alpha_2 \int_t^{(1+x)t} \frac{du}{u}.
$$

Using the monotonic transformation $y \to 1-e^{-y}$ we obtain the stated result.

For the next theorem, it is more convenient to use the notation $\Pi_{0,c}$. For $c \geq 0$ the distribution functions $\Pi_{0,c}$ have been defined in the introduction. For $c < 0$ we set

$$
\Pi_{0,c}(x) = \begin{cases} 
0 & \text{if } x \leq 0 \\
\frac{1}{1-(1+cx)^c} & \text{if } 0 \leq x \leq |c|^{-1} \\
1 & \text{if } x \geq |c|^{-1}.
\end{cases}
$$

Theorem 5. Let $F$ be a distribution function which has a positive, differentiable density $F'$ for $t \geq t_0$. Let $c_1$ and $c_2$ be real numbers such that

$$
c_1 \leq \frac{d}{dt}\left|\frac{1-F(t)}{F'(t)}\right| \leq c_2 \quad \text{for } t \geq t_0.
$$

If $X$ is a random variable with distribution $F$, then for $t \geq t_0$ and all $x$

$$
\Pi_{0,c_2}(x) \leq P\{\frac{X-t}{\sigma(t)} \leq x | X > t\} \leq \Pi_{0,c}(x)
$$
where

$$a(t) = \frac{1-F(t)}{F'(t)}.$$

**Proof.** For $t \geq t_0$ and $x \geq 0$ integration between $t$ and $t + xa(t)$ gives

$$c_1 xa(t) \leq a(t+xa(t)) - a(t) \leq c_2 xa(t)$$

or equivalently

$$1 + c_1 x \leq \frac{a(t+xa(t))}{a(t)} \leq 1 + c_2 x.$$  \hfill (13)

By taking the logarithmic derivative with respect to $x$ one readily checks that

$$P\{\frac{X-t}{a(t)} > x | X > t\} = \frac{1-F(t+xa(t))}{1-F(t)} = \exp - \int_0^x \frac{a(t)}{a(t+sa(t))} \, ds \hfill (14)$$

and the theorem follows from the inequality (13) above.

**Corollary 1.** Setting $G_t(x) = P\{(a(t))^{-1}(X-t) \leq x | X > t\}$ we see from (14) that

$$\frac{G_t'(x)}{1-G_t(x)} = \frac{a(t)}{a(t+xa(t))} \quad \text{for } x > 0.$$ 

Also for $x > 0$ (and $x < |c|^{-1}$ if $c < 0$) we have

$$\Pi_{0,c}'(x) = (1+cx)^{-1} \cdot (1-\Pi_{0,c}(x)).$$

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Hence under the hypotheses of theorem 5, we have the following inequality for the density of the properly normed residual lifetime distribution:

\[
\frac{1+c_1x}{1+c_2x} \Pi_{0,c_1}(x) \leq G_t'(t) \leq \frac{1+c_2x}{1+c_1x} \Pi_{0,c_2}(x)
\]

which is valid for all \( x \) if \( c_1 \geq 0 \) and for \( x < |c_1|^{-1} \) if \( c_1 < 0 \).

**Corollary 2.** If \( c_1 = y-\epsilon \) and \( c_2 = y+\epsilon \) with \( y \geq 0 \) and \( \epsilon > 0 \) in theorem 5, then

\[
\Pi_{0,y-\epsilon}(x) \leq P\left[ \frac{X-t}{a(t)} \leq x \mid X > t \right] \leq \Pi_{0,y}(x)+\epsilon
\]

for all \( x \) and \( t \geq t_0 \).

**Proof.** It suffices to prove

\[
|\Pi_{0,y}(x) - \Pi_{0,\delta}(x)| \leq |y-\delta|
\]

for \( y \geq 0 \) and all real \( \delta \) and \( x \). This follows from the two lemma's below.

**Lemma 11.**

\[
0 \leq \frac{\partial}{\partial y} \Pi_{0,y}(x) \leq 1 \quad \text{for} \quad y \geq 0.
\]

**Proof.** Set \( u(y) = \frac{\log(1+y)}{y} = 1 - \frac{y}{2} + \frac{y^2}{3} - \ldots \), then
(15) \[-u'(y) = \frac{\log(1+y)}{y} - \frac{1}{y(1+y)} \geq 0\]

since \(-u'(0) = \frac{1}{2} > 0\) and \(\frac{d}{dy} \{(1+y)\log(1+y) - y\} = \log(1+y) > 0\).

Now \(1 - \Pi_{0,y}(x) = (1 + \gamma x)/y = e^{-ux(yx)}\) and

\[
\frac{3}{3y} \Pi_{0,y}(x) = \left\{ \frac{\log(1+\gamma x)}{\gamma^2} - \frac{x}{\gamma(1+\gamma x)} \right\} \{1 - \Pi_{0,y}(x)\}.
\]

We shall prove

\[0 \leq -u'(\gamma x).x^2 = \frac{\log(1+\gamma x)}{\gamma^2} - \frac{x}{\gamma(1+\gamma x)} \leq e^{ux(\gamma x)}.
\]

The first inequality follows from (15); to prove the second one, we need the inequalities

\[x^2 \leq \exp \frac{2x}{e}\] for \(x > 0\)

and \(e > \frac{11}{4}\), which implies

\[
\frac{4}{e^2} \leq \frac{2}{e} \cdot \frac{3}{4} \leq \frac{6}{11} < 0.559\ldots = \log \frac{7}{4}.
\]

Suppose first \(\gamma x > \frac{3}{4}\). Then \(\log(1+\gamma x) > \frac{4}{e^2}\) and

\[
\frac{\log(1+x)}{\gamma^2} = \frac{(xu)^2}{\log(1+\gamma x)} \leq \frac{(e/2)^2}{xu} \leq e^{ux}.
\]

Suppose next \(\gamma x \leq \frac{3}{4}\). Then \(u(\gamma x) \geq u(\frac{3}{4}) > \frac{2}{e}\) by (15) and (16).
Moreover

\[-u'(y) = \frac{1}{2} - \frac{2y}{3} + \frac{3y^2}{4} - \ldots\]

\[= (1 - y + y^2 - \ldots) + (- \frac{1}{2} + \frac{y}{3} - \frac{y^2}{4} + \ldots)\]

\[\leq \frac{1}{1+y} \leq 1 \quad \text{if} \quad 0 \leq y \leq 1.\]

Hence

\[-u'(yx)x^2 \leq x^2 \leq e^{\frac{2x}{e}} \leq e^{xu(\frac{3}{4})} \leq e^{xu(yx)}.\]

**Lemma 12.**

\[0 \leq \Pi_{0,0}(x) - \Pi_{0,\gamma}(x) \leq -\gamma \quad \text{for} \quad \gamma < 0.\]

**Proof.** Set \(\alpha = -\gamma\), then

\[0 \leq (1-\alpha x)^{\frac{1}{\alpha}} \leq \exp\left\{\frac{1}{\alpha} \log(1-\alpha x)\right\} \leq e^{-x}.\]

We shall prove

\[0 \leq e^{-x-(1-\alpha x)^{\frac{1}{\alpha}}} \leq \alpha \quad \text{for} \quad x, \alpha > 0, \alpha x \leq 1.\]

The inequality is trivial for \(\alpha \geq e^{-x}\). If \(\alpha \leq e^{-x}\), then

\[ax \leq xe^{-x} \leq e^{-1}\]

and

\[-x - \frac{1}{\alpha} \log(1-\alpha x) = \frac{ax^2}{2} + \frac{ax^3}{3} + \ldots \leq \frac{ax^2}{2} \frac{1}{1-e^{-1}} \leq ax^2.\]
Hence
\[ e^{-x} - \exp\left\{-\frac{1}{\alpha} \log(1-\alpha x)\right\} \leq \alpha x^2 e^{-x} \leq \alpha. \]

We conclude this section with two theorems which are analogous to theorem 4 and theorem 5 for distribution functions which possibly are not differentiable.

**Theorem 6.** Let \( F \) be a distribution function \((F(x) < x \text{ for all } x)\) such that

\[ (17) \quad \int_1^\infty \frac{1}{x} \{1-F(x)\} \, dx < \infty. \]

and let \( \alpha_1 \) and \( \alpha_2 \) be positive real numbers such that

\[ \alpha_1 \leq \frac{\int_s^t \{1-F(s)\} \, ds}{s} \leq \alpha_2 \quad \text{for } t \geq t_0. \]

If \( X \) is a random variable with distribution \( F \), then

\[ 1 - \frac{\alpha_1}{\alpha_2} \{1 - \bar{\varepsilon}_{\alpha_1}(x)\} \leq P\{\frac{X-t}{t} \leq x \mid X > t\} \leq 1 - \frac{\alpha_1}{\alpha_2} \{1 - \bar{\varepsilon}_{\alpha_1}(x)\} \]

for all \( x \) for \( t \geq t_0 \).

**Remark.** The condition (17) is fulfilled for all distribution functions \( F \in D_r(\bar{\varepsilon}_{\alpha}) \) for any \( \alpha > 0 \).
Proof. It is not difficult to see that (see de Haan 1970, p. 72)

\[
\frac{1-F(t+tx)}{1-F(t)} = \frac{c(t+tx)}{c(t)} \exp \left( -\int_1^{1+x} \frac{c(ts)}{s} \, ds \right),
\]

where

\[
c(t) = \frac{1-F(t)}{\int_0^t (1-F(s))ds}.
\]

The proof of the inequalities is obvious.

Theorem 7. Let \( F \) be a distribution function \( (F(x) < 1 \text{ for all } x) \) such that

\[
(18) \quad \int_0^\infty x^2 dF(x) < \infty.
\]

We define the function \( b(t) \) by

\[
b(t) = -1 + (1-F(t))^2 \frac{1}{(1-F(s))ds} \int_0^\infty (1-F(u))du ds.
\]

Let \( c_1 \) and \( c_2 \) be real numbers such that

\[
c_1 \leq b(t) \leq c_2 \quad \text{for } t \geq t_0.
\]

If \( X \) is a random variable with distribution \( F \), then for \( t \geq t_0 \) and all \( x \)

\[
\frac{1+c_2}{1-c_1} \left\{ \frac{1}{0, c_2^{-1}(1+2c_1)} \left( \frac{X}{a(t)} \right) \right\} \leq P \left\{ X \leq t \right\} \leq \frac{1+c_1}{1-c_2} \left\{ \frac{1}{0, c_1^{-1}(1+2c_2)} \left( \frac{X}{a(t)} \right) \right\}
\]

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where \( a(t) = \left( \int (1 - F(s)) ds \right)^{-1} \int \int (1 - F(u)) du ds. \)

**Remark.** The condition (18) is fulfilled for all distribution functions \( F \in D_r(\Pi) \). Then also \( \lim_{t \to \infty} b(t) = 0. \)

**Proof.** It can be proved (see de Haan 1970 theorem 2.5.2 and the part b) \( \Rightarrow \) c) of the proof) that

\[
\frac{1 - F(t + xa(t))}{1 - F(t)} = \frac{1 + b(t + xa(t))}{1 + b(t)} \exp - \int_0^x \frac{1 + 2b(t + sa(t))}{a(t + sa(t))} ds.
\]

The inequalities now can be proved in the same way as in the proof of theorem 5.
5. **Equivalence of weak convergence and moment convergence.**

In this section we state the conditions for the domains of attractions in an alternative form using certain moments of the distributions.

**Lemma 13.** Suppose $F$ is a distribution function with $F(x) < 1$ for all $x$. Let $\alpha$ and $\xi$ be positive constants and $\xi < \alpha$. The following statements are equivalent.

a) $1 - F$ is regularly varying at infinity with exponent $-\alpha$.

b) The integral $\int_0^\infty x^\xi dF(x)$ converges and

$$\lim_{x \to \infty} \frac{\int_0^x y^\xi dF(y)}{x^\xi (1 - F(x))} = (1 - \xi/\alpha)^{-1}.$$

**Remark.** A result of this lemma is that if the last statement holds for one $\xi < \alpha$, the statement also holds for any other $\xi < \alpha$.

**Proof.** Partial integration of the numerator gives
\[
\frac{\int y^\xi dF(y)}{x^\xi(1-F(x))} = \frac{\int y^{\xi-1}(1-F(y))dy}{x^\xi(1-F(x))} + 1.
\]

The statement of the lemma now follows from Karamata's theorem for regularly varying functions (see e.g. de Haan 1970 theorem 1.2.1 and remark 1.2.1).

**Theorem 8.** Suppose $X$ is a real-valued random variable with distribution function $F$ and $F(x) < 1$ for all real $x$.

a) (i) If $F \in D_\alpha(\Xi)$, i.e. if

\[
\lim_{t \to \infty} \{ \frac{X}{t} \leq x \mid X > t \} = \Xi_\alpha(x-1)
\]

for all $x > 0$, then for all $0 < \xi < \alpha$ the integral $\int_0^\infty y^\xi dF(y)$ converges and

\[
\lim_{t \to \infty} E((\frac{X}{t})^\xi \mid X > t) = \int_0^\infty x^\xi d\Xi_\alpha(x-1) = (1-\xi/\alpha)^{-1}.
\]

(ii) If for some $\xi > 0$ the integral $\int_0^\infty x^\xi dF(y)$ converges and for some $c > 1$

\[
\lim_{t \to \infty} E((\frac{X}{t})^\xi \mid X > t) = c,
\]

then $F \in D_\alpha(\Xi)$ with $\alpha = \xi c(c-1)^{-1}$.

b) We have $F \in D_\alpha(\Xi)$, i.e.
\[
\lim_{t \to \infty} P\left( \frac{X-t}{a(t)} > x \mid X > t \right) = \Pi(x)
\]

for all positive \( x \) with (by corollary 2 to theorem 2)
\[
a(t) = \frac{t}{1-F(t)} = E(X-t \mid X > t)
\]

if and only if \( \int x^2 dF(x) \) converges and \( E((\frac{X-t}{a(t)})^2 \mid X > t) = \int_0^\infty x^2 d\Pi(x) = 2. \)

**Proof.**

a) This part is a simple consequence of theorem 2 and lemma 13.

b) By theorem 2 and the theorem 2.5.1 and 2.5.2 of de Haan (1970) we have \( F \in D_r(\Pi) \) if and only if
\[
\lim_{t \to \infty} \frac{\{1-F(t)\}\{\int \int (1-F(s))dsv\}}{\int \{1-F(s)\}ds} = 1.
\]

By partial integration we obtain
\[
\int \int (1-F(s))dsv = \frac{1}{2} \int (s-t)^2 dF(s)
\]

and
\[
\lim_{t \to \infty} \mathbb{E}(X-t^2 \mid X > t) = \lim_{t \to \infty} \frac{\int_{t}^{\infty} (s-t)^2 dF(s)}{1-F(t)} \cdot \frac{(1-F(t))^2}{\int_{t}^{\infty} (1-F(s)) ds} = 2.
\]

**Corollary.** We thus have \( F \in D_r(\Pi) \) if and only if

\[
\lim_{t \to \infty} \frac{\text{Var}(X-t \mid X > t)}{(E(X-t \mid X > t))^2} = 1.
\]

**Acknowledgements.** The work by the second author has been done partly at Michigan State University and partly at Stanford University. The support by NSF Contract No. GP-23\#80 through the Statistical Laboratory at Michigan State and by NSF Contract No. GP-30711X through the Statistics Department at Stanford is gratefully acknowledged.

**References.**

