ON THE ESTIMATION OF FAULT SLIP IN SPACE AND TIME

BY

MARK VINCENT MATTHEWS

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STANFORD UNIVERSITY
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Abstract

This dissertation develops models and estimation methods to invert crustal deformation measurements for slip at depth on a fault. Using linear elastic dislocation theory to define a forward model relating measurable deformation to deep slip, we propose and investigate physically-motivated principles by which to construct optimal, finite-dimensional representations of source functions for inversion. The principles applied all involve minimization of a norm that is quadratic in slip but directly expressed in terms of either slip and fault-surface traction or fault-surface traction alone. The three norms that we consider are the self-energy, which measures a component of the change in crustal strain energy accompanying slip, the stress magnitude, which measures the squared $L^2$ norm of the fault-surface traction distribution, and the stress variability, which measures the squared $L^2$ norm of the derivatives of fault-surface traction.

In Chapter 1, the three norms are motivated, introduced, and analyzed in the context of a simple, antiplane faulting model. In this setting, we show that each of the norms we define in terms of fault-surface traction has an equivalent form involving some order of derivative of slip. Precisely, the stress magnitude and stress variability functionals are equivalent, respectively, to the squared $L^2$ norms of the first and second derivatives of an antiplane slip distribution. The self-energy is found to be equivalent to a fractional, $\frac{1}{2}$-derivative of slip.

We take advantage of the fact that each of these norms is quadratic in slip by using simple geometric principles that identify functions minimizing quadratic forms subject to
linear constraints. In each instance, these principles lead to equations implicitly defining fundamental components of the source representations we seek. For the self-energy norm the orthogonality principle leads to a *traction-matching* equation, and for the stress variability and stress magnitude it leads to simple, linear differential equations. We use these equations to compute basis functions that represent linear functionals of interest in inverting for fault slip, comparing results among various norms and considering the properties of each norm individually.

In Chapter 2, we apply the representation theory of Chapter 1 to a practical problem: estimating the distribution of slip in the 1906 San Francisco earthquake by inverting coseismic angle change measured in a triangulation network at Point Arena, California. We describe statistical principles that may be used in estimating coefficients of basis functions, once they are identified, and we use these principles to look at the coseismic slip distribution of interest. We compute estimates of this distribution in all three of the norms defined in Chapter 1, and at a range of assumed faulting depths. Using cross-validation to assess goodness-of-fit, we find that the data prefer slip to depths of 15 to 20 km, somewhat in excess of earlier estimates of the faulting depth parameter. This discrepancy may be due, in part, to the fact that the source representations we use have the property that slip is everywhere continuous in space. In contrast to conventional, uniform rectangular dislocation models, our estimation procedure produces slip distributions that grade smoothly to zero slip at the base of the slip zone, possibly allowing better fits to slip that varies with depth.

Chapter 3 extends one of the representation principles of Chapter 1, minimizing the stress-variability norm, from the antiplane faulting model to more general distributions of slip varying in both directions on a planar fault. The approach is somewhat similar to that in the antiplane model, as we use Fourier analysis to illuminate the relationship of stress variability to slip in an infinite space. Again, we find that the stress variability is equivalent to a positive-definite quadratic form in second derivatives of slip, and we specify a differential equation that generates slip distributions on a vertical strike-slip fault that minimize this form subject to a linear constraint. We apply these ideas to analyze the
apparent steady-state rates of slip in the vicinity of the San Andreas fault at Parkfield, California, as determined by measurements on the two-frequency geodimeter in operation there since 1984.

In the final Chapter of this dissertation, we apply the representation methods of Chapter 3 to the problem of estimating a distribution of fault slip at depth as a function of space and time. The space-time estimation method we develop uses Gaussian processes as a priori models for components of frequently-measured deformation signals. In particular, we model local deformation signals as Brownian motion, and we model time-dependent coefficients of orthogonal basis functions spanning the estimable component of deep slip as integrated Brownian motions. This enables us to specify a linear state-space model describing the relationship of the observed data to all of the assumed components of signal and noise. Applying this space-time model to the Parkfield two-color data, we find that there is apparent time-variation in the rate of deep slip on this part of the San Andreas fault. Our results also suggest that some of this apparent rate variation may be associated with seismic activity on or near the San Andreas fault.
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Chapter 1

Source function representations

1.1 Introduction

By measuring deformation of the Earth’s surface in the vicinity of an active fault and relating observed deformation to motion on the fault surface, we may hope to gain valuable insights into fault behavior at seismogenic depths. The monitoring of static deformation in a time period intervening large earthquakes, for instance, may provide a spatial, or spatio-temporal, picture of motion at depth that is useful in estimating rates of stress, strain energy, and moment-deficit accumulation. These estimates may, in turn, lead to inference of earthquake potential or even to earthquake predictions. Likewise, static deformation measurements made over a time interval containing a large earthquake may be used to image coseismic fault slip, providing a complement or, for historic earthquakes, alternative, to dynamic modeling and wave-form inversion.

To construct a picture of motion at depth from measurements made at the surface, we must take a physical model mapping what can’t be seen to what can and, somehow, “undo” it. In other words, we require an inversion algorithm that maps observed effects back to plausible causes. The construction of such an algorithm is the subject of this paper. More accurately stated, our subject is only the first steps in construction of an
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inversion algorithm: defining the model space and obtaining finite-dimensional representations of potential solutions. Subsequent work [37, 33, 34] tells the rest of the story, dealing with statistical questions and data-analytic issues that are of ultimate concern. Presently, we suppose that we have a set of crustal deformation measurements related by an elastic dislocation model to an unknown distribution of slip on a fault plane, and we confine our attention to these questions: What physical principles may be used to pose the inverse problem?; and, having posed the problem, How may we represent its solution in an efficient and informative fashion? Before summarizing our work on these questions, we will elaborate on our motives and review previous endeavors in this area.

1.1.1 Background and motivation

The model we will develop expresses the relationship between a slip distribution, \( s \), and a collection of deformation measurements, \( Y_1, Y_2, \ldots, Y_n \), as

\[
Y_i = \phi_i(s) + \epsilon_i. \tag{1.1.1}
\]

Each \( \phi_i \) is a linear functional that takes a function (field) as input and produces a real number. In linear dislocation models, all commonly measured deformations — displacement, line length change, angle change, tilt, strain, and stress — are expressed as linear integral mappings of slip. The \( \epsilon_i \) in (1.1.1) represent perturbations of the signal that may be due to nontectonic contaminants or to random measurement error, for instance. Our goal will be to take a set of deformation measurements and to use a relationship as modeled in (1.1.1) to construct an estimate of \( s \) or some property of \( s \). Presently, we will limit ourselves to consideration of either coseismic or steady-state slip distributions where only the spatial, not temporal, character of slip is at issue. See [35] for treatment of the space-time inverse problem.

The abstract model in which observable data are the result of random perturbations added to signals that depend linearly on an unobservable function of interest is ubiquitous in geophysics. As thoroughly discussed by Backus and Gilbert [5] and Parker [42], linear
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_inverse problems_ resulting from these models generally are egregiously ill-posed. If admissible solutions are required _a priori_ only to lie in some high dimensional linear space, then to any solution that gives an acceptable fit to the approximate linear conditions imposed by a set of observations may be added any member of a high dimensional hyperplane to produce an equally acceptable solution.

The earliest attempts to invert crustal deformation measurements for fault slip pursued modest aims and intentionally avoided the issue of nonuniqueness. When Savage et al. [45] inverted trilateration measurements from the vicinity of Hollister, CA by least squares to estimate a slip distribution on the San Andreas and Calaveras faults they used only four parameters to describe the set of admissible slip distributions. Thatcher [58] studied fault slip in roughly the same area as Savage _et al._, and he formulated a slightly more ambitious inverse problem, using an eleven parameter family of slip distributions. In each of these instances, slip was described by a collection of laterally segmented, shallow blocks of constant slip rate overlying a very large block representing deep slip. Savage _et al._ partitioned shallow slip over three blocks of effectively infinite lateral extent, while Thatcher segmented the upper 15 km of fault surface into 10 blocks of average lateral dimension 20 km. The number of data constraining slip in each case was greater than the number of degrees of freedom in the family of possible solutions, so least squares estimation problems were well-posed.

There are several shortcomings in estimation approaches that segment fault surfaces into huge blocks of constant slip. Most obvious among these, perhaps, is the coarseness of the result. Slip estimates resulting from highly discretized models are able, at best, to reveal only gross spatial averages of slip; no precise insight into the nature of variation of slip with fault surface position is likely to emerge. In order to overcome this difficulty, several workers in this area have proposed means to allow substantial spatial variability in members of the class of admissible slip distributions. All of these attempts rely primarily on discretization of the fault surface into patches of spatially uniform slip, but they generally start with such a large number of uniform patches that the full set of parameters is undetermined. Various _ad hoc_ auxiliary criteria have been proposed to supplement
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empirical information by enforcing enough "regularity" to produce unique solutions to the problem of estimating a spatially-variable slip distribution. S. Ward and coworkers have, for instance, proposed iterative schemes which trade off the sum of squared misfit to the data against the Euclidean norm of the discretized solution while constraining the solution always to be nonnegative. They have applied these techniques to invert deformation measurements for coseismic slip in the 1983 Borah Peak earthquake [61], the 1960 Chilean earthquake [8], and the 1915 Avezzano, Italy earthquake [62].

In their investigation of slip on the San Andreas fault near Parkfield, CA, Segall and Harris [47, 16, 48] adapted to slip estimation the general technique of formulating regularized, quadratic optimization problems as well-posed modifications of ill-posed linear inverse problems. They inverted trilateration data to estimate slip on a patch of the SAF by discretizing the patch into disjoint rectangular elements of constant slip rate and then setting rates on these elements to satisfy the data and certain "smoothness" conditions. Imposition of a "roughness penalty", in this case proportional to a discretized estimate of the Laplacian, compensated for the low dimensionality of data space vis a vis the model space, and it also enforced a natural belief that slip should not vary too much over small spatial distances.

Heretofore, all endeavors to invert crustal deformation data for fault slip have commenced with discretization of the fault surface into small or large elements over which slip is taken to be uniform. Piecewise constant functions may be able to approximate true slip distributions well in the sense of some measure of average pointwise discrepancy between the actual and estimated distributions, but they are unphysical and inefficient. The performance of a procedure using this kind of discretization depends not only on the quality and quantity of data that go into the inversion but on how the fault surface is partitioned and on the choice of a roughness penalty. A "natural" partition of the fault surface into identical rectangular patches necessitates compromise between accuracy and computational burden. In any particular area where data may have high resolution, it is desirable to have a fine partition with many elements available to fit local features in the slip distribution. But a locally fine uniform partition must be globally fine, so local
accuracy may require a global partition with many basis elements. Increasing the number of basis functions increases both the computational burden — for conventional linear calculations it grows like the cube of the number of basis functions — and indeterminacy in the solution.

Regardless of the choice of a fault surface partition, any slip estimate derived from a piecewise-constant representation is unphysical in certain fundamental respects. The convenience of a finite-dimensional, but unphysical, class of admissible slip distributions comes at the price of complications that persist after the putative slip distribution is in hand. For instance, slip discontinuities on the fault surface are accompanied by implausible stress and energy singularities so stress and energy estimates do not follow from the slip estimate without further ad hoc qualifications or approximations. Stress may be approximated as a piecewise constant function with values determined by stresses calculated at the center points of patches of constant slip, but then stress and slip distributions that must be compatible in principle are not so in practice.

1.1.2 An overview of methods and results

In this paper, we seek to address some shortcomings of conventional approaches to dislocation inverse problems by appropriately defining these problems from a physical perspective. Definition of a well-posed inverse problem requires specification both of the set of possible solutions and an a priori criterion for uniquely identifying the “best” solution among those fitting empirical data equally well. These two elements are essentially one in our work, for we we identify norms measuring physical properties associated with slip and then argue that these norms must be finite and ought to be small. This leads us to define possible solutions as those with finite norm and the best solution as the one with smallest norm.

The physical properties that we use to define norms on slip are the self-energy, measuring a component of potential energy change accompanying slip, the stress magnitude, measuring the change in the $L^2$ norm of the the fault surface traction distribution, and the stress variability, measuring the change in the $L^2$ norm of the derivatives of the fault.
surface traction. Looking in turn at the norms defined by each of these functionals, we ask, "What slip distribution producing the measured value of a single linear property of the deformation field has smallest norm?" This question is related to the observation model, (1.1.1), by viewing the values of linear functionals as the "properties" of which we speak. For instance, measuring the moment (integral) of slip and using self-energy to define the norm, would lead us to ask, "What slip distribution with this moment has least self-energy?" Or if we measure the norm by stress variability and observe the displacement of a single point on the surface, then the question is, "Among all slip distributions moving this point by the required amount, which has the most nearly uniform distribution of fault surface traction?"

Because the three functional we consider as norms are all quadratic in slip, we are able to solve the optimization problems we pose with relative ease, specifying in each instance a linear condition that defines the solution. When the norm is the self-energy, this linear condition instructs us to find the slip distribution with fault surface traction in the slipping zone equal to the function weighting slip to produce the value of the measured linear functional. If we measure the moment, for instance, the weight function is constant and our condition, the traction-matching condition, says that energy is minimized by the slip distribution giving constant stress change over the slipping zone. For the stress magnitude and stress variability functionals, conditions defining the desired minimizing functions are given by linear differential equations.

In Appendix A, we summarize the general approach to the problem we encounter repeatedly: minimize a quadratic form subject to a linear constraint. For the intended applications, we define and solve this problem on infinite-dimensional spaces, but the conceptual essence of the solutions we obtain is hardly more complicated in general than in the very simplest instance. Suppose that we want to estimate a "function" with only two components, \( f = (f_1, f_2) \); in other words, the function is really a vector in two dimensions. Let's say we know only that a quantity, \( \phi \), depending linearly on \( f \), has the value 28, and that we want to what \( f \) produced this value. There are infinitely many \( f \)'s that satisfy the single imposed constraint, all lying on the line shown in Figure 1.1. To identify this line,
Figure 1.1: Norm minimization problem in two dimensions.
note that since $\phi$ is linear in $f$ it must be defined by $\phi(f) = \Phi_1 f_1 + \Phi_2 f_2$ for some fixed $\Phi$. Now, if we measure the squared norm of $f$ by $f_1^2 + f_2^2$ and ask for $f$ with smallest norm subject to the constraint imposed by $\phi$, we find the solution at once by recognizing that $\Phi$ points in a direction orthogonal to the set of possible solutions and that the solution with smallest norm is, therefore, proportional to $\Phi$:

$$\hat{f} = 28 \frac{\Phi}{\|\Phi\|^2}. \quad (1.1.2)$$

Again, see Figure 1.1. The abstraction of this problem that we use repeatedly is this: the function with smallest value of a positive quadratic form subject to a constraint given by a linear functional is proportional to the function representing the linear functional in the inner product associated with the quadratic form.

As a forward mapping taking slip to observable deformation, we use the familiar linear elastic dislocation model, first introduced to this context by Stekete [53, 54]. The essentials of this model are described in Section 2.1 and quickly specialized in Section 2.2 to the simplest useful instance: antiplane slip on a vertical fault in a homogeneous, isotropic half-space. This model, and a set of coseismic deformation measurements from the 1906 San Francisco earthquake to which the antiplane model may apply, are used throughout this paper to illustrate calculations and results. The principles that we invoke and methods we apply are not, for the most part, limited in scope to this simple one-dimensional model. We concentrate on this model simply because it facilitates computations and provides clear illustration of various points that might otherwise be obscured by technicalities. See [33] and [34] for extensions to two-dimensional faulting.

After summarizing forward models and a motivating example in Section 2, we begin looking at norm-minimization by motivating the use of self-energy, defining the relevant energy-minimization problem, and giving its solution in Section 3. We review a computational method for solving the singular integral equation produced by the traction-matching condition in the antiplane case, and we illustrate the results of applying this method with various examples. The desire to find a natural measure of smoothness of slip functions was part of our original motivation for looking at the self-energy functional, and we use the
Section 1.2. Forward models

Fourier transforms of slip and traction in unbounded space to show, at the end of Section 3, that this functional is interpretable as the squared $L^2$ magnitude of the "1/2" derivative of slip and that it fails to enforce smoothness in a precise and important manner. Some technicalities pertinent to the issue of when a bound on the norm of a function bounds the function pointwise are discussed in Appendix B. In Section 4, we define the stress magnitude and stress variability functionals as alternatives to the self-energy that do enforce smoothness. Taking Fourier transforms, again, in the antiplane model in unbounded space, we show that, though these functionals are defined in terms of the distribution of surface traction, they are equivalent to functionals defined in terms of derivatives of slip, and we then use the derivative functionals defined in unbounded space as norms for slip on a half-space, the rationale being that smoothness is a "local" property, a measure of which ought not to be overly sensitive to presence or absence of a boundary. For both functionals depending on traction, we give closed expressions for the norm-minimizing source functions given constraints on displacement or shear strain in the antiplane model. From these expressions, norm-minimizing representations are available for all commonly-measured geodetic functionals. This paper closes with a discussion of the potential and limitations of the methods we describe and some pointers toward further work.

1.2 Forward models and an example

Elastic dislocation theory has been widely applied in the present context since its introduction to geophysics by Stekete [53], [54]. This theory provides a convenient description of the elastic displacement field around a fault as a linear integral mapping of fault motion. As we have mentioned, we will expound mostly on results for a simple, one-dimensional dislocation model, but the principles we invoke are more generally applicable. Hence, we begin with a brief description of the general (linear) dislocation model before specializing to the simple model and motivating data set to be used for illustrative purposes.
1.2.1 The linear elastic dislocation model

Let $\Sigma$ denote the surface of a fault in the Earth’s crust, or more generally, a smooth, two-dimensional dislocation surface embedded in a linearly elastic body, $\mathcal{X}$. Three elastostatic states on $\mathcal{V}$, differing in function, if not in fact, will be distinguished superscripts "0", denoting the "relaxed" (unstressed) state, "r" denoting a "reference" state for displacements, and "f" denoting a "final" state of with displacement field (from the reference state) $\{u(x) : x \in \mathcal{X}\}$. Define slip at a point $\xi \in \Sigma$ by

$$s(\xi) \triangleq u(\xi+) - u(\xi-), \quad (1.2.3)$$

where $\xi+$ and $\xi-$ represent limits as a point approaches $\xi$ along the direction normal to $\Sigma$ at $\xi$ from the "positive" and "negative" sides, respectively. Under Hooke’s law, with stress, $\sigma_{ij}$, and strain, $\epsilon_{pq}$, related by

$$\sigma_{ij}(x) = C_{ijpq}(x)\epsilon_{pq}(x), \quad (1.2.4)$$

the displacement field off $\Sigma$ is linear mapping of the slip field on $\Sigma$:

$$u_k(x; s) = \int s_i(\xi)C_{ijpq}(x)\frac{\partial \Gamma_{kp}(x, \xi)}{\partial x_q}\nu_j(\xi)d\Sigma(\xi) \quad (1.2.5)$$

$$= \int s_i(\xi)G_{ij}^k(x, \xi)\nu_j(\xi)d\Sigma(\xi). \quad (1.2.6)$$

In (1.2.5), known as Volterra's formula, $\Gamma_{kp}(x, \xi)$ is the displacement in the $k$ direction at $x$ due to application of a unit point force in the $p$ direction at $\xi$, and the Green’s tensors in (1.2.6) are

$$G_{ij}^k(x, \xi) \triangleq C_{ijpq}(x)\frac{\partial \Gamma_{kp}(x, \xi)}{\partial x_q}. \quad (1.2.7)$$

See [30, 53, 54, 32, 19] [40] for discussion of dislocation models and specification of the Green’s tensors for particular fault geometries.

The strain field due to slip, $s$, has components

$$\epsilon_{ij}(x; s) \triangleq \frac{1}{2} \left\{ \frac{\partial u_i(x; s)}{\partial x_j} + \frac{\partial u_j(x; s)}{\partial x_i} \right\}, \quad (1.2.8)$$

and the stress field is given as in (1.2.4).
Section 1.2. Forward models

We will make frequent use of the fault surface self-traction field, with components to be denoted by $\tau_i(\xi; s), \xi \in \Sigma$. The surface tractions are are well-defined only when the slip distribution satisfies certain smoothness conditions to be mentioned below, in Section 5. Assuming that these conditions hold, the surface self-traction components are given by

$$\tau_i(\xi; s) \overset{\Delta}{=} \lim_{x \to \xi} \sigma_{ij}(x; s) \nu_j(\xi)$$  \hspace{1cm} (1.2.9)

where, again, $\nu(\xi)$ is the positive unit normal to $\Sigma$ at $\xi$. The change in crustal strain energy accompanying slip may be written in terms of slip and surface traction (see [2])

$$\Delta E(s) \overset{\Delta}{=} E^I - E^r = -\frac{1}{2} \int_{\Sigma} s_i(\xi)\{\tau^r_i(\xi) + \tau_i(\xi; s)\}d\Sigma(\xi).$$  \hspace{1cm} (1.2.10)

In anticipation of arguments to follow, we find it convenient to define the interaction energy between two slip distributions, $s_1, s_2$, as

$$E_{\text{int}}(s_1, s_2) \overset{\Delta}{=} -\frac{1}{2} \int_{\Sigma} s_{1i}(\xi) \cdot \tau_i(\xi; s_2) d\Sigma(\xi).$$  \hspace{1cm} (1.2.11)

By Betti's reciprocity theorem, the bilinear interaction energy form is symmetric, i.e., $E_{\text{int}}(s_1, s_2) = E_{\text{int}}(s_2, s_1)$. The associated quadratic form, giving the so-called self-energy,

$$E_{\text{self}}(s) \overset{\Delta}{=} E_{\text{int}}(s, s),$$  \hspace{1cm} (1.2.12)

measures elastic potential energy created in the quasi-static transformation from an unstrained state to the state defined by $s$, is essentially positive definite, with $E_{\text{self}}(s) > 0$ unless $s$ corresponds to rigid motion. Note that the energy change due to slip, as given in (1.2.10), is appropriately written in the terms just defined as

$$\Delta E(s, s') = 2E_{\text{int}}(s, s') + E_{\text{self}}(s).$$  \hspace{1cm} (1.2.13)

1.2.2 Coseismic slip in the 1906 San Francisco earthquake: The antiplane fault

The great San Francisco earthquake of 1906 may have occasioned the first use of geodetic survey data to describe crustal deformation associated with an earthquake. Several triangulation networks, that had been surveyed in the decades before the earthquake were
resurveyed shortly after. Survey to survey angle changes provide measurements related to coseismic displacements, and they may be used to determine, modulo a linear "null space" (see [49]), the motions of surveyed benchmarks around a fault.

Figure 1.2 shows the locations of benchmarks near the San Andreas fault in the vicinity of Point Arena, CA. Thirty azimuths between pairs of points in this network were surveyed in 1891 and the again in 1907 (see Hayford and Baldwin [18]). Pre-1906 angle measurements were subtracted from post-1906 data to produce the thirty angle-change measurements listed in Table 1.1. If we assume that the primary cause of pre- to post-seismic angle changes is slip on the San Andreas fault that occurred during the earthquake, then we may use the data in Table 1.1 to estimate the spatial distribution of coseismic slip.

To relate observed angle changes to the coseismic slip distribution, we will make use of the fact that the rupture length, $L$, was on the order of 400 km, much larger than the faulting depth, $D$, which was on the order of 10 km. Combining the knowledge that $L >> D$ with the assumption that coseismic slip was purely horizontal "strike slip", parallel to the free surface, we obtain a simple forward model that postulates antiplane slip on a vertical plane in a homogeneous, isotropic, elastic half-space. Precisely, we assume that slip was entirely fault-parallel, that it varied only as a function of depth, and that it was zero below depth $D$. In the coordinate system shown in figure 1.3, with the 1, 2, 3, axes respectively referring to fault-parallel, depth, and fault-normal coordinates, the displacement field is expressed in terms of slip by

$$u_1(x_2, x_3) = \frac{1}{2\pi} \int_{-D}^{0} s(\xi) \left\{ \frac{x_3}{x_3^2 + (x_2 - \xi)^2} + \frac{x_3}{x_3^2 + (x_2 + \xi)^2} \right\} d\xi.$$  

(1.2.14)

For points on the free surface, $x_2 = 0$, where all survey benchmarks are taken to lie, (1.2.14) may be written as

$$u_1(x_3) = \frac{1}{\pi} \int_{-D}^{0} s(\xi) \left\{ \frac{x_3}{x_3^2 + \xi^2} \right\} d\xi.$$  

(1.2.15)

Letting $Y_i$ denote the $i^{th}$ measured angle change and $\delta \theta_i$, the true angle change, our model is

$$Y_i = \delta \theta_i(s) + \epsilon_i,$$  

(1.2.16)
Figure 1.2: Benchmark location in the Point Arena triangulation network.
<table>
<thead>
<tr>
<th>From</th>
<th>Through</th>
<th>To</th>
<th>Observed change</th>
</tr>
</thead>
<tbody>
<tr>
<td>COLD SPRING</td>
<td>FISHER</td>
<td>DUNN</td>
<td>-18.0</td>
</tr>
<tr>
<td>FISHER</td>
<td>DUNN</td>
<td>COLD SPRING</td>
<td>7.9</td>
</tr>
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Table 1.1: Measured changes from 1891 to 1907 of azimuths among benchmarks in the Point Arena network
Section 1.2. Forward models

Figure 1.3: Coordinate system for antiplane slip in a half-space.
where the noise terms, $\epsilon_i$, are assumed to be independent random variables with mean zero and variance $\sigma^2$. With $x^{(j)}$ denoting the coordinates of the $j^{th}$ benchmark, the fault-parallel station displacements due to anitplane slip distribution, $s$, are given by

$$u^{(j)}(s) = \frac{1}{\pi} \int_{-D}^{0} s(\xi) \cdot \frac{x_1^{(j)}}{(x_1^{(j)})^2 + \xi^2} d\xi.$$

Since displacements are small compared to the distances between benchmarks (on the order of 1 part in $10^3$) we may accurately relate angle changes linearly to benchmark displacements. Consider an angle defined by two vectors emanating from benchmark number $j_0$, one to benchmark $j_1$ and the other to benchmark $j_2$, and let

$$v_k = (z_1^{(j_k)} - z_1^{(j_0)}, z_2^{(j_k)} - z_2^{(j_0)}), \quad k = 1, 2.$$

Then, expanding to linear terms the exact relation

$$\delta \theta = \cos^{-1} \left( \frac{<v_1 + u^{(j_1)} - u^{(j_0)}, v_2 + u^{(j_2)} - u^{(j_0)}>}{\|v_1 + u^{(j_1)} - u^{(j_0)}\| \cdot \|v_2 + u^{(j_2)} - u^{(j_0)}\|} \right) \approx \cos^{-1} \left( \frac{<v_1, v_2>}{\|v_1\| \cdot \|v_2\|} \right),$$

and

$$\delta \theta \approx (\|v_1\|^2 \cdot \|v_2\|^2 - <v_1, v_2>^2)^{-\frac{1}{2}} \cdot \left\{ <v_1, v_2> \cdot \left[ \frac{<v_1, u^{(j_1)} - u^{(j_0)}>}{\|v_1\|^2} + \frac{<v_2, u^{(j_2)} - u^{(j_0)}>}{\|v_2\|^2} \right] \right. \left. - \frac{<v_2, u^{(j_2)} - u^{(j_0)}>}{\|v_2\|^2} \right\}. \quad (1.2.17)$$

Applying (1.2.18) to each measurement in the data set gives an accurate linear approximation to the functionals measured in (1.2.16). From these approximations, we define linear functionals, $\phi_i(s), i = 1, \ldots, n$ by

$$\phi_i(s) \triangleq a_{ij} u^{(j)}(s), \quad (1.2.19)$$

with the coefficients, $a_{ij}$, coming from (1.2.18).

In order to make sense of energy changes associated with faulting for which the antiplane approximation is used, we must multiply the energy per unit fault length,
\[ \int s_1 \tau(\cdot; s_1), \] by the actual (finite) length of faulting. The interaction energy between two antiplane slip distributions is thus defined by

\[ E_{\text{int}}(s_1, s_2) = L \int_{\xi=\infty}^{\xi=-\infty} s_1(\xi) \tau(\xi; s_2) d\xi, \quad (1.2.20) \]

where \( L \) is the assumed lateral extent of faulting.

### 1.3 Representations from strain energy minimization

In searching for a variational principle to apply toward solution of a physical inverse problem, we naturally think of energy minimization principles in the first instance. Analytical arguments based on the principle of virtual work [27] are not rigorously applicable to a dissipative system such as that defined by a fault slipping against frictional resistance, but we may nevertheless consider the usefulness of a measure of crustal strain energy as a regularizing functional for slip estimation. Processes of accumulation and release of strain energy due to elastic deformation around faults are of essential interest and are sufficient motivation for using an explicit form of the elastic potential energy functional to generate a variational criterion for the inverse problem. This section provides a look at minimum-norm representations for slip with norms defined by elastic potential energy.

In subsection 3.1, we pose the energy-minimization problem of interest, and we discuss its motivation and significance beyond that which has been stated; in subsection 3.2, we give a condition that defines energy-minimizing source functions implicitly as solutions of certain integral equations; and in subsection 3.3, we show how to solve these integral equations in the antiplane case and we give examples of the resulting functions.

As an aside, it is worth noting, in regard to energy minimization principles in nonparametric functions estimation, a connection between the problem of current interest and a seminal problem in the mathematical theory of interpolation and smoothing. In 1946, the mathematician I. J. Schoenberg [46] asked what shape would be assumed by a "spline", a flexible rod deformed to pass through a specified collection of points in a plane. He answered this question by recognizing that the spline in equilibrium with applied forces would
be in a state of least possible potential energy, so an explicit form for the shape is found by solving a constrained variational problem with minimization of the potential energy functional as the objective. Schoenberg’s original spline functions for one-dimensional interpolation, which are piecewise cubic polynomials, are now widely-used in function estimation problems [60], and using deformation of a thin plate as the two-dimensional analog of a deformed rod, they have been generalized to higher-dimensional surface fitting problems by Duchon [13] and Meinguet [39].

1.3.1 Motivation for energy-minimizing estimates

In order to know how much strain energy is stored in a crustal volume at any particular instant, we must essentially know the entire history of local strain accumulation and release for the volume of interest. Lacking such knowledge, we may choose to focus on the part of the energy functional defined in (1.2.10) that depends only slip about which we presume to have information in the form of deformation measurements. That component of the energy change is given by the self-energy of $s$ as defined in (1.2.12).

Given the value of a linear functional of slip, say, $\phi(s) = y$, we will seek the slip distribution with least self-energy among those satisfying the imposed constraint. If we restrict ourselves to slip distributions supported on $\Sigma_0 \subset \Sigma$, then we may define a linear space of functions with finite energy

$$\mathcal{H}_0(\Sigma_0) \overset{\Delta}{=} \{ s : \Sigma_0 \rightarrow \mathbb{R}^3 \mid E_{\text{self}}(s) < \infty \},$$

(1.3.21)

and pose the problem

$$\text{minimize}_{s \in \mathcal{H}} \ E_{\text{self}}(s) \text{ subject to } \phi(s) = y.$$

(1.3.22)

Before adapting the argument of Appendix A to give the solution to (1.3.22), we may further motivate a desire to solve this problem by considering interpretations of the self-energy of distributions of fault slip during and between large earthquakes.

Looking first at energy-minimizing estimates of coseismic slip, let $s(\xi), \xi \in \Sigma$ represent the static field of fault displacement discontinuity on $\Sigma$ coincident with an earthquake.
Section 1.3. Energy-minimization

Let \( \dot{s}(\xi, t) \) denote the dynamic, coseismic slip velocity field, so that

\[
s(\xi) = \int_0^T \dot{s}(\xi, t) dt,
\]

where \( T \) is the earthquake duration. The total coseismic change in crustal strain energy is given by

\[
\Delta E = -\frac{1}{2} \int_\Sigma s_i(\xi) (\tau_i^0(\xi) + \tau_i^f(\xi)) d\Sigma(\xi)
\]

\[
= -\left\{ \int_\Sigma s_i(\xi) \tau_i^f(\xi) d\Sigma(\xi) + E_{\text{self}}(s) \right\},
\]

(1.3.23)

where \( \tau^0 \) and \( \tau^f \) are the pre- and post-seismic fault surface traction fields. Relation (1.3.23) is depicted in the stress–slip diagram of Figure 1.4, where the area of polygon ABCO represents total strain energy change, the area of triangle ABD is the self-energy, and the interaction energy between slip and the final static traction is the area of rectangle BCOD. The static potential energy change in (1.3.23) is equal to minus the coseismic kinetic energy release, \( W_{\text{tot}} \), given by

\[
W_{\text{tot}} = W_{\text{tric}} + E_{\text{rad}}.
\]

(1.3.24)

Here,

\[
W_{\text{tric}} \triangleq -\int_0^T \int_\Sigma \dot{s}_i(\xi, t) f_i(\xi, t) d\Sigma(\xi) dt,
\]

(1.3.25)

is the frictional work on the fault surface, and \( E_{\text{rad}} \) is the radiated seismic energy; \( f_i(\xi, t) \) are components of the friction stress field during dynamic sliding. Kinetic energy release is represented in Figure 1.5, where the curve from A to B signifies the dynamic stress-displacement path, the area under the curve gives the \( W_{\text{tric}} \), and the difference between the total energy change and \( W_{\text{tric}} \) is \( E_{\text{rad}} \). Setting \( W_{\text{tot}} = -\Delta E \), we see that

\[
W_{\text{tric}} + E_{\text{rad}} = -(2E_{\text{int}} + E_{\text{self}}),
\]

which implies that

\[
E_{\text{self}} - E_{\text{rad}} = W_{\text{tric}} + 2E_{\text{int}}
\]

\[
= \int_\Sigma \left\{ \int_0^T \dot{s}_i(\xi, t) f_i(\xi, t) dt - s_i(\xi) \tau_i^f(\xi) \right\} d\Sigma(\xi).
\]

(1.3.26)
Figure 1.4: Schematic depiction of coseismic change in crustal strain energy. Total energy change is area in ABCO; self-energy is area under ABD; interaction energy is area in BCOD.

For some value, \( \bar{f}_i(\xi) \), the "mean" friction stress, we have

\[
\int_0^T \dot{s}_i(\xi,t) f_i(\xi,t) dt = \bar{f}_i(\xi) \int_0^T \dot{s}_i(\xi,t) dt = \bar{f}_i(\xi)s_i(\xi),
\]

so (1.3.26) becomes

\[
E_{\text{rad}} - E_{\text{self}} = \int_{\Sigma} s_i(\xi)[\bar{f}_i(\xi) + \tau^I_i(\xi)]d\Sigma(\xi).
\]  (1.3.28)

If the mean frictional traction during sliding, as defined by (1.3.27), is close to minus the final static fault surface traction, then (1.3.28) shows that \( E_{\text{rad}} = E_{\text{self}} \). According to Orowan [41] and Kanamori and Anderson [21], the sliding friction is related to the
Figure 1.5: Schematic depiction of coseismic kinetic energy release. Total energy release is area in ABCO; work against frictional resistance energy is area under AB; difference is radiated seismic energy.
Section 1.3. Energy-minimization

postseismic traction in such a way that the approximation $E_{\text{rad}} \approx E_{\text{self}}$ holds, so, using static deformation data to bound the self-energy of a coseismic slip distribution has the effect of approximately bounding the radiated seismic energy. This energy bound gives a physical bound on the size of an earthquake and translates directly to a magnitude bound, for instance, by the Gutenberg-Richter relation.

Now we consider the energetics of slip between large earthquakes. For interseismic slip on faults in areas where tectonic forces are active, we want not to minimize the energy of slip, but of slip deficient, the difference between rigid plate motion and actual slip. Suppose we have a deformation measurement, $\phi(s) = y$, where $s$ is the spatial distribution of fault slip over a fixed time interval in which there are no large earthquakes. Let $p$ stand for the slip distribution on $\Sigma$ corresponding to rigid plate motion at the expected rate, and let $d = p - s$ be the slip deficit function. Then we want to modify (1.3.22) and to minimize $E_{\text{self}}(d)$ subject to $\phi(p - d) = y$.

From (1.2.10), the total change in strain energy accompanying interseismic slip, $s = p - d$, is given by

\[
\Delta E(s) = \Delta E(-d) = -\frac{1}{2} \int -d_i(\xi)(2\tau_i^*(\xi) + \Delta\tau_i(\xi; -d))d\Sigma
= \int d_i(\xi)\tau_i^*(\xi)d\Sigma - \frac{1}{2} \int d_i(\xi)\Delta\tau_i(\xi; d)d\Sigma
= -2E_{\text{int}}(d, s^*) + E_{\text{self}}(d)
\]

Unless the reference state, which we think of as the postseismic state of the last large earthquake, was produced by substantial dynamic overshoot, we expect the reference traction field and the interseismic slip deficit field to point in the same direction. Hence, we expect that the interaction energy term in (1.3.29) will be negative and that the change in crustal strain energy will be given by subtracting a negative quantity from the self-energy of the slip deficit. Thus, by minimizing the self-energy of the slip deficit, we effectively place a lower bound on the crustal strain energy accumulated in the interseismic period. Such a lower bound may be useful in bounding the seismic potential either in the possible size of an earthquake (the “slip predictable” model) at a given time or the possible
time of an earthquake of given size (the “time-predictable” model).

1.3.2 The traction-matching condition

Problem (1.3.22) represents a particular instance of the generic norm-minimization problem, (A.1.2), discussed in Appendix A. The space $\mathcal{H}_0(\Sigma_0)$ is a Hilbert space with inner product given by the interaction energy

$$< s_1, s_2 > \overset{\Delta}{=} E_{int}(s_1, s_2),$$

and norm given by the square root of the self energy

$$||s|| \overset{\Delta}{=} E_{self}^{\frac{1}{2}}(s).$$

With this perspective, we know that we can solve (1.3.22) if and only if $\phi$ is bounded on $\mathcal{H}_0(\Sigma_0)$, in which case the solution is given by

$$s^*(\xi) = \frac{y}{\phi(\Phi^E)} \Phi^E(\xi).$$

In (1.3.32), the function $\Phi^E \in \mathcal{H}_0(\Sigma_0)$ is the energy representer of the $\phi$, and is defined as the unique solution to the energy representation equation

$$< \Phi^E, s >_E = \phi(s) \quad \forall s \in \mathcal{H}_0(\Sigma_0).$$

In geophysical applications, most functionals of interest are expressed by variants of Volterra’s formula and, so, are given by linear integral operations on slip. Consider a generic functional of this type:

$$\phi(s) = \int_{\Sigma} s_i(\xi) w_i(\xi) d\Sigma(\xi)$$

where $w_i$ are the components of the weighting function. Letting $W^E$ denote the energy representer of the functional defined by the weighting function, $w$ in (1.3.34), we may put the inner product form, (1.3.30), on the left hand side and (1.3.34) on the righthand side of the representation equation, (1.3.33), thereby expressing the required condition as

$$-\frac{1}{2} \int s_i(\xi) \Delta \tau_i(\xi; W^E) d\Sigma(\xi) = \int s_i(\xi) w_i(\xi) d\Sigma(\xi) \quad \forall s \in \mathcal{H}_0(\Sigma_0).$$
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Rewriting this as

\[ \int s_i(\xi)[w_i(\xi) + \frac{1}{2} \Delta \tau_i(\xi; W^b)]d\Sigma(\xi) = 0 \quad \forall s \in H_0(\Sigma_0), \]  

(1.3.36)

we have the weak condition that defines \( W^b \). By the so-called "fundamental theorem of the calculus of variations" [12], it may be shown that the weak condition imposed by (1.3.36) is equivalent to the strong condition

\[ w_i(\xi) + \frac{1}{2} \Delta \tau_i(\xi; W^b) = 0 \quad \forall \xi \in \Sigma_0. \]  

(1.3.37)

We may re-express (1.3.37) in a more transparent form via traction matching. Given any vector field, \( f \), on \( \Sigma \) that defines an admissible distribution of fault surface traction, let \( F \) be the traction-matching slip distribution related to \( f \) by the mixed boundary conditions that require

\[
\begin{align*}
\Delta \tau(\xi; F) &= f(\xi) \quad \xi \in \Sigma_0, \\
F(\xi) &= 0 \quad \xi \notin \Sigma_0.
\end{align*}
\]  

(1.3.38)

Let \( W \) be the slip distribution with fault surface traction matching the weight function \( w \) in (1.3.34). Then the energy representer, \( W^b \), is given by

\[ W^b = -\frac{1}{2} W. \]  

(1.3.39)

The condition given in (1.3.39) that says that the energy representer of the functional in (1.3.34) is proportional to the traction-matching slip distribution, defined by (1.3.38) will called, simply, the traction-matching condition (TMC).

An illustration of the what may be learned from the TMC is available when the functional in (1.3.34) is the moment:

\[ M(s) \overset{\Delta}{=} \int_{\Sigma_0} s(\xi)d\Sigma(\xi) = m. \]  

(1.3.40)

Viewing this constraint as a weighted integral of slip, we see that the weight function is constant on the slipping patch \( w(\xi) = 1, \quad \xi \in \Sigma_0 \). By the traction-matching condition, the energy-minimizing slip distribution subject to a moment constraint must have constant
traction on the slipping part of the fault. In the solution to this moment constraint problem, we may see the essential simplicity of the reasoning underlying our energy minimization results. Let \( s^*(\xi) \) be the slip distribution with constant traction satisfying (1.3.40). Then \textit{any} slip distribution satisfying (1.3.40) may be written as \( s(\xi) = s^*(\xi) + \bar{s}(\xi) \) with
\[
M(\bar{s}) = 0.
\]
The self-energy is then
\[
E_{\text{self}}(s) = \int_{\Sigma_0} s(\xi) \Delta \tau(\xi; s) d\Sigma
= \int_{\Sigma_0} s^*(\xi) \Delta \tau(\xi; s^*) d\Sigma + \int_{\Sigma_0} s^*(\xi) \Delta \tau(\xi; \bar{s}) d\Sigma
+ \int_{\Sigma_0} \bar{s}(\xi) \Delta \tau(\xi; s^*) d\Sigma
+ \int_{\Sigma_0} \bar{s}(\xi) \Delta \tau(\xi; \bar{s}) d\Sigma
= \int_{\Sigma_0} s^*(\xi) \Delta \tau(\xi; s^*) d\Sigma + 2 \int_{\Sigma_0} \bar{s}(\xi) \Delta \tau(\xi; s^*) d\Sigma
+ \int_{\Sigma_0} s^*(\xi) \Delta \tau(\xi; \bar{s}) d\Sigma
\]
\[(1.3.41)\]
Since \( s^* \) has constant self-traction, the middle term in (1.3.41) is proportional to \( M(\bar{s}) \), which must vanish. Hence,
\[
E_{\text{self}}(s) = E_{\text{self}}(s^*) + E_{\text{self}}(\bar{s})
\geq E_{\text{self}}(s^*),
\]
the last inequality due to positive definiteness of self-energy. From (1.3.42) we may conclude that \( \bar{s} \) does indeed minimize the self-energy.

Note that the above argument yields a rather intuitive result in the familiar context of fracture mechanics: the elliptical, "Griffith", crack produces least elastic potential energy among all cracks with the same moment. In other words, if the \( x \)-axis represent a one-dimensional slip surface in an infinite, homogeneous, two-dimensional elastic body and slip is limited to the interval \(-a \leq x \leq a\), then the energy-minimizing slip distribution must have constant shear traction on \([-a, a]\) and, hence, is given by
\[
\bar{s}(x) = \begin{cases} 
\frac{c\sqrt{a^2 - x^2}}{2}, & |x| \leq a \\
0, & |x| > a 
\end{cases}
\]
An intuitive expalantion of traction-matching as a means to energy minimization in dislocation models follows from consideration of a point displacement functional. In this
case, Volterra’s formula, (1.2.5), reveals that the weighting functions acting on slip to produce the value of a displacement functional are given by the surface traction distribution equivalent to that due to a point source located at the point where the observation is made. Hence, when we invoke the TMC to find the energy-minimizing slip distribution with specified displacement at a point, we get a solution that is equivalent, in the sense of fault surface tractions, to putting a pont force at the observation point.

1.3.3 Antiplane traction-matching by inversion of the finite Hilbert transform

Solution of the traction-matching equation often may be difficult, essentially requiring inversion of a hypersingular integral operator. In the antiplane model, however, Tricomi’s [59] formula for inverting the finite Hilbert transform may be used. Consider an infinite homogeneous body with differentiable antiplane slip confined to the interval [−1, 1]. The 1, 3 shear stress is given by

\[
\sigma_{13}(x_2, x_3) = \frac{\mu}{2} \frac{\partial u_1(x_2, x_3)}{\partial x_3} = \frac{\mu}{2\pi} \frac{\partial}{\partial x_3} \int_{-1}^{1} s(\xi) \frac{x_3}{(x_2 - \xi)^2 + x_3^2} d\xi.
\]

For points off the fault, we may differentiate under the integral sign, thereby writing

\[
\sigma_{13}(x_2, x_3) = \frac{\mu}{2\pi} \int_{-1}^{1} s(\xi) \frac{(x_2 - \xi)^2 - x_3^2}{[(x_2 - \xi)^2 + x_3^2]^2} d\xi.
\]

Stress at points on the fault surface is obtained formally by taking limits as \(x_1\) goes to zero in (1.3.43). By careful exchange of order of integration and differentiation, we may use a Cauchy principal-value integral to express the limiting stress distribution on the dislocation surface essentially as the finite Hilbert transform of the derivative of slip [63]:

\[
\Delta \tau(x_2; s) \triangleq \sigma_{13}(x_2, 0) = \frac{\mu}{2\pi} \int_{-1}^{1} s'(\xi) \frac{d\xi}{x_2 - \xi}.
\]
Section 1.3. Energy-minimization

Substituting the stress expression in (1.3.45) for the left hand side of the traction matching equation gives

\[
\frac{\mu}{2\pi} \int_{-1}^{1} \frac{W'(\xi)}{x_2 - \xi} d\xi = w(\xi) \quad \forall \xi \in (-1, 1),
\]
(1.3.46)

to be solved for the representer \( W^* \). By Tricomi's inversions formulas, we may deduce that, up to constants of proportionality,

\[
W'(x) = \frac{1}{\pi} \frac{1}{\sqrt{1 - x^2}} \int_{-1}^{1} \frac{w(y)\sqrt{1 - y^2}}{x - y} dy,
\]

whence we get the desired solution by integrating:

\[
W(\xi) = \int_{-1}^{\xi} W^*(x) dx.
\]
(1.3.47)

Whenever the weight function on the right hand side of (1.3.46) is symmetric about 0, as it will be, for instance, when the observed functional is displacement or strain in the surface \( x_2 = 0 \), then we can find the representer for that functional on a half space domain by taking the function given by (1.3.47) on \( -1 \leq \xi \leq 0 \).

1.3.4 Examples and properties of antiplane energy representers

We have implemented the numerical procedure outlined in the previous section to calculate minimum energy representers for certain functionals of interest in the antiplane model. As has been noted, the assumed source depth is the characteristic distance that defines scale in the antiplane model, so distances in this section are given in units of source depth (sd). Figures 1.6 and 1.7 show representers for displacement and shear strain functionals for a point at fixed normal distance of 0.1 sd and at depths of 0, 0.2, 0.5, 2, and 10 sd. The qualitative behavior here of these functions is as we expect. For observation points with fault surface projections inside the slipping zone, the representers are peaked around the projected point. For points outside the slipping zone, there is a warping of the representer that lessens as the measurement point becomes more distant and, the representers converge to the moment representer as the measurement point recedes from the slipping zone. Figures 1.8 and 1.9 show the representers for the displacement and shear strain functionals, respectively, for a point on the free surface of a half space at
Figure 1.6: Antiplane energy representers for displacement functional at a point of varying fault-normal position.
Figure 1.7: Antiplane energy representers for shear strain functional at a point of varying fault-normal position.
fault normal distance 0.01, 0.1, 1, and 10 sd. Again we see qualitatively that when an observation is made close to the fault, the representer is quite sharply peaked in a neighborhood of the projection of the observation point onto the fault plane. The peakedness of the representer for a point close to the fault is more pronounced when the observation functional is strain than when it's displacement, but in either case, the peak smooths out as the observation point moves away from the fault surface, and in the limit as the point gets infinitely far from the fault the representers for displacement and strain both converge to the elliptical shape that gives the representer of the moment functional varying position on the free surface of a half-space. The energy-minimizing representers for two angle changes measured in the Point Arena network are shown in
Figure 1.9: Antiplane energy representers for shear strain functional at a point of varying position on the free surface of a half-space.
Figure 1.10. The two functions are rather like one another in shape, though the representer for the SHOEMAKER-SPUR-LANE measurement puts comparatively more slip near the surface. Since the points defining this azimuth are slightly close to the fault, we expect more peakedness in its representer. Figure 1.11 shows four orthonormal basis functions derived from the 30 representers as described in [37]. The slip estimate obtained by taking the appropriate linear combination of these four functions is shown in Figure 1.12. The estimate has the basic shape of the representer in Figure 1.8 corresponding to a point 0.1 source depths from the fault. This is to be expected given the nature of the observations, which all depend on displacements at points roughly between one and ten kilometers from a fault with assumed depth of 15 km.

1.3.5 Roughness of minimum-energy solutions

The character of the self-energy functional and of the representers it generates are clarified if we look at the self-energy in wave number space. The Fourier transform of the antiplane fault surface traction is given by

\[
\tau(\xi; s) = \mu \lim_{\epsilon \to 0} \left[ \frac{\partial}{\partial x_3} u_1(x_2, x_3) \right]_{x_3 = \epsilon} \\
= \mu \lim_{\epsilon \to 0} \left[ \frac{\partial}{\partial x_3} \int_{-\infty}^{\infty} \hat{s}(\omega) e^{-\xi (x_1 |\omega| + 2\pi i x_3)} d\omega \right]_{x_3 = \epsilon} \\
= \mu \lim_{\epsilon \to 0} \left[ \int_{-\infty}^{\infty} \hat{s}(\omega) e^{2\pi i x_1 \xi} e^{-x_1 |\omega|} d\omega \right]_{x_1 = \epsilon} \\
= \mu \int_{-\infty}^{\infty} (-|\omega| \hat{s}(\omega)) e^{i \omega \xi} d\omega.
\]  

(1.3.48)

Equation (1.3.48) gives the fault surface traction as the inverse Fourier transform of \((-|\omega| \hat{s}(\omega))\). By the uniqueness theorem for Fourier transforms, we may then conclude that

\[
\hat{\tau}(\omega; s) = -\mu |\omega| \hat{s}(\omega).
\]  

(1.3.49)
Figure 1.10: Self energy representers for the angle changes measured from Cold Spring to Fisher to Clark and from Shoemaker to Spur to Lane in the Point Arena triangulation network.
Figure 1.11: Orthonormalized minimum energy basis functions from Point Arena net.
Figure 1.12: Minimum-energy estimate of coseismic slip in the 1906 San Francisco earthquake from inversion of Point Arena data.
Using (1.3.49), we may express the self-energy by

\[ E_{\text{self}}(s) = -\frac{1}{2} \int \tilde{s}(\omega) \overline{\tilde{r}(\omega; s)} d\omega \]
\[ = -\frac{1}{2} \int \tilde{s}(\omega) \cdot -\mu |\omega| \tilde{s}(\omega) d\omega \]
\[ = \frac{\mu}{2} \int |\omega| |\tilde{s}(\omega)|^2 d\omega \]  \hspace{1cm} (1.3.50)

For \( k \geq 0 \), define the \( k^{\text{th}} \)-derivative norm of a function by

\[ \|f\|_k \Delta \left\{ \int |\omega|^{2k} |\hat{f}(\omega)|^2 d\omega \right\}^{\frac{1}{2}}, \] \hspace{1cm} (1.3.51)

the name justified by the fact that for integer \( k \), \( \|f\|_k \) is the \( L_2 \) norm of \( f^{(k)} \). With this definition, we may compare (1.3.50) and (1.3.51) to see that

\[ E_{\text{self}}(s) = \frac{\mu}{2} \|s\|_\frac{3}{2}, \]

\( i.e., \) the anitplane self energy is proportional to the squared \( \frac{1}{2} \)-derivative norm of slip.

The interpretation of the self-energy as a norm involving a fractional derivative of slip is corroborated by certain properties of the representers. For instance, consider again the moment functional for a function on the interval \([-1, 1] \):

\[ M(f) = \int_{-1}^{1} f(x) dx. \]

Let \( \mathcal{H}^k[-1, 1] \) be the linear space of functions on \([-1, 1]\) with square integrable \( k^{\text{th}} \) derivative and with \( f^{(j)}, j = 0, 1, \ldots, k - 1 \) vanishing at the endpoints of the interval. Then it may be shown that the representer, \( M_k \), of the functional, \( M \), on the space \( \mathcal{H}^k[-1, 1] \) with norm \( \|f\|_k \) is proportional to \( (1 - x^2)^k \). \( \mathcal{H}^k \) representers for the moment functional at \( k = 1, 2, \) and \( 3 \) are shown in Figure 1.13, along with the self-energy representer of the moment functional, which we have already seen to be proportional to \( (1 - x^2)^{\frac{3}{2}} \). When we imbed the self-energy representer into the sequence of moment representers in spaces with norms determined by derivatives of integer order, we again see that the self-energy behaves like a half derivative. The fact that the self-energy is only a half derivative is quite significant as it has serious implications about the smoothness properties of functions with
Figure 1.13: Moment functional representers in $\mathcal{H}^1$, $\mathcal{H}^2$, and $\mathcal{H}^3$, along with $\mathcal{H}^{1/2}$, the squared normed of which is identified with the self-energy.
finite self-energy. We now turn to look at these properties and to consider issues pertinent to the characterization, description, and exploitation of smoothness in slip distributions on faults.

Using Fourier arguments, it may be shown [29] that Sobolev conditions for smoothing of norms defined by pseudo-differential operators of fractional order are essentially the same as for operators of integer order, i.e., twice the order of the differential operator in the norm must be strictly larger than the dimension of the domain for the smoothing property to obtain. On one-dimensional domain, such as we have in the antiplane model, the Sobolev condition implies that we need on bound on a norm of order strictly greater than $\frac{1}{2}$ for smoothness. We saw in the previous section that the self-energy norm is precisely of order $\frac{1}{2}$, so we may draw the significant conclusion that elastic strain energy is not a smoothing norm in the antiplane model. We can exhibit a sequence of antiplane slip distributions for which the ratio of slip at a point to strain energy is unbounded. Consider slip distributions, $s_n$, with Fourier transforms

$$
\hat{s}_n(\omega) = \begin{cases} 
(\log(n))^{-7/4} \frac{\log(|\omega|)}{|\omega|}, & 1 \leq |\omega| \leq n; \\
0, & \text{otherwise.}
\end{cases} \quad (1.3.52)
$$

As $n$ increases, these slip distributions include "bumps" of increasing wave number. The reader may verify that the superposition of these bumps at zero causes $s_n(0)$ to go to infinity like $(\log(n))^{1/4}$ while the damping of the side lobes as $n$ increases leads to decreasing strain energy. In fact, the strain energy goes to zero like $(\log(n))^{-1/2}$, so the energy norm, obtained by taking the square root of the self-energy, behaves like rate $(\log(n))^{-1/4}$. The function $s_8(\xi)$ is shown in Figure 1.14.

### 1.4 Representations from smoothing

As we show in [33], the conclusion that the self-energy of a slip distribution is quadratic in a fractional derivative of order $1/2$ applies more generally than to just the antiplane case. According to Sobolev's criterion, we need to require that admissible slip distributions be smoother than are some with finite self-energy, and we may seek to formulate such a
Figure 1.14: A function with high slip at $\xi = 0$ and low self-energy.
requirement by defining linear spaces of functions with finite norms that depend on higher
derivatives. In so doing, we wish still to draw motivation from physical considerations of
underlying mechanical processes, and we find such motivation by looking at properties of
the fault surface traction field resulting from a given distribution of slip.

1.4.1 The stress magnitude functional

From the fact that point evaluation is not continuous in the self-energy norm, we may
gather that the elastic strain energy functional does not, for present purposes, pay enough
attention to local properties of the stress distribution. Certainly, stress singularities are
permitted in cases where the energy is finite. Not only are they permitted, in fact, they
are of essential import in the development of models for fracture and crack propagation
(Broeck [10]). But force singularities are physically impossible. A shortcoming of the self-
energy norm is, perhaps, that it permits slip distributions which, in reality, would lead
to significant inelastic deformation, the work of which is unaccounted for in a measure of
elastic energy change.

We address the problem of limiting force concentration on the fault surface, by defining
a quadratic functional giving the integrated, squared magnitude of fault surface traction.
Let $\tau(\xi; s), \xi \in \Sigma$ denote a fault surface traction field as given in (2.x), and define the stress
magnitude functional as the sum of squared $L^2(\Sigma)$ norms of the traction components:

$$\text{SM}(s) \overset{\Delta}{=} \int_{\Sigma} \tau_i(\xi; s) \tau_i(\xi; s) d\xi.$$  \hspace{1cm} (1.4.53)

Proceeding as we did with strain energy norms in Section 3, we may associate SM with a
bilinear stress magnitude inner product, $<\cdot, \cdot>_{\text{SM}}$, given by

$$<s_1, s_2>_{\text{SM}} \overset{\Delta}{=} \int_{\Sigma} \tau_i(\xi; s_1) \tau_i(\xi; s_2) d\xi,$$  \hspace{1cm} (1.4.54)

and then define a Hilbert space, $\mathcal{H}_M(\Sigma)$, of slip distributions on $\Sigma$ with finite stress
magnitude. If we observe a linear functional, $\phi$, and we wish to minimize $\text{SM}(s + s')$
subject to $\phi(s) = y$, where, again, $s'$ is the reference slip distribution, then we may mimic
the argument of Appendix B and write the solution when \( \phi \) is bounded on \( \mathcal{H}_M(\Sigma) \) as
\[
\phi(x) = \left\{ \frac{y + \phi(s^r)}{\phi(\Phi^M)} \right\} \Phi^M(s^r) - s^r(x)
\] (1.4.55)

In (1.4.55), \( \Phi^M \) is the stress magnitude representer of \( \phi \), defined as the solution to the representation equation
\[
< s, \Phi^M >_{\text{sm}} = \phi(s) \quad \forall s \in \mathcal{H}_M.
\]

1.4.2 SM representers for the antiplane fault

In the antiplane model, the multiplier in the stress magnitude functional is of asymptotic order 1, and on a one-dimensional domain this is enough for smoothing. Since the functional has the desired smoothing property in the antiplane case, we will pursue its use a bit further. The antiplane stress magnitude inner product is given by
\[
< s_1, s_2 >_{\text{sm}} = \int \tau(x; s_1) \tau(x; s_2) dx
= \mu^2 \int \omega^2 \bar{s}_1(\omega) \bar{s}_2(\omega) d\omega
= \frac{\mu^2}{4\pi^2} \int 2\pi i\omega \bar{s}_1(\omega) \cdot 2\pi i\omega \bar{s}_2(\omega)
= \frac{\mu^2}{4\pi^2} \int \bar{s}_1'(\xi) s_2'(\xi) dx,
\] (1.4.56)

so the stress magnitude inner product is the \( L^2 \) inner product of the first derivatives of slip, and we may equate \( \mathcal{H}_M \) with \( \mathcal{H}_1 \), the Hilbert space of functions with square integrable derivatives defined in Appendix B. Having recognized this equivalence, we may use the \( \mathcal{H}_1 \) reproducing kernel (Appendix B) to solve the representation equation with relative ease for most functionals of interest. Confining attention to observations made on the free surface of a half-space with slip restricted to the interval \((-D, 0)\), let \( U_{\text{sm}}(\cdot; y) \) and \( V_{\text{sm}}(\cdot; y) \) denote the SM representers the displacement and 1,3 shear stress functionals measured at fault normal coordinate \( y \). Letting \( R_1(x, y) = x \wedge y \), be the reproducing kernel given in Appendix B, we have, up to multiplicative constants,
\[
U_{\text{sm}}(\xi; y) = \int_{-D}^{0} R_1 \left( \frac{\xi + D, \eta + D}{D} \right) \cdot \frac{y}{\eta^2 + y^2} d\eta
\]
Section 1.4. Smoothing

\[
= \frac{y}{2} \log \left( \frac{y^2 + \xi^2}{y^2 + D^2} \right) \\
- \left\{ D \tan^{-1} \left( \frac{-D}{y} \right) + \xi \tan^{-1} \left( \frac{\xi}{y} \right) \right\}
\]  

(1.4.57)

and

\[
V_{SM}(\xi; y) = \int_{-D}^{0} R_1 \left( \frac{\xi + D}{D}, \frac{\eta}{D} \right) \frac{\eta^2 - y^2}{(\eta^2 + y^2)^2} d\eta \\
= \log \left( \frac{y^2 + \xi^2}{y^2 + D^2} \right) \\
= \frac{\partial U_{SM}(\xi; y)}{\partial y}
\]  

(1.4.58)

Figures 1.15 and 1.16 show SM representers for displacement and strain functionals measured at points on the free surface at distances 0.01, 0.1, 1, and 10 source depths from the fault. There is some qualitative difference between the behavior of these functions and that of the minimum-energy representers of Figures refig:SEDispRepsHs and 1.9. Most notably, for displacements measured at points close to the fault, functions minimizing stress magnitude are linear, not highly peaked as they are for energy minimization. This is a consequence of boundedness of point-evaluation in the SM norm which implies that the representers of a displacement measurement for point approaching the fault do not generate a delta sequence. Representers for strain functionals, on the other hand, do become more peaked as the measurement point approaches the fault. Going in the other direction, as a point moves away from the fault, the representers of both displacement and strain functionals converge to a parabolic shape.

Figure 1.17 show representers for two angle changes in the Point Arena network. Both look very similar to the displacement representor in Figure 1.15 for the point 0.1 source depths from the fault. Figure 1.18 show four orthonormalized basis functions used to construct the minimum stress magnitude estimate of coseismic slip in the 1906 earthquake shown in Figure 1.19. This estimate again has the shape of the representor for displacement at a moderately distant point: linear growth at depth, leveling off close to the surface.
Figure 1.15: Antiplane stress magnitude representers for the displacement functional measured various points on the free-surface of a half-space.
Figure 1.16: Antiplane stress magnitude representers for the shear strain functional measured various points on the free-surface of a half-space.
Figure 1.17: Stress magnitude representers for the angle changes measured from Cold Spring to Fisher to Clark and from Shoemaker to Spur to Lane in the Point Arena triangulation network.
Section 1.4. Smoothing

Minimum SM orthobasis functions, $D = 15$

Figure 1.18: Orthonormalized SM basis functions from Point Arena net.
Section 1.4. Smoothing

Minimum SM slip estimate, $D = 15$

Figure 1.19: Minimum SM estimate of coseismic slip in the 1906 San Francisco earthquake from inversion of Point Arena data.
1.4.3 The stress variability functional

Having found that limiting neither the elastic strain energy nor the magnitude of fault surface traction dictates sufficient regularity to enable general formulation of well-posed minimization problems with point-value constraints on slip, we now move another step up the smoothing ladder and consider the effects of limiting the variability of the fault surface traction distribution. Physical motivation for seeking slip estimates that distribute stress smoothly on the fault plane comes from the fact that tectonic forces acting at a distance and accommodated on the fault plane by aseismic slip or by large earthquakes will tend to produce smooth stress fields. Andrews (1980) has modeled fault surface stress spectrally as the sum of a coherent distribution in wave number space, produced by tectonic loading, and a self-similar “1/f” component that results from and leads to a fractal distribution of small earthquakes. We may, and do, choose to aim for slip estimates that are grossly consistent with a fault surface traction field that is viewed, to first order, as being uniform.

To measure deviation from uniformity in a surface traction distribution, we define the stress variability (SV) functional by

$$\text{SV}(s) \triangleq \int \frac{\partial \tau_i(\xi; s)}{\partial \xi_j} \cdot \frac{\partial \tau_i(\xi; s)}{\partial \xi_j} d\xi.$$  \hspace{1cm} (1.4.59)

Examination of the form of (1.4.59) in relation to the Fourier transforms given in Appendix B reveals that the SV norm will, indeed, be a smoothing norm on a two-dimensional fault, for the multiplier in its Fourier representation will be of asymptotic order two.

Let $\mathcal{H}_2(\Sigma)$ denote the space of slip fields on $\Sigma$ with finite stress variability, and define the SV inner product on $\mathcal{H}_2$ by

$$< s_1, s_2 >_{\text{SV}} \triangleq \int \frac{\partial \tau_i(\xi; s_1)}{\partial \xi_j} \cdot \frac{\partial \tau_i(\xi; s_2)}{\partial \xi_j} d\xi.$$  \hspace{1cm} (1.4.60)

Supposing that we have measured a functional of slip, $\phi(s) = y$, we may proceed as with the SE and SM and pose a norm-minimization problem like (A.1.2) with $\mathcal{H}_V(\Sigma)$ as the space of admissible solutions and with norm given by the square root of the stress variability defined in (1.4.59). According to the principles of Appendix B, we may solve this optimization problem by finding the SV representor, $\Phi^{\text{SV}}$, of $\phi$. We look in turn at SV
representation equations for an antiplane fault and a vertical strike-slip fault, providing methods general computational methods and specific examples of representers in each case.

1.4.4 SV representers for the antiplane fault

The antiplane SV inner product is given by

\[
<s_1, s_2>_{SV} = \int \tau'(\xi; s_1) \tau'(\xi; s_2) d\xi = \mu^2 \int 2\pi i \omega |\omega| \dot{s}_1(\omega) \cdot \overline{2\pi i \omega |\omega| \dot{s}_2(\omega)} d\omega = \frac{\mu^2}{4\pi^2} \int 4\pi^2 \omega^2 \dot{s}_1(\omega) \cdot 4\pi^2 \omega^2 \dot{s}_2(\omega) d\omega = \frac{\mu^2}{4\pi^2} \int s_1''(\xi) s_2''(\xi) d\xi, \tag{1.4.61}
\]

so the SV inner product is the same as that on the space \( \mathcal{H}_2 \) defined in Appendix B. Just as it did for the SM inner product, the recognition that the SV inner product has a simple form depending only on derivatives of slip brings with it a convenient computational device for solving representation equations. For antiplane slip in a half space on \((-D, 0)\), and observation points on the free surface, let \( U_{SV}(\cdot; y) \) and \( V_{SV}(\cdot; y) \) be the SV representers of the displacement and 1, 3 shear strain functionals measured at fault-normal coordinate, \( y \). Then, it terms of the reproducing kernel, \( R_2(x, y) \triangleq (x \wedge y)^2(3(x \vee y) - (x \wedge y)) \), we have

\[
U_{SV}(\xi; y) = \int_{-D}^{0} R_2 \left( \frac{\xi + D}{D}, \frac{\eta + D}{D} \right) \cdot \frac{1}{\pi} \frac{y}{\eta^2 + y^2} d\eta = y(\xi + D)(5\xi + D) + 2\left\{3z(y^2 - d^2) - 2d^3\right\} \tan^{-1} \left( \frac{-D}{y} \right) + 2\xi(\xi^2 - 3y^2) \tan^{-1} \left( \frac{\xi}{y} \right) + y^2(y + 3\xi^2) \log \left( \frac{y^2 + \xi^2}{y^2 + D^2} \right) - 3y(\xi + D)^2 \log \left( 1 + \frac{D^2}{y^2} \right), \tag{1.4.62}
\]

and

\[
V_{SM}(\xi; y) = \int_{-D}^{0} R_2 \left( \frac{\xi + D}{D}, \frac{\eta + D}{D} \right) \cdot \frac{\mu}{\pi} \frac{\eta^2 - y^2}{(\eta^2 + y^2)^2} d\eta.
\]
\[ V_{sv}(\xi) = 3\xi^2 + 4D\xi + 4y\xi \left\{ \tan^{-1}\left( \frac{-D}{y} \right) - \tan^{-1}\left( \frac{\xi}{y} \right) \right\} \]
\[ + (y^2 + 2D\xi + D^2) \log \left( \frac{y^2 + \xi^2}{y^2 + D^2} \right) \]
\[ - (\xi + D)^2 \log \left( 1 + \frac{\xi^2}{y^2} \right) \]  \hspace{1cm} (1.4.63)

Figures 1.20 and 1.21 show SV representers for displacement and strain functionals measured at points on the free surface at distances 0.01, 0.1, 1, and 10 source depths from the fault. In their behavior for a point approaching the fault, these function are more similar to the SM representers than to the energy representers; they do not become peaked. In fact, the minimum SV representers become parabolic in shape for points close to the fault, and quartic (i.e., fourth degree polynomial) for distance points.

Figure 1.22 show representers for two angle changes in the Point Arena network. Both look very similar to the displacement representor in Figure 1.20 for the point 0.1 source depths from the fault. Figure 1.23 show four orthonormalized basis functions from the Point Arena network, and the minimum stress variability estimate of coseismic slip in the 1906 earthquake constructed from these four functions is shown in Figure 1.24. As for the energy and stress magnitude, the estimate again has the shape of the representor for displacement at a moderately distant point.

1.5 Discussion and conclusions

We have proposed and illustrated methods for representing source functions in inverse problems requiring estimation of fault slip from crustal deformation measurements. These methods are based on the intent to minimize various norms of the solution, norms chosen by \textit{a priori} reasoning about physical properties of a slipping fault loaded by remotely-acting tectonic forces. Our illustrations of these techniques have been limited to the very simplest model: antiplane slip on a fault in a homogeneous, isotropic half-space. The principles underlying these illustrations are, however, much more general though, perhaps, messier in execution.
Figure 1.20: Antiplane stress variability representers for the displacement functional measured various points on the free-surface of a half-space.
Figure 1.21: Antiplane stress variability representers for the shear strain functional measured various points on the free-surface of a half-space.
Figure 1.22: Stress variability representers for the angle changes measured from Cold Spring to Fisher to Clark and from Shoemaker to Spur to Lane in the Point Arena triangulation network.
Figure 1.23: Orthonormalized SM basis functions from Point Arena net.
Section 1.5. Discussion

Minimum SV slip estimate, D = 15

Figure 1.24: Minimum SV estimate of coseismic slip in the 1906 San Francisco earthquake from inversion of Point Arena data.
Section 1.5. Discussion

1.5.1 Comparison of methods

Figure 1.25 shows estimates of coseismic slip in the 1906 earthquake minimizing each of the three norms we have defined. The estimates are obviously very similar to one another, which may reassure us, to a degree that the Point Arena triangulation data are revealing at least a component of the true coseismic slip distribution. It is heartening to find no strongly varying artifacts of analytical showing up in the solution and also to find that the estimates meet certain first-order consistency checks with observations that have not gone into producing the fits. For instance, using all of the different norms that we have defined and varying the assumed faulting depth over some range, as discussed in [37], we find that the data consistently prefer estimates with surface slip in the 5 to 6 meter range, which is consistent with the size of measured offsets of fault-crossing landmarks in the vicinity of Point Arena [57].

Though the estimates derived from the three different norms we have defined do not differ much from one another in this case, there is potential that they could in another data set with greater spatial resolution and higher signal-to-noise ratio [37]. We may make some qualitative observations on the expected form of solutions likely to emerge from each of the methods. Minimum-energy solutions from geodetic observations made near a fault will tend to be highly peaked, with most of the estimated moment release pushed up close to the surface. For observations made in the far-field, these estimates will produce “crack-like”, constant stress-drop distributions. In contrast, estimates minimizing stress magnitude and stress variability will tend to be smoother, assuming linear and parabolic shapes, respectively, from near field observations, and parabolic and quartic shapes from far field data.

1.5.2 Extensions and further work

In extending the results we have given to more complicated models, where slip is allowed to vary over a two-dimensional surface, the smoothing conditions discussed in Section 4 become even more stringent. On a two-dimensional domain, the requirement that only
Figure 1.25: Three norm-minimizing estimates of coseismic slip in the 1906 San Francisco earthquake obtained by inversion of the Point Arena triangulation data.
first derivative be square integrable is *not sufficient* to define norms with the smoothing property. Hence, the stress magnitude functional for slip on a 2-d fault has the same shortcoming as the self-energy on the antiplane fault. The importance of using smoothing norms in slip estimation is evident when we wish to construct a function satisfying specified boundary conditions. If the boundary conditions are nonhomogeneous, we will generally determine a single “boundary component” of the solution that satisfies these conditions and then use the norm-minimization principles we have advocated to construct basis functions with homogeneous boundary conditions that are used to estimate true slip minus the boundary component. This type of analysis is carried out in [33], in which we use the stress variability functional to generate basis functions for slip on the Parkfield segment of the San Andreas fault. In this analysis, we corroborate the results of Segall and Harris [16], who found a “locked zone” of low slip rate that is nearly coincident with the rupture zone of the 1966 Parkfield earthquake. The salient point at present, however, is not to discuss this result but to point out that the analysis required satisfaction of boundary conditions that it was mostly easily carried out using the stress variability functional we have described.

In addition to the conditions on boundedness of point evaluation, of which we have spoken, there is another property of the stress variability that recommends it as a functional requiring the “right” amount of smoothness. If we view the elastic displacement field around a slipping fault as the potential field of a surface distribution of double couple sources, then potential theoretic results on properties of the normal derivative of a potential field may be brought to bear on the question of existence of the surface traction distribution. The usual (though not absolute weakest) sufficient condition for existence of the limits defining surface tractions is $L^2$ integrability of the second derivatives of the source function. (Cf. Jaswon and Symm [20]) This is the condition imposed by the stress variability functional.

Besides, thinking about how our results apply in higher dimensions, we may consider their role in models complicated by more realistic physical assumptions. Extensions to include nonhomogeneous constitutive laws, perhaps incorporating a compliant fault zone
or a layered stiffness model, are straightforward in principle. The traction-matching condition still provides minimum-energy solutions, though they will sometimes differ from those obtained under assumed homogeneity. If we observe a displacement functional, for instance, then the TMC would imply that the energy-minimizing source function would have surface traction equal, on the slipping zone, to the Green's function for the non-homogeneous model. For a moment functional, on the other hand, the kernel weighting slip is still constant and the energy-minimizing source function is still given by the slip distribution with constant stress drop in the the slipping zone.

Finally, we have, of course, neglected to say what to do with an set of basis functions once they are in hand. In describing procedures for finding a basis functions, we have provided an efficient means to turn an estimation problem on an infinite-dimensional function space into a finite-dimensional linear model, requiring only the specification of a finite number of coefficients. This requirement, however raises scores of statistical and data-analytic issues that we will not here discuss, but which are treated at length in [37]. Presently, we merely state that if norm-minimizing estimate make sense, then the presence of measurement errors and external contaminants of the signal presumed to exist in a set of observations described by a linear model such as (1.1.1) does not alter the desire to pose and solve the sorts of problems we have discussed in this paper.

Questions of generality and extension aside, we believe that the principles and methods we have presented are part of a procedure for analysis of crustal deformation which, combined with the statistical models introduced in [37] and [34], offers a genuine improvement over existing approaches. We advocate the notions that one ought to reason physically toward solution of inverse problems, much as one might in the construction of a forward model, and that one sometimes can and should explicitly solve the representational problems raised in this endeavor. As instrumental technology improves and deformation measurements become available in increasing quantity and accuracy, we may hope that some of the ideas put forth herein will contribute toward fuller understanding of fault mechanics and seismogenic processes.
Chapter 2

Statistical methods and estimation of slip in the 1906 earthquake

2.1 Introduction

The measurement of crustal deformation on the Earth’s surface around active faults provides data that may be of use in the study of deep fault behavior and of seismogenic processes. In order best to mine whatever information may lie buried within a particular collection of raw deformation measurements, we must devise data analytic methods that account well for various uncertainties when mapping measurements back to putative physical sources. Uncertainty in the process of inverting crustal deformation data to obtain an estimate of fault slip may be partitioned into two sorts arising from two fundamental sources: physical complexity and measurement inaccuracy. The inability to furnish a low-dimensional, parametric description for the family of possible physical sources producing the signal in a given set of measurements gives rise to the first, which may be call a priori uncertainty, and it leads to a multiplicity of potential solutions that all are optimally good in terms of a particular empirical goodness-of-fit criterion. The second
Section 2.1. Introduction

type of uncertainty, to be referred to as empirical, is a product of inaccuracy in the measurement process. This inaccuracy, which may be attributed to instrumental imprecision or to external noise sources that obscure the signal to be discerned, necessitates the use of statistical techniques both for estimation of the source and for inference about its properties.

In [36] we proposed and investigated methods that contend with a priori uncertainty in the process of inverting crustal deformation data for fault slip. These methods rely on physically-motivated regularization principles that replace the ill-posed inverse problem: “Find the source function that fits the data.” with a well-posed optimization problem: “Find the extreme source function that fits the data.” In the well-posed problem, physical arguments are used to define a meaningful sense in which to require that the solution be extreme. For instance, we may seek the smoothest possible source function, measuring smoothness by the stress magnitude or stress variability functional introduced in [36]. Or we may seek the solution that minimizes some measure of potential energy, as was also investigated in [36]. Within the context of fault slip estimation, or of any linear inverse problem for that matter, the use of quadratic functionals of the solution to formulate well-posed optimization problems leads to finite-dimensional representations in which the estimable part of the source function is given as a linear combination of specific basis functions. In [36], we gave algorithms for each proposed functional that would produce the desired basis function corresponding to a particular deformation measurement. Presently, we turn attention toward how to proceed with the inversion and analysis once a set of basis functions is in hand.

Our analysis in this paper concentrates on a relationship between data and fault slip described by a simple antiplane faulting model. This model is summarized in Section 2. Along with the model, we describe a data set of geodetic measurements giving pre- to post-1906 angle changes in a triangulation network near Point Arena, California to for the antiplane model may be applied to estimate coseismic fault motion in the 1906 San Francisco earthquake. In Section 3, we summarize variational methods, discussed at length in [36], that produce optimal, finite-dimensional representations of norm-minimizing source
functions to be used in construction of estimates by analysis of geodetic observations. We give closed form expressions for functions producing point displacements and minimizing either the magnitude or the variability of stress acting on the fault plane. From these expressions, or from numerical solutions of integral equations that produce energy-minimizing representations, we may compute ideal basis functions to act as fundamental components in linear inversion for fault slip. In Section 4, we begin to describe what to do with a finite-dimensional set of basis functions, regardless of its origin. The procedures we describe involve simple, generalized least squares estimation of coefficients, and they are best understood using the eigen and singular value analyses of Section 4.2 to orthogonalize the solution components and to appreciate the effects of the signal-noise-ratio in determining what signal may be visible in a given collection of data. In Section 5, we discuss the signal-to-noise effect at greater length, showing why the damping parameter in linear inversion should be interpreted as a signal to noise ratio and then describing the method of cross-validation which may be to estimate the signal-to-noise ratio by minimizing an estimate of prediction error. In Section 6, we apply our full inversion algorithm to analysis of the Point Arena triangulation data and estimation of slip in the 1906 earthquake. We use the finite-dimensional source representations described in Section 3 and the statistical methods of Section 4 and 5 to construct a raft of minimum-norm slip estimates, differing in choice of norm and faulting depth. Our best fitting estimates are similar to those shown in Figure 2.1, where we see three estimates, each minimizing one of the norms defined in Section 3, all allowed to slip to a depth of 15 km. Using source functions tapered to zero at the base of the slipping zone, we find that the geodetic data prefer faulting depths that are somewhat larger than the shallow depths favored in most previous analyses of these data. Our best estimates are produced when slip is permitted to a depth of about 20 km, and the deep slip in our models leads to correspondingly higher moment estimates, in the range of 5 to $6 \times 10^{27}$ dyne-cm. The lower bound in this range is about the same as the upper bound produced by Thatcher [57] in his inversions using uniform dislocation representations.
Figure 2.1: Three estimates of slip in 1906 San Francisco earthquake obtained by minimum-norm inversions of the Point Arena triangulation data with $D = 15$. 
Section 2.2. Introduction

2.2 The data and model

The San Francisco earthquake of 1906 predates the wide deployment of seismographic networks, so instrumental data on this earthquake is exists mainly in the form of static deformation measurements. Several triangulation networks that had been surveyed in the decades before the earthquake were resurveyed shortly after the event, and pre- to post-seismic angle changes may be linearly related to coseismic displacements of survey benchmarks, as discussed in [49].

Of the large collection of geodetic data assembled by Hayford and Baldwin [18], we will use a subset from the Point Arena triangulation network situated near the northern end of the 1906 rupture zone, as shown in the inset of Figure 2.2 Table 2.1 lists fault-normal and fault-parallel station coordinates, the parallel coordinates being measured from an arbitrary origin to the southwest of the network, and normal coordinates being negative on the Pacific side of the fault. Thirty azimuths between pairs of points in this network were surveyed in 1891 and the again in 1907 (see [18]). Pre-1906 angle measurements were subtracted from post-1906 data to produce the thirty angle-change measurements listed in Table 2.2. These data are part of a larger set of geodetic observations measured areas around the San Andreas fault spanning a period of several decades before and after the 1906 earthquake. The entire data set has been analyzed in detail by Thatcher [56, 57].

To relate observed angle changes to the coseismic slip distribution, we will make use of the fact that the rupture length, $L$, was on the order of 400km, much larger than the faulting depth, $D$, which was on the order of 10km. Combining the knowledge that $L \gg D$ with the assumption that coseismic slip was purely horizontal "strike slip", parallel to the free surface, we obtain a simple forward model that postulates antiplane slip on a vertical plane in a homogeneous, isotropic, elastic half-space. Precisely, we assume that slip was entirely fault-parallel, that it varied only as a function of depth, and that it was zero below depth $D$. In the coordinate system with the 1, 2, 3, axes referring respectively to fault-parallel, depth, and fault-normal coordinates, the displacement field
Figure 2.2: Point Arena triangulation network.

<table>
<thead>
<tr>
<th>Station</th>
<th>Normal, km</th>
<th>Parallel, km</th>
</tr>
</thead>
<tbody>
<tr>
<td>FISHER</td>
<td>11.0840</td>
<td>9.8898</td>
</tr>
<tr>
<td>COLD SPRING</td>
<td>13.4835</td>
<td>2.1633</td>
</tr>
<tr>
<td>DUNN</td>
<td>4.1057</td>
<td>7.3137</td>
</tr>
<tr>
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<td>9.6390</td>
</tr>
<tr>
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<td>5.0951</td>
</tr>
<tr>
<td>SPUR</td>
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<td>6.5343</td>
</tr>
<tr>
<td>SHOEMAKER</td>
<td>-1.4880</td>
<td>5.1935</td>
</tr>
<tr>
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<td>7.0989</td>
</tr>
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<td>ARENA</td>
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</tr>
<tr>
<td>SINCLAIR</td>
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</tr>
<tr>
<td>HIGH BLUFF</td>
<td>-6.7946</td>
<td>0.0984</td>
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Table 2.1: Fault-normal and fault-parallel coordinates of stations in the Point Arena triangulation network.
<table>
<thead>
<tr>
<th>From</th>
<th>Azimuth Through</th>
<th>To</th>
<th>Observed change</th>
</tr>
</thead>
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<tr>
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<tr>
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<td>CLARK</td>
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</tr>
<tr>
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</tr>
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<td>SPUR</td>
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</tr>
<tr>
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<td>SHOEMAKER</td>
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</tr>
</tbody>
</table>

Table 2.2: Measured changes from 1891 to 1907 of azimuths among benchmarks in the Point Arena network.
Section 2.2. Introduction

is expressed in terms of slip by

\[ u_1(x_2, x_3) = \frac{1}{2\pi} \int_{-D}^{0} s(\xi) \left\{ \frac{x_3}{x_3^2 + (x_2 - \xi)^2} + \frac{x_3}{x_3^2 + (x_2 + \xi)^2} \right\} d\xi. \]  \tag{2.2.1}

For points on the free surface, \( x_2 = 0 \), where all survey benchmarks are taken to lie, (2.2.1) may be written as

\[ u_1(x_3) = \frac{1}{\pi} \int_{-D}^{0} s(\xi) \left\{ \frac{x_3}{x_3^2 + \xi^2} \right\} d\xi. \]  \tag{2.2.2}

Differentiating the displacements given in (2.2.1) produces the components of strain that are not identically zero:

\[ \epsilon_{ij} \equiv \epsilon_{j1} \triangleq \frac{1}{2} \frac{\partial u_1(x_2, x_3)}{\partial x_j}, \quad j = 2, 3. \]  \tag{2.2.3}

Shear stresses are given by

\[ \sigma_{ij} \equiv \sigma_{j1} \triangleq \mu \epsilon_{ij}, \]  \tag{2.2.4}

where \( \mu \) is the shear modulus, to which we assign the value \( 3 \times 10^{11} \) dynes per cm\(^2\).

Because of the displacement discontinuity at the fault, (2.2.3) and (2.2.4) do not hold at \( x_3 = 0 \). But the assumption that traction is continuous across the fault implies that the \textit{fault surface tractions}, \( \tau(\xi) = \sigma_{13}(x_2, 0)|_{x_2=\xi} \), exist and are given by

\[ \tau(\xi) \triangleq \lim_{x_3 \to 0} \sigma_{13}(x_2, x_3)|_{x_2=\xi}. \]  \tag{2.2.5}

To relate the free surface displacement field in (2.2.2) to the measured angle changes at Point Arena, let \( Y_i \) denote the \( i^{th} \) measurement and \( \delta \theta_i \), the true angle change. The model for the data is

\[ Y_i = \delta \theta_i(s) + \epsilon_i, \]  \tag{2.2.6}

where the noise terms, \( \epsilon_i \), are assumed to be independent random variables with mean zero and variance \( \sigma^2 \). With \( x^{(j)} \) denoting the coordinates of the \( j^{th} \) benchmark, the fault-parallel station displacements are given by

\[ u^{(j)}(s) = \frac{1}{\pi} \int_{-D}^{0} s(\xi) \cdot \frac{x_1^{(j)}}{(x_1^{(j)})^2 + \xi^2} d\xi. \]
Section 2.3. Basis functions

Since displacements are small compared to the distances between benchmarks (on the order of 1 part in $10^3$) we may accurately relate angle changes linearly to benchmark displacements by the approximation given in [36], the result being that we can rewrite (2.2.6) in vector form as

$$ Y = Au(s) + \epsilon, \quad (2.2.7) $$

where $Y$ is the vector of observations, $u(s)$ is the vector of coseismic benchmark displacements, and $A$ is the matrix of coefficients in the linear approximation relating displacement to angle change. As discussed by Segall and Matthews [49], $A$ is always rank deficient, with null space corresponding at least to rigid body motion. We use $\phi_i(s)$ to refer to the linear functionals of slip giving the signal component in the $i^{th}$ datum, i.e.,

$$ \phi_i(s) \triangleq a_{ij} u_j^{(i)}(s). \quad (2.2.8) $$

2.3 Regularizing functionals and basis functions

We have alluded to the fact that "optimal" estimates of fault slip may be nonuniquely defined if quality is judged only on how well a solution explains a finite set of observations. This is typical of estimation problems in which the set of possible solutions is an infinite-dimensional function space. Problems of this sort are described as nonparametric, though, ultimately, their solutions are parametrized, one way or another. In linear problems, a parametrization is generally given by expanding the unknown function in a finite-dimensional basis, with the choice of basis functions guided by reason, by convenience, or by some combination of the two.

Convenience has been the apparent dictator of parametrizations in the majority of estimation problems involving inversion of crustal-deformation measurements; "step-functions" of uniform, rectangular dislocations producing discontinuous, piecewise constant estimates have been most commonly used to represent the class of possible function estimates. Solutions derived from a representation of this sort are fundamentally unphysical in certain respects. Whether or not these are important respects is an issue that may be subject to debate and that probably can be settled not in the abstract, but only on a situational
basis. Irrespective of the anticipated uses of an estimate, however, it would seem difficult to argue for an estimation procedure that is guaranteed to produce physically impossible results when an equally (or more) powerful procedure is available to generate estimates that are at least physically plausible. For estimating antiplane fault slip, physically sound alternatives to step functions certainly are available for representing solutions. Most obvious among these alternatives are low order Chebyshev polynomials, which facilitate stress and energy computation (see Barnett and Freund [7]; Mavko [38]) in the antiplane model, or minimum-norm basis functions derived from the physical arguments and techniques presented in [36]. We now describe certain minimum-norm representations and exhibit them as they are determined by the Point Arena triangulation network.

In [36], we considered three particular functionals for defining the norm of a slip distribution in a useful fashion. Letting \( \tau(\xi; s) \) denote the change in fault surface traction at \( \xi \) due to slip, \( s \), the functionals are defined for the antiplane model by

- **self-energy:**

\[
SE(s) = L \int_{\xi=-D}^{0} s(\xi) \tau(\xi; s) d\xi, \tag{2.3.9}
\]

- **stress magnitude:**

\[
SM(s) \triangleq \int_{-\infty}^{0} \tau^2(\xi; s) d\xi. \tag{2.3.10}
\]

- **stress variability:**

\[
SV(s) \triangleq \int_{-\infty}^{0} \left( \frac{d\tau(\xi; s)}{d\xi} \right)^2 d\xi. \tag{2.3.11}
\]

The first of these functionals measures a component of the change in crustal strain energy accompanying slip; the second measures the squared \( L^2 \) norm of the change in the fault surface traction distribution; and the third measures the squared \( L^2 \) norm of the first derivative of change in surface traction. Though these three functionals measure different physical characteristics of elastic deformation associated with slip, have a common mathematical form; each is quadratic in \( s \). The fact that the norms defined by these functionals are obtained by taking square roots of quadratic forms is significant, for it leads to easily
specified conditions that define norm-minimizing fits to linear data as given in (2.2.7). The reasoning leading to these conditions and the steps required to find functions that satisfy them are discussed in detail in [36]. The basic idea is to use the inner product associated with a particular quadratic form to define a function that represents a given linear functional in the linear space of functions with finite norm. Letting \( \| \cdot \|^2 \) stand for a generic positive definite quadratic form, which may be one of the three we have defined or any other, we take \( \langle \cdot, \cdot \rangle \) to be the associated inner product so that \( \|s\|^2 = \langle s, s \rangle \), and we ask how we can make \( \|s\|^2 \) as small as possible while honoring the information provided by observation of a single linear functional, \( \phi_i \). The answer is provided by solving the representation equation, which requires a function, \( \Phi \), that produces values of \( \phi_i \) through the inner product:

\[
\langle \Phi_i, s \rangle = \phi_i(s) \quad \forall s \in \mathcal{F}. \tag{2.3.12}
\]

If we can find \( \Phi_i \) satisfying (2.3.12), then multiplication of \( \Phi_i \) by the appropriate scalar coefficient gives the desired norm-minimizing solution. When inexact values of a set of functionals, \( \phi_i, i = 1, \ldots, n \), are measured, as in (2.2.7), we may ideally solve the representation equation \( n \) times, once for each \( \phi_i \), to produce an "optimal" set of \( n \) basis functions \( \{\Phi_i\}_{i=1}^{n} \). Coefficients of these functions may then be determined as described in Sections 4 and 5.

The inner products associated with the three functionals defined in (2.3.9)-(2.3.11) are

- **self-energy**: 

\[
\langle s_1, s_2 \rangle_{SE} \triangleq \int_{-D}^{0} s_1(\xi) \tau(\xi; s_2) d\xi, \tag{2.3.13}
\]

- **stress magnitude**: 

\[
\langle s_1, s_2 \rangle_{SM} \triangleq \int_{-\infty}^{0} \tau(\xi; s_1) \tau(\xi; s_2) d\xi. \tag{2.3.14}
\]

- **stress variability**: 


\begin{equation}
\langle s_1, s_2 \rangle_{SV} \triangleq \int_{-\infty}^{0} \left\{ \frac{d\tau(\xi; s_1)}{d\xi} \right\} \cdot \left\{ \frac{d\tau(\xi; s_2)}{d\xi} \right\} d\xi. \tag{2.3.15} \end{equation}

The representation equation for the energy inner product is solved by satisfying the traction matching condition (TMC) given in [36]. This condition dictates that we find a slip distribution with fault surface traction equal to the weight function in the integral defining the linear functional to be observed, and its implementation generally involves the solution of singular integral equations. It is much easier to solve the representation equations for the SM and SV inner products making use of the fact, shown in [36], that they are equivalent to inner products involving first and second derivatives of slip, respectively. Recognizing this equivalence, we may use reproducing kernels [4] to write down closed forms for the the representor of the functional defined by displacement of a fixed point. Letting $U_{SM}(\xi; y)$ and $U_{SV}(\xi; y)$ denote the values at fault depth $\xi$ of the representors in the SM and SV norms, respectively, for the functional measuring displacement at fault-normal coordinate $y$, we find that

\begin{equation}
U_{SM}(\xi; y) = \frac{y}{2} \log \left( \frac{y^2 + \xi^2}{y^2 + D^2} \right) \tag{2.3.16}
\end{equation}

and

\begin{equation}
U_{SV}(\xi; y) = y(\xi + D)(5\xi + D) \\
+ 2 \left\{ 3z(y^2 - d^2) - 2d^3 \right\} \tan^{-1} \left( \frac{-D}{y} \right) \\
+ 2\xi(\xi^2 - 3y^2) \tan^{-1} \left( \frac{\xi}{y} \right) \\
+ y^2(y + 3\xi^2) \log \left( \frac{y^2 + \xi^2}{y^2 - y^2} \right) \\
- 3y(\xi + D)^2 \log \left( 1 + \frac{D^2}{y^2} \right) \tag{2.3.17} \\
\end{equation}

By taking linear combinations of the appropriate displacement representors, we get the angle-change representors as required for the Point Arena data. Figures 2.3-2.5 show representors in each of the three norms defined for measurements of the COLD SPRING–FISHER–CLARK and SHOEMAKER–SPUR–LANE angles. In all three instances, the
two representers are very similar to one another, being virtually identical in the stress variability norm. The two representers are most dissimilar in the energy norm, due to the fact that this norm requires less smoothness and is therefore, in a sense, more sensitive to positions of measurement points in the near field. The representers shown in these figures are but two of the thirty we get, in each norm, by solving the required representation equations for each measured azimuth in the Point Arena network. We now describe an algorithm by which to construct a slip estimate from the basis functions so derived, or from any finite-dimensional representation given in terms of a linear expansion in specified basis functions.
Figure 2.4: Minimum SM representer for angle changes measured from Cold Spring to Fisher to Clark and from Shoemaker to Spur to Lane, $D = 15$. 

Minimum SM representer for two angle changes, $D = 15$ km
Minimum energy representers for two angle changes, $D = 15$ km

Figure 2.5: Minimum SV representers for angle changes measured from Cold Spring to Fisher to Clark and from Shoemaker to Spur to Lane, $D = 15$. 
2.4 The inversion algorithm

In this section, we describe an estimation procedure to be carried out once a finite-dimensional representation of the desired solution is at hand. The procedure is not tailored to the particular model or data set we have described, but is applicable in any function-estimation problem in which a linear model akin to (2.2.7) describes the relationship between the data and the unknown function.

2.4.1 The basic algorithm

Generally, suppose that we observe, for $i = 1, \ldots, n$,

$$Y_i = \phi_i(f) + \epsilon_i,$$  \hspace{1cm} (2.4.18)

where the $\phi_i$ are linear functionals and the $\epsilon_i$ are random disturbance terms with mean zero and covariance matrix $\Omega$, and that we wish to construct an estimate of $f$. We assume that $f$ lies in some Hilbert space, $\mathcal{F}$, with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, and that we desire an estimate with $\|f\|$ as small as is possible subject to agreement with the observed data. We also assume that the $\phi_i$ are continuous on $(\mathcal{F}, \| \cdot \|)$. See [36] for discussion of this last requirement.

Let $\hat{Y}(f)$ denote the $n$-vector with $i^{th}$ component equal to $\phi_i(f)$. Because of the assumed signal perturbations $\epsilon_i$, in (2.4.18), we generally do not wish to require that $\hat{Y}(f) = Y$, exactly. Instead, we will tolerate a residual to the data that is of magnitude consistent with the assumed probability distribution of $\epsilon$. Measuring the residual norm in a generalized least squares sense, we will say that a candidate function, $f$, fits the data satisfactorily if the residual satisfies the bound

$$(Y - \hat{Y}(f))^T \Omega^{-1} (Y - \hat{Y}(f)) < r^2.$$  \hspace{1cm} (2.4.19)

The value of the parameter, $r$, in (2.4.19) may be an important determinant of the form of the solution. In Section 5, we will discuss empirical methods to guide the setting of this critical value, but for now we'll treat it as a parameter that indexes things that depend
on the bound in (2.4.19). For instance, we will refer to the subset of $\mathcal{F}$ satisfying the requisite constraint as $\mathcal{F}_r$:

$$\mathcal{F}_r \triangleq \{ f \in \mathcal{F} : \ f \text{ satisfies } (2.4.19) \}.$$  

(2.4.20)

The minimum-norm estimate, $\hat{f}_r$, that we seek is defined by

$$\hat{f}_r \triangleq \min_{f \in \mathcal{F}_r} \| f \|.$$  

(2.4.21)

Following Reinsch [44], we may introduce two parameters to account for the inequality defining the set of admissible solutions, and then show that if either there is a degenerate solution with $\|f\| = 0$ in $\mathcal{F}_r$, or that the desired solution is given by

$$\hat{f}_\alpha = \min_{f \in \mathcal{F}_r} \| f \| Q(f; Y, \Omega, \alpha),$$  

(2.4.22)

where $Q$ is a quadratic form,

$$Q(f; Y, \Omega, \alpha) \triangleq (Y - \hat{Y}(f))^T \Omega^{-1} (Y - \hat{Y}(f)) + \alpha^{-2} \| f \|^2,$$  

(2.4.23)

and the Lagrange multiplier, $\alpha = \alpha(r)$, determined implicitly by $r$:

$$(Y - \hat{Y}(\hat{f}_\alpha))^T \Omega^{-1} (Y - \hat{Y}(\hat{f}_\alpha)) = r^2.$$  

(2.4.24)

Henceforth, we will assume that no degenerate solution exists.

Let $\{ \tilde{B}_j \}_{j=1}^M$ be a set of basis functions from which an estimate will be constructed; the tildes will distinguish this basis and quantities related to it from those defined in a more convenient basis to be defined shortly. Ideally, basis functions are obtained by applying methods such as those described in [36] and summarized in Section 3 to find the representers in $\mathcal{F}$ of the functionals giving the signal in (2.4.18). The set of representers, $\{ \Phi_j \}_{j=1}^n$, the $j^{th}$ member of which is defined by

$$< f, \Phi_j > = \phi_j(f) \ \forall f \in \mathcal{F},$$  

(2.4.25)

is ideal in the sense that $f_\alpha$, as defined by (2.4.22), is in the subspace

$$\tilde{\mathcal{F}} \triangleq \text{Span}\{ \Phi_1, \ldots, \Phi_n \}.$$  

(2.4.26)
Section 2.4. The algorithm

If \( \mathcal{B} \in \mathcal{F} \) is the subspace spanned by \( \{ \tilde{B}_j \} \), then we can get the exact solution, \( f_\alpha \), only if \( \mathcal{F} \subset \mathcal{B} \); if there are functions in \( \mathcal{F} \) orthogonal to \( \mathcal{B} \), then the solution based on the chosen expansion will be only approximate, missing the component of \( f_\alpha \) in the subspace of \( \mathcal{F} \) orthogonal to \( \mathcal{B} \) and compensating by having larger norm per unit misfit to the data.

The Gram matrix, or simply Grammian, of the functions \( \{ \tilde{B}_i \} \) is a symmetric, positive semidefinite, \( M \times M \) matrix formed by taking pairwise inner products. The \( ij^{th} \) entry, \( \gamma_{ij} \), of \( \Gamma \) is

\[
\gamma_{ij} = \langle \tilde{B}_i, \tilde{B}_j \rangle. \tag{2.4.27}
\]

Writing \( f \in \mathcal{B} \) as \( f(x) = \sum_{j=1}^{M} \tilde{c}_j \tilde{B}_j(x) \), we must compute the \( n \times M \) matrix, \( \tilde{G} \) with entries

\[
\tilde{g}_{ij} = \phi_i(\tilde{B}_j), \tag{2.4.28}
\]

with which the observation equation, (2.4.18), is written in terms of the coefficient vector, \( \tilde{c} \), as

\[
Y = \tilde{G}\tilde{c} + \epsilon. \tag{2.4.29}
\]

The quadratic form defined in (2.4.23) is

\[
Q(f; Y, \alpha, \Omega) = (Y - \tilde{G}\tilde{c})\Omega^{-1}(Y - \tilde{G}\tilde{c}) + \alpha^{-2}\tilde{c}^T\Gamma\tilde{c}. \tag{2.4.30}
\]

Taking the gradient of \( Q \) with respect to the coefficient vector and setting it to zero gives the condition for the coefficients producing the desired minimum as

\[
(\tilde{G}^T\Omega^{-1}\tilde{G} + \alpha^{-2}\Gamma)\tilde{c} = \tilde{G}^T\Omega^{-1}Y. \tag{2.4.31}
\]

If the matrix on the lefthand side of (2.4.31) is full-rank, then the desired estimate of \( f \) is uniquely defined by

\[
\hat{f} = \sum_{j=1}^{M} \tilde{c}_j \tilde{B}_j, \tag{2.4.32}
\]

with

\[
\tilde{c} = (\tilde{G}^T\Omega^{-1}\tilde{G} + \alpha^{-2}\Gamma)^{-1}\tilde{G}^T\Omega^{-1}Y. \tag{2.4.33}
\]

In order to simplify further exposition, we will henceforth assume that a transformation of the finite-dimensional model, (2.4.29), has been made so that the error distribution
Section 2.4. The algorithm

has covariance matrix \( \sigma^2 I \). This will save us from having to carry \( \Omega \) around and will serve to illustrate the importance of the signal-to-noise ratio. Indeed, having made this transformation, (2.4.33) becomes

\[
\hat{c} = (\sigma^{-2} \tilde{G}^T \tilde{G} + \alpha^{-2} I)^{-1} \tilde{G}^T \frac{1}{\sigma^2} Y \\
= (\tilde{G}^T \tilde{G} + \frac{\sigma^2}{\alpha^2} I)^{-1} \tilde{G}^T Y \\
= (\tilde{G}^T \tilde{G} + \rho^2 I)^{-1} \tilde{G}^T Y,
\]

(2.4.34)

where \( \rho^{-1} \) is the signal-to-noise ratio:

\[
\rho^{-1} \triangleq \frac{\alpha}{\sigma}.
\]

(2.4.35)

The reason for calling \( \rho^{-1} \) the signal-to-noise ratio will be explained in Section 6.

2.4.2 Matrix decompositions and an orthonbasis

Having chosen to measure a particular collection of functionals, to represent the solution in a particular basis, and to find the solution by minimizing a norm with an associated inner product, all of the information about what may be “seen” is contained in the matrices \( \Gamma \) and \( \tilde{G} \). Eigen and singular value decompositions of these matrices facilitate the use of this information by revealing how many linearly independent observations may come from a given configuration of measurement points and how strong a particular signal component must be to show up in the solution.

Being symmetric, the Gram matrix has eigen decomposition

\[
\Gamma = \Psi \Lambda \Psi^T,
\]

(2.4.36)

where \( \Lambda \) is diagonal with eigenvalue \( \lambda_j \) in the \( jj^{th} \) position, and \( \Psi \) is an orthogonal matrix with \( jj^{th} \) column, \( \psi_j \), being the eigenvector associated with \( \lambda_j \). From (2.4.36), we may construct an orthonbasis, \( \{ B_j \} \), by setting

\[
B_j \triangleq \lambda_j^{-\frac{1}{2}} \sum_{k=1}^{M} \psi_{kj} \tilde{B}_k.
\]

(2.4.37)
Section 2.4. The algorithm

This basis will have \( p \) members, \( p \) being the number of strictly positive eigenvalues of \( \Gamma \).

Writing \( f \) in terms of \( B_j \) as

\[
    f = \sum_{j=1}^{p} c_j B_j
\]

and the observation equation as

\[
    Y = Gc + \epsilon,
\]

we have

\[
    G = \tilde{G} \Psi \Lambda^{-\frac{1}{2}}.
\]

The Gram matrix for \( B_j \) is the \( p \times p \) identity, so the quadratic to be minimized is

\[
    Q = \|Y - Gc\|^2 + \rho^2 \|c\|^2,
\]

and the solution, now guaranteed to be unique, is given by

\[
    \hat{c} = (G^T G + \rho^2 I)^{-1} G^T Y.
\]

If the original basis, \( \tilde{B}_j \), is formed by the representers as defined in (2.3.12), then

\[
    \tilde{G} = \Gamma, \quad \text{and, from (2.4.39),}
\]

\[
    G = \Gamma \Psi \Lambda^{-\frac{1}{2}}
\]

\[
    = \Psi \Lambda^{\frac{1}{2}},
\]

and

\[
    \hat{c} = (\Lambda + \rho^2 I)^{-1} \Lambda^{\frac{1}{2}} \Psi^T Y
\]

\[
    = \text{Diag} \left( \frac{\sqrt{\lambda_i}}{\lambda_i + \rho^2} \right) \cdot \Psi^T Y.
\]

Taking the expected value of \( \hat{c}_j \), i.e., averaging over the distribution of \( \epsilon \), we find that

\[
    E\hat{c}_j = \frac{\lambda_j c_j}{\lambda_j + \rho^2},
\]

c\(_j\) being the "true" component of \( s \) in direction \( B_j \). In this equation, we see that when \( \rho^2 > 0 \), the estimated coefficients are biased toward zero. The effect is greater for coefficients of
Section 2.4. The algorithm

the orthobasis functions belonging to smaller eigenvalues. On average, when \( \rho^2 \) is much larger than \( \lambda_j \) there is a large damping effect on the \( j^{th} \) component and very little of \( B_j \) shows up in the solution; conversely, when \( \rho^2 \) is small compared to \( \lambda_j^2 \), the bias toward zero is relatively slight and, we see essentially all that there is to see of \( B_j \). Only part of the story is in the bias, however. Computing the covariance matrix of the estimated coefficients, we find that it is diagonal with

\[
\text{Var}(\hat{c}_j) = \sigma^2 \frac{\lambda_j}{(\lambda_j + \rho)^2}.
\]

(2.4.46)

From this expression, we see why it is not necessarily wise to try to make the bias too small by setting \( \rho^2 \) low. Decreasing the bias increases the variance in the estimated coefficients, with the variance of the coefficients corresponding to small eigenvalues potentially blowing up for very small values of \( \rho^2 \). We will revisit this so-called “bias–variance” tradeoff in Section 5.

To generalize the results just given to the case of an arbitrary initial basis, not necessarily given by the representers, we need to use the singular value decomposition (SVD). Assuming that the transformation to the basis \( B_j \) has been made so that (2.4.39) is the observation equation, we take the SVD of \( G \) as

\[
G = UDV^T,
\]

(2.4.47)

where \( U \) and \( V \) are orthogonal matrices with dimensions \( n \times n \) and \( p \times p \), respectively, and \( D \) is an \( n \times p \) diagonal matrix of singular values, \( d_j \). In terms of the components of the SVD, (2.4.42) is

\[
\hat{c} = (VD^TUDV^T + \rho^2 I)^{-1} VD^T U^T Y
\]

\[
= V \cdot \text{Diag} \left( \frac{d_j}{d_j^2 + \rho^2} \right) U^T Y
\]

\[
= V \cdot \text{Diag} \left( \frac{\tilde{y}_j}{\tilde{y}_j^2 + \rho^2} \right),
\]

(2.4.48)

where \( \tilde{y}_j \) is the \( j^{th} \) component of \( \tilde{Y} \overset{\Delta}{=} U^T Y \). This result is much the same as in (2.4.44), with the singular values, \( d_j \), replacing the square roots of the eigenvalues and the rotation
Section 2.4. The algorithm

of \( Y \) being defined by the singular vectors in \( U \) rather than the eigenvectors in \( \Psi \). For an analog to (2.4.45), let \( c' = V^T c \). The expected values of the components of \( c' \) are then given by

\[
E\hat{c}'_j = \frac{d_j^2 c'_j}{d_j^2 + \rho^2},
\]

(2.4.49)

\( c'_j \) again denoting the "true" value of the coefficient.

2.4.3 Eigen analysis of Point Arena network

Applying the arguments just presented to the representers of the observation functionals in the Point Arena network, we may investigate what is likely to be seen by inversion of these data. From the fact that the thirty angle changes measured in this network depend on the displacements of twelve stations, each with one degree of free motion in the antiplane model, we may conclude that twelve is an upper bound on the number of positive eigenvalues or singular values of any observation matrix. In fact, representers for angle changes are computed by finding the displacement representers for each station using formulas (2.3.16) and (2.3.17) and then taking linear combinations as prescribed by the matrix, \( A \), in (2.2.7) that maps displacements to angle changes. If \( \Gamma_D \) is the Gram matrix of the displacement representers, then \( \Gamma = A \Gamma_D A^T \).

The Gram matrices formed from the representers of the measured functionals in each of the three norms defined in Section 3, have the property that the first eigenvalue is always much greater than all others, by a factor of nearly 100 times even in the best case. Table 2.3 lists the ratios of eigenvalues 2 through 8 to the first eigenvalue for each Gram matrix. This table reveals what we might have guessed from Figures 2.3-2.5, namely, that similarity of the individual representers to one another leads to high concentration of observational power in one component of the source function, and that this effect is progressively greater as we move up the ladder of smoothness in the regularizing norm. In other words, the concentration in the first eigendirection is weakest in the norm requiring the least smoothness: energy, and greatest in the norm requiring the most smoothness: stress variability.
Section 2.4. The algorithm

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<th>Stress magnitude</th>
<th>Stress variability</th>
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<td>$2.13 \times 10^{-11}$</td>
<td>$1.17 \times 10^{-13}$</td>
</tr>
</tbody>
</table>

Table 2.3: Ratios of eigenvalues of Gram matrices for Point Arena network. Entries are $\lambda_k/\lambda_1, k = 1, \ldots, 8.$

The first four orthonormal functions for the Point Arena network in each of the three norms are shown in Figures 2.6–2.8. Comparing basis functions within each set, we see that the first is always monotonic, growing steadily from zero slip at the base of the assumed slip zone and reaching its highest value at the surface. Moving to directions associated with smaller eigenvalues, we find that the basis functions become progressively more “undulatory” as the index of the eigenvalue goes up. Counting the number of zero crossings, for instance, we find that for both the energy and SM functions, the first has no crossings, the second has one, the third has two and the fourth has three, though the undulations of the energy-minimizing functions are comparatively larger in each case. Similarly, the first three SV basis functions have zero, one, and two zero-crossings, respectively, but the oscillations about zero are relatively small. The fourth SV basis function seems almost, but not quite, to have a third zero-crossing. It is worth noting that, because they tend to fluctuate around zero, the higher order basis functions have very small moment and, hence, will not contribute much to fitting geodetic observations that primarily impose moment constraints. Most of the moment will be taken up in the first basis function; higher order functions will come into the picture, if at all, only to accommodate variation of slip with depth that may be demanded by geodetic observations with sufficient spatial resolution to pick up any such variation that may exist. The fact that the orthonormal functions corresponding to the first eigenvalue in each of the three norms are similar to one another implies that the three different norm-minimizing slip estimates that may be constructed from observations with low signal-to-noise ratio will
Figure 2.6: First four minimum-energy orthobasis functions from Point Arena triangulation network.

not differ significantly from one another. We now turn to the question of determining the apparent signal-to-noise ratio in a given set of data.

2.5 Determination of the signal-to-noise ratio

The inversion algorithm described to this point produces not one solution, but rather a family of solutions indexed by the error bound, $r^2$, set in (2.4.19), or, equivalently, by $\alpha^2$ or $\rho^2$. Figure 2.9 shows three different estimates of the antiplane distribution of coseismic slip in the 1906 earthquake. All three are obtained by linear combination of the first four orthobasis functions from the minimum-energy representation as shown in Figure 2.6;
Figure 2.7: First four minimum-SM orthobasis functions from Point Arena triangulation network.
Figure 2.8: First four minimum-SV orthobasis functions from Point Arena triangulation network.
they differ only in specification of $\rho^2$. The most "wiggly" function, with $\rho^2/\lambda_2 = 0.002$, produces the best fit to the observed angle changes, giving a residual sum of squares of 4317.4. The estimates with $\rho^2/\lambda_2 = 0.888$ and $\rho^2/\lambda_2 = 10$ give error sums of squares of 4362.7 and 6090.2. On the other hand, the squared norms of these three solutions are, respectively, 1.63, 1.18, and 0.99. The same qualitative phenomenon is evident in Figures 2.10 and 2.11. As we would expect, increasing $\rho^2$ damps the solution toward zero, making the norm smaller, as is desired, but at the expense of increasing misfit to the data. As has been pointed out by Backus and Gilbert [5], for instance, this trade-off is universal in the sense that it is not possible simultaneously to reduce misfit and norm; one may be reduced only by increasing the other. Obviously, there is potential for great variation among solutions differing only in choice where to set the trade-off between solution norm and misfit, so it is essential to the production of accurate and reliable results that the trade-off parameter be chosen judiciously. In this section, we give a brief theoretical argument that aids in interpretation of this parameter, and we describe an empirical method that may serve to determine its appropriate value.

2.5.1 A theoretical optimum

The minimum-norm inversion algorithm derived in Section 4 is based on the stated intent to minimize some function space norm, $\|s\|_{\mathcal{P}}$, subject to satisfying a quadratic bound on goodness-of-fit to a given set of data. What we are actually trying to do, of course, is to estimate $s$ accurately. The goal of minimizing the quadratic form defined in (2.4.30) with respect to the coefficients, $c_i$, of the orthobasis functions is equivalent to that of minimizing the expected squared function space norm of the difference between the true function and its estimate, with the \textit{a priori} assumption that the component of $s$ orthogonal to $\mathcal{B}$, the span of the chosen basis functions, is small. In other words, the goal of the inversion procedure may be viewed as minimization of the \textit{mean squared norm error}:

\[
\text{MNSE}(s, \delta) \triangleq E\|s - \delta\|^2_{\mathcal{P}}. \tag{2.5.50}
\]
Section 2.5. Signal-to-noise

Minimum energy slip estimates, various alpha, D = 15

Figure 2.9: Three energy-minimizing slip estimates differing in choice of $\rho^2$. 
Section 2.5. Signal-to-noise

Minimum SM slip estimates, various alpha, D = 15

Figure 2.10: Three SM-minimizing slip estimates differing in choice of $\rho^2$. 
Figure 2.11: Three SV-minimizing slip estimates differing in choice of $\rho^2$. 
Section 2.5. Signal-to-noise

Letting $P$ denote the projection operator onto $B$ in $\mathcal{F}$, and $P^\perp$ the projection operator onto $B^\perp$,

\[
\text{MNSE}(s, \hat{s}) = E\left\{\|P(s - \hat{s})\|_F^2 + \|P^\perp(s - \hat{s})\|_F^2\right\}
= E\|P(s - \hat{s})\|_F^2 + \|P^\perp s\|_F^2,
\]

since $P^\perp \hat{s} = 0$. Define the mean squared norm of the projected error by

\[
\text{MSNPE}(s, \hat{s}) \triangleq E\|P(s - \hat{s})\|_F^2.
\]  \hspace{1cm} (2.5.51)

Then we have

\[
\text{MSNPE} = E\|c - \hat{c}\|_2^2,
\]  \hspace{1cm} (2.5.52)

the expected squared Euclidean norm of the difference between the true and estimated orthobasis coefficients. From (2.4.39) and (2.4.44), letting

\[
\Delta \triangleq \text{Diag}\left(\frac{\sqrt{\lambda_i}}{\lambda_i + \rho^2}\right),
\]  \hspace{1cm} (2.5.53)

\[
c - \hat{c} = c - \Delta\Psi^T \mathbf{y}
= c - \Delta\Psi^T \left\{\Psi \Delta \frac{1}{2} c + \epsilon\right\}
= \text{Diag}\left(\frac{\rho^2}{\lambda_i + \rho^2}\right) c - \Delta\Psi^T \epsilon.
\]  \hspace{1cm} (2.5.54)

Taking the expectation of $\|c - \hat{c}\|_2^2$ with respect to $\epsilon$ gives

\[
\text{MSNPE} = \rho^2 \sum_{j=1}^{p} \frac{c_j}{(\lambda_j + \rho^2)^2} + E\epsilon^T\Psi \Delta^2 \Psi^T \epsilon
= \rho^2 \sum_{j=1}^{p} \frac{c_j}{(\lambda_j + \rho^2)^2} + \sigma^2 \text{Tr}(\Delta^2)
= \rho^2 \sum_{j=1}^{p} \frac{c_j}{(\lambda_j + \rho^2)^2} + \sigma^2 \sum_{j=1}^{p} \frac{\lambda_j}{(\lambda_j + \rho^2)^2}
\]  \hspace{1cm} (2.5.55)

To minimize this quantity with respect to $\rho^2$, we set the derivative to zero and simplify the resulting expression to find that the optimal value of $\rho^2$ is defined by the condition

\[
\rho^2 \sum_{j=1}^{p} \frac{c_j\lambda_j}{(\lambda_j + \rho^2)^3} = \sigma^2 \sum_{j=1}^{p} \frac{\lambda_j}{(\lambda_j + \rho^2)^3},
\]  \hspace{1cm} (2.5.56)
or, in terms of $\alpha^2$,

$$\alpha^2 = \sum_{j=1}^{p} c_j^2 w_j.$$  \hfill (2.5.57)

This result says that the optimal value of $\alpha^2$ is given by a weighted average of the squared magnitudes of the signal components, the weights defined in the obvious way from (2.5.56). Generally, (2.5.57) is an implicit equation for $\alpha^2$, since the weights depend on $\alpha^2$ through $\rho^2$. Taking the simple case where all of the $c_j$ are of the same magnitude, $|c|$, say, implies that we want $\alpha = |c|$. This gives $\rho^{-1} = |c|/\sigma$, and justifies reference to $\rho^{-1}$ as the signal to noise ratio.

### 2.5.2 Cross validation

Knowing the form of the theoretically optimal value of $\alpha^2$ or $\rho^2$ is useful for interpretive purpose, but generally is of small practical consequence. Good data analysis requires a method that does not necessarily depend on a priori knowledge of the scale of the signal and/or noise, knowledge that may often be vague at best. One such method that is widely used in statistical function-estimation, as well as in other data-analytic contexts, is founded on the principle that a good model will do well at predicting data that have not been used to fit the model. The method of cross-validation (CV), as described by G. Wahba [60] and others, prescribes that we make use of this principle by systematically setting aside subsets of a given collection of data, fitting a variety of models to the remaining data, and then using each fitted model to "predict" values of the observations that have been set aside. By carrying out this procedure, various models may be compared with one another on the basis of how well they predict independent observations. This sort of comparison tends to save us from "overfitting" by trying to make the residuals from a fit to an entire data set too small, for two sets of observations produced by a combination of signal and independent, random errors will tend to have only the signal component in common. If we fit "noise" in one set of observations, then we will tend to inflate the errors incurred in predicting another set since the noise in the second set will not cohere with that in the first.
The principle of cross-validation may be applied in various forms, differing from one
another in the choice of subsets for fitting and testing. We will investigate the use of a
common and convenient form for linear models, the so-called "leave one out" form, which
compares models by fitting each \( n \) times to \( n - 1 \) observations obtained by leaving out each
data point in succession. For each of the \( n \) fits, the estimated model is used to predict
the omitted observation, and the resulting prediction errors are squared and summed
to provide an assessment of model quality. The preferred model, by this criterion, is the
one producing the smallest sum of squared prediction errors.

The algorithm for using CV to guide the choice of the damping parameter in the
present context is as follows. Let \( \mathbf{Y}_{-i} \) be the \( n - 1 \) vector formed by leaving out the
\( i^{th} \) observation in (2.4.39), and \( \mathbf{G}_{-i} \) be the corresponding design matrix with the \( i^{th} \) row
deleted. For a particular value of \( \rho^2 \), use \( \mathbf{Y}_{-i} \) to estimate the coefficients, \( \mathbf{c} \), as in (2.4.42)

\[
\hat{\mathbf{c}}_{-i}(\rho) \triangleq (\mathbf{G}_{-i}^T \mathbf{G}_{-i} + \rho^2 \mathbf{I})^{-1} \mathbf{G}_{-i} \mathbf{Y}_{-i}.
\]  

(2.5.58)

From \( \hat{\mathbf{c}}_{-i}(\rho) \), get the predicted value of \( Y_i \),

\[
\hat{Y}_{-i}(\rho) \triangleq \mathbf{g}_i^T \hat{\mathbf{c}}_{-i}(\rho),
\]  

(2.5.59)

and the prediction error

\[
e_{-i}(\rho) \triangleq Y_i - \hat{Y}_{-i}(\rho).
\]  

(2.5.60)

The \textit{cross-validated sum of squared errors} is formed by

\[
\text{CVSS}(\rho) \triangleq \sum_{i=1}^{n} e_{-i}^2(\rho).
\]  

(2.5.61)

The prescription is to vary \( \rho \) and compute CVSS in order to find the value of \( \rho \) giving the
smallest sum of squared prediction errors.

Carrying out the leave-one-out CV procedure just described may appear computa-
tionally demanding, particularly for large data sets. Trying enough different values of \( \rho \)
to find accurately the minimum of the CVSS function, and fitting a linear model to \( n - 1 \)
points \( n \) times for each value of \( \rho \) could, indeed, be a burden if \( n \) is large. But a neat
algorithmic simplification enables us, in fact, to compute CVSS by fitting only the model
only once for each choice of $\rho$. This simplification is available through the so-called "hat matrix", $H(\rho)$,

$$H(\rho) \define G(G^T G + \rho^2 I)^{-1} G^T, \quad (2.5.62)$$

which maps observed to predicted values of $\mathbf{Y}$ in the full data set:

$$\hat{\mathbf{Y}}(\rho) = H(\rho) \mathbf{Y}. \quad (2.5.63)$$

By recognizing that the cross-validated predictions, $\hat{Y}_{-i}(\rho)$, differ systematically from $\hat{Y}_i(\rho)$, we may use updating formulas relating the inverses of two matrices differing by a matrix of rank one to derive the "leave one out lemma" (Wahba, [60]), the upshot being that, if $\mathbf{e}(\rho)$ is the residual vector from the full fit,

$$\mathbf{e}(\rho) \define \mathbf{Y} - \hat{\mathbf{Y}}(\rho)$$

$$= (I - H(\rho)) \mathbf{Y}, \quad (2.5.64)$$

then

$$CVSS(\rho) = \sum_{i=1}^{n} \left\{ \frac{e_i(\rho)}{1 - h_{ii}(\rho)} \right\}^2, \quad (2.5.65)$$

$h_{ii}(\rho)$ being the $i^{th}$ diagonal element of the hat matrix. Thus, to compute the cross-validated sum of squares for a given value of $\rho$, we simply fit model once using all of the data, pick out the residuals and diagonal elements of the hat matrix from this fit, and then compute the sum of squared ratios dictated in (2.5.65).

Results of carrying out cross-validation on the Point Arena data are depicted in Figures 2.12-2.14. Each figure contains four plots, the top two showing the behavior of the squared norm (the sum of squared coefficient estimates) and the residual sum of squares as $\rho$ varies, the bottom left showing the behavior of CVSS, and the bottom right showing the trade-off between solution norm and data misfit parametrized by $\rho^2$. As we would expect, larger values of $\rho^2$ "damp" the size of the estimated coefficients toward zero, producing smaller solution norms and larger apparent misfits to the data as measured by the residual sum of squares. Over some range, however, increasing $\rho$ decreases the cross-validated error sum of squares. There is, in each instance, a value of $\rho$ producing a clear minimum
Section 2.6. Point Arena analysis

Figure 2.12: Behavior of energy, residual sum of squares, and cross-validated sum of squares as a function of $\rho^2$ for minimum energy estimate.

of the CVSS function; changing $\rho$ from this value in either direction leads to degradation of model quality as measured by CVSS.

2.6 Estimation of coseismic slip by inversion of Point Arena data

Combining methods for generating finite-dimensional representations of solutions, as described in [36] and summarized in Section 3, with the statistical tools developed in Sections 4 and 5, we have the means to perform inversions and interpret the results. In this section, we apply our inversion algorithms to the Point Arena triangulation data in order to
Figure 2.13: Behavior of stress magnitude, residual sum of squares, and cross-validated sum of squares as a function of \( \rho^2 \) for minimum SM estimate.
Figure 2.14: Behavior of stress variability, residual sum of squares, and cross-validated sum of squares as a function of $\rho^2$ for minimum SV estimate.
produce estimates of coseismic slip in the 1906 earthquake. As part of this application, we consider an issue that has not been raised to this point: faulting depth. When describing how to produce basis functions minimizing the various norms of interest, we have always stated, "If the assumed faulting depth is ..., then the minimum-norm representer is given by ..." Our strategy for this practical application is simply to construct basis functions in each norm for each of several assumed faulting depths, spanning a reasonable range, and then to compare results produced by the various depth assumptions. The depths used in our analyses are: 5, 10, 15, 20, and 30 km. Orthobasis functions for the extreme depths, 5, and 30 km, in each norm are shown in Figures 2.15–2.20. Our a priori belief was that 5 km would be too shallow, 30 km too deep, and the intermediate depths all plausible. Based upon our analyses of the data, we prefer depths in the 15–20 km range. This preference is somewhat at odds with the conclusions of earlier analyses, in which faulting depths in the 5 to 10 km range have been found to produce the best fits by some empirical criterion (see [57] and references therein). We will discuss this apparent discrepancy below.

2.6.1 Cross-validation and noise estimates

Table 2.4 summarizes the results of applying the cross-validation procedure of Section 5.2 to the Point Arena data with basis functions generated by the SE, SM, and SV norms at assumed faulting depths of 5, 10, 15, 20, and 30 km. For each combination of norm and depth, the third column of the table, labeled “Minimum CVSS” reports the smallest cross-validated sum of squared prediction errors; the fourth column lists the value of \( \rho^2/\lambda_2 \) at which the minimum is attained.

Our description of cross-validation in Section 5 was focused on its use in estimating the signal-to-noise ratio for any specific finite-dimensional representation of the solution, but there is no reason not to use the prediction principle underlying the method to compare various representations. In so doing, we find that, at all but the shallowest depth, the SV norm representation produces the smallest value of CVSS, followed in order by the SM and SE representations. When slip is allowed only to 5 km, the SM norm does
Figure 2.15: Minimum-energy orthobasis functions, $D = 5$ km.
Section 2.6. Point Arena analysis

Minimum SM orthobasis functions, $D = 5$ km

Figure 2.16: Minimum-SM orthobasis functions, $D = 5$ km.
Figure 2.17: Minimum-SV orthobasis functions, $D = 5$ km.
Section 2.6. Point Arena analysis

Minimum SM orthobasis functions, D = 30 km

Figure 2.18: Minimum-energy orthobasis functions, D = 30 km.
Section 2.6. Point Arena analysis

Minimum energy orthobasis functions, $D = 30$ km

Figure 2.19: Minimum-SM orthobasis functions, $D = 30$ km.
Section 2.6. Point Arena analysis

Minimum SV orthobasis functions, D = 30 km

Figure 2.20: Minimum-SV orthobasis functions, D = 30 km.
marginally better than SV. Comparing CVSS across depths, the orderings from best to worst fits in each of the norms are as follows. SE: \((20,30) > 15 > 10 > 5\); SM: \(15 > 10 > 20 > 30 > 5\); and SV: \(20 > 15 > 10 > 30 > 5\). As measured by CVSS, the best fit among all combinations of norm and depth is given by the SV norm at a faulting depth of 20 km. In every norm, faulting to only 5 km is unable to produce an acceptable fit, supporting the belief that the earthquake rupture certainly extended below this depth.

In addition to the cross-validation results, Table 2.4 lists the degrees of freedom used to fit estimate the signal (Signal df), the sum of squared misfits to the measured angle changes (SSE), and an estimate of the standard deviation of the errors in the data \(\hat{\sigma}\). Degrees of freedom are calculated by analogy to ordinary least squares estimation where, when a model with \(p\) parameters is fit to \(n\) observations, the expected sum of squared errors is \(\sigma^2 \cdot (n - p)\), with \(p\) interpreted as the degrees of freedom used in the fit and \(n - p\) as the degrees of freedom for error. In the present situation, the residual vector, defined in terms of the hat matrix in equation (2.5.64) has expected squared norm

\[
ESSE \equiv E\|e\|^2 = E\|(I - H)Y\|^2 = c^T(I - H)^2c + \sigma^2Tr((I - H)^2). \tag{2.6.66}
\]

If the bias term, \(c^T(I - H)^2c\), in (2.6.66) is small, then the expected error sum of squares will be about \(\sigma^2Tr((I - H)^2)\), which leads us to interpret \(Tr((I - H)^2)\) (which equals \(n - p\) in ordinary least squares) as the error degrees of freedom and \(n\) minus this quantity as the degrees of freedom used up in fitting the signal. Thus, the quantity listed in column 5 of Table 2.4 is 30 minus the trace of \((I - H)^2\) each particular fit, and is consistently in the 1.5–2.5 range. This indicates that the estimated signals have only about two linearly-independent components.

The estimated standard deviations in column 7 are given by

\[
\hat{\sigma} = \sqrt{\frac{SSE}{30 - \text{Signal df}}} \tag{2.6.67}
\]

The model-based estimates of the noise level are consistently on the order of 12.5, slightly
larger in the 5 km fits. If we remove the antiplane assumption, and allow 2 degrees of free motion for each benchmark, we may compute model-independent error estimates using the triangle misclosure method of Thatcher [57] or the pure error method of Segall and Matthews [49]. There are two complete triangles in the Point Arena network, CLARK-DUNN-LANE with angles measuring \(-39.8''\), \(9.3''\), and \(21.8''\), giving misclosure \(M_1 = 8.7''\), and LANE-SHOEMAKER-SPUR with angles \(-357.6''\), \(119.8''\), and \(231.9''\) giving misclosure \(M_2 = 5.9\). The misclosure estimate is thus \(\hat{\sigma}_{\text{misclosure}} = \sqrt{(M_1^2 + M_2^2)/6} = 4.29\). This is slightly larger than the pure error estimate, obtained by dividing the pure error sum of squares, SSPE = 145.75, by the degrees of freedom for pure error, DFPE = 12, and taking the square root, yielding \(\hat{\sigma}_{\text{PE}} = 3.49\). Both model-independent estimates are on the order of three to four times smaller than the estimates based on antiplane inversion, the discrepancy presumably attributable to some combination of departure from model idealizations, inadequacy of the antiplane approximation, and poor resolution of the triangulation network.

Table 2.5 lists observed angle changes are their predictions based on the best-fitting minimum SV model with slip to 20 km. The largest absolute misfits tend to occur on the angles with the largest observed changes, indicating that the assumption of identically distributed errors in (2.2.7) is not really appropriate. Thatcher [57] remarked on a similar pattern in misfits to his constant dislocation models and identified the probable cause of this pattern in the fact systematic departures from the idealized model assumptions are likely to show up most strongly where the signal is largest. Using the inner-coordinate generalized inverse of \(A\) as described by Segall and Matthews [49], we have directly calculated estimates of two-dimensional station displacements that minimize magnitudes of fault-normal motion. These estimates reveal an appreciable component of convergence, particularly in the motions of near-fault benchmarks LANE, SPUR, and SHOEMAKER. This convergent component, which can’t be fit in the antiplane model, is probably the source of the relatively large errors in predicted change for angles involving benchmarks closest to the fault. The measurement from SHOEMAKER through LANE to ARENA LIGHTHOUSE supports this belief as the large discrepancy between the observed and
### Table 2.4: Cross-validation results and estimated noise levels from inverting Point Arena data in various norms and at various assumed faulting depths.

<table>
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<tr>
<th>Faulting depth, km</th>
<th>Basis minimizes</th>
<th>Minimum CVSS</th>
<th>( \hat{\rho}^2 )</th>
<th>Signal df</th>
<th>SSE</th>
<th>( \hat{\sigma} )</th>
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<td>0.2765</td>
<td>2.666</td>
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<td>2.440</td>
<td>4685.02</td>
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<td></td>
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predicted angle changes may be attributable to convergent components in the motion of SHOEMAKER and LANE.

2.6.2 Estimates of slip and moment

Table 2.6 summarizes the results of inverting for the coseismic slip distribution by estimating the coefficients of the first four orthobasis functions in each norm and at each assumed faulting depth using the values of $\rho^2$ recommended by cross-validation. The columns of the table index the number of the basis function, number 1 being that associated with the largest eigenvalue of the Grammian, 2 with the second largest, and so on. Rows are paired, the top one at each combination of depth and norm giving the coefficient estimates and the bottom giving the estimated standard errors of the estimates. The estimated standard errors are obtained by plugging the estimates of $\sigma^2$ given in Table 2.4 into equation (2.4.46). Examining Table 2.6, we typically find that the ratios of the estimated coefficients to their estimated standard errors decrease in order of the index of the corresponding basis function. The fact that the coefficients of the third and fourth basis functions tend to be small when measured in units of standard error is yet another indication that the signal in these directions is apparently below the noise level and is severly damped in the resulting estimates. This means that either the signal in these directions is actually small or that it is lost on account of poor network coverage and large measurement error.

Figures 2.21–2.24 show the estimated slip distributions in each of the three norms at faulting depths of 5, 10, 20, and 30 km. The similarity of the results among each of the three norms at a fixed depth is apparent, though the differences with varying depth are interesting. Estimates in which slip is confined to depths shallower than 5 km grow rapidly from the base of the slip zone to reach maximum slip in the 8 to 9 meter range at depths of 2 to 3 km and then descend back to around 6 meters at the surface. These estimates are apparently trying to accommodate both moment constraints supplied by observations more distant from the fault and shallow slip constraints provided by near-fault measurements. With slip constrained by assumption at the base of the slip zone and by near-field data at
<table>
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<th>To</th>
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<td>HIGH BLUFF</td>
<td>37.3</td>
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Table 2.5: Point Arena angle changes observed and predicted by minimum-SV inversion with D = 20 km.
the surface, the only place to put the required moment is in the middle. Estimates with other assumed faulting depths do not have to vary as much to accommodate moment as do those at 5 km. It is noteworthy that the estimates from 10 km are, for the most part, concave, i.e., negatively curved, while estimates from 20 and 30 km are mostly convex (positively curved). The concave estimates grow rapidly at depth and tend to level off near the surface, and the convex estimates do the opposite — grow slowly at depth and rapidly nearer the surface. There is an apparent transition from concavity to convexity at about 15 km, where the estimates are nearly linear, as may be seen in Figure 2.1. Estimates produced with 30 km of vertical faulting tend to put very little slip below about 20 km. In fact, the minimum SV estimate at 30 km is even slightly left-lateral between 30 and 20 km. This behavior seems to indicate that the data do not require slip below about 20 km, as might be expected either because there was no coseismic slip at those depths or because the benchmarks in this network are too close to the fault to pick up such deep slip if it was present.

From estimates of the coseismic slip distribution, we may derive estimates of quantities depending on slip. Two such quantities of obvious interest are the value of slip at the surface, which provides some measure of consistency with independent observations, and the moment, which provides an overall measure of the size of the earthquake. Estimates and standard errors of surface slip and moment per unit rupture length derived from each of the models fit in Table 2.6 are listed in Table 2.7. Estimates of surface slip are in the five to seven meter range, concentrating in the vicinity of six meters for the best fitting models. These values are slightly higher than the reported offsets of fault-crossing landmarks, which were close to five meters (see Thatcher [57]).

Moment estimates tend to grow with faulting depth, as might be expected, though they are fairly consistent over the 15–20 km range of preferred depths. Letting \( m \) denote the moment per unit rupture length, as reported in Table 2.7, the seismic moment may be computed as \( M_0 = m \times L \times \mu \), \( L \) being the rupture length and \( \mu \) the shear modulus. Taking \( L = 400 \) km and \( \mu = 3 \times 10^{11} \) dynes/cm\(^2\), and plugging in the highest and lowest estimates of \( m \) from models slipping to depths of 15 or 20 km, we find the range of seismic
Figure 2.21: Three estimates of slip in 1906 San Francisco earthquake obtained by minimum-norm inversions of the Point Arena triangulation data with $D = 5$.
Figure 2.22: Three estimates of slip in 1906 San Francisco earthquake obtained by minimum-norm inversions of the Point Arena triangulation data with $D = 10$. 

Slip estimates minimizing different norms, $D = 10$
Figure 2.23: Three estimates of slip in 1906 San Francisco earthquake obtained by minimum-norm inversions of the Point Arena triangulation data with $D = 20$. 
Figure 2.24: Three estimates of slip in 1906 San Francisco earthquake obtained by minimum-norm inversions of the Point Arena triangulation data with $D = 30$. 
moments as $4.97 \times 10^{27}$ to $6.11 \times 10^{27}$ dyne-cm. These moment estimates are slightly larger than Thatcher’s [57], who produced moments ranging from $1.6$ to $4.5 \times 10^{27}$ dyne-cm based on his inversions of the Point Arena data. The discrepancy is accounted for by the deep slip in our models.

2.7 Discussion and conclusions

We have described statistical methods for estimation of fault slip that may be applied to analyze geodetic data once a finite-dimensional representation for the solution is in hand. Combined with the variational methods described in [36] and summarized in Section 3, these methods produce a complete linear inversion algorithm for estimating slip distributions related to geodetic observations by linear elastic dislocation models. It may be noted that the basic ideas of our analysis – variational methods to produce optimal representations, eigen analysis to orthonormalize the derived representation and to gain insight in to the “visibility” of signal components, and empirical determination of the apparent signal-to-noise-ratio – are potentially applicable to any linear inverse problem.

The convenience of linearity in our inversion algorithms promotes the thorough analysis of signal components and noise levels described in Sections 4 and 5. We have been fortunate, perhaps, in finding that linear methods have performed quite well in application to inverting the Point Arena triangulation data for 1906 coseismic slip. It is conceivable that other applications may demand imposition of nonlinear constraints such as positivity, though we have found that linear, minimum-norm solutions tend to satisfy natural constraints when the signal-to-noise ratio is sufficiently high and the estimates are adequately smoothed.

Linear network analysis by examination of eigenvalues or singular values of the so-called design matrix, $G$, in equation (2.4.39) is potentially useful in addressing the issue of network design. Faced with the choice of how to configure a new network or where to add benchmarks to an existing network, one may, for any hypothetical configuration, examine the eigen or singular values of $G$ in order to determine how well-conditioned
### Table 2.6: Estimates of coefficients and their standard errors, in parentheses, for the first four orthobasis functions from inverting Point Arena data in various norms and at various assumed faulting depths.

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<tr>
<th>Faulting depth, km</th>
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<th>3</th>
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<td>s.d.</td>
<td>Moment per unit length, m Estimate</td>
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Table 2.7: Estimates and standard errors of surface slip and moment per unit lateral rupture length in the 1906 earthquake from inversion of Point Arena angle changes.
the signal-estimation problem for that network would be. Network configuration may be
guided by the principle that we would like the singular values of the design matrix to be
as large as possible, since, as we have seen, it is easiest to pick up signal components in
directions corresponding to the largest singular values. Various notions of optimal design
for linear experiments, (Cf. [14, 50]) based on criteria such as maximizing the product
of the singular values, so-called D-optimality, or the sum of the singular values, so-called
A-optimality, may and should be of use in this regard.

Our analysis of the Point Arena triangulation data has produced results that are
somewhat at odds with conclusions drawn by others who have looked at the same data. On
the question of faulting depth, Thatcher [57] cites work by Kasahara [22], Chinnery [11],
and Stacey [52] that brackets faulting depth in the 2 to 6 km range. Thatcher’s own analysis
using a χ² goodness-of-fit statistic confines the range of admissible faulting depths for a
uniform dislocation fit to the Point Arena data in the 5–12 km range, and he concludes
essentially that geodetic data constrain slip in the 1906 earthquake to depths of 10 km or
less. In contrast, our best-fitting estimates allow slip to depths in the 15 to 20 km range,
putting small, but nonnegligible moment below 10 km. Inversions with deep faulting
consistently attain values in the vicinity of two meters right-lateral slip by the time they
reach a depth of 10 km, in fact, regardless of the depth at which they start. Hence, we
find that the argument in favor of slip at depths greater than 10 km is reasonably strong,
being supported both by goodness of fit to the geodetic observations and consistency of
the deep slip estimates obtained by various choices in the inversion algorithms.

A primary difference between the algorithms we have proposed and those that have
been previously applied to inversion for fault slip is the replacement of unphysical, uni-
form, rectangular dislocations by solution representations that satisfy physical constraints
on stress and energy concentration and, in so doing, produce smooth estimates with slip
tapered to zero at depth. This difference apparently leads to the discrepancy between
our results and earlier conclusions on faulting depth and moment release in the 1906
earthquake. Our representations for solutions are not, in our opinion, any less “simple”
or “intuitive” than uniform dislocation representations. Furthermore, we believe that by
using physical principles in the formulation of the slip inverse problem and by applying useful data-analytic tools within the context of physically-derived representations for solutions, we may produce more accurate insights into the nature of deep slip than have been obtained by cruder methods. This belief may have been realized in our analysis of slip in the 1906 earthquake, at least to the extent that we have been able to shed new light on some old questions that have been the subject of some controversy. Of course, one example does not entirely justify faith in any given data-analytic procedure. In order to understand when our methods do or do not work, we must generalize them to more complicated situations, as we have done in [33], producing minimum-SV estimates for a two-dimensional distribution of strike slip on a vertical fault. In future work, we hope to use variations on the methods we have described to analyze other collections of geodetic data, both real and simulated. In the course of such analyses, we may be able to draw stronger conclusions on how well our methods perform both absolutely and in comparison to popular alternatives.
Chapter 3

Two-dimensional, steady-state problems

3.1 Introduction

The desire to use crustal deformation data as a tool in the study of fault behavior often leads to a geophysical inverse problem. Essentially, if we conceive of fault motion as the physical source of a spatially and temporally coherent component in a collection of deformation measurements, we may try to identify this component and map it back to the source, thereby learning something of the nature of the mechanical processes of fundamental interest. The inverse problem relating crustal deformation to fault slip, like many inverse problems from geophysics and elsewhere, raises both theoretical issues, such as specifying the model and data spaces and analyzing properties of the implied association between points in these spaces, and practical issues regarding algorithms for removing noise and computing estimates.

In Chapters 1 and 2, we have developed inversion algorithms for estimating fault slip from crustal deformation measurements, and we have illustrated the performance of these algorithms by applying them to a simple antiplane dislocation model in which slip is assumed to vary only with depth on a fault of infinite lateral extent in a half-space. The
Section 3.1. Introduction

basic goal of the inversion algorithms we described was to find, from among all slip distributions consistent with a given collection of deformation measurements, those minimizing various norms chosen on physical grounds. We investigated three particular norms, the self energy, the stress magnitude, and the stress variability, the first of which depends on the change in crustal strain energy accompanying slip, and the second two on changes in the magnitude and roughness of the fault-surface traction distribution, respectively. In Chapter 1, we described the manner in which optimal linear representations of source functions minimizing any of the three chosen norms may be derived by applying a simple geometric principle, and we specified either numerical methods or closed form expressions for computing the desired representations in the antiplane model. In Chapter 2, we built upon the framework of Chapter 1 by applying statistical arguments to determine appropriate values of the coefficients weighting the basis functions derived from the optimization principles of Chapter 1, and we applied the resulting finite-dimensional, linear, statistical model to the problem of estimating the coseismic slip distribution in the 1906 San Francisco earthquake by inverting triangulation measurements from Point Arena, California.

Our present goal is to generalize some of the results of Chapter 1 to more complicated faulting models. As in Chapter 1, we use linear elastic dislocation models to relate fault slip to surface deformation, but instead of the one-dimensional antiplane faulting, we consider slip distributions varying with both depth and lateral position over a planar fault surface. Only one of the three functionals defined in Chapter 1 extends usefully to two-dimensional faulting regimes: the stress-variability (SV) functional. The shortcomings of the self-energy and stress-magnitude functionals lie in their failure to smooth slip distributions sufficiently, in the sense discussed at length in Chapter 1. In contrast, the SV functional does impose smoothness requirements that are stringent enough to enable formulation and solution of well-posed boundary value problems for slip estimation.

In Section 2, after introducing the forward model from linear elastic dislocation theory, we motivate and define the stress variability functional and we state the inverse problem that we wish to solve: among all slip distributions consistent with a set of deformation
measurements, find the one with the smoothest distribution of fault surface traction. We address this problem with methods similar to those applied to the antiplane model in Chapter 1, using Fourier transforms to express the stress-variability due to slip in unbounded space, where the operator acting on slip to produce the displacement field is of convolution type. In Section 2.3, we state a result directly analogous to a result given in Chapter 1 for antiplane slip, namely, that stress-variability in unbounded space has an equivalent form involving $L^2$ norms in second derivatives of slip. Details of the arguments leading to this result are given in the appendix. For geophysical applications, in which a crustal volume of interest is modeled as a half-space, we adopt without modification the differential form of the stress-variability derived in unbounded space and use the resulting expression in terms of derivatives of slip as an approximate measure of stress-variability. The rationale for this approximation lies in the belief that a measure of smoothness is sensitive primarily to local properties of a function and is not greatly influenced by the presence or absence of a boundary. In Section 2.4, we use the derived approximation to the stress-variability functional to look at the problem of optimal representations for strike-slip sources on a vertical fault. After making a small, additional simplification in the form of the functional to be minimized, we find that optimal source functions for the vertical, strike-slip fault may be found by solving elliptic boundary-value problems involving a biharmonic operator. Using a numerical biharmonic equation solver, we are able to find these optimal functions, as we illustrate with examples in Section 2.4.

In Section 3, we apply the source representations derived for a vertical, strike-slip fault in Section 2 to a practical problem: estimating the interseismic slip rate on a segment of the San Andreas fault from measured trilateration data. We use data from the two-color geodimeter network that has been in operation near Parkfield, California since 1984. In using geodetic data from Parkfield to estimate interseismic slip rates at depth, we revisit ground covered by Segall and Harris [47, 48, 16], who used a set of geodetic measurements collected over a twenty-year period to invert for the interseismic rate at Parkfield and for the coseismic slip distribution in the 1966 Parkfield earthquake.

The Parkfield segment of the San Andreas is a region of transition between a "creeping"
zone to the northwest and a “locked” zone to the southwest. It is of primary interest in the inversion of geodetic data from Parkfield to learn what we can about the nature of the transition from creeping to locked regimes. In their analyses, Segall and Harris [47, 16] found that geodetic observations are not entirely consistent with a “smooth” model for Parkfield slip, i.e., an interseismic slip distribution that smoothly interpolates boundary values imposed by surface creep measurements and assumed deep slip rates. Segall and Harris concluded that a locked zone at depth, roughly colocated with the inferred rupture zone of the 1966 Parkfield mainshock, produced a better fit to the rates of line-length change determined by repeated geodetic measurement. Using the inversion methods of Section 2 and new geodetic data from Parkfield, we find some agreement with the results of Segall and Harris. Our preferred estimate for the interseismic slip rate, the genesis of which we describe in Section 3, is shown in Figure 3.1. Though there appear to be features of the two-color data that are inconsistent with the idealized model of pure strike-slip on a vertical plane, there are also some recognizable, systematic effects that are would be expected with a slip model exhibiting retardation from observed rates of surface motion, as was found by Segall and Harris. In Figure 3.1, the zone of low slip appearing just to the southwest of Car Hill at a depth of about 5 to 8 km is consistent in location and slip magnitude with the zone of low slip identified by Segall and Harris, and the moment-deficit rate of $2.9 \times 10^{24}$ dyne-cm/year in the Parkfield seismogenic zone implied by this estimated slip rate distribution gives a time-predictable expected interval between Parkfield mainshocks as 25.2 years, in agreement with, though not necessarily in support of, the 22 year expected recurrence interval assumed in the Characteristic Parkfield Earthquake model of Bakun and McEvilly [6].
Figure 3.1: Estimated interseismic slip rate in the Parkfield seismogenic zone from inversion of two-color data.
3.2 Representation of source functions

3.2.1 The forward model

Let \( \mathcal{X} \) be a homogeneous, isotropic, linearly elastic volume with an embedded planar fault, \( \Sigma \). From a displacement field, \( u(x), x \in \mathcal{X} \), define slip at a point \( \xi \in \Sigma \) by

\[
s(\xi) \triangleq u(\xi^+) - u(\xi^-),
\]

where \( \xi^+ \) and \( \xi^- \) represent limits as a point approaches \( \xi \) along the direction normal to \( \Sigma \) at \( \xi \) from the "positive" and "negative" sides, respectively. Hooke's law relating stress, \( \sigma_{ij} \), and strain, \( \epsilon_{pq} \), is

\[
\sigma_{ij}(x) = C_{ijpq} \epsilon_{pq}(x),
\]

with the stiffness tensor for isotropic media expressed in terms of the Lamé constants, \( \lambda, \mu \), as

\[
C_{ijpq} = \lambda \delta_{ij} \delta_{pq} + \mu (\delta_{ip} \delta_{jq} + \delta_{iq} \delta_{jp}).
\]

The relationship between slip and displacement at an arbitrary body point is given by Volterra's formula:

\[
u_k(x; s) = \int s_i(\xi) C_{ijpq} \frac{\partial \Gamma_{kp}(x, \xi)}{\partial x_q} \nu_j(\xi) d\xi = \int s_i(\xi) G_{ij}^k(x, \xi) \nu_j(\xi) d\xi.
\]

In (3.2.4), \( \Gamma_{kp}(x, \xi) \) is the displacement in the \( k \) direction at \( x \) due to application of a unit point force in the \( p \) direction at \( \xi \), and the Green's tensors in (3.2.5) are

\[
G_{ij}^k(x, \xi) \triangleq C_{ijpq} \frac{\partial \Gamma_{kp}(x, \xi)}{\partial x_q}.
\]

See Love [30], Stekete [53, 54], Maruyama [32], and Iwasaki and Sato [19]) for discussion of dislocation models and specification of the Green's tensors for particular fault geometries.

In geophysical applications, a common model approximates the Earth's crust as a half-space with a stress-free surface. Dislocation models in a half-space are reasonably tractable, but less so than in unbounded space, where the Green's tensors are of convolution type and Fourier analysis may be used to examine the nature of the singular integral.
operator propagating slip to displacement. When $\mathcal{X}$ is unbounded, the components of $\Gamma_{kp}$ in (3.2.4) form the so-called Somigliana tensor, to be denoted by $S_{kp}$, with components given [30] by

$$
S_{kp}(x, \xi) = S_{kp}(x - \xi) \triangleq \frac{1}{8\pi\mu} \left[ 2r^{-1} \delta_{kp} - \frac{\lambda + \mu}{\lambda + 2\mu} \frac{\partial^2 r}{\partial x_k \partial x_p} \right],
$$

(3.2.7)

with $r = \sqrt{(x_j - \xi_j) \cdot (x_j - \xi_j)}$ being the Euclidean distance from the source point to the field point.

Though there is a displacement discontinuity at $\Sigma$, the dislocation field equations are derived under the assumption that traction is continuous across the dislocation surface. Under this assumption, we define the fault surface traction field in a dislocation model by

$$
\tau_j(\xi) \triangleq \lim_{x \to \xi} \sigma_{ij}(x) \nu(\xi)
$$

(3.2.8)

Certain smoothness conditions must be imposed on the slip distribution in order that the required limits exist in (3.2.8). These conditions are well-known in potential theory, where the elastostatic displacement field in unbounded space is seen as a generalized vector potential field generated by a surface distribution of double sources with density $s$. In potential theoretic terms, the surface traction is analogous to the normal derivative of a vector potential field, and theorems of Günter [15] and Kupradze [26] (see also, [20]) imply that continuity of $s$ is sufficient to insure that

$$
\lim_{\epsilon \to 0} \left\{ \sigma_{ij}(\xi_1, \xi_2, \epsilon) - \sigma_{ij}(\xi_1, \xi_2, -\epsilon) \right\} \nu(\xi_1, \xi_2) = 0.
$$

(3.2.9)

A sufficient condition for the existence of the individual limits in (3.2.9) and for the surface traction distribution to be well-defined is that $s$ and its partial derivatives of order up to and including 2 be continuous. Functions on $\Sigma$ satisfying this smoothness condition will be referred to as being in the class $\mathcal{C}^2(\Sigma)$.

### 3.2.2 The inverse problem and a smoothness principle

Dislocation theory is used in geophysical data analysis to model the dependency of measured crustal deformation on a physical cause: fault motion. Suppose that we have a
Section 3.2. Source representation

collection, \( y = (y_1, \ldots, y_n)^T \), of deformation measurements. These may be measurements of any field values – displacements, strains, stresses, etc., – measured at any collection of points in \( \mathcal{X} \). Using the expression relating slip to displacement and, as necessary, expressions relating displacement to other field properties, the dependence of the value of the \( i^{th} \) measured field property on a hypothetical distribution of slip has a functional form, \( \phi_i(s) \). Viewing the data as produced by adding perturbations to the deformations predicted by some distribution of slip, we have the observational model

\[ y_i = \phi_i(s) + \epsilon_i, \quad i = 1, \ldots, n. \]  

(3.2.10)

The perturbations, \( \epsilon_i \), in (3.2.10) may have both a random component due to chance measurement errors, for instance, and a systematic component reflecting coherent departures from idealized model assumptions.

Writing the relationship in (3.2.10) as

\[ y = F(s) + \epsilon, \]  

(3.2.11)

where the forward mapping, \( F \) takes \( s \) to the vector of predicted values, \( (\phi_1(s), \ldots, \phi_n(s))^T \), the inverse problem of interest requires mapping the observed values, \( y \), back to an estimated source function, \( s \), effectively inverting the forward mapping. In order to construct an inverse, we must identify the model space, which is the domain of the the forward mapping and will be the range of the inverse mapping. Identification of the model space requires description of the spatial area over which slip is allowed to vary, usually a bounded subset of a fault surface that may be infinite in some directions, specification of boundary conditions at the edges of the slip zone, and recognition of physical requirements that must be imposed on admissible source functions. As we have stated, we will generally conceive of the fault surfaces as infinite planes in unbounded spaces, or semi-infinite planes in half-spaces, but we will be interested in applying inversion algorithms to image slip distributions supported on rectangular subregions of these fault planes. We will impose smoothness requirements, alluded to at the end of the previous subsection, that are essentially necessary to insure existence of the fault surface traction distribution. Specifically,
we will generate a model space by requiring that all components of slip have continuous second derivatives in the slip zone, and first derivatives that are continuous across the boundaries of the slip zone. Letting $\Sigma_0$ denote the assumed slip zone, the model space will essentially coincide with the space of functions in $C^2(\Sigma_0)$ which, along with their boundary-normal derivatives, satisfy homogeneous boundary conditions. We will refer to this space as $C_0^2(\Sigma_0)$.

The inverse problem requiring mapping of $y$ to some slip function in $C^2(\Sigma_0)$ is ill-posed regardless of the number of observation points in $y$. If $y$ is finite, then the linear forward mapping for $C_0^2(\Sigma_0)$ to the finite-dimensional data space has an infinite dimensional null space. Even if $y$ is infinite and the relationship between points in the model space and data space is one-to-one, the inverse problem is ill-posed due to the fact that points in the model space that are far apart, in some natural sense, map to data sets that are close together. In order to pose stable inverse problems that have unique solutions even with finite collections of data, we will appeal to a physically-motivated smoothing principle for slip on faults loaded by remotely-acting tectonic forces. This principle is based on the assumption that tectonic forces acting at a distance and accommodated on the fault plane by aseismic slip or by large earthquakes will tend to produce smooth stress fields. Following Andrews [3], for instance, who has modeled fault surface stress spectrally as the sum of a coherent distribution in wave number space, produced by tectonic loading, and a small component of self-similar "$1/f$" component that results from and leads to a fractal distribution of small earthquakes, we may, choose to to aim for slip estimates that are, as nearly as possible, consistent with a fault surface traction field that is viewed as being uniform to first order. We apply this principle to the inverse problem by defining a functional that measures the deviation from uniformity of the fault-surface traction field produced by any given distribution of slip and then minimizing this functional over the class of admissible slip distributions that are consistent with a given set of observations.

To measure deviation from uniformity in a surface traction distribution, we use the stress variability (SV) functional, defined for antiplane slip distributions in Chapter 1,
and more generally by
\[
SV(s) \triangleq \int \frac{\partial \tau_i(\xi; s)}{\partial \xi_j} \cdot \frac{\partial \tau_i(\xi; s)}{\partial \xi_j} d\xi. \tag{3.2.12}
\]
This functional measures the squared $L^2$ magnitude of the plane first derivatives of the surface traction distribution. Clearly, it is a nonnegative definite quadratic functional in slip that evaluates to zero only for slip distributions producing everywhere constant since the integral in (3.2.12) extends over the entire fault plane, not just over the slip zone. The only uniform stress field that may be produced on an infinite or semi-infinite plane by slip on a bounded subregion and with homogeneous boundary conditions is the trivial field that is everywhere zero. Hence, the SV functional is, in fact, positive definite on $C_0^2(\Sigma_0)$.

In order to obtain maximally smooth slip estimates consistent with a given set of data, with "roughness" measured by the SV functional, we may appeal to the same arguments applied in Chapter 1 to derive source function representations for antiplane slip distributions. These arguments are based on a simple geometric principle that identifies the source function minimizing a positive definite quadratic form subject to a linear constraint as being proportional to the function representing the constraint functional in the inner product associated with the quadratic form to be minimized. Let $\mathcal{H}_0^2(\Sigma_0)$ be the linear space of slip admissible distributions obtained by completing $C_0^2(\Sigma_0)$ in the SV norm. This is a Hilbert space on which, for two slip distributions $s_1, s_2 \in \mathcal{H}_0^2(\Sigma)$, the SV inner product is given by
\[
\langle s_1, s_2 \rangle_{sv} \triangleq \int \frac{\partial \tau_i(\xi; s_1)}{\partial \xi_j} \cdot \frac{\partial \tau_i(\xi; s_2)}{\partial \xi_j} d\xi. \tag{3.2.13}
\]
Subject to reproducing the value of a single linear functional, say $\phi(s) = y$, the source function minimizing the stress variability will be proportional to the distribution, $\Phi$, satisfying the SV representation equation:
\[
\langle s, \Phi \rangle = \phi(s) \quad \forall s \in \mathcal{H}_0^2(\Sigma_0). \tag{3.2.14}
\]
As explained in Chapters 1 and 2, an optimal source function representation for inverting data given by a model as in (3.2.10) may be obtained by solving the representation equation once for each observed functional, thereby constructing an $n$-dimensional basis $\{\Phi_j\}_{j=1}^n$, in which to express the desired estimate.
3.2.3 The SV functional in unbounded space

Parseval's relation says that the Fourier transform is a Hilbert space isomorphism on $L^2(\mathbb{R})$, meaning that it preserves inner products. This relation will aid us in understanding the nature of the linear mapping taking slip to fault surface traction, and will enable us to derive for the SV functional in an elastic volume without boundaries an expression directly in terms of slip.

Let "\( \hat{\cdot} \)" denote the Fourier transform on $L^2(\Sigma)$, so that $f \in L^2(\Sigma)$ has Fourier transform, $\hat{f}$, at wave number $\omega$ given by

$$\hat{f}(\omega) \triangleq \int_{\Sigma} e^{-2\pi i \omega \cdot \xi} f(\xi) d\xi,$$  \hspace{1cm} (3.2.15)

By assumption, the derivatives of all components of the fault-surface traction field are in $L^2(\Sigma)$, so the SV inner product in (3.2.13) may be written as the inner product in wave-number domain of the Fourier transforms of the derivatives of the components of traction. Furthermore, the Fourier transform of the derivative of a function is just a constant times wave number times the Fourier transform of the functions. Hence, the SV inner product in unbounded space is equivalent to

$$\langle s_1, s_2 \rangle_{SV} = \int 2\pi i \hat{\tau}_k(\omega; s_1) \cdot \overline{2\pi i \hat{\tau}_k(\omega; s_2)} d\omega$$

$$= 4\pi^2 \int |\omega|^2 \hat{\tau}_k(\omega; s_1) \cdot \overline{\hat{\tau}_k(\omega; s_2)} d\omega. \hspace{1cm} (3.2.16)$$

In order to express the Fourier-transformed SV inner product in terms of slip, we need to know the relationship between the Fourier transforms of slip and traction. We now summarize a derivation of this relationship, details of which are given in the appendix.

Assume, without loss of generality, that the fault plane is defined by $x_3 = 0$, and, for $x \notin \Sigma$, let $T_k(x) = \sigma_{k3}(x)$, so that $\tau_k(\xi_1, \xi_2)$ is the limiting value of $T_k(\xi_1, \xi_2, \epsilon)$ as $\epsilon$ goes to zero. The components of $T$ are related to slip by

$$T_k(x) = C_{k3mn} \frac{\partial u_n}{\partial x_m}$$

$$= C_{k3mn} \frac{\partial}{\partial x_n} \int s^j C_{j3pq} \frac{\partial S_{np}}{\partial x_q} d\xi.$$
Section 3.2. Source representation

Since everything is well-behaved for $x \notin \Sigma$, we can differentiate inside the integral to get

$$T_k(x) = \int s_j(\eta)h_{jk}(x - \eta)d\eta,$$  \hspace{1cm} (3.2.17)

where

$$h_{jk}(x - \eta) \triangleq C_{k3mn}C_{j3pq} \frac{\partial^2 S_{np}(x - \eta)}{\partial x_m \partial x_q},$$  \hspace{1cm} (3.2.18)

$S_{np}$ being the Somigliana tensor defined in (3.2.7). Taking second derivatives of the components of the Somigliana tensor produces $r^{-5}$ terms in $h_{jk}$, which are hypersingular on the fault surface. Since we have assumed that slip has bounded support and continuous second derivatives, however, we may integrate by parts twice in (3.2.17) to obtain expressions for $T_k$ involving second derivatives of slip convolved with a kernel having only (integrable) $r^{-1}$ singularities on $\Sigma$. In these relations, we may take the limit as $x$ approaches $\xi$ inside the integral, thereby expressing the fault surface tractions in terms of a convolution operator acting on derivatives of slip. To obtain the desired Fourier transforms of the traction components, we then simply multiply the Fourier transforms of the derivatives of slip by the Fourier transforms of the kernels in the convolutions producing the tractions. The results of this procedure are expressions of the form

$$\hat{T}_k(\omega) = H_{kj}(\omega)\hat{s}_j(\omega),$$  \hspace{1cm} (3.2.19)

where the components of $H$ satisfy $H_{jk} = H_{kj}$, and are given by

$$H_{11}(\omega) = \frac{\pi \mu}{1 - \nu} (\omega_1^2 + (1 - \nu)\omega_2^2)|\omega|^{-1}$$  \hspace{1cm} (3.2.20)

$$H_{12}(\omega) = \frac{\pi \mu \nu}{1 - \nu} \omega_1 \omega_2 |\omega|^{-1}$$  \hspace{1cm} (3.2.21)

$$H_{13}(\omega) = 0$$  \hspace{1cm} (3.2.22)

$$H_{22}(\omega) = \frac{\pi \mu}{1 - \nu} ((1 - \nu)\omega_1^2 + \omega_2^2)|\omega|^{-1}$$  \hspace{1cm} (3.2.23)

$$H_{23}(\omega) = 0$$  \hspace{1cm} (3.2.24)

$$H_{33}(\omega) = \frac{2\pi \mu}{1 - \nu} |\omega|.$$  \hspace{1cm} (3.2.25)

Plugging these expressions back into (3.2.16), we have the desired form of the SV inner product in wave number domain:

$$< s^1, s^2 >_{SV} = 4\pi^2 \int |\omega|^2 H_{kj}(\omega)\hat{s}_j^1(\omega) \cdot \overline{H_{kj}(\omega)\hat{s}_j^2(\omega)} d\omega.$$  \hspace{1cm} (3.2.26)
3.2.4 Approximate SV representers for a vertical, strike slip fault

The SV functional for slip on a two-dimensional fault turns out to have a convenient form. Essentially, the differentiation of each traction component by each spatial direction on the fault leads to terms which, in wave number space, exactly cancel the $|\omega|^{-1}$ terms that arise from the pseudo-differential action of convolution with a singular kernel. Generally, the SV inner product is expressible in terms of second order differential operators acting spatially on slip. Taking the strike slip fault as an example, assume that slip is nonzero only in the $x_1$ direction. Let $s^1, s^2$ be two scalar, strike-slip fields on the plane. Then we have

\[
< s^1, s^2 >_{SV} = 4\pi^2 \int |\omega|^2 \cdot \left\{ \left| \tilde{H}_{11}(\omega) \right|^2 + \left| \tilde{H}_{12}(\omega) \right|^2 + \left| \tilde{H}_{13}(\omega) \right|^2 \right\} \cdot \tilde{s}^1(\omega) \tilde{s}^2(\omega) d\omega
\]

\[
= 4\pi^4 \mu^2 \int |\omega|^2 \cdot \left\{ \left[ \left( \frac{\omega_1^2}{1-\nu} + \omega_2^2 \right) |\omega|^{-1} \right]^2 + \left[ \nu \omega_1 \omega_2 \right] \left( 1 - \nu \right) \cdot |\omega|^{-1} \right\} \cdot \tilde{s}^1(\omega) \tilde{s}^2(\omega) d\omega
\]

\[
= 4\pi^4 \mu^2 \int \left\{ \frac{\omega_1^4}{(1-\nu)^2} + \omega_2^4 + \left[ \frac{2}{1-\nu} + \left( \frac{\nu}{1-\nu} \right)^2 \right] \omega_1^2 \omega_2^2 \right\} \cdot \tilde{s}^1(\omega) \tilde{s}^2(\omega) d\omega
\]

\[
= \pi^2 \mu^2 \int \left( \frac{1}{1-\nu} \right)^2 \left[ \frac{\partial^2 s^1}{\partial \xi_1^2} \cdot \frac{\partial^2 s^2}{\partial \xi_2^2} + \frac{\partial^2 s^1}{\partial \xi_2^2} \cdot \frac{\partial^2 s^2}{\partial \xi_1^2} \right] + \left[ \frac{2}{1-\nu} + \left( \frac{\nu}{1-\nu} \right)^2 \right] \cdot \frac{\partial^2 s^1}{\partial \xi_1 \partial \xi_2} \frac{\partial^2 s^2}{\partial \xi_1 \partial \xi_2} d\xi. \quad (3.2.27)
\]

Equation (3.2.27) gives the SV inner product for a strike slip fault in an unbounded space in a form involving second-order differential operators acting on slip. This is analogous to the antiplane result given in Chapter 1, where the SV functional was determined to be exactly equivalent to the squared $L^2$ norm of the second derivative of slip.

In order to obtain SV representers for estimating slip supported on a bounded region, $\Sigma_0$, we may equip $H^2_0(\Sigma_0)$ with the inner product given in (3.2.27) and then solve the representation equation given in (3.2.14). If we measure a function $\phi$, assumed to be
bounded on $\mathcal{H}_0^2(\Sigma_0)$, the SV representation equation for $\Phi^{sv}$ is obtained from (3.2.27) as

$$
\int_{\Sigma_0} \left( \frac{1}{1-\nu} \right)^2 s_{11} \Phi^{sv}_{,11} + s_{22} \Phi^{sv}_{,22} + \left[ \frac{2}{1-\nu} + \left( \frac{\nu}{1-\nu} \right)^2 \right] s_{12} \Phi^{sv}_{,12} d\xi = \phi(s) \quad \forall \, s \in \mathcal{H}_2.
$$

(3.2.28)

Substituting the usual integral form for the functional,

$$
\phi(s) = \int_{\Sigma_0} s(\xi) w(\xi) \, d\xi,
$$

(3.2.29)

on the right hand side of (3.2.28), we see that if we can find a function, $W \in \mathcal{H}_2$ for which the fourth derivatives, $W_{iijj}$ exist and are continuous for all $i, j$ (including $i = j$) then we may integrate by parts on the left hand side of (3.2.28) to obtain

$$
\int_{\Sigma_0} \left( \frac{1}{1-\nu} \right)^2 s \cdot W_{,1111} + s \cdot W_{,2222} + \left[ \frac{2}{1-\nu} + \left( \frac{\nu}{1-\nu} \right)^2 \right] s \cdot W_{,1122} d\xi = \int_{\Sigma_0} s \cdot w d\xi \quad \forall \, s \in \mathcal{H}_2.
$$

(3.2.30)

By the fundamental theorem of the calculus of variations (Courant and Hilbert [12], the SV representation equation for the functional in (3.2.29) has the same solution as the fourth-order, elliptic boundary value problem on $\Sigma_0$:

$$
\begin{align*}
\left( \frac{1}{1-\nu} \right)^2 W_{,1111} + W_{,2222} + \left[ \frac{2}{1-\nu} + \left( \frac{\nu}{1-\nu} \right)^2 \right] W_{,1122} & = w \\
W = W_{,1} = W_{,2} = W_{,11} = W_{,22} = W_{,12} & = 0 \quad \text{on} \quad \partial \Sigma_0.
\end{align*}
$$

(3.2.31)

The appearance of an elliptic partial differential operator in a reformulation of the representation equation is not peculiar to the case of a strike slip fault. Analyzing the contribution of the other components of slip to the SV functional, we find that we can always recast the SV representation equation into a differential equation similar to (3.2.31). We gain something by rewriting the representation equation as in (3.2.30), as this transformation produces an equation that is amenable, at least, to numerical solution. The same cannot be said for the representation equation in its original form involving integro-differential equations with hypersingular kernels. We will avail ourselves of the opportunity to solve (3.2.31) after a small bit of additional simplification.
Define a linear differential operator, $L$, by

$$L = \pi^2 \mu^2 \left\{ \left[ \frac{1}{1 - \nu} \right]^2 \frac{\partial^4 s_1}{\partial \xi_1^4} + \frac{\partial^4 s_1}{\partial \xi_2^4} + \left[ \frac{2}{1 - \nu} + \left( \frac{\nu}{1 - \nu} \right)^2 \right] \frac{\partial^4 s_1}{\partial \xi_1^2 \partial \xi_2^2} \right\}$$

so that the condition in (3.2.31) becomes $LW = w$. The coefficient of the cross-product term in $L$ is

$$\frac{2}{1 - \nu} + \left( \frac{\nu}{1 - \nu} \right)^2 = \frac{2}{1 - \nu} \left( 1 + \frac{\nu^2}{2(1 - \nu)} \right),$$

and satisfies

$$1 \leq \left( 1 + \frac{\nu^2}{2(1 - \nu)} \right) \leq \frac{5}{4},$$

by virtue of the fact that $0 \leq \nu \leq \frac{1}{2}$, and for a Poisson solid, with $\nu = 1/4$, the term $\frac{\nu^2}{2(1 - \nu)}$ contributes a factor of only $1/9$. If we may neglect this factor in (3.2.33), then we approximate $L$ by

$$\tilde{L} = \pi^2 \mu^2 \left\{ \left[ \frac{1}{1 - \nu} \right]^2 \frac{\partial^4 s_1}{\partial \xi_1^4} + \frac{\partial^4 s_1}{\partial \xi_2^4} + \left[ \frac{2}{1 - \nu} \right] \frac{\partial^4 s_1}{\partial \xi_1^2 \partial \xi_2^2} \frac{\partial^2 s_1}{\partial \xi_1 \partial \xi_2} \right\}$$

$$= \tilde{\Delta}^2,$$

where $\tilde{\Delta}$ is the scaled Laplacian:

$$\tilde{\Delta} = \pi \mu \left( \frac{1}{1 - \nu} \frac{\partial^2 s_1}{\partial \xi_1^2} + \frac{\partial^2 s_1}{\partial \xi_2^2} \right).$$

Hence, we see that the approximating operator in (3.2.34) is essentially a biharmonic differential operator, and that, to the degree to which the operator, $\tilde{\Delta}^2$, approximates the operator in the SV norm, the SV representers for a functional of the type given in (3.2.29) may be found by solving a biharmonic equation.

We have carried out the prescribed procedure for finding approximate SV representers for functionals of strike slip supported on a bounded, rectangular. For a Poisson solid, i.e., $\nu = 1/4$, we have solved the biharmonic equations resulting from (3.2.34) by using Bjorstad's Biharmonic Equation Solver [9]. Figures 3.2–3.9 show the SV representers for measurements of fault-parallel and fault-normal displacements at surface points variously situated with respect to the source region, which is taken to be a vertical plane extending two units laterally from and to a depth of one unit. Figures 3.2–3.5 exhibit the effects
Section 3.3. Parkfield steady-state

changing the fault-normal position of the measuring point on the representers of parallel displacement functionals. In each of these figures, the fault-parallel position of the measurement point is fixed, while the normal component varies from 0.01 to 0.1 units in the upper left and right panels and 1 to 10 units in the bottom left and right. In Figure 3.2, the strike coordinate of the measurement point is at zero, right in the middle of the source region. In Figures 3.3–3.5, the strike coordinate moves progressively close to the left edge of the source region, being set at positions -0.2, -0.5, and -0.8 in the respective figures. In every case, we see that, when the observation point is very close to the fault, the strike displacement representer grows sharply right below the lateral position of the observation point. As the point moves away from the fault, the representer flattens out, losing its steepness near the surface. In every instance, the representer for the strike displacement functional has a unique maximum situated with respect to the depth almost precisely in the middle of the source region. (It should be noted that the appearance of two maxima in the bottom left panel of Figure 3.2 is simply an artifact of the contouring algorithm.) The maxima are quite sharp for observation points close to the fault, and they broaden as the points move away. The appearance of a single maximum in the strike displacement representers may be contrasted with the behavior of the normal displacement representers, shown in Figures 3.6–3.9 at the same receiver point positions as depicted in Figures 3.2–3.5. These functions tend to have two lobes of opposite sign, the sign change taking place at the same lateral position as the observation point. Though it may not be apparent in the contour plots as drawn, the normal displacement representers do exhibit the tendency to have sharper peaks when observation points are closer to the fault. They also have the property that the peaks (and valleys) show up right in the middle of the source region with respect to the depth coordinate.

3.3 Steady-state analysis of Parkfield two-color data

As an application of the source representation theory developed in Section 2, we will examine the problem of inverting rates of line-length change in the Parkfield network to
Figure 3.2: SV representers for measurement of fault-parallel displacement at a point at zero along strike and at 0.01, 0.1, 1.0, and 10 source depths normal to the fault.
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Figure 3.3: SV representers for measurement of fault-parallel displacement at a point at -0.2 along strike and at 0.01, 0.1, 1.0, and 10 source depths normal to the fault.
Figure 3.4: SV representers for measurement of fault-parallel displacement at a point at -0.5 along strike and at 0.01, 0.1, 1.0, and 10 source depths normal to the fault.
Figure 3.5: SV representers for measurement of fault-parallel displacement at a point at -0.8 along strike and at 0.01, 0.1, 1.0, and 10 source depths normal to the fault.
Figure 3.6: SV representers for measurement of fault-normal displacement at a point at zero along strike and at 0.01, 0.1, 1.0, and 10 source depths normal to the fault.
Figure 3.7: SV representers for measurement of fault-normal displacement at a point at -0.2 along strike and at 0.01, 0.1, 1.0, and 10 source depths normal to the fault.
Figure 3.8: SV representers for measurement of fault-normal displacement at a point at -0.5 along strike and at 0.01, 0.1, 1.0, and 10 source depths normal to the fault.
Figure 3.9: SV representers for measurement of fault-normal displacement at a point at -0.8 along strike and at 0.01, 0.1, 1.0, and 10 source depths normal to the fault.
Table 3.1: Fault-centered coordinates for stations in Parkfield two-color network.

<table>
<thead>
<tr>
<th>Line to</th>
<th>Line length</th>
<th>Azimuth</th>
<th>Strike</th>
<th>Normal</th>
</tr>
</thead>
<tbody>
<tr>
<td>Car Hill</td>
<td>-</td>
<td>-</td>
<td>0</td>
<td>0.02</td>
</tr>
<tr>
<td>Bare</td>
<td>4.824</td>
<td>330.19</td>
<td>-4.18</td>
<td>-2.33</td>
</tr>
<tr>
<td>Buck</td>
<td>3.050</td>
<td>350.50</td>
<td>-3.00</td>
<td>-0.55</td>
</tr>
<tr>
<td>Can</td>
<td>5.666</td>
<td>320.59</td>
<td>-4.37</td>
<td>-3.51</td>
</tr>
<tr>
<td>Creek</td>
<td>5.620</td>
<td>173.70</td>
<td>5.58</td>
<td>0.66</td>
</tr>
<tr>
<td>Gold</td>
<td>9.236</td>
<td>186.89</td>
<td>9.16</td>
<td>-1.14</td>
</tr>
<tr>
<td>Hogs</td>
<td>5.000</td>
<td>75.99</td>
<td>-1.21</td>
<td>4.84</td>
</tr>
<tr>
<td>Hunt</td>
<td>2.720</td>
<td>210.49</td>
<td>2.34</td>
<td>-1.34</td>
</tr>
<tr>
<td>Lang</td>
<td>4.080</td>
<td>29.59</td>
<td>-3.54</td>
<td>2.00</td>
</tr>
<tr>
<td>Mason</td>
<td>6.270</td>
<td>127.30</td>
<td>3.79</td>
<td>4.94</td>
</tr>
<tr>
<td>Mels</td>
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<td>0.02</td>
</tr>
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<td>-4.53</td>
<td>-0.55</td>
</tr>
<tr>
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<td>271.99</td>
<td>-0.03</td>
<td>-1.06</td>
</tr>
<tr>
<td>Pomo</td>
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<td>10.19</td>
<td>-5.52</td>
<td>0.94</td>
</tr>
<tr>
<td>Table</td>
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<td>2.21</td>
<td>-5.89</td>
</tr>
<tr>
<td>Todd</td>
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<td>125.00</td>
<td>2.12</td>
<td>3.03</td>
</tr>
<tr>
<td>Turk</td>
<td>2.315</td>
<td>194.00</td>
<td>2.13</td>
<td>-0.53</td>
</tr>
</tbody>
</table>

estimate the interseismic slip rate distribution on the Parkfield segment of the San Andreas fault. We first describe the two-color data and a "smooth transition" slip model derived from boundary conditions believed to obtain at the edges of the Parkfield seismogenic zone. In Section 3.2, we describe the orthogonal basis for estimating deep slip that is obtained from the numerically-determined SV representers of the measured line-length change functionals, and in Section 3.3, we present the result our our analysis.

3.3.1 The data and a smooth transition model

Seventeen baselines emanating from a point on Car Hill, just on the Pacific side of the San Andreas and in the middle of the lateral extent of the 1966 Parkfield rupture zone, have been surveyed frequently since the mid-1980's using a two-frequency laser geodimeter. Table 3.1 lists the fault-centered coordinates of benchmarks in the two-color network shown in Figure 3.10. Time series of measured line-length changes are shown in Figure 3.11.
Figure 3.10: Two-color geodimeter network near Parkfield, California.
Figure 3.11: Time series of measured baseline lengths in the Parkfield two-color geodimeter network.
Section 3.3. Parkfield steady-state

As we mentioned in the introduction, the Parkfield fault segment is transitional between a zone showing steady surface creep, to the northwest, and a locked zone to the southeast. We model this transition zone as a rectangular region on a vertical fault plane, extending 40 km laterally, with Car Hill defining the lateral midpoint, and extending to a depth of 20 km. We will refer to this rectangular region as the "Parkfield Seismogenic Zone" (PSZ). Conditions on three boundaries of the PSZ are determined by the assumption. We set the slip rate at the base of the PSZ to 34 km/year, roughly the rate of plate motion accomodated in the San Andreas deformation zone. On the southeast and northwest vertical boundaries, we require a constant gradient of slip with respect to depth, linearly interpolating the assumed deep slip rate of 34 mm/year to the observed surface rate of zero at the southeast and 25 mm/year at the northwest. Surface creep measurements in Parkfield zone essentially indicate a smooth, monotonic transition in surface slip rates between the creeping and locked regions. Using the same measurements as Segall and Harris [47], we set surface slip rates by taking constant gradients interpolating 25 mm/year at the northwest boundary of the PSZ to 12 mm/year at Midd to 9 mm/year at Gold Hill and to zero at the southeast boundary.

A smooth distribution of slip rates satisfying the imposed boundary conditions may be viewed as a "null" model for the transition from creeping to locked regimes. Our plan is to construct such a "boundary-determined" distribution, and then to add to it, as demanded by the data, some linear combination of line-length change representer satisfying homogeneous boundary conditions. In order to find this distribution, in keeping with the representation theory of Section 2, we use the differential form of the SV innerproduct given in equation (3.2.27). Let $B(\xi)$ denote the boundary-determined component and $s$ any slip distribution in $H_0^2(\Sigma_0), \Sigma_0$ now denoting the rectangular PSZ. The inner product associated with the form in (3.2.34) is

$$< s, B > = \pi^2 \mu^2 \int_{\Sigma_0} \left( \frac{1}{1-\nu} \right)^2 \left( \frac{\partial^2 s}{\partial \xi_1^2} \cdot \frac{\partial^2 B}{\partial \xi_1^2} + \frac{\partial^2 s}{\partial \xi_2^2} \cdot \frac{\partial^2 B}{\partial \xi_2^2} \right) + \frac{2}{1-\nu} \cdot \frac{\partial^2 s}{\partial \xi_1 \partial \xi_2} \frac{\partial^2 B}{\partial \xi_1 \partial \xi_2} d\xi$$
\[ = \int_{\Sigma_0} s \cdot \left\{ \frac{1}{1-\nu} B_{1111} + B_{2222} + \frac{2}{1-\nu} B_{1122} \right\} d\xi \]

\[ = \int_{\Sigma_0} s \cdot \tilde{A}^2 B d\xi. \quad (3.3.36) \]

The form given in (3.3.36) comes, again, from integrating twice by parts, using the assumed homogeneous boundary conditions on $s$ and its derivatives. From this equation, we see that if $B$ satisfies

\[ \tilde{A}^2 B = 0, \quad (3.3.37) \]

then it will be orthogonal to distributions in $\mathcal{H}_0^2(\Sigma_0)$. Hence, to obtain slip estimates that are optimally smooth in the desired sense, we determine $B$ to satisfy (3.3.37) with boundary conditions on $B$ given by the assumptions we have stated and with the derivatives of $B$ normal to the boundaries set everywhere to zero. The resulting slip component is shown in Figure 3.12.

From the null slip model obtained from boundary conditions, we may compute null-model predicted slip rates by integration against the line-length change kernels given by the dislocation model. These predicted rates of line-length change are given in Table 3.2, along with "observed" rates and residual rates obtained by subtracting predicted from observed. The rates labeled "Observed" in Table 3.2 are the slopes of straight lines fit to the data in Figure 3.11. The volume and time span of the two-color data—several hundred observations per line over five to seven years—are sufficiently large to determine the trends in the baselines to within a very narrow range. It is difficult to produce changes of more than a few tenths of a millimeter per year in the estimated average rates of line-length change listed in Table 3.2 using any reasonable data-fitting criterion. Hence, statistical errors in direct estimation of trends in the data are essentially negligible.

In Figure 3.11, we have plotted lines with trends predicted by the smooth transition slip model along with the observed data. The agreement between the predictions of the smooth model and the observed data does not appear to be satisfactory. In most cases, the residual rate of extension or contraction that is unexplained by the model determined from boundary conditions is at least 20% as large as the predicted rate. There are also
Figure 3.12: Smooth transition between boundary slip rates.
### Table 3.2: Observed and estimated rates of line-length change for lines from Car Hill to benchmarks in Parkfield two-color geodimeter network.

<table>
<thead>
<tr>
<th>Line to</th>
<th>Observed</th>
<th>Predicted</th>
<th>Observed - Predicted</th>
</tr>
</thead>
<tbody>
<tr>
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<td>-10.61</td>
<td>-11.84</td>
<td>1.23</td>
</tr>
<tr>
<td>Buck</td>
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<td>Can</td>
<td>-8.94</td>
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</tr>
<tr>
<td>Creek</td>
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</tr>
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<td>0.53</td>
<td>-1.99</td>
</tr>
<tr>
<td>Mide</td>
<td>-15.36</td>
<td>-13.97</td>
<td>-1.39</td>
</tr>
<tr>
<td>Nore</td>
<td>-2.27</td>
<td>-0.65</td>
<td>-1.61</td>
</tr>
<tr>
<td>Pomo</td>
<td>3.59</td>
<td>-0.01</td>
<td>3.60</td>
</tr>
<tr>
<td>Table</td>
<td>5.30</td>
<td>4.72</td>
<td>0.58</td>
</tr>
<tr>
<td>Todd</td>
<td>1.19</td>
<td>0.69</td>
<td>0.50</td>
</tr>
<tr>
<td>Turk</td>
<td>9.80</td>
<td>12.83</td>
<td>-3.03</td>
</tr>
</tbody>
</table>
some peculiar sign differences. The line from Car Hill to Lang, for instance, is predicted to show .4 mm/year contraction, but it is observed to be extending at nearly 1.5 mm/year. Similarly, the line to Pomo should show virtually no change in length, according to the model, but it has an observed rate of more than 3.5mm/year of extension.

### 3.3.2 Representers and orthobasis functions

To investigate the extent to which we may explain the discrepancies between the observed and predicted trends listed in Table 3.2 by adding components of slip to the boundary-determined model, we will use the methods of Section 2 to obtain SV representers for the line-length change functionals measured in the Parkfield two-color network. As in the examples of Section 2.4, these representers will satisfy homogeneous boundary conditions, so slip rate estimates obtained by adding a linear combination of these representers to the smooth boundary-determined component in Figure 3.12 will continue to have the required values on the boundaries of the PSZ. When displacements are small compared to nominal baseline-length, the change in length is almost exactly linear in the displacements of the end points of the line. Letting \( u^0 \) and \( u^1 \) denote the displacements of the two end points of a baseline, and assuming that vertical displacement differences are negligible, the change in length of a baseline with azimuth \( \theta \) in the \( x_2 = 0 \) plane is given by

\[
\delta L = \cos \theta \cdot (u^1_1 - u^0_1) + \sin \theta \cdot (u^1_2 - u^0_2).
\]  

(3.3.38)

From this expression, the kernel against which slip is integrated to produce line-length change is given by

\[
K(\xi; x^0, x^1) = \cos \theta \cdot (G^{1}_{13}(x^1, \xi) - G^{1}_{13}(x^0, \xi)) \\
+ \sin \theta \cdot (G^{2}_{13}(x^1, \xi) - G^{2}_{13}(x^0, \xi)),
\]  

(3.3.39)

where the Green’s functions, as in equation (3.2.6), are given for points on the free surface of a half-space by Okada [40]. In order to obtain representers for the line-length functionals measured in the two-color network, we plug the coordinates of the end points of each
baseline into (3.3.39) and solve the boundary-value problems

\[
\begin{align*}
\bar{\Delta}^2 \Phi_j &= K_j & \text{on } \Sigma_0 \\
\Phi_j &= \partial_{\text{norm}} \Phi_j = 0 & \text{on } \partial \Sigma_0.
\end{align*}
\] (3.3.40)

In (3.3.40), \(K_j\) is the line-length kernel for the \(j^{th}\) baseline, \(\Phi_j\) is the SV representer for the corresponding functional, and \(\partial_{\text{norm}}\) denotes the boundary-normal derivative. Using Bjorstad’s biharmonic solver, as we did in Section 2.4, we obtain representers two of which are shown in Figures 3.13 and 3.14. In the contour plots of the representers, the projections onto the fault trace of the end points of each baseline are shown at the top, a “+” denoting a point east of the fault and an “x” a point to the west. Of course, an end point at Car Hill is common to all baselines. The Parkfield two-color network representers exhibit some of the properties noted in the displacement representers shown in Figures 3.2-3.9. The representers tend to grow most steeply right beneath the locations of the observation points on the surface, and to be more sharply peaked for stations close to the fault. In most cases, the representers have unique maxima or minima located between 5 and 10 km below the surface, a property that is shared with the fault-parallel displacement representers in Figures 3.2-3.5. The representers for the lines to Hogs, Nore, and Table, i.e., the lines with most nearly fault-normal orientations, have the “bi-lobed” form of the normal displacement representers in Figures 3.6-3.9. We expect this sort of behavior, both on intuitive grounds and by considering the form of the line-length kernel given in (3.3.39). In the extreme cases, a line that is exactly fault-parallel sees only the fault-parallel components of the displacements of its endpoints, and a line that is exactly fault-normal sees only the normal displacements. There is a smooth transition between these extremes, which accounts for the observation that the more fault-parallel lines have representers that look like those for a parallel displacement and the more fault-normal lines have representers resembling those for normal displacement.

For reasons discussed in Chapter 2, it is convenient to take transform a basis formed by a collection of representers into an equivalent orthonormal basis. We may do this by
Figure 3.13: SV representer for change in length of baseline from Car Hill to Gold Hill.
Figure 3.14: SV representer or change in length of baseline from Car Hill to Nore.
Figure 3.15: The first four SV orthobasis functions for the Parkfield two-color network.
Figure 3.16: The second four SV orthobasis functions for the Parkfield two-color network.
eigen-decomposition of the *Gram matrix* of pairwise innerproducts:

\[
\Gamma = ((\gamma_{ij})) = < \Phi_i, \Phi_j > .
\]  

(3.3.41)

Let \( \Lambda \) be the diagonal matrix of eigenvalues of \( \Gamma \), and \( \Psi \) be the matrix with the eigenvectors, normalized to have unit Euclidean length, in its columns. From the representers, \( \Phi_j \), the \( k^{th} \) in a set of \( p \) orthobasis functions, \( p \) being the number of strictly positive eigenvalues of \( \Gamma \), is given by

\[
\beta_k = \lambda_k^{-\frac{1}{2}} \sum_j \psi_{jk} \Phi_j .
\]  

(3.3.42)

In Chapter 2, we remarked on the insights that may be gained into the observational power of a network by examination of the eigenvalues, eigenvectors, and orthobasis functions produced by a set of representers. The first eight eigenvalues of the Gram matrix obtained from the SV representers of the seventeen line-length changes measured in the two-color network are listed in Table 3.3, and the corresponding orthobasis functions are shown in Figures 3.15 and 3.16. The first orthobasis function of the Parkfield network looks like a representer for a fault-parallel displacement functional, and this basis function produces a displacement field that is mostly fault-parallel. The second orthobasis function has the form of a representer of a fault-normal displacement functional, and it produces mostly fault-normal displacement field. Higher order orthobasis functions are progressively more complicated, and do not have such obvious interpretations in terms of normal and parallel displacements.

### 3.3.3 Slip estimates

Any member of the class of admissible slip estimates is determined by adding a linear combination of the orthobasis functions described in the last section to the boundary-determined component described in Section 3.1. Writing \( s \) as

\[
s(\xi) = B(\xi) + \sum_{j=1}^{17} c_j \beta_j(\xi),
\]  

(3.3.43)

the rates of line-length change predicted by \( s \) are

\[
\delta L_j(s) = \delta L_j(B) + \Psi \Lambda^{\frac{1}{2}} c,
\]  

(3.3.44)
Section 3.3. Parkfield steady-state

| 1 | 1.223 × 10^{-1} |
| 2 | 1.965 × 10^{-2} |
| 3 | 3.530 × 10^{-3} |
| 4 | 2.187 × 10^{-3} |
| 5 | 1.699 × 10^{-3} |
| 6 | 7.005 × 10^{-4} |
| 7 | 4.632 × 10^{-4} |
| 8 | 2.906 × 10^{-4} |

Table 3.3: Eigenvalues of Gram matrix of SV representers for line-length observations in Parkfield two-color network.

we obtain values for the coefficients, \( c \), of the orthobasis functions by applying a damped inverse of \( \Psi A^{\frac{1}{2}} \), to the residual rates of line-length change remaining after subtracting the boundary-determined predictions from the observed rates, as reported in Table 3.2. Damping is necessary to suppress instabilities due to amplifying effect of the smallest eigenvalues on noise and model biases. Letting \( r \) denote the vector of residual rates of change, \( \hat{c} \), an estimate of \( c \), is obtained by

\[
\hat{c} = \text{Diag} \left( \frac{\lambda_j^{\frac{1}{2}}}{\lambda_j + \rho^2} \right) \Psi^T r,
\]

\( \rho^2 \) being the damping parameter. Figures 3.17 and 3.18 show two different estimates of the slip rate that differ in the choice of \( \rho^2 \) in (3.3.45). In both figures, the top panel shows the residual slip component obtained by taking the dictated linear combination of orthobasis functions satisfying homogeneous boundary conditions, and the bottom panel shows the slip rate estimate resulting from addition of the boundary component. The residual component of the first estimate, with \( \rho^2 = 0.0037 \), has a region of high slip at depth to the northwest of Car Hill, and a shallow region of low slip to the southeast. Increasing \( \rho^2 \) to 0.0074 in Figure 3.18, we find that magnitudes of the high and low slip regions have been damped down, and that the resulting estimate, now everywhere between zero and 34 mm/year, still exhibits a small region of increased slip rate at depth and a “locked” region to to the southeast, between car Hill and Gold Hill.

The slip estimate in Figure 3.17 is rather implausible, given our assumptions about
Figure 3.17: Estimates of Parkfield interseismic slip rate with $\rho^2 = 0.0037$. 
Figure 3.18: Estimates of Parkfield interseismic slip rate with $\rho^2 = 0.0074$. 
the driving forces causing motion on the fault. It is difficult to a cause for a rate of fault motion that would exceed the driving rate of deep slip at the base of the seismogenic zone, so the appearance of slip rates nearly 10 mm/year higher than the assumed deep rate of 34 mm/year is not credible. The pattern of right lateral slip at high rate to the northwest and low rate to the southeast has the effect of producing a large fault-normal displacement, and of moving the central benchmark at Car Hill to the southwest, thus extending most baselines with azimuths to the north. The retardation of slip to the southeast also produces contraction on fault-crossing lines with southwest azimuths: Gold, Mels, and Turk.

By increasing the damping parameter, as we have done in going from Figure 3.17 to 3.18, we have damped away the appearance of physically implausible features such as left lateral slip and slip rates exceeding the assumed deep rate. In so doing, we degrade the fit to the observations. The slip rate estimates in Figure 3.18) are obtained with the smallest possible damping parameter that produces an estimate that is consistent with the assumed deep slip rate. The observed residual rates of line-length change from the predictions of the boundary model, and the predictions of residual rates of change from the slip estimate in Figure 3.18 are shown in Table 3.4. The effect of adding the components of deep slip shown in Figure 3.18 is apparently small, as the rms misfit after addition of these components is only about 6% smaller than is the rms misfit to the boundary model.

3.4 Discussion and conclusion

We have developed source representation methods that may be used to obtain optimally smooth estimates of the distribution of slip on a fault by inverting geodetic observations. These methods are based on a measure of smoothness that evaluates the deviation from a globally uniform distribution of fault-surface traction produced by a given distribution of slip. By deriving the Fourier transforms of the fault-surface traction distribution, we have been able to write the stress-variability functional measuring smoothness directly in terms of slip in the case where slip is distributed over a planar fault in an unbounded,
## Table 3.4

Residual rates of line-length change from boundary-determined model and residual rates predicted by model with added component of slip.

<table>
<thead>
<tr>
<th>Line to</th>
<th>Boundary residual</th>
<th>Model predicted</th>
<th>Model residual</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bare</td>
<td>1.230</td>
<td>0.507</td>
<td>0.723</td>
</tr>
<tr>
<td>Buck</td>
<td>2.459</td>
<td>0.186</td>
<td>2.273</td>
</tr>
<tr>
<td>Can</td>
<td>1.465</td>
<td>0.593</td>
<td>0.872</td>
</tr>
<tr>
<td>Creek</td>
<td>-0.185</td>
<td>0.433</td>
<td>-0.618</td>
</tr>
<tr>
<td>Gold</td>
<td>-2.602</td>
<td>-0.505</td>
<td>-2.097</td>
</tr>
<tr>
<td>Hogs</td>
<td>1.201</td>
<td>-0.488</td>
<td>1.689</td>
</tr>
<tr>
<td>Hunt</td>
<td>-2.970</td>
<td>-0.536</td>
<td>-2.436</td>
</tr>
<tr>
<td>Lang</td>
<td>1.888</td>
<td>-0.458</td>
<td>2.346</td>
</tr>
<tr>
<td>Mason</td>
<td>-1.521</td>
<td>0.275</td>
<td>-1.796</td>
</tr>
<tr>
<td>Mels</td>
<td>-2.930</td>
<td>-0.720</td>
<td>-2.200</td>
</tr>
<tr>
<td>Midd</td>
<td>-1.986</td>
<td>0.016</td>
<td>-2.002</td>
</tr>
<tr>
<td>Mide</td>
<td>-1.394</td>
<td>0.143</td>
<td>-1.537</td>
</tr>
<tr>
<td>Nore</td>
<td>-1.615</td>
<td>0.169</td>
<td>-1.786</td>
</tr>
<tr>
<td>Pomo</td>
<td>3.603</td>
<td>-0.125</td>
<td>3.728</td>
</tr>
<tr>
<td>Table</td>
<td>0.583</td>
<td>-0.034</td>
<td>0.617</td>
</tr>
<tr>
<td>Todd</td>
<td>0.503</td>
<td>0.358</td>
<td>0.145</td>
</tr>
<tr>
<td>Turk</td>
<td>-3.035</td>
<td>-0.325</td>
<td>-2.710</td>
</tr>
<tr>
<td>RMS</td>
<td>2.065</td>
<td>0.399</td>
<td>1.953</td>
</tr>
</tbody>
</table>
uniform, elastic space. If a more realistic physical model were used that incorporated nonhomogeneity, for instance, then the principle of determining representations based on making the stress distribution optimally smooth should, perhaps, be modified to take account of assumed local variations in compliance. Using elastostatic Green's functions that take account of nonhomogeneity has the effect of changing the form of the weight function, $w$, in (3.2.29), and, so, would give different representers satisfying (3.2.31), when the differential form of the SV inner product is used. We have not investigated the nature of these change, though one would expect that they would be very small for assumed constitutive relations that do not radically differ from uniformity.

The differential form of the SV functional in (3.2.27) is essentially the same (with $\nu = 0$) as the thin-plate norm inner product that is obtained by approximating the quadratic form measuring the elastic strain energy produced by lateral deformation of a homogeneous, thin plate. This norm was first proposed for use in function estimation by Duchon [13], and has been often used for smoothing problems by Wahba [60], for instance. We may remark in passing that the use of a numerical, biharmonic solver will also produce representers in the thin-plate spline inner product for bounded functions obtained by integrating the unknown function to be estimated against a known kernel.

Our application source representation methods to the inverse problem requiring estimation of slip from the Parkfield two-color data has revealed, essentially, that the two-color data do not resolve a components of deep slip that are significant in the sense of producing appreciably better fits to the data than a smooth slip model that interpolates assumed boundary conditions. These data are, perhaps, best suited to bear on the question of the steady-state slip rate distribution at Parkfield in conjunction with other data from the era of the Parkfield prediction experiment and earlier. In appropriately smoothed estimates of Parkfield slip rates, we have seen suggestions of the locked zone at depth that Harris and Segall [16] found to be required to fit rates of line-length change collected over a much broader area around the Parkfield fault segment than is covered by the two-color network. We cannot say that the two-color data unequivocally demand this feature, for there is a great deal of equivocation in the data. Some of the two-color observations do
not seem to be consistent with any reasonable distribution of right-lateral strike slip on a vertical plane following the surface trace of the San Andreas fault. Stations Mide and Midd, for instance are almost directly on opposite sides of the fault, the line from Car Hill to Mide crossing the fault and the line to Midd, not. In Table 3.2, we see that Mide is contracting at a rate of more than 15 mm/year, nearly 1.5 mm/year greater than the 14 mm/year predicted by the boundary-determined model. To fit this observation, we would need to increase the slip gradient north of Car Hill, but the observation on the line to Midd requires precisely the opposite remedy. This line should, by all reason, be extending, due to the positive slip gradient to the positive slip northeast, but it is observed to be contracting at 1.5 mm/year. It is difficult to distribute slip on the fault in a manner that will improve the fit to either Midd or Mide without worsening the fit to the other. Similar difficulties are found in trying to reconcile observation such as the 3.5 mm/year externsion of Pomo and the 1.47 mm/year extension of Lang with boundary conditions. Though these stations have azimuths to the north of Car Hill, they are sufficiently distant from the fault that the smooth model predicts that they will be in contraction since Car Hill, being very close to the fault, will be displaced to the northwest than either Lang or Pomo. Some of the additional deep slip to the north of Car Hill required to fit rates on Lang and Pomo, showed up in our estimated slip rates, but, as we have remarked, this additional slip to the north has the opposite of the desired effect on station Midd.

A variety of plausible explanation exist for the apparent discrepancies between the observed staeady-state rates of line-length change in the two-color network and the rates predicted by any consistent distribution of right-lateral strike slip. Firstly, it is possible that there is a large component of fault-normal motion at Car Hill that is due to some departure from pure strike motion on the fault. This possibility may be investigated by examining other geodetic observations collected in the Car Hill area. the GPS benchmark now sitting within a few yards of the sight of the two-color instrument on Car Hill may, over sufficient time, portray a picture of the motion of Car Hill that will be useful in explaining some of the properties of deformation observed in the two-color network. Understanding the motion of Car Hill may be crucial, since this sight is common to all baselines. Any
displacement signal at Car Hill, even if it is due to local, nontectonic causes, will produce a network-wide signal that is, in some sense, spatially coherent. By applying inverse methods to signals so generated, we may be trying to map to strike slip on the fault a signal with a different origin altogether.

Though local motion at Car Hill is a particularly acute concern in modeling and analyzing the two-color data, the potential for local motion in a subset of the network is also an issue. The four stations we mentioned – Midd, Mide, Lang and Pomo – as producing somewhat suspect observed rates of change all lie on the side of Middle Mountain, where local topography is steep and where the benchmarks may be subject to the effects of erosion or landslide.

In our steady-state analysis of the two-color data, we have ignored the information on time-dependency slip rates that may be contained in these frequently-measured data. Despite the apparent discrepancies between some observed average rates of line-length change in the two-color network, it may be possible to resolve coherent in changes over time that are due to true variations in slip rate, if these variations exist. In Chapter 4, we look at the problem of extending the purely spatial inversion algorithms we have described thus far to the problem requiring estimates over time of a slip rate distribution that may vary with time.
Chapter 4

Estimation of slip in space-time

4.1 Introduction

The Parkfield two-color geodimeter data described in Chapter 3 has been, and continues to be, collected as part of an earthquake-prediction experiment. One of the primary goals of this experiment is to ascertain whether or not one can "see an earthquake coming" before it actually happens. Laboratory studies of rock fracture that mimic earthquake rupture processes on a very small scale, have revealed that failure is preceded by a brief period of nucleation marked by unstable, accelerating slip. Similar instability and accelerated motion probably precedes seismic rupturing of faults in the Earth's crust, but the scale of the effect, either in space or time, is unknown. Certain anecdotal evidence concerning the appearance of surface cracks and the displacement of landmarks hours to days before mainshocks may suggest that slip accelerates before mainshocks in a manner that may be detectable by in deformation measurements with sufficient spatial and temporal resolution.

In the steady-state analysis of the Parkfield two-color data carried out in Chapter 3, temporal information in the data was ignored. In this Chapter, our goal is to develop algorithms to estimate slip distributions in space-time and to use these algorithms for spatio-temporal modeling of the Parkfield data. The algorithms we develop are based on stochastic models for the coefficients in time-dependent linear expansions for slip using
minimum-SV basis functions derived by the methods outlined in Chapter 3.

4.2 Components of the data

According to its designer, L.E. Slater [51], the two-frequency distance measuring instrument at Parkfield has a theoretical accuracy of 0.1 parts per million in measuring lengths of lines in the 1 to 10 km range. Previous analyses using independent and repeated measurements against which to check the reading of the two-color instrument have confirmed that individual measurements appear to have accuracy in the theoretical range. Langbein et al., [28], for instance have modeled the apparent measurement error variances as containing a line-length-dependent and a length-independent component. Letting $\sigma^2(L)$ denote the measurement error variance in measurements on a line of nominal length, $L$, they fit the theoretically-based model

$$\sigma^2(L) = a^2 + b^2 L^2,$$

and estimated values at $a = 0.3$ mm and $b = 0.12 \times 10^{-6}$. As we described in Chapter 3, steady-state response to tectonic forces will produce constant-rate extension or contraction of each baseline in the two-color network, leading to expected linear trends in the measured lengths of the baselines. Table 4.1 shows, for each baseline, the measurement error variances appropriate for each baseline using (4.2.1) with $a, b$ set to the values estimated by Langbein et al.. It also shows mean squared deviations about linear trends fit to all of the lines in the network. These trends are fit directly, by least squares, not through a tectonic model, so they provide the best possible steady-state fit to the data. Clearly, the data are inconsistent with a model describing them as the sum of a linear trend plus measurement error. The mean squared residual about the estimated trend is too large by about a factor of three in the best case, and is off by a factor of nearly 40 for some stations.

Figure 4.1 shows the residual time series on each baseline after detrending. The presence of seasonal variations in the measured length on many of the baselines is clear, as are other systematic departures from random variation about zero.
Figure 4.1: Detrended time series of two-color distance measurements.
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<table>
<thead>
<tr>
<th>Line</th>
<th>Theoretical variance</th>
<th>Mean squared residual</th>
<th>Observed Theoretical</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bare</td>
<td>0.2788</td>
<td>1.968</td>
<td>7.058</td>
</tr>
<tr>
<td>Buck</td>
<td>0.1538</td>
<td>0.997</td>
<td>6.487</td>
</tr>
<tr>
<td>Can</td>
<td>0.3769</td>
<td>1.701</td>
<td>4.513</td>
</tr>
<tr>
<td>Creek</td>
<td>0.3809</td>
<td>2.437</td>
<td>6.398</td>
</tr>
<tr>
<td>Gold</td>
<td>1.3246</td>
<td>3.767</td>
<td>2.844</td>
</tr>
<tr>
<td>Hogs</td>
<td>0.3134</td>
<td>2.317</td>
<td>7.395</td>
</tr>
<tr>
<td>Hunt</td>
<td>0.1379</td>
<td>1.118</td>
<td>8.113</td>
</tr>
<tr>
<td>Lang</td>
<td>0.2215</td>
<td>1.600</td>
<td>7.225</td>
</tr>
<tr>
<td>Mason</td>
<td>0.4945</td>
<td>1.923</td>
<td>3.889</td>
</tr>
<tr>
<td>Mels</td>
<td>0.3470</td>
<td>2.469</td>
<td>7.115</td>
</tr>
<tr>
<td>Midd</td>
<td>0.3094</td>
<td>2.092</td>
<td>6.761</td>
</tr>
<tr>
<td>Mide</td>
<td>0.2578</td>
<td>10.036</td>
<td>38.929</td>
</tr>
<tr>
<td>Nore</td>
<td>0.3806</td>
<td>1.722</td>
<td>4.526</td>
</tr>
<tr>
<td>Pomo</td>
<td>0.3811</td>
<td>4.839</td>
<td>12.698</td>
</tr>
<tr>
<td>Table</td>
<td>0.4586</td>
<td>8.660</td>
<td>18.883</td>
</tr>
<tr>
<td>Todd</td>
<td>0.1957</td>
<td>6.890</td>
<td>35.211</td>
</tr>
<tr>
<td>Turk</td>
<td>0.1260</td>
<td>1.181</td>
<td>9.380</td>
</tr>
</tbody>
</table>

Table 4.1: Theoretical variances and mean squared residuals about a linear trend.

The distance-measuring instrument at Parkfield looks at phase differences between transmitted and reflected laser beams and, so, measures distance modulo the wavelength of the transmitted beam. Since absolute distance is not measured, only changes in length of individual baselines with time are meaningful. We will thus consider that the baselines have origin times, $t^0_j$, defined as the times of first measurement on each line, and we let $\delta L_j(t) \triangleq L_j(t) - L_j(t^0_j)$ denote the change in the length, $L_j$, of the $j^{th}$ baseline from the line origin time to time $t$. Let $y_j(t_i), j = 1, \ldots, 17, i = 1, \ldots, n_j$, denote the time series of measurements on each of the baselines in the two-color network, with $i$ indexing baseline. We model these series as the sum of an unknown constant "offset", $O_j$, in the length scale plus true changes in line-length plus measurement errors, $\epsilon_{ij}$:

$$y_j(t_i) = O_j + \delta L_j(t_i) + \epsilon_{ij}. \quad (4.2.2)$$

We are interested in true changes in line-length due to tectonic signals, and we model line-length change over time as the sum of a tectonic signal, and a residual that we will
Section 4.2. Introduction

refer to as a "local" component

$$\delta L_j(t) = \delta L_j^{tec}(t) + \delta L_j^{loc}(t).$$ (4.2.3)

The idea is that there is a signal related to the spatio-temporal distribution of slip on the San Andreas fault that produces a global, spatially-coherent signal in the collection of measured time series and that distance measurements see this signal plus signals from processes that are local to the sites of benchmarks defining measurement points. The local signals are due to sources of motion that are acting on individual benchmarks in the network rather than over a broad areal scale. We conceive of these signals as smooth functions of time attributable to such effects as soil erosion or landslide around a benchmark that may produce a trend in the line-length signal, "wobble" of a benchmark about a fixed point, that may produce a random-walk-like signal, and alternating periods of soil saturation in the winter and desication in the summer that may produce seasonal signals in the data.

4.2.1 A model for local deformation

We have mentioned that a question of primary interest regarding the temporal nature of deformation due to tectonic effects concerns the presence or absence of acceleration in fault motion. To address this question, we take an empirical Bayesian approach, using a priori beliefs to construct a parametrized family of probability models for the space-time distribution of fault slip, and for local benchmark motion.

Let $w^j(t)$ denote the vector of cumulative locally-produced displacement of the $j^{th}$ benchmark from time $t^0_j$ to time $t$, with $j = 0$ indexing Car Hill. We model each component of the local displacement vector as the sum of a random walk, to account for benchmark wobble, and a sinusoid with annual period, to account for seasonal effects. Thus, we have

$$w^j_k(t) = \eta_{jk} \hat{B}^j_k(t - t^0_j) + \alpha_{jk} \cos(2\pi t + \omega_{jk}), \frac{\lambda + \mu}{\lambda + 2\mu} = 1, 2$$ (4.2.4)

where the $\hat{B}^j_k(\cdot)$, representing benchmark wobble, are modeled a priori as Brownian motions. The changes in line-length due to the local displacements are determined, to good
approximation, by
\[ \delta L_j^{\text{loc}}(t) = \cos(\theta_j) \cdot \{ w_1^j(t) - w_1^0(t) \} + \sin(\theta_j) \cdot \{ w_2^j(t) - w_2^0(t) \}, \] (4.2.5)

where \( \theta_j \) is the azimuth of the \( j^{th} \) baseline. Generally, we will assume that the 36 components of benchmark wobble, one in each of 2 coordinate directions in the horizontal plane at each of the eighteen measurement sites in the network are independent and identically distributed Brownian sample paths with common scale parameter, \( \eta \). Under this assumption, the benchmark wobble components, the \( j^{th} \) of which is given by

\[ B_j(t) \triangleq \cos(\theta_j) \cdot \{ \bar{B}_1^j(t - t_j^0) - B_1^0(t - t_j^0) \} + \sin(\theta_j) \cdot \{ \bar{B}_2^j(t - t_j^0) - B_2^0(t - t_j^0) \}, \] (4.2.6)

are mean zero processes with covariances

\[ EB_i(s)B_j(t) = (s \land t) \cdot \eta^2 \cdot \{ \delta_{ij} + \cos(\theta_i - \theta_j) \}. \] (4.2.7)

Likewise, the annual signals in (4.2.4) will be given by a simple cosine wave:

\[ A_j(t) = \alpha_j \cos(2\pi t + \omega_j). \] (4.2.8)

### 4.2.2 A stochastic model for the signal

To model the space-time distribution of fault slip in an efficient and useful manner, we appeal to the arguments that we presented in deriving steady-state models and source representations in Chapters 1 and 3. To describe our model, we require some basic notions from the theory of probability and stochastic processes in linear function spaces, as may be found in [25], for instance. We will use Hilbert space-valued Gaussian processes in modeling a time-dependent component of deep slip, to be added to a steady-state contribution from the boundary-determined slip component, thus constructing a prior probability model for the spatio-temporal slip distribution, \( s(\xi, t) \). Let

\[ f(\xi, t) = s(\xi, t) - ts_{\text{bar}}(\xi), \] (4.2.9)

where \( s_{\text{bar}}(\cdot) \) is the boundary-determined component of slip rate described in Chapter 3. \( f(\cdot, t) \) represents the time-dependent distribution of deep slip. We require \( f(\cdot, t) \in \)
Section 4.2. Introduction

\( \mathcal{H}_0^2(\Sigma_0) \) \( \forall t \in T \), where the inner product on \( \mathcal{H}_0^2 \) is defined by the differential form of the stress variability inner product derived in Chapter 3, and we view \( f(\cdot, t) \) \textit{a priori} as a Gaussian process in this space. More precisely, we model \( f \) as distributed like a \( \mathcal{H}_0^2 \)-valued sample path of a Gaussian process, \( \tilde{F}(\cdot, t) \equiv \tilde{F}_t \). We specify a family of distributions for \( \tilde{F}_t \) by the following considerations regarding the temporal and spatial processes at work.

Spatially, we still wish to maintain smoothness of the fault surface traction distribution, as we did in our steady-state analysis. The natural stochastic equivalent to the minimum-SV models in the steady-state case would have us model \( \tilde{F}_t \) as

\[
\tilde{F}_t(\xi) = t\tilde{V}(\xi)
\]  

(4.2.10)

where \( \tilde{V}(\xi) \), the residual deep slip rate to be added to the boundary-determined component to produce the steady-state rate slip estimate, is a Gaussian r.v. with mean zero and \textit{isotropic covariance} in \( \mathcal{H}_0^2 \). This is the stochastic equivalent to our steady-state model of Chapter 3 in the sense of Kimeldorf and Wahba [23, 24], who have described the close connections between variational problems requiring constrained minimization of quadratic forms and Bayesian problems involving calculation of the posterior mean of a Gaussian measure on function space when data comes as the value of a bounded linear functional.

There is a problem with this suggestive equivalence, however, stemming from the fact that isotropic probability measures can’t live on infinite-dimensional spaces (Cf. Kuo [25]), but this problem is circumvented in the empirical Bayes approach, where we do not claim the ability, nor do we face the need, to specify uniquely a prior probability model for the function of interest. By repeatedly observing only a finite number of linear functionals of an evolving space-time function with values in a infinite-dimensional Hilbert space, we gain empirical information regarding the behavior of the function only in the low-dimensional subspace spanned by the representers of the observed functionals. On this basis, we may replace the assumption of isotropy with the assumption of orthogonality between the space that we see through the data and the space that we do not see. In other words, we write \( \tilde{F}_t \) as the sum of two indepent processes, \( F_t \), which takes values in the finite-dimensionaonal space about which we have empirical information, and \( F_t^0 \), which
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takes values in the infinite-dimensional space orthogonal to the span of the representers and, so, is never "seen" by in the data. The orthogonality assumption amounts to saying only that, lacking the possibility to obtain evidence that suggests another course, we choose to act as if the component of the process that is invisible does not covary with the "estimable" component that is visible in the data.

Reliance on the orthogonality assumption simplifies the problem of specifying a prior, or a family of priors, by reducing it to finite spatial dimensionality. Let $B$ denote the subspace of $\mathcal{H}_0^2(\Sigma_0)$ spanned by the representers of the functionals observed in the data. Then we write

\[
F_t(\xi) = \sum_{j=1}^{17} \hat{Z}_t^j \Phi_j(\xi) = \sum_{j=1}^{17} Z_t^j \beta_j(\xi),
\]

(4.2.11)

(4.2.12)

where the $\Phi_j$ are the representers of the observation functionals, the $\beta_j$ are orthobasis functions derived from the representers as in Chapter 3, and $\hat{Z}_t \triangleq (\hat{Z}_t^1, \ldots, \hat{Z}_t^{17})^T$ and $Z_t \triangleq (Z_t^1, \ldots, Z_t^{17})^T$ are vector Gaussian processes related by the eigenvectors and eigenvalues of the Gram matrix, $\Gamma$, of the representers as

\[
Z_t = \Psi \Lambda^{\frac{1}{2}} \hat{Z}_t.
\]

(4.2.13)

We specify the temporal nature of the coefficient processes in (4.2.11) and (4.2.12) based on the assumption that tectonic driving forces are smoothly distributed in time. In particular, we assume that the velocity process,

\[
\dot{F}_t = \sum \dot{Z}_t^j \beta_j,
\]

(4.2.14)

exists, and that the velocity might change from some random initial value, $V_0(\xi)$, by the action of random perturbations that are stationary in time and that are independent in nonoverlapping time intervals. This leads us to model the velocity as a Gaussian process with stationary, independent increments, a model that implies that the components of $\dot{Z}_t$ are Wiener processes. Thus, with $W^j$ being ordinary Wiener processes, we have

\[
\dot{Z}_t^j = \tau_j V_0^j + \alpha_j \int_0^t dW_s^j,
\]

(4.2.15)
Section 4.3. State-space model

which implies that $Z_t$ is a vector integrated Wiener process

$$
Z_t = \int_0^t \dot{Z}_u \, du
= t \cdot \tau_j V_0^j + \alpha_j \int_0^t \int_0^u dW_j^j \, du.
$$

Equation (4.2.16), together with the previously stated orthogonality assumption, defines a stochastic space-time model for the signal we wish to estimate up to specification of the scale parameters, $\tau_j$, $\alpha_j$, and covariances of the components of the random initial velocity, $V_0$, and the vector of Wiener processes $W_t$ which are integrated in (4.2.15) to produce time variation in $Z_t$. Generally, we will continue to rely only on the belief that the SV norm should be small and, so, we will model the coefficients of basis functions that are orthogonal in the SV inner product as being independent and identically distributed. Thus, the stochastic space-time model we will use expresses the spatio-temporal distribution of slip in terms of independent, standard normal r.v.'s, $V_0^j$, $j = 1, \ldots, 17$, and independent, standard Wiener processes, $W_s^j$, $s \geq 0$, as

$$
S(\xi, t) = t B(\xi) + \sum_j Z_t^j \beta_j(\xi)
= t B(\xi) + \sum_j \left\{ t \cdot \tau_j V_0^j + \alpha \int_0^t \int_0^u dW_j^j(s) \, du \right\} \cdot \beta_j(\xi)
$$

(4.2.17)

4.3 A state-space model and linear filter

The models given in equations (4.2.17) and (4.2.7) for the contributions to the data of the tectonic and local signals provide a convenient means for computing both posterior distributions of the signal parameters given the observed data, and the mariginal likelihood of parameters and hyperparameters, "marginal" meaning integrated with respect to the prior distribution on the signal. The convenience derives from representing the model in a linear state-space form that allows application of recursive filtering and smoothing algorithms. We now describe the state-space model and recursive estimation algorithms applicable to this model.
4.3.1 State-space model

A stochastic state-space model is defined by two linear equations: an *state transition equation* that describing the temporal evolution of a stochastic *state vector* in terms of linear propagation and random, independent perturbations, and an *observation equation* describing the linear relationship between the state and data that are observed over time. In our description of the state-space model for the processes we are trying to estimate, we will discuss only the components that are modeled as stochastic processes. For practical purposes, we have carried out most linear filtering and smoothing on data that have already been seasonally adjusted, *i.e.*, the seasonal components local motion have already been subtracted. These seasonal signals are actually estimated using variants of the state space models and we will describe in this section, but we will not further discuss the seasonal variations in the data. Also, following the procedure of Chapter 3, we will subtract from the data the component of deformation predicted by the steady state action of fault slip at the rate determined by boundary conditions, as we described in Chapter 3. Henceforth, when we talk about "the data" we shall be refer to the data minus the seasonal and boundary-determined components. In the space-time estimation procedure to be described, we will also treat observations made to different baselines on the same day as having been measured simultaneously. Thus, when we speak of a vector $\mathbf{y}_k$, observations made at time $t_k$, we refer to all observations made on the $k^{th}$ observation day.

Combining the models, (4.2.7) and (4.2.17), for local and tectonic deformation and having subtracted the contributions from the assumed steady-state rate of tectonic motion and an estimate of the seasonal terms, we wish to construct a state-space form for the model

$$y_j(t_i) = O_j + (\tilde{\gamma}_j)^T Z_t + B_j(t - t_i) + \epsilon_{ij}.$$  \hspace{1cm} (4.3.18)

We obtain the required evolutionary models for the stochastic elements of the signal quite simply. To model the components of $Z_t$, which are independent, random initial values plus independent integrated Wiener processes, we need to keep track of both $Z_t$,
and $\dot{Z}_t$. The $k^{th}$ component $Z_t$ is described by

$$
\begin{pmatrix}
Z_t^k \\
\dot{Z}_t^k
\end{pmatrix}
= 
\begin{pmatrix}
1 & t - s \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
Z_s^k \\
\dot{Z}_s^k
\end{pmatrix}
+ q_{s,t},
$$

(4.3.19)

where $q_{s,t}$ is Gaussian with mean zero, and covariance matrix

$$
E q_{s,t} q_{s,t}^T = \alpha \begin{pmatrix}
\frac{(t-s)^3}{3} & \frac{(t-s)^2}{2} \\
\frac{(t-s)^2}{2} & t - s
\end{pmatrix};
$$

(4.3.20)

the increments, $q_{s_1,t_1}, q_{s_2,t_2}$ also are independent for $s_1 < t_1 \leq s_2 < t_2$. The vector of Brownian motion terms, $B_t$, satisfies

$$
B_t = B_s + r_{s,t},
$$

(4.3.21)

where the increment vector has mean zero and covariance matrix

$$
\text{Cov}(r_{s,t}) = (t - s) \left( (\delta_{kl} + \cos(\theta_k - \theta_l)) \right).
$$

(4.3.22)

We define the state vector, $X(t)$, as the stochastic process with components

$$
X(t) = \begin{pmatrix}
Z_t \\
\dot{Z}_t \\
B_t
\end{pmatrix}.
$$

(4.3.23)

Since the are collected at discrete (but not equally-spaced) time intervals, we will actually need values for the state vector only in discrete time. We will write $X_k$ for $X(t_k)$, write the discrete-time state-space model as

$$
X_k = F_k X_{k-1} + \delta_k, \quad \delta_k \sim N(0, Q_k)
$$

(4.3.24)

$$
Y_k = H_k X_k + \epsilon_k, \quad \epsilon_k \sim N(0, R_k).
$$

(4.3.25)

In (4.3.24), the state vector and transition matrix are as above, and the increment, $\delta_k$, is the vector with containing the increments$q_k$ of the integrated Wiener processes, $Z_t$, and the Brownian motions $B_t$. The observation, $Y_k$, is the vector of data measured made at the $k^{th}$ epoch, and $R_k$, the covarinace matrix of the measurement errors, is diagonal with variances given by the appropriate theoretical values from (4.2.1).
4.3.2 Filtering and smoothing

In the context of the state-space model in (4.3.25), estimation of components of the signal is carried out by recursive linear filtering and smoothing. Let

\[
X_{k \mid j} = E[X_k \mid y_1, \ldots, y_j], \tag{4.3.26}
\]

\[
\Sigma_{k \mid j} = \text{Cov}[X_k \mid y_1, \ldots, y_j] \tag{4.3.27}
\]

be the conditional mean and covariance matrix of the state at epoch \(k\) given data through epoch \(j\). The conditional means and covariances satisfy the one-step-ahead prediction equations,

\[
\dot{X}_{k \mid k-1} = F_k \dot{X}_{k-1 \mid k-1} \tag{4.3.28}
\]

\[
\Sigma_{k \mid k-1} = F_k \Sigma_{k-1 \mid k-1} F_k^T + Q_k \tag{4.3.29}
\]

and the update equations

\[
\dot{X}_{k \mid k} = \dot{X}_{k \mid k-1} + G_k \left( y_k - H_k \dot{X}_{k \mid k-1} \right) \tag{4.3.30}
\]

\[
= \dot{X}_{k \mid k-1} + G_k \nu_k
\]

\[
\Sigma_{k \mid k} = \Sigma_{k \mid k-1} - G_k H_k \Sigma_{k \mid k-1}.
\tag{4.3.31}
\]

In these equations,

\[
G_k \triangleq \Sigma_{k \mid k-1} H_k^T \left[ H_k \Sigma_{k \mid k-1} + R_k \right]^{-1}, \tag{4.3.32}
\]

is the *Kalman gain*, and

\[
\nu_k \triangleq y_k - H_k \dot{X}_{k \mid k-1}, \tag{4.3.33}
\]

is the *innovation*.

Estimation proceeds by starting with a prior mean \(\dot{X}_{1 \mid 0}\) and covariance matrix, \(\Sigma_{0 \mid 0}\), for the state at the time zero. We take \(\dot{X}_{0 \mid 0} = 0\). The covariance matrix of the components of \(X\) corresponding to \(\dot{Z}\), the velocities of the added slip components is set to \(\tau^2 I\), as specified in (4.2.17), and the components of the local signal modeled as Brownian motion have a diffuse prior on their starting values. By using a diffuse prior, we handle the
arbitrary distant offsets, $O_j$, on each line effectively by using them to define the origins of the local benchmark motions.

Beginning with the specified priors, we step through the data one observation epoch at a time. At the conclusion of the the $k - 1$st, we have computed values for $\hat{X}_{k-1|k-1}$ and $\Sigma_{k-1|k-1}$, to which we apply the prediction equations, (4.3.28), (4.3.29) to obtain a mean and covariance, $\hat{X}_{k|k-1}, \Sigma_{k|k-1}$, which act as a "prior" at time $t_k$. The update equations compute the posterior mean and covariance matrix for the state at time $t_k$ by taking into account the information that becomes available through the innovation. In the case where the observation at $t_k$ agrees exactly with what is predicted based on the prior, there is no innovation and the conditional mean of the state vector does not change, though the covariance may.

Having stepped through all of the data going forward in time, we end up with the conditional mean and covariance

$$\hat{X}_{n|n} = E\hat{X}(t_n | Y),$$

$$\Sigma_{n|n} = \text{Cov}(\hat{X}(t_n | Y)).$$

To complete the estimation procedure we need to compute the posterior means and covariance of the state at all times of interest and conditioned on all of the available data. We may accomplish this using the same recursive prediction and updating structure as in the filtering equations, but now going backwards in time. The recursive smoothing algorithm due to Rauch, Tung and Striebel [43] has the form

$$\hat{X}_{k|n} = \hat{X}_{k|k} + S_k \left\{ \hat{X}_{k+1|n} - F_{k+1} \hat{X}_{k|k} \right\},$$

(4.3.34)

$$\Sigma_{k|n} = \Sigma_{k|k} + S_k \left\{ \Sigma_{k+1|n} - \Sigma_{k+1|k} \right\} S_k^T,$$

(4.3.35)

where the smoothing matrix at the $k^{th}$ epoch is given by

$$S_k = \Sigma_{k|k} F_k^T \Sigma_k^{-1} \Sigma_{k|k}.$$

(4.3.36)
4.3.3 Recursive computation of the marginal likelihood

There are three as yet unspecified scale parameters in our stochastic model: $\alpha$, the "acceleration" parameter weighting the integrated Brownian motion terms that model time-variation in the slip rates, $\tau$, the scale of the random initial slip velocities, and $\eta$, the scale parameter in the local Brownian motion processes. We wish to use the marginal distribution of the data, integrated over the signal, to estimate these parameters. It is convenient to do this, in the context of the linear state-space model described in the previous section, by using the filtering equations to evaluate recursively the marginal likelihood and its derivative, and then searching the parameter space for the point at which the likelihood is maximized.

Recursive computation of the likelihood is based on the so-called prediction-error decomposition (Harvey [17]), in which the joint density of $n$ ordered observations is written as the product of the densities of each observation given the parameter, $\Theta$, and all previous observations:

$$ f(y_1, \ldots, y_n \mid \Theta) = \prod_{k=1}^{n} f_k(y_k \mid \Theta; y_1, \ldots, y_{k-1}). $$

From this decomposition, the log-likelihood for $\Theta$ given $Y$ is

$$ \mathcal{L}(\Theta \mid Y) = \sum_{k=1}^{n} \log \{f_k(y_k \mid \Theta; y_1, \ldots, y_{k-1})\} $$

For Gaussian data, the conditional densities are Gaussian with means

$$ E_{\Theta}\{y_k \mid Y_1 = y_1, \ldots, Y_{k-1} = y_{k-1}\} = \hat{y}_k \mid k-1, $$

and covariances

$$ E\{(y_k - \hat{y}_k \mid k-1) \cdot (y_k - \hat{y}_k \mid k-1)^T\} = \Omega_k. $$

so the log-likelihood is

$$ \mathcal{L}(\Theta \mid Y) = -\frac{1}{2} \sum_{k=1}^{n} \log(|\Omega_k|) + \{y_k - \hat{y}_k \mid k-1\}^T \Omega_k^{-1} \{y_k - \hat{y}_k \mid k-1\}. $$

In the state-space model, the conditional mean of the $k^{th}$ observation given the previous $k-1$ is

$$ E_{\Theta}\{y_k \mid y_1, \ldots, y_{k-1}\} = H_k \hat{X}_k \mid k-1, $$
where both $H_k$ and $\bar{X}_k | k$ on the right hand side may depend on $Θ$. Hence, the difference between the $k^{th}$ observation and its expectation given previous data is precisely the innovation in (4.3.33) The covariance matrix of the $k^{th}$ innovation is

$$Ω_k = R_k + H_k Σ_{k | k-1} H_k^T.$$  \hspace{1cm} (4.3.43)

Inserting the innovation and its covariance matrix into (4.3.41), noting the dependence of each on $Θ$, gives the innovations form for the log-likelihood as

$$L(Θ | y_1, \ldots, y_n) = -\frac{1}{2} \sum_{k=1}^{n} \log(||Ω_k(Θ)||) + ν_k^T(Θ)Ω_k^{-1}(Θ)ν_k(Θ).$$  \hspace{1cm} (4.3.44)

Note that this expression may also be differentiated with respect to the parameter to obtain recursively-computable forms for the gradient and, if differentiated twice, the Hessian of the log-likelihood. The ability to compute first and second derivatives efficiently is, of course, useful in carrying out a numerical search for the MLE using Newton methods, as we do, and in computing the information matrix once the MLE is found.

### 4.4 Results on Parkfield data

In this section, we present results obtained by applying to the Parkfield two-color data the space-time estimation procedure we have described.

Using the procedure described in Section 4.3.3, we have obtained maximum likelihood estimates for some of the parameters in the space-time model for the Parkfield data. In order to simplify the computation involved in searching numerically for the maximium of the likelihood, we have set the parameter $τ^2$ defining the variance of the random initial values of slip rate to agree with the smallest consistent value the damping parameter that we found in the steady-stae analysis of Chapter 3. That value for the damping parameter was $ρ^2 = .0074$, which translates in the present model to setting $τ = 1/ρ = 11.6$. With this value of $τ$ fixed, we used a quasi-Newton algorithm to search for the value of $(α^2, η^2)$ at which the maximum is obtained. Though it doesn't seem that it would necessarily be so, there appears to be a unique maximum at roughly $α^2 = 150, η^2 = 6.3$. The maximum is not
Section 4.4. Parkfield space-time

well determined in the $\alpha^2$ direction, with changes in $\alpha^2$ by a factor of two only producing about a five percent decrease in the log-likelihood from its value at the maximum.

Figures 4.2 and 4.3 show the estimated time-dependent rate coefficients on the first six orthobasis functions derived from the SV-representers of the line-length functionals in the Parkfield network. To interpret these figures, recall that the model represents the spatial distribution of the rate of strike slip in the PSZ as the sum of a constant contribution from the boundary-determined component and a linear combination of the orthobasis functions:

$$\dot{s}(\xi, t) = s_{\text{bary}}(\xi) + \sum_{k=1}^{p} \dot{Z}_j^i \beta_j(\xi).$$

(4.4.45)

The functions depicted in Figures 4.2 and 4.3 are the estimated, time-dependent slip rates on each of the first six orthobasis functions. The estimates are given by the posterior means, $E[\dot{Z}_j^i | Y]$, where $Y$ is the entire collection of Parkfield two-color data, and the scale parameters in the model are set at the MLE’s. In examining Figures 4.2 and 4.3 an immediately striking fact is that they are by no means constant! There is certainly an apparent time-varying signal that is picked up by the estimation procedure we have used, a signal that may be an artifact of some non-tectonic signal in the data that maps to fault slip or may, indeed, reflect some true temporal variation of deep slip rates. The manner in which the time variation depicted in the figures translates to slip is most easily understood in the first two coefficients. Recall that the first orthobasis function is a smooth distribution of left-lateral strike slip with a maximum value of about 1 mm at a depth of about 10 km. Thus, positive values on the first coefficient translate to left-lateral slip at depth which, when added to the constant rate contribution from the boundary component, presumably represents retardation of the rate of right-lateral slip. The second orthobasis function produces a signal that is left lateral to the north of Car Hill and right lateral to the south, yielding compression toward the center of the network on the west side of the San Andreas and extension on the east side of the fault, and producing large fault-normal disaplements.

The signal in the top panel of Figure 4.2 is initially right-lateral at a constant rate
Section 4.4. Parkfield space-time

of about 2 mm/year from the onset of observation in mid-1984 through about mid-1985, at which time it begins to turn around and go left-lateral, reaching a peak of about 4 mm/year in late 1986, then decreasing slightly before increasing again in 1988 up to about 5 mm/year in late 1989, at which time it begins to decrease back to nearly zero in early 1991. One may be tempted to associate the observed rate changes with large earthquakes that have occurred in the vicinity of Parkfield or on the San Andreas fault. The onset of the shift from right-lateral to left-lateral in the first coefficient, for instance, coincides almost exactly with the Kettleman Hills earthquake (M = 5.5) of August, 1985; the largest rate change in the other direction, going from left-lateral to right-lateral comes in late 1989, at about the time of the Loma Prieta earthquake (M = 7.0) on (or near) the San Andreas fault but 200 km north of Parkfield. In the second basis function, we see a constant decrease in the rate beginning in early 1985 and lasting until the beginning of 1987, at which time the rate levels off to about -15 mm/year. The third through sixth estimated rate functions also show time variations with the most notable feature being the apparent changes in mid to late 1985.

Figures 4.4-4.20 show the decompositions of the data on each baseline as the sum of the estimated, time dependent signal, the estimated local deformation signal, and a residual that is unexplained by any of the signal components. If the model is accurate, then the residuals ought to have the appearance of white noise, i.e., independent random errors with constant variance over time. In the figures, we note that the residuals are not entirely consistent with what we expect, as there are time periods of noteworthy increase in residual variance. The most obvious period is in early 1990, when there is an apparent increase in the dispersion of the residuals on most of the baselines in the network. The appearance of large residual variation about the estimated model actually coincides, in this instance, with a time period in which the two-color instrument was not operating reliably and for which the data are suspect (J. Langbein, personal communication). Thus, the appearance of this systematic effect in the residuals is, in some sense, a good sign. The effect is attributable to an unmodeled component in the data, but this component is in the measuring instrument, not in the ground. Thus, we actually prefer that it show up in
Figure 4.2: Estimated time-dependent rates on first three orthobasis functions.
Figure 4.3: Estimated time-dependent rates on second three orthobasis functions.
<table>
<thead>
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<th>Line</th>
<th>Theoretical variance</th>
<th>Mean squared residual</th>
<th>Observed Theoretical</th>
</tr>
</thead>
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<td>0.9866</td>
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<td>Hunt</td>
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<td>0.1414</td>
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</tr>
<tr>
<td>Lang</td>
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<td>0.2949</td>
<td>1.3315</td>
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<td>0.6043</td>
<td>1.2221</td>
</tr>
<tr>
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<td>0.4817</td>
<td>1.3881</td>
</tr>
<tr>
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<td>0.3094</td>
<td>0.3137</td>
<td>1.0138</td>
</tr>
<tr>
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<td>0.9637</td>
</tr>
<tr>
<td>Nore</td>
<td>0.3806</td>
<td>0.0769</td>
<td>0.2021</td>
</tr>
<tr>
<td>Pomo</td>
<td>0.3811</td>
<td>0.4390</td>
<td>1.1521</td>
</tr>
<tr>
<td>Table</td>
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<td>0.5959</td>
<td>1.2993</td>
</tr>
<tr>
<td>Todd</td>
<td>0.1957</td>
<td>0.3450</td>
<td>1.7623</td>
</tr>
<tr>
<td>Turk</td>
<td>0.1260</td>
<td>0.0799</td>
<td>0.6344</td>
</tr>
</tbody>
</table>

Table 4.2: Theoretical variances and mean squared residuals about the predicted model.

the residuals rather than the signal components, though it is possible that some of it does show up there as well. Table 4.2 shows the theoretical variance, the mean squared residual about the estimated model, and the ratio of the two for each baseline. The agreement between the expected and observed variances is quite good, though for the line to Nore, the expected variance is about 5 times larger than the mean squared residual about the model predictions.

The estimated local signals in Figures 4.4-4.20 actually do resemble, in their erratic behavior, the stochastic process that we used to model there appearance a priori: Brownian motion. Of course, we do not expect to see trends in Brownian motion, as we do on many of the baselines. It is interesting to note the directions of the trends on some of the baselines with observed trends in extension or contraction that are difficult to explain in the steady-state model. For instance, Pomo should have a tectonic signal that produces contraction, but its steady-state motion is extension. In Figure 4.17, we see that the observed extension is mapped into the local signal and that the estimated tectonic signal
Section 4.4. Parkfield space-time

Figure 4.4: Estimated decomposition of data on baseline from Car Hill to Bare into tectonic signal, local signal, and residual.
Figure 4.5: Estimated decomposition of data on baseline from Car Hill to Buck into tectonic signal, local signal, and residual.
Figure 4.6: Estimated decomposition of data on baseline from Car Hill to Can into tectonic signal, local signal, and residual.
Figure 4.7: Estimated decomposition of data on baseline from Car Hill to Creek into tectonic signal, local signal, and residual.
Figure 4.8: Estimated decomposition of data on baseline from Car Hill to Gold Hill into tectonic signal, local signal, and residual.
Figure 4.9: Estimated decomposition of data on baseline from Car Hill to Hogs into tectonic signal, local signal, and residual.
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Figure 4.10: Estimated decomposition of data on baseline from Car Hill to Hunt into tectonic signal, local signal, and residual.
Figure 4.11: Estimated decomposition of data on baseline from Car Hill to Lang into tectonic signal, local signal, and residual.

shows the contraction that we expect. Similarly, the line to Hogs has an observed steady-state of extension that is of opposite sign to what should be produced by the pattern of right-lateral slip thought to exist at Parkfield. In Figure 4.9, we see that the aberrant trend is mapped to local motion and the tectonic signal is of the expected sign.

4.4.1 Effects of smoothing parameters

We have examined the effects of varying the model parameters of primary interest, the scale parameters on the local and tectonic signal processes, by running the recursive smoothing algorithm with a variety of parametrizations. In Figures 4.21 and 4.22, we have plotted the estimated time-dependent coefficients in the expansion of the the tectonic
Figure 4.12: Estimated decomposition of data on baseline from Car Hill to Mason into tectonic signal, local signal, and residual.
Figure 4.13: Estimated decomposition of data on baseline from Car Hill to Mels into tectonic signal, local signal, and residual.
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Figure 4.14: Estimated decomposition of data on baseline from Car Hill to Midd into tectonic signal, local signal, and residual.
Figure 4.15: Estimated decomposition of data on baseline from Car Hill to Mide into tectonic signal, local signal, and residual.
Figure 4.16: Estimated decomposition of data on baseline from Car Hill to Nore into tectonic signal, local signal, and residual.
Figure 4.17: Estimated decomposition of data on baseline from Car Hill to Pomo into tectonic signal, local signal, and residual.
Section 4.4. Parkfield space-time

Figure 4.18: Estimated decomposition of data on baseline from Car Hill to Table into tectonic signal, local signal, and residual.
Figure 4.19: Estimated decomposition of data on baseline from Car Hill to Todd into tectonic signal, local signal, and residual.
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Figure 4.20: Estimated decomposition of data on baseline from Car Hill to Turk into tectonic signal, local signal, and residual.
signal obtained by increasing the acceleration parameter, \( \alpha^2 \), by two orders of magnitude, going from the MLE of 150 up to 15,000. This has the expected effect of producing much more erratic looking estimates than were produced with a lower value of this scale parameter. Note that there is a very large negative jump in the coefficient of the first basis function in mid-1985, right at the time of the Kettleman Hills earthquake. This maps to a large right lateral signal on the fault at the time of the earthquake followed by abrupt left-lateral rebound. By “undersmoothing” the model relative to what is recommended by the maximum likelihood principle, we have learned that the smooth, left-lateral increase after the Kettleman Hills earthquake is actually preceded by a large, right-lateral coseismic signal. It would be an interesting exercise to compare some of the other large, transient variations in these functions with the seismic record in the Parkfield area to see if there is an association with seismicity.

We might expect that if we set the acceleration parameter to zero, we would produce estimates that would look like the steady-state estimates of the previous chapter. This is not so, however, because the local estimates of local motion pick up mush of the signal, including the trend, if it is not allowed to go into time-varying slip. The mean values of the time-varying slip rates are actually quite close to their steady-state values when we smooth at the ML parameter values.

We may do a formal test for the statistical significance of time variation in the tectonic signal by computing the difference in values of the log-likelihood at the MLE for the full model and at the MLE for \( \eta^2 \) when \( \alpha^2 \) is set equal to zero. The MLE for \( \eta^2 \) when \( \alpha^2 \) is set to zero increases only slightly, from 6.3 to 6.9. The values of the loglikelihood are \( L_1 = l(150, 6.3) = -3,205.97 \), and \( L_0 = l(0, 6.9) = -3,207.93 \), giving a likelihood ratio statistic of \( \lambda = 2 \cdot (L_1 - L_0) = 3.91 \). This value is slightly larger than the 95th percentile of the \( \chi^2_1 \) distribution, leading to the conclusion that, based on this analysis, acceleration of deep slip is a statistically significant effect in the Parkfield two-color data.
Figure 4.21: Estimated time-dependent coefficients of first three slip components smoothed with $\alpha^2 = 15,000, \eta^2 = 6.0.$
Figure 4.22: Estimated time-dependent coefficients of second three slip components smoothed with $\alpha^2 = 15,000, \eta^2 = 6.0$. 
4.5 Discussion and conclusions

In this chapter, we have extended the inverse theory and source representation principles of Chapters 1 and 3 to address the problem of estimating time-varying tectonic signals contaminated by both random measurement error and by local deformation signals. Estimation in carried out by modeling the various model components a priori as Gaussian processes with smoothness properties dictated by assumptions regarding the nature of their causes.

Though our results are preliminary, requiring comparison with other data collected in the Parkfield experiment and requiring some thoughtful, geophysical interpretation, the estimation method appears to perform well. It is somewhat surprising that apparent tectonic signals were picked out so well in the Parkfield data, given the fact that these signals were free to map into local motion. We had anticipated that it would be difficult to separate local from tectonic effects because of the poor configuration of the two-color network. The central site at Car Hill, which is common to all baselines in the network, is only about 200 meters from the fault trace. Presumably, shallow fault motion maps into the two-color network its large effect on Car Hill. Perhaps by relying on the methods of Chapters 1 and 3 to produce representation for deep sources that do not allow slip at the surface, we have succeeded in picking out signals due to slip at depth, as was our objective.

In a broader view, the methods we have developed for estimating functions in space-time in the presence of complex contaminants, rather than only independent errors, may be of use in other function-estimation problems in which data pertinent to some spatial process is accumulated over time. Space-time function estimation problems abound in geophysics, and it is the hope of this writer to apply some of the methods developed in this dissertation to other data sets and examples, both from tectonophysics and elsewhere.
Appendix A

Appendix

A.1 Minimization of a quadratic form subject to a linear constraint

Let $\langle \cdot, \cdot \rangle$ denote a bilinear form in slip with positive definite associated quadratic form that generates a norm:

$$\|s\| \triangleq \sqrt{\langle s, s \rangle}.$$  

We will use $\| \cdot \|$ to define a normed linear space of admissible slip distributions by

$$\mathcal{S} \triangleq \{ s : \Sigma \to \mathbb{R}^2 : \|s\| < \infty \}. \quad (A.1.1)$$

Given only that some linear functional $\phi(s)$, has some value $y$, we will seek a minimum norm estimate of $s$ which will solve the variational problem

$$\text{minimize}_{s \in \mathcal{S}} \|s + s^r\| \quad \text{subject to} \quad \phi(s) = y, \quad (A.1.2)$$

The reference slip distribution, $s^r$, in (A.1.2) is a fixed slip distribution that may define the elastic state of the crust at the time of initiation of slip.

The pair $(\mathcal{S}, \langle \cdot, \cdot \rangle)$ will generally be a Hilbert space of functions on $\Sigma$, and we will say that a linear functional, $\phi$, is bounded (continuous) on $(\mathcal{S}, \langle \cdot, \cdot \rangle)$ if there is a constant, $c$, such that

$$|\phi(s)| \leq c\|s\| \quad \forall s \in \mathcal{S}.$$  

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From elementary Hilbert space theory, we may deduce that problem (A.1.2) has a unique solution when $\phi$ is bounded and otherwise has no solution. Furthermore, when $\phi$ is bounded on $(\mathcal{S}, < \cdot, \cdot >)$, the Fréchet-Riesz representation theorem [55] says that there is a unique function, $\Phi \in \mathcal{S}$, satisfying the representation equation:

$$\phi(s) = < s, \Phi > \quad \forall s \in \mathcal{S}. \quad (A.1.3)$$

This function, to be referred to as the representer of $\phi$ in $\mathcal{S}$, is of essential interest, for, if $\Phi$ is known, then the solution to (A.1.2) may be written as

$$s^*(\xi) = \left\{ \frac{y + \phi(s^r)}{\phi(\Phi)} \right\} \Phi(\xi) - s^r(\xi). \quad (A.1.4)$$

To see that (A.1.4) is the sought-after solution to (A.1.2), first introduce a Lagrange multiplier, $\lambda$, to recast the constrained minimization problem, (A.1.2), as an unconstrained problem:

$$\text{minimize} \quad F(s) = ||s|| - \lambda \cdot \{ < s, \Phi > - y \},$$

where minimization is now with respect to $s \in \mathcal{S}(\Sigma_0)$, and to $\lambda \in \mathbb{R}$. Then, we have

$$F(s) = < s_r + s, s_r + s > - \lambda \{ < s, \Phi > - y \} = < s, s > + < s, 2s_r - \lambda \Phi > + \lambda y.$$  

The part of this expression depending on $s, < s, s > + < s, 2s_r - \lambda \Phi >$, is minimized when

$$s = \frac{\lambda}{2} \Phi - s_0. \quad (A.1.5)$$

Among such $s$, the one satisfying (A.1.2) has

$$\lambda = \frac{y + \phi(s_r)}{\phi(\Phi)}. \quad (A.1.6)$$

Combining (A.1.5) and (A.1.6) yields (A.1.4).

### A.2 Smoothing theory

If we know values of a function at some, but not all, points in a domain and we can assume that the function is smooth, then we may, in fact, have reasonably accurate knowledge
of the function globally. Mathematicians working on function interpolation and estimation often use smoothness principles to fill in the gaps left by incomplete or imperfect knowledge, and there is a connection between elastic strain energy and smoothness that is evident in the seminal spline problem of Schoenberg, to which we have alluded. There, if \( f \) is a function describing the lateral deflection of a thin elastic rod, the functional \( \int (f'')^2 \) is proportional to elastic strain energy. In a broader interpretation, \( \int (f''')^2 \) is viewed as a measure of roughness, and is thus freed from the narrow mechanical context in which it was originally used so that it may be applied in a wide variety of problems having nothing to do with elastic deformation or any other physical phenomenon. In fact, our own study of self-energy as a regularizing functional for problems involving fault slip began as a quest for a "natural" measure of smoothness of slip distributions that might be preferred, on physical grounds, to the arbitrarily chosen norms of Segall and Harris. By analogy to the spline problem, the self-energy seemed to be a good candidate for the office of "natural smoother" in inverse dislocation problems, but we have found, to the contrary, that the self-energy does not impose smoothness in an important and precise sense that we will now describe.

A.2.1 Smoothing norms and reproducing kernels

To define a useful notion of smoothness that dictates when a global bound on a norm of a function in a linear space translates into any local constraint on the behavior of the function at a point, we will use the definition of a bounded linear functional as given in Section 1. Let \((\mathcal{F}, \| \cdot \|)\) be a normed linear space on a domain \(\mathcal{X}\). For fixed \(x \in \mathcal{X}\), let \(\delta_x\) be the point-evaluation functional at \(x\), defined by

\[
\delta_x(f) = f(x).
\]

We will call \(\| \cdot \|\) a smoothing norm on \(\mathcal{F}\) if \(\delta_x\) is bounded in \((\mathcal{F}, \| \cdot \|)\) for all \(x \in \mathcal{X}\). If \(\phi\) is any linear functional, we may define its norm by

\[
\|\phi\| \triangleq \max_{\{f : \|f\| = 1\}} |\phi(f)|.
\]
and note that $\phi$ is bounded if and only if its norm is finite, in which case the constant, $k$, in (2.6) may be set to $\|\phi\|$. From this definition of the norm, we have, for any $x_0, x_1 \in \mathcal{X}$,

$$
|f(x_0) - f(x_1)| = |(\delta_{x_0} - \delta_{x_1})(f)| \\
\leq \|\delta_{x_0} - \delta_{x_1}\| \cdot \|f\| \\
\leq (\|\delta_{x_0}\| + \|\delta_{x_1}\|) \cdot \|f\|,
$$

so a bound on $\|f\|$ bounds the difference between values of $f$ at two points if point evaluation is bounded at both points. On the other hand, if $\delta_{x_0}$ is unbounded for some $x_0 \in \mathcal{X}$, then it’s possible to specify an arbitrarily “small” $f$, in the sense of the norm, for which $|f(x_0) - f(x_1)|$ is arbitrarily large, no matter how close $x_1$ is to $x_0$.

A Hilbert space with a smoothing norm is known as a reproducing kernel Hilbert space (rkHs), with reproducing kernel (rk), $R(x, y), x, y \in \mathcal{X}$, defined by

$$
R(x, y) \triangleq \delta_x(y), \quad (A.2.7)
$$

where $\delta_x(\cdot)$ is the representer of point evaluation at $x$. Among the important properties of the rk, as given in [4], for instance, are that it is is symmetric, positive definite function, that $R(x, \cdot) \in \mathcal{F} \forall x \in \mathcal{X}$, and that, for any $f \in \mathcal{F}$,

$$
<R(x, \cdot), f> = f(x). \quad (A.2.8)
$$

Plugging $R(x', \cdot)$ in for $f$ in (A.2.8) reveals the “reproducing” nature of $R$:

$$
<R(x', \cdot), R(x, \cdot)> = R(x', x).
$$

In function estimation problems, the availability of a simple, closed form expression for a reproducing kernel may be extremely helpful. From the rk, we may find the representer of any bounded linear functional by simply applying the functional to the rk. If $\phi$ is a bounded linear functional on $\mathcal{F}$, the its representer, $\Phi$, is given by

$$
\Phi(x) = \phi(R(x, \cdot)). \quad (A.2.9)
$$

For further discussion of reproducing kernels and their role in function estimation problems, see [60].
We close this section with two examples of reproducing kernels for Hilbert spaces of functions on one-dimensional intervals, satisfying either one-point or two-point boundary conditions. It is left to the reader to verify that these functions have all of the aforestated properties. In these examples (and subsequently), $x \land y$ is the minimum of $x, y$, and $x \lor y$ is the maximum.

**Example 1.** Let

$$\mathcal{H}_1(a, b) = \{ f : (a, b) \to \mathbb{R} : \int (f')^2 < \infty, f(a) = 0 \}. \quad (A.2.10)$$

The reproducing kernels for $\mathcal{H}_1(0, 1)$ and $\mathcal{H}_1(a, b)$ are, respectively,

$$R_1(x, y) \triangleq x \land y, \quad (A.2.11)$$

$$R_1(x, y; a, b) = R_1\left( \frac{x-a}{b}, \frac{y-a}{b} \right) \quad (A.2.12)$$

**Example 2.** Let

$$\mathcal{H}_2(a, b) = \{ f : (a, b) \to \mathbb{R} : \int (f'')^2 < \infty, f(a) = f'(a) = 0 \} \quad (A.2.13)$$

The reproducing kernels are

$$R_2(x, y) \triangleq (x \land y)^2 \cdot (3(x \lor y) - (x \land y)), \quad (A.2.14)$$

$$R_2(x, y; a, b) = R_1\left( \frac{x-a}{b}, \frac{y-a}{b} \right) \quad (A.2.15)$$

### A.2.2 Smoothing properties of norms in derivatives and pseudo-derivatives

We will make some use of an important set of necessary and sufficient conditions for norms defined by linear combinations of $L^2$ norms of derivatives to have the smoothing property. These conditions, usually referred to as *Sobolev imbedding theorems* (see [29, 1, 31]) say essentially (for our purposes) that an $L^2$ norm in $m^{th}$ (partial) derivatives of a function on an $n$-dimensional domain is a smoothing norm if and only if $m > n/2$.

Smoothness requirements are most easily understood, perhaps, in the wave number domain, where all hinges on the asymptotic behavior of Fourier transforms as $|\omega| \to \infty$. 
The relevant heuristics are straightforward when we consider norms that may be written symbolically as

$$\|s\|^2 = \|GD_k s\|^2,$$

(A.2.16)

where $D_k$ is a differential operator of order $k$, $G$ is an integral convolution operator bounded on $L^2(\Sigma)$, and $\| \cdot \|_2$ is the $L^2(\Sigma)$ norm. This class of norms is meant symbolically to encompass quadratic forms involving pseudo-differential operators acting on functions through integration against singular kernels. For simplicity, suppose that $s$ has Fourier transform asymptotic to $|\omega|^{-h}$ for some $h$, that $D_k s$ has Fourier transform asymptotic to $|\omega|^k \hat{s}(\omega) = |\omega|^{k-h}$, and that the kernel of $G$ has Fourier transform asymptotic to $|\omega|^g$. Now, the norm in (A.2.16) has the smoothing property if and only if a bound on the norm imposes pointwise bounds on the function. Thus we are led to consider a function, $s$, with a singularity at the origin due to divergence at infinity of the integral in

$$s(0) = \int \hat{s}(\omega) d\omega.$$  

(A.2.17)

For smoothing, we require that divergence in (A.2.17) imply divergence the right hand side of the expression

$$\|s\|^2 = \int |\hat{G}(\omega)|^2 \cdot |\hat{D_k s}(\omega)|^2 d\omega,$$

(A.2.18)

which depends asymptotically on $|\omega|^{2(g+k-h)}$. In $\mathbb{R}^d$, $d\omega = c_d |\omega|^{d-1} d|\omega|$, the necessary condition is that

$$\int |\omega|^{-h} \cdot |\omega|^{d-1} \to \infty \implies \int |\omega|^{2(g+k-h)} \cdot |\omega|^{d-1} \to \infty.$$

The first integral diverges when $d-1-h \geq -1$, i.e., $h \leq d$, hence we require that

$$2(g+k-h) + d - 1 \geq -1$$

(A.2.19)

whenever $h \leq d$. Plugging $h = d$ into (A.2.19), the condition for smoothing becomes

$$g + k \geq d/2.$$  

(A.2.20)

When $G$ is the identity, we get a pure differential operator acting in (A.2.16) and (A.2.20) is close to the Sobolev imbedding condition. More generally, (A.2.20) gives a lower bound on the order of a pseudo-differential expression to generate a smoothing norm in (A.2.20).
A.3 Fourier transforms of the fault-surface traction field

In this appendix, we derive the expressions given in Section 2.3 for the Fourier transforms of the components of the fault surface traction field. We begin by writing \( T_k(x) = \sigma_{k3}(x), x \notin \Sigma \), as

\[
T_k(x) = \frac{1}{8\pi \mu} \int s_j C_{k3mn} C_{j3pq} \frac{\partial}{\partial x_m \partial x_q} \left\{ 2r^{-1} \delta_{np} - \frac{\lambda + \mu}{\lambda + 2\mu} \frac{\partial^2 r}{\partial x_n \partial x_p} \right\} d\xi
\]

\[
= \frac{1}{8\pi \mu} \int s_j \left\{ 2A_{jk} r^{-1} - \frac{\lambda + \mu}{\lambda + 2\mu} B_{jk} r \right\} d\xi
\]

\[
= \frac{1}{16\pi \mu} \int s_j \left\{ 4A_{jk} r^{-1} - \frac{1}{(1 - \nu)} B_{jk} r \right\} d\xi, \tag{A.3.21}
\]

where \( A_{jk} \) and \( B_{jk} \) are differential operators defined by

\[
A_{jk} \triangleq C_{k3mn} C_{j3pq} \delta_{np} \frac{\partial^2}{\partial x_m \partial x_q} \tag{A.3.22}
\]

and

\[
B_{jk} \triangleq C_{k3mn} C_{j3pq} \frac{\partial^4}{\partial x_m \partial x_q \partial x_n \partial x_p}. \tag{A.3.23}
\]

and \( \nu = \frac{\lambda}{2(\lambda + \mu)} \) is Poisson's ratio.

Inserting the isotropic stiffness tensor, (3.2.3), into (A.3.22) and (A.3.23) leads to

\[
A_{jk} = \lambda^2 \delta_{j3} \delta_{k3} \Delta + \mu(\mu + 2\lambda) \left\{ \delta_{k3} \frac{\partial^2}{\partial x_j \partial x_3} + \delta_{j3} \frac{\partial^2}{\partial x_k \partial x_3} \right\}
\]

\[
+ \mu^2 \left\{ \frac{\partial^2}{\partial x_j \partial x_k} + \delta_{jk} \frac{\partial^2}{\partial x_3^2} \right\}, \tag{A.3.24}
\]

and

\[
B_{jk} = \lambda^2 \delta_{j3} \delta_{k3} \Delta^2 + 2\lambda \mu \left\{ \delta_{k3} \frac{\partial^2}{\partial x_j \partial x_3} + \delta_{j3} \frac{\partial^2}{\partial x_k \partial x_3} \right\} \Delta
\]

\[
+ 4\mu^2 \frac{\partial^4}{\partial x_j \partial x_k \partial x_3^2}. \tag{A.3.25}
\]

In (A.3.24) and (A.3.25), \( \Delta \) is the Laplace operator,

\[
\Delta \triangleq \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}.
\]
and $\Delta^2$, the squared Laplacian, is the biharmonic operator. Applying $A_{jk}$ to $r^{-1}$, which is harmonic, and $B_{jk}$ to $r$, which is biharmonic, we find that

$$A_{jk} r^{-1} = \mu(\mu + 2\lambda) \left\{ \delta_{k3} \frac{\partial^2 r^{-1}}{\partial x_j \partial x_3} + \delta_{j3} \frac{\partial^2 r^{-1}}{\partial x_k \partial x_3} \right\} + \mu^2 \left\{ \frac{\partial^4 r^{-1}}{\partial x_j \partial x_k} + \frac{\partial^2 r^{-1}}{\partial x_3^2} \right\}$$  \hspace{1cm} (A.3.26)

and

$$B_{jk} r = 4\mu \lambda \left\{ \delta_{k3} \frac{\partial^2 r^{-1}}{\partial x_j \partial x_3} + \delta_{j3} \frac{\partial^2 r^{-1}}{\partial x_k \partial x_3} \right\} + 4\mu^2 \left\{ \frac{\partial^4 r}{\partial x_j \partial x_k \partial x_3^2} \right\}$$  \hspace{1cm} (A.3.27)

Plugging (A.3.26) and (A.3.27) back into (A.3.21), we have expressions that we may integrate by parts, based on the fact that $s$ is in $C^2$ and has bounded support. When twice integrated by parts the singular terms in (A.3.26) and (A.3.27) become integrable, permitting us to take inside the integral the limit of $T_k(\mathbf{x})$ as $\mathbf{x}$ approaches $\Sigma$. The terms in (A.3.26) and (A.3.27), are variants of two basic types: $\partial^2 r^{-1}/\partial x_i \partial x_k$, and $\partial^4/\partial x_i \partial x_k \partial x_3^2$.

For each of these types, we will derive the required limiting form and its Fourier transform on a case by case basis. Let $\eta = (\eta_1, \eta_2)$, and let $f$ be a $C^2$ function of $\eta$ with derivatives denoted by subscripts after a comma, i.e., $f_{,1} = \partial f/\partial \eta_1$, $f_{,12} = \partial^2 f/\partial \eta_1 \partial \eta_2$, etc.. Assume that $f$ and its derivatives vanish at infinity. Finally, let $r = \{(\xi_1 - \eta_1)^2 + (\xi_2 - \eta_2)^2 + x_3^2\}^{1/2}$, and let $r^{-1} = \{(\xi_1 - \eta_1)^2 + (\xi_2 - \eta_2)^2\}$. Using integration by parts, the Gauss flux theorem (see, e.g., Jaswon and Symm [20]), and the fact that $r^{-1}$ has Fourier transform at wave number $(\omega_1, \omega_2)$ equal to $|\omega|^{-1}$, we have the following results:

**Case 1.a.:** $\frac{\partial^2 r^{-1}}{\partial \xi_i \partial \xi_k}$

$$g(\xi) = \lim_{x_3 \to 0} \int f(\eta) \frac{\partial^2 r^{-1}}{\partial \xi_i \partial \xi_k} d\eta$$

$$= \lim_{x_3 \to 0} \int f_{,ik}(\eta) r^{-1} d\eta$$

$$= \int f_{,ik}(\eta) r^{-1} d\eta$$

$$\hat{g}(\omega) = -4\pi^2 \omega_i \omega_k |\omega|^{-1} \hat{f}(\omega).$$

**Case 1.b.:** $\frac{\partial^2 r^{-1}}{\partial \xi_k \partial x_3}$
\[
g(\xi) = \lim_{x_3 \to 0} \int f(\eta) \frac{\partial^2 r^{-1}}{\partial \xi_k \partial x_3} d\eta \\
= \lim_{x_3 \to 0} \int f_{,k}(\eta) \frac{\partial r^{-1}}{\partial x_3} d\eta \\
= -2\pi f_{,k}(\xi) \\
\hat{g}(\omega) = -4\pi^2 i\omega_k \hat{f}(\omega).
\]

**Case 1.c.:** \(\frac{\partial^2 r^{-1}}{\partial x_3^2}\)

Using the fact that \(r^{-1}\) is harmonic when \(x_3 \neq 0\),

\[
g(\xi) = \lim_{x_3 \to 0} \int f(\eta) \frac{\partial^2 r^{-1}}{\partial x_3^2} d\eta \\
= \lim_{x_3 \to 0} -\int f(\eta) \left\{ \frac{\partial^2 r^{-1}}{\partial \xi_1^2} + \frac{\partial^2 r^{-1}}{\partial \xi_2^2} \right\} d\eta \\
= \lim_{x_3 \to 0} -\int \{f_{,11}(\eta) + f_{,22}(\eta)\} r^{-1} d\eta \\
= -\int \{f_{,11}(\eta) + f_{,22}(\eta)\} r_2^{-1} d\eta \\
\hat{g}(\omega) = 4\pi^2 |\omega| \hat{f}(\omega).
\]

**Case 2.a.:** \(\frac{\partial^4 r}{\partial \xi_i \partial \xi_k \partial x_3^2}\)

\[
g(\xi) = \lim_{x_3 \to 0} \int f(\eta) \frac{\partial^4 r}{\partial \xi_i \partial \xi_k \partial x_3^2} d\eta \\
= \lim_{x_3 \to 0} \int f_{,ik}(\eta) \frac{\partial^2 r}{\partial x_3^2} d\eta \\
= \lim_{x_3 \to 0} \int f_{,ik}(\eta) \left\{ \frac{1}{r} - \frac{x_3^2}{r^3} \right\} d\eta \\
= \lim_{x_3 \to 0} \int f_{,ik}(\eta) \left\{ \frac{(\xi_1 - \eta_1)^2 + (\xi_2 - \eta_2)^2}{r^3} \right\} d\eta \\
= \int f_{,ik}(\eta) r_2^{-1} d\eta \\
\hat{g}(\omega) = -4\pi^2 \omega_i \omega_k |\omega|^{-1} \hat{f}(\omega).
\]

**Case 2.b.:** \(\frac{\partial^4 r}{\partial \xi_k \partial x_3^2}\)
\[ g(\xi) = \lim_{x_3 \to 0} \int f(\eta) \frac{\partial^4 r}{\partial \xi_k \partial x_3^2} d\eta \]

\[ = \lim_{x_3 \to 0} \int f,\xi_k(\eta) \frac{\partial^2}{\partial \xi \partial x_3} \left\{ \frac{2}{r} + \frac{\partial^2 r}{\partial \eta_1^2} + \frac{\partial^2 r}{\partial \eta_2^2} \right\} d\eta \]

\[ = \lim_{x_3 \to 0} 2 \int f(\eta) \frac{\partial^{2r-1}}{\partial x_3^2} d\eta \]

\[ + \lim_{x_3 \to 0} \int \{f,\xi_i(\eta) + f,\xi_k(\eta)\} \frac{\partial^2 r}{\partial x_i \partial x_3} d\eta. \]

Use Case 1.b. to evaluate the first line of this last expression. In the second line,

\[ \lim_{x_3 \to 0} \int \{f,\xi_i(\eta) + f,\xi_k(\eta)\} \frac{\partial}{\partial x_3} \left( \frac{\xi_k - \eta_k}{r} \right) d\eta = 0 \]

by the Gauss flux theorem.

Case 2.c.: \( \frac{\partial^4 r}{\partial x_3^2} \)

\[ g(\xi) = \lim_{x_3 \to 0} \int f(\eta) \frac{\partial^4 r}{\partial x_3^2} d\eta \]

\[ = \lim_{x_3 \to 0} \int f(\eta) \frac{\partial^2}{\partial x_3^2} \left\{ \frac{2}{r} + \frac{\partial^2 r}{\partial \eta_1^2} + \frac{\partial^2 r}{\partial \eta_2^2} \right\} d\eta \]

\[ = \lim_{x_3 \to 0} 2 \int f(\eta) \frac{\partial^{2r-1}}{\partial x_3^2} d\eta \]

\[ + \lim_{x_3 \to 0} \int \{f,\xi_i(\eta) + f,\xi_k(\eta)\} \frac{\partial^2 r}{\partial x_3^2} d\eta. \]

Case 1.c. may be used to evaluate the first line of this last expression. The second line reduces to,

\[ \lim_{x_3 \to 0} \int \{f,\xi_i(\eta) + f,\xi_k(\eta)\} \frac{\partial^2 r}{\partial x_3^2} d\eta \]

\[ = \lim_{x_3 \to 0} \int \{f,\xi_i(\eta) + f,\xi_k(\eta)\} \left\{ \frac{1}{r} - \frac{x_3^2}{r^3} \right\} d\eta \]

\[ = \lim_{x_3 \to 0} \int \{f,\xi_i(\eta) + f,\xi_k(\eta)\} \cdot \frac{(\xi_1 - \eta_1)^2 + (\xi_2 - \eta_2)^2}{r^3} d\eta \]

\[ = \int \{f,\xi_i(\eta) + f,\xi_k(\eta)\} r_2^{-1} d\eta. \]
The Fourier transform of the last part is given by

\[-4\pi|\omega| \hat{f}(\omega)\]
Bibliography


BIBLIOGRAPHY


