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{X_n, n \geq 1} are i.i.d. unbounded random variables with continuous d.f. \( F(x) = 1 - e^{-R(x)} \). \( X_j \) is a record value of this sequence if \( X_j > \max\{X_1, \ldots, X_{j-1}\} \). The almost sure behavior of the sequence of record values \( \{X_{R_n}\} \) is studied. Sufficient conditions are given for
\[
\limsup_{n \to \infty} \frac{X_{R_n}}{R^{-1}(n)} = e^c, \quad \liminf_{n \to \infty} \frac{X_{R_n}}{R^{-1}(n)} = e^{-c}, \quad \text{a.s.,} \quad 0 \leq c \leq \infty,
\]
and also for
\[
\limsup_{n \to \infty} \frac{X_{R_n} - R^{-1}(n)}{\alpha_n} = 1, \quad \liminf_{n \to \infty} \frac{X_{R_n} - R^{-1}(n)}{\alpha_n} = -1, \quad \text{a.s.,}
\]
for suitably chosen constants \( \alpha_n \). The a.s. behavior of \( \{X_{R_n}\} \) is compared to that of the sequence \( \{M_n\} \), where \( M_n = \max\{X_1, \ldots, X_n\} \).
The method is to translate results for the case where the \( X_n \)'s are exponential to the general case by means of an extended theory of regular variation.

**Key words:** record values, maxima, extreme values, regular variation, iterated logarithm, a.s. limit points.
1. Introduction and Preliminaries.

Let \( \{X_n, n \geq 1\} \) be a sequence of independent, identically distributed (i.i.d.) random variables with common distribution function \( F(\cdot) \). \( X_j \) is a record value of this sequence iff

\[
X_j > \max(X_1, \ldots, X_{j-1}).
\]

By convention \( X_1 \) is a record value. The indices at which record values occur are given by the random variables \( \{L_n, n \geq 0\} \) defined by

\[
L_0 = 1, \quad L_n = \min\{j | j > L_{n-1}, X_j > X_{L_{n-1}}\}.
\]

We assume throughout this paper that \( F(x) \) is continuous and \( F(x) < 1 \) for all \( x \). This insures that the sequence \( \{L_n\} \) is well defined and that \( X_{L_n} \to \infty \) a.s. A further comment on the necessity of the continuity assumption is made after (2).

The basic structure of the record value sequence is given by the following lemma (see [8], Lemma 1.1 and [11]).

**Lemma 1.** Suppose \( \{X_n, n \geq 1\} \) is an i.i.d. sequence with

\[
P[X_n \leq x] = 1 - e^{-x}, \quad x \geq 0.
\]

Then \( X_{L_n} = X_{L_0} + \sum_{j=1}^{n} (X_{L_j} - X_{L_{j-1}}) \equiv \sum_{j=0}^{n} Y_j \)

where \( \{Y_j, j \geq 0\} \) is an i.i.d. sequence distributed according to the same exponential distribution.
Rather than deal with the distribution \( F(x) \) it is more convenient to consider its \( R \)-function defined for \(-\infty < x < \infty\) by:

\[
R(x) = -\log(1-F(x)) \quad \text{or} \quad 1-F(x) = e^{-R(x)}.
\]

It is convenient to employ inverse functions so we define

\[
R^{-1}(x) = \inf\{y|R(y) \geq x\}
\]

for \( 0 \leq x \leq \infty \).

It is quickly checked that if \( X \) is a random variable with continuous distribution \( F(x) \), then \( R(X) \) has an exponential distribution with parameter 1. Since \( R \) and \( R^{-1} \) are monotone functions, Lemma 1 can be recast to give the following representations:

\[
R(X_{n}^{(1)}) = \sum_{j=0}^{n} Y_{j}, \quad X_{n}^{(1)} = R^{-1}(\sum_{j=0}^{n} Y_{j})
\]

where \( X_{n}^{(1)} \) is the \( n \)th record value of the i.i.d. sequence with continuous distribution \( F(x) \) and \( \{Y_{j}, j \geq 1\} \) are i.i.d. exponentially distributed random variables. Note that (2) is not valid without the assumption of continuity.

Representation (2) leads to consideration of special cases of the functional limit problem: Given information about the behavior of \( \sum_{j=0}^{n} Y_{j} \), extract information about the corresponding behavior of
The limit law problem for $\{X_n\}$ was solved in [8] and a.s. stability considered in [9]. Here we make further remarks about stability and present sufficient conditions for iterated logarithm theorems for $X_n$.

Techniques useful in dealing with a.s. functional limit questions were developed in [4] and [5] to derive iterated logarithm results for order statistics.

This paper presumes familiarity with the theory of regularly varying functions. Good expositions are found in [3] or [7]. We will also need some facts about functions of the classes $\Pi$ and $\Gamma$ as developed in [3]. Certain of the relevant ideas are summarized below.

A non-decreasing function $U(t)$, defined for $t \geq 0$, belongs to the class $\Pi$ (written $U \in \Pi$) if there exist real functions $a(t) > 0, b(t)$ such that for all $x > 0$:

$$\lim_{t \to \infty} \frac{U(tx) - b(t)}{a(t)} = \log x. \quad (3)$$

It is easy to see that the functions $a(t), b(t)$ can be chosen as $b(t) = U(t), a(t) = U(te) - U(t)$.

A positive non-decreasing function $U(t)$ belongs to the class $\Gamma$ (written $U \in \Gamma$) if there exists a function $f: \mathbb{R} \to \mathbb{R}^+$ (called the auxiliary function of $U$) such that for all $x$:

$$f(x) = \log x.$$
(4) \[ \lim_{t \to \infty} \frac{U(t + xf(t))}{U(t)} = e^x. \]

In this case \( \lim_{x \to \infty} f(x)/x = 0 \) and \( f \) may be chosen as

(5) \[ f(x) = \int_0^x \frac{U(t)dt}{U(x)}. \]

If \( U_1, U_2 \) are in \( \Gamma \) with auxiliary functions \( f_1 \) and \( f_2 \) respectively, and if \( f_1(x) - f_2(x) \) as \( x \to \infty \) then

(6) \[ U_1(x) = V(U_2(x)) \]

where \( V \) is a regularly varying function of exponent 1 [6].

Remark 1: From (6) comes the useful implication that to delineate the class of functions in \( \Gamma \) with particular auxiliary function \( f \), one needs to find only one function \( U \in \Gamma \) with this \( f \). The remaining functions of this class are given by (6).

Remark 2: The classes \( \Pi \) and \( \Gamma \) are related as follows: Define all inverse functions as in (1). If \( U \in \Gamma \), with auxiliary function \( f(t) \), then \( U^{-1} \in \Pi \) and the function \( a(t) \) of (3) can be taken as \( f(U^{-1}(t)) \). Conversely if a function \( U_1 \in \Pi \) and \( \lim_{x \to \infty} U_1(x) = \infty \), then \( U_1^{-1} \in \Gamma \) ([3], p. 47).
2. Limit Points for Record Values

In [9], necessary and sufficient conditions were found for the existence of constants \( b_n \) such that \( X_{L_n}/b_n \to 1 \) a.s. as \( n \to \infty \). It was also shown that if such constants exist, then: \( b_n \sim R^{-1}(n) \).

The open question, raised in [9], of whether or not other limit points exist for the sequence \( \{X_{L_n}/R^{-1}(n)\} \) can now be answered affirmatively:

**Theorem 1:** Suppose \( F(x) \) is continuous. If \( R^{-1} \in \Gamma \) with auxiliary function \( f(t) = c^{-1}(2t \log \log t)^{-\frac{1}{2}} \), \( 0 < c < \infty \), then almost surely:

\[
\limsup_{n \to \infty} \frac{X_{L_n}}{R^{-1}(n)} = e^c, \quad \liminf_{n \to \infty} \frac{X_{L_n}}{R^{-1}(n)} = e^{-c}.
\]

The class of functions in \( \Gamma \) with the given auxiliary function is of the form

\[
R^{-1}(x) = V(H_c(x))
\]

where \( V(x) \) is 1-varying and \( H_c(x) = \exp(\int_0^x c(2t \log \log t)^{-\frac{1}{2}} \, dt) \).

The following is fully equivalent to (7):

\[
\lim_{t \to \infty} (R(tx) - R(t))/\left(2R(t) \log R(t)\right)^{\frac{1}{2}} = c^{-1} \log x
\]

for all \( x > 0 \).
Proof: Set 
\[ Z_n = \left( \sum_{j=0}^{n} Y_j - n \right)(2n \log \log n)^{-\frac{1}{2}} \]
where the \( Y_j \)'s are as in (2). Then from the Hartman-Wintnor Law of the Iterated Logarithm [2]:

\[
\limsup_{n \to \infty} Z_n = 1, \quad \liminf_{n \to \infty} Z_n = -1
\]
a.s. If \( R_{-1}^{-1} \in \Gamma \) with the given auxiliary function then

(9) \[
\lim_{t \to \infty} R_{-1}^{-1}(t + x(2t \log t)^{\frac{1}{2}}) / R_{-1}^{-1}(t) = e^{cx}.
\]

The left side of (9) is monotone in \( x \) and the right side is continuous in \( x \) and therefore the convergence must be uniform for \( x \) in finite intervals.

Considering (2) and the uniform convergence in (9), we have that:

\[
\limsup_{n \to \infty} \frac{X_{L_n}}{R_{-1}^{-1}(n)} = \limsup_{n \to \infty} R_{-1}^{-1}\left( \sum_{j=0}^{n} Y_j \right) / R_{-1}^{-1}(n)
\]

\[
= \limsup_{n \to \infty} R_{-1}^{-1}(n + Z_n (2n \log \log n)^{\frac{1}{2}}) / R_{-1}^{-1}(n)
\]

\[
= \exp\{c \limsup_{n \to \infty} Z_n\}
\]

\[
= e^{c}.
\]

A similar argument holds for \( \liminf \) and this shows the first part of the theorem. The second part follows from Remark 1 after verifying
that \( H_c \in \Gamma \) with the given auxiliary function. Remark 2 shows the equivalence of (7) and (8) and the proof is complete.

A more convenient sufficient condition is the following:

**Theorem 1':** Suppose \( F(x) \) has a density so that \( R(x) \) has the density \( R'(x) = r(x) \). If

\[
(10) \quad \lim_{x \to \infty} x r(x)(2R(x) \log \log R(x))^{-\frac{1}{K}} = c^{-1}
\]

then almost surely:

\[
\limsup_{n \to \infty} \frac{X_n}{R^{-1}(n)} = e^c, \quad \liminf_{n \to \infty} \frac{X_n}{R^{-1}(n)} = e^{-c}.
\]

**Proof:** By Theorem 1, it suffices to show that (10) implies that \( R^{-1} \) is of the form \( V(H_c(x)) \). This is the case iff \( V(x) = R^{-1}(H_c^{-1}(x)) \) is 1-varying or equivalently iff

\[
V^{-1}(x) \equiv V_1(x) = H_c(R(x))
\]

is 1-varying [3, p. 22]. A sufficient condition that \( V_1 \) be 1-varying is

\[
\lim_{x \to \infty} \frac{x V'_1(x)}{V_1(x)} = 1
\]
[3, p. 109] and after performing the necessary differentiations we arrive to (10) and the proof is complete.

**Remark 3:** It is quickly seen that the sufficient conditions of Theorems 1 and 1' also imply $X_n / R^{-1}(n) \xrightarrow{P} 1, \ n \to \infty$. A necessary and sufficient condition for stability in probability of $\{X_n\}$ was shown in [9] to be

$$\lim_{t \to \infty} R(tx) - R(t)/R^2(t) = \infty$$

for all $x > 1$. This condition is implied by (8).

**Examples** of distributions for which Theorem 1 is valid: Applying asymptotic inversion formulas for regularly varying functions one obtains that the following is asymptotic to the inverse of the function $H_c(t)$ of (3):

$$R_c(x) = (\sqrt{c})^{-2}(\log x)^2 \log \log \log x$$

for $x > \exp(e^x)$. The function $R_c(x)$ satisfies (10) so that the conclusion of Theorems 1 and 1' is valid. Inverting (7) shows that distributions with $R$-functions of the class $R_c(V_1(x))$, where $V_1$ is $1$-varying, give rise to record value sequences whose behavior is described by the conclusion of Theorems 1 and 1'.
There is a companion result to Theorems 1 and 1' which treats the case \( c = \infty \), or \( c = 0 \). We need the following definition: Call a nondecreasing function \( W: \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) **rapidly varying** at infinity if

\[
\lim_{t \to \infty} \frac{W(tx)}{W(t)} = \begin{cases} 
\infty & \text{for } x > 1 \\
0 & \text{for } x < 1 
\end{cases}
\]

**Theorem 2:** Suppose \( F(x) \) is continuous and \( c = 0 \) or \( +\infty \). If

\[
\lim_{s \to \infty} R^{-1}(s + x(2s \log \log s)^{k_2})/R^{-1}(s) = e^c
\]

for all \( x > 0 \) then

\[
\limsup_{n \to \infty} \frac{X_L}{R^{-1}(n)} = e^c, \quad \liminf_{n \to \infty} \frac{X_L}{R^{-1}(n)} = e^{-c}.
\]

The class of functions satisfying (11) is of the form:

\[
R^{-1}(s) = W(H_1(s))
\]

where \( H_1 \) is as in Theorem 1 and for

\[
c = \infty, \ W \text{ is rapidly varying,} \]

\[
c = 0, \ W \text{ is slowly varying.}
\]
The following is fully equivalent to (11) or (13):

\[ \lim_{t \to \infty} \frac{R(tx) - R(t)}{(2R(t) \log \log R(t))^{\frac{1}{2}}} = c^{-1} \]

for all \( x > 0 \). A condition sufficient for (11) or (13) or (14) and hence for (12) is that \( R(x) \) have a density \( r(x) \) such that

\[ \lim_{x \to \infty} \frac{x r(x)}{(2R(x) \log \log R(x))^{\frac{1}{2}}} = c^{-1} \]

**Proof:** Throughout the proof we assume \( c = \infty \). The proof for the case \( c = 0 \) is almost the same.

First we show the equivalence of (11) and (13). If (13) holds then for any \( x > 0 \):

\[ \lim_{s \to \infty} \frac{H_1(s + x(2s \log \log s)^{\frac{1}{2}})}{H_1(s)} = e^x > 1 \]

and hence there exists \( \varepsilon > 0 \) such that for \( s \) sufficiently large:

\[ H_1(s + x(2s \log \log s)^{\frac{1}{2}}) \geq (1 + \varepsilon)H_1(s) \].

Therefore since \( W \) must be non-decreasing we have that (11) can be written as

\[ \lim_{s \to \infty} \frac{W(H_1(s + x(2s \log \log s)^{\frac{1}{2}}))}{W(H_1(s))} \]

\[ \geq \lim_{s \to \infty} \frac{W((1 + \varepsilon)H_1(s))}{W(H_1(s))} = \infty \]
since $W$ is rapidly varying. The proof of the converse is similar.

Next we observe that (11) implies

$$\lim_{s \to \infty} R^{-1}(s - x(2s \log \log s)^{1/2})/R^{-1}(s) = 0$$

for all $x > 0$. To prove this we use (13). For $x > 0$:

$$\lim_{s \to \infty} H_{1}(s - x(2s \log \log s)^{1/2})/H_{1}(s) = e^{-x} < 1.$$ 

Hence there exists $\eta > 0$ such that $e^{-x} + \eta < 1$ and for $s$ sufficiently large $H_{1}(s - x(2s \log \log s)^{1/2}) \leq (e^{-x} + \eta)H_{1}(s)$. Therefore, the ratio in (16) becomes for sufficiently large $s$:

$$W(H_{1}(s - x(2s \log \log s)^{1/2}))/W(H_{1}(s))$$

$$\leq W((e^{-x} + \eta)H_{1}(s))/W(H_{1}(s)) \to 0$$

as $s \to \infty$ since $e^{-x} + \eta < 1$ and $W$ is rapidly varying.

By the method of Theorem 1, (12) follows from (11) and (16). To prove the equivalency of (13) and (14) we make use of the fact that $W^{-1}$ (the inverse of a rapidly varying function) is slowly varying ([3], p. 22). Condition (13) then translates into either:

$H_{1}(R(x)) = L(x)$ or $R(x) = H_{1}^{-1}(L(x))$, where $L(x)$ is slowly varying. The equivalency of these conditions with (14) is then proven using the same techniques as the proof of the equivalency of (11) and
and (13).

To check (15), note that (13) holds iff \( \frac{R^{-1}(H^{-1}_1(s))}{W(s)} = \infty \) is rapidly varying. A sufficient condition for this is [3, p. 118]

\[
\lim_{x \to \infty} xW'(x)/W(x) = \infty
\]

and performing the differentiations implicitly gives (14).

Remark 4: Using the methods of [9] Section 3, the following can be shown: For \( c > 0 \) fixed, suppose for all \( s > e^c \) that

\[
\frac{(R(st)-R(t))/R^k_s(t)}{R^h_s(t)}
\]

and

\[
\frac{(R(t)-R(s^{-1}t))/R^k_s(t)}{R^h_s(t)}
\]

are non-decreasing functions of \( t \). Then \( \limsup_{n \to \infty} X_{L_n}/R^{-1}(n) \leq e^c \) a.s. iff for all \( s > e^c \)

\[
(17) \quad \int_{-\infty}^{\infty} \frac{R(sy)-R(y)}{R^k_s(y)} \exp \left\{ -\frac{1}{2} \left| \frac{R(sy)-R(y)}{R^k_s(y)} \right|^2 \right\} d \log R(y) < \infty
\]

and \( \liminf_{n \to \infty} X_{L_n}/R^{-1}(n) \geq e^{-c} \) a.s. iff for all \( t < e^{-c} \):

\[
(18) \quad \int_{-\infty}^{\infty} \frac{R(y)-R(ty)}{R^k_s(y)} \exp \left\{ -\frac{1}{2} \left| \frac{R(y)-R(ty)}{R^k_s(y)} \right|^2 \right\} d \log R(y) < \infty.
\]
Furthermore (18) implies (17) and from this follows the conclusion that

\[
\limsup_{n \to \infty} \frac{X_n}{R^{-1}(n)} = \infty \quad \text{a.s.}
\]

\[
\Rightarrow \liminf_{n \to \infty} \frac{X_n}{R^{-1}(n)} = 0 \quad \text{a.s.}
\]

**Examples:** Consider those distributions for which there exist \( \alpha > 0 \) and normalizing constants \( \alpha_n > 0, \beta_n \), \( n \geq 1 \) such that for all \( x \):

\[
P\left[ X_n \leq \alpha_n x + \beta_n \right] \to N(\alpha \log x)
\]

where \( N(x) \) is the standard normal distribution. According to [8], Theorem 4.2, the above convergence holds iff:

\[
\lim_{t \to \infty} \frac{R(tx) - R(t)}{R^2(t)} = \alpha \log x.
\]

However this condition implies (14) and hence for this class of distributions (which includes, among others, the log-Normal) we have almost surely:

\[
\limsup_{n \to \infty} \frac{X_n}{R^{-1}(n)} = \infty, \quad \liminf_{n \to \infty} \frac{X_n}{R^{-1}(n)} = 0.
\]
Take \( \alpha = 2 \) and \( R(x) = (\log x)^2 \) for \( x \geq e \). Then the previous conclusion holds but it is not true that \( \frac{X_{ln}}{R^{-1}(n)} \xrightarrow{P} 1 \) (see [8], examples, Section 2). Hence the analogue of remark 3 for the case \( c = \infty \) does not hold true.

We now investigate sufficient conditions for the Law of the Iterated Logarithm:

**Theorem 3:** Let \( F(x) \) be continuous.

(i) Suppose for all \( x \):

\[
(19) \quad \lim_{s \to \infty} \frac{R^{-1}(s + x(2s \log \log s)^{1/2}) - R^{-1}(s)}{R^{-1}(s + (2s \log \log s)^{1/2}) - R^{-1}(s)} = x.
\]

Then almost surely:

\[
(20) \quad \limsup_{n \to \infty} \frac{X_{ln} - R^{-1}(n)}{R^{-1}(n + (2n \log \log n)^{1/2}) - R^{-1}(n)} = 1, \quad \liminf_{n \to \infty} \frac{X_{ln} - R^{-1}(n)}{R^{-1}(n + (2n \log \log n)^{1/2}) - R^{-1}(n)} = -1.
\]

(ii) The class of functions satisfying (19) is of the form

\[
(21) \quad R^{-1}(x) = U(H_1(x))
\]

where \( U \in \Pi \) and \( H_1 \) is as in Theorem 1.
Proof: (i) The left side of (19) is monotone in \( x \) and the right side is continuous in \( X \) so that the convergence must be uniform on finite \( x \)-intervals. Again setting \( Z_n = \left( \sum_{j=0}^{n} Y_j \right) (2n \log \log n)^{-\frac{1}{3}} \) and recalling (2) and the Hartman-Wintner Law of the Iterated Logarithm give

\[
\limsup_{n \to \infty} \frac{X_{\frac{1}{n}} - R^{-1}(n)}{R^{-1}(n + (2n \log \log n)^{\frac{1}{3}}) - R^{-1}(n)} = \limsup_{n \to \infty} \frac{R^{-1}(n + Z_n (2n \log \log n)^{\frac{1}{3}}) - R^{-1}(n)}{R^{-1}(n + (2n \log \log n)^{\frac{1}{3}}) - R^{-1}(n)} = \limsup_{n \to \infty} Z_n = 1 \quad \text{a.s.,}
\]

the next to last equality following by uniform convergence. The procedure for \( \liminf \) is identical.

(ii) Suppose \( R^{-1}(x) = U(H_1(x)), U \in \Pi \). We must show (19) holds with \( R^{-1}(x) \) replaced by \( U(H_1(x)) \). Since \( U \in \Pi \), (3) must hold and as above, the convergence must be uniform on finite \( x \)-intervals. Setting \( f(t) = (2t \log \log t)^{\frac{1}{3}} \) and remembering that \( \lim H_1(t + xf(t))/H_1(t) = e^x \) for all \( x \), gives for (11)
\[
\lim_{t \to \infty} \frac{U(H_t(1+xf(t)) - U(H_t(1))}{U(H_t(1+f(t)) - U(H_t(1))}
\]

\[
= \lim_{t \to \infty} \frac{U(H_t(1+xf(t))/H_t(1)) - U(H_t(1))}{U(eH_t(1)) - U(H_t(1))}
\]

\[
= \log(\lim_{t \to \infty} \frac{H_t(1+xf(t))/H_t(1)}{\log \left( \lim_{t \to \infty} \frac{H_t(1+xf(t))/H_t(1)}{H_t(1)} \right)} = \frac{\log e^{x}}{\log e} = x.
\]

Conversely, given that \( R^{-1} \) satisfies (18), one can check that the function \( U \) defined by \( R^{-1}(H^{-1}_1(x)) = U(x) \) belongs to \( \Pi \).

As with Theorem 1, more convenient conditions can be given in terms of densities:

**Theorem 3'**: Suppose \( R(x) \) has a differentiable density \( r(x) \) with derivative \( r'(x) \). If

\[
\lim_{x \to \infty} \frac{r'(x)}{r^2(x)}(R(x) \log \log R(x))^{1/2} = 0
\]

then almost surely:

\[
\limsup_{n \to \infty} \frac{X_{L_n} - R^{-1}(n)}{(2n \log \log n)^{1/2}/r(R^{-1}(n))} = 1, \quad \liminf_{n \to \infty} \frac{X_{L_n} - R^{-1}(n)}{(2n \log \log n)^{1/2}/r(R^{-1}(n))} = -1.
\]
Proof: Given (22) we verify that (21) holds. This entails showing that \( U(x) = R^{-1}(H_1^{-1}(x)) \in \mathbb{N} \) or equivalently that \( U_1(x) \equiv U^{-1}(x) \) = \( H_1(R(x)) \in \Gamma \). According to [3] Theorem 2.7.4, or [12], a sufficient condition for \( U_1 \in \Gamma \) is \( \lim_{x \to \infty} U''_1(x)U_1(x)/(U'_1(x))^2 = 1 \). Performing the required differentiations, one obtains (22). Hence (20) holds.

It now remains to prove that

\[
(24) \quad R^{-1}(n+(2n \log \log n)^{1/2}) - R^{-1}(n) \rightarrow (2n \log \log n)^{1/2}/r(R^{-1}(n))
\]

as \( n \to \infty \). In terms of the representation (21), (24) may be replaced by

\[
(25) \quad U(H_1(s+(2s \log \log s)^{1/2})) - U(H_1(s)) \sim U'(H(s))H(s)
\]

as \( s \to \infty \). Remembering that \( H_1(s+(2s \log \log s)^{1/2}) \sim eH_1(s) \) and taking into account that if \( U \in \mathbb{N} \) then (3) holds uniformly for \( x \) in finite intervals gives after the change of variable \( y = H_1(s) \) that (25) is fully equivalent to

\[
(26) \quad U(ey) - U(y) \sim yU'(y)
\]

as \( y \to \infty \). We now show (26).

Define functions \( R_1 \) and \( F_1 \) by \( R_1^{-1}(\log s) = U(s) \), \( F_1(t) = 1 - e^{-R_1(t)} \), \( -\infty < t < \infty \), \( 0 \leq s \leq \infty \). Since \( \lim_{x \to \infty} U''_1(x)U_1(x)/(U'_1(x))^2 = 1 \) we have \( \lim_{x \to \infty} r_1'(x)/r_1^2(x) = 0 \). This is the Von Mises sufficient condition.

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for attraction to the double exponential distribution ([3], Theorem 2.7.4, or [12]) so:

\[
\lim_{s \to \infty} F_1^{R}(a(s)x+b(s)) = \exp\{-e^{-x}\}
\]

for suitably chosen normalizing functions \(a(s) > 0\) and \(b(s)\). It is known that when the Von Mises sufficient condition holds:

\[a(s) - R^{-1}_1(\log s) - R^{-1}_1(\log s) - 1/r_1(R^{-1}_1(\log s))\]

as \(s \to \infty\) ([12]) and this is precisely (26). The proof is complete.

Remark 5: As expected, the conditions of either Theorem 3 or 3' imply that \(\lim_{n \to \infty} X_{L_n}/R^{-1}_n(n) = 1\) a.s. In order to show this we first prove

\[(27) \quad R^{-1}_n(n+(2n \log \log n)^{1/2}) - R^{-1}_n(n)\]

as \(n \to \infty\). The conditions of Theorem 3 or 3' imply (21). Hence taking the ratio of the two sides of (27) gives

\[U(H_1(n+(2n \log \log n)^{1/2}))/U(H_1(n)).\]  

Now \(H_1(n+(2n \log \log n)^{1/2})/H_1(n) \to e\) as \(n \to \infty\) and \(U \in \Pi\) entails \(U\) is slowly varying ([3], Corollary 1.4.1) so that by [3], Corollary 1.2.1-2 (27) follows.

Next, divide numerator and denominator of (20) by \(R^{-1}_1(n)\). Since
the resulting denominators approach zero we must have
\[ \lim_{n \to \infty} \frac{X_{l_n} - R^{-1}(n)}{R^{-1}(n)} = 0 \quad \text{a.s. as asserted.} \]

Our simplest condition for the Iterated Logarithm Theorem is the following:

**Theorem 4:** If \( R(x) \) has a regularly varying density \( r(x) \) with exponent of variation \( \rho > -1 \) then

\[ \limsup_{n \to \infty} \frac{X_{l_n} - R^{-1}(n)}{(2n \log \log n)^{\frac{1}{\rho}}/r(R^{-1}(n))} = 1 \]

\[ \liminf_{n \to \infty} \frac{X_{l_n} - R^{-1}(n)}{(2n \log \log n)^{\frac{1}{\rho}}/r(R^{-1}(n))} = -1 \]

**Remark:** \( \rho \) cannot be \( < 1 \) since \( R(x) \to \infty \) as \( x \to \infty \).

**Proof:** From the mean value theorem the ratio in (19) becomes:

\[ \frac{\sqrt{r} x (R^{-1}(n+\theta_2 (2n \log \log n)^{\frac{1}{\rho}}))}{r(R^{-1}(n+\theta_1 (2n \log \log n)^{\frac{1}{\rho}}))} \]

(28)

where \( 0 \leq \theta_1, \theta_2 \leq 1 \). Since \( r(x) \) is \( \rho \)-varying, \( \rho > -1 \), \( R(x) \) is \( \rho + 1 \)-varying, \( \rho + 1 > 0 \) ([3], Lemma 1.2.2) and hence \( R^{-1}(x) \) is \( (\rho + 1)^{-1} \)-varying ([3], Corollary 1.2.1-5). This necessitates that the composite function \( r(R^{-1}(x)) \) be regularly varying ([3], Corollary
1.2.1-3). Since

\[
(n + \theta_2 (2n \log \log n)^{1/2})^t / (n + \theta_1 (2n \log \log n)^{1/2}) - 1
\]

as \( n \to \infty \), the expressions in (20) approaches \( x \) because of [3],

Corollary 1.2.1-2.

Since (19) holds so does (20). By the above argument it is clear that

\[
R^{-1}(n + (2n \log \log n)^{1/2}) - R^{-1}(n)
\]

\[
= (2n \log \log n)^{1/2} / r(R^{-1}(n + \theta(2n \log \log n)^{1/2}))
\]

\[
= (2n \log \log n)^{1/2} / r(R^{-1}(n))
\]

as \( n \to \infty \), where \( |\theta| \leq 1 \). This completes the proof.

Distributions whose \( R \)-functions are of the form

\[
R(x) = x^\alpha, \quad x > 0
\]

for \( \alpha > 0 \), all satisfy the condition of Theorem 4. Moreover so do the normal and gamma distributions. To see this for the standard normal distribution \( N(x) \) use the well known relation \( 1 - N(x) = n(x)/x \) which implies

\[-\log 1 - N(x) = -\log n(x) + \log x - x^2/2 \]

as \( x \to \infty \).
so that \( r(x) = x \). For the gamma distribution note that for \( \alpha > 0 \)

\[
\int_{t}^{\infty} e^{-x}/\Gamma(\alpha)dx = e^{-t}\frac{\alpha-1}{\Gamma(\alpha)}, \quad t \to \infty
\]

so that the R-function is asymptotic to \( t \) and \( r(t) \to 1 \).
3. Comparison with Sample Maxima; Examples

Along with the record values \( \{X_{L_n}\} \) of the i.i.d. sequence \( \{X_n, n \geq 1\} \) one may consider the sample maxima \( \{M_n\} \) defined by

\[
M_n = \max\{X_1, \ldots, X_n\}.
\]

It is of interest to compare the behavior of the two sequences \( \{X_{L_n}\} \) and \( \{M_n\} \), especially in view of the fact that the first is a subsequence of the second. Generally \( \{X_{L_n}\} \) is less well-behaved than \( \{M_n\} \) (see [9]) so that if a limit result holds for \( \{X_{L_n}\} \), an equivalent or stronger result holds for \( \{M_n\} \).

Consider the following statements, where it is assumed that \( R \) has a differentiable density \( r(x) \):

(i) If

\[
(29) \quad \lim_{x \to \infty} \frac{r'(x)}{r^2(x)} = 0, \quad \lim_{x \to \infty} \frac{r'(x)}{r^2(x)} = 0,
\]

then there exist normalizing constants \( a_n > 0, b_n, n \geq 1 \) such that for all \( x \):

\[
\lim_{n \to \infty} P[M_n \leq a_n x + b_n] = A(x) = \exp\{-x\}
\]

(ii) If

\[
(30) \quad \lim_{x \to \infty} \frac{\log R(x) r'(x)}{r^2(x)} = 0
\]
Then almost surely:

\[
\limsup_{n \to \infty} \frac{M_n R^{-1}(\log n)}{\log \log n / r(R^{-1}(\log n))} = 1, \quad \liminf_{n \to \infty} \frac{M_n R^{-1}(\log n)}{\log \log n / r(R^{-1}(\log n))} = 0
\]

(iii) If

\[\lim_{x \to \infty} (R^{\frac{1}{2}}(x))r'(x)/r^2(x) = 0\]

then there exist normalizing constants \(a_n > 0, \beta_n, n \geq 1\) such that for all \(x\):

\[
\lim_{n \to \infty} P[X_n \leq a_n x + \beta_n] = N(x),
\]

where \(N(x)\) is the standard normal distribution.

(iv) If

\[\lim_{x \to \infty} \frac{R(x) \log \log R(x)}{(2n \log \log n)^{\frac{1}{2}}/r(R^{-1}(n))} = 0\]

Then almost surely:

\[
\limsup_{n \to \infty} \frac{X_n R^{-1}(n)}{(2n \log \log n)^{\frac{1}{2}}/r(R^{-1}(n))} = 1, \quad \liminf_{n \to \infty} \frac{X_n R^{-1}(n)}{(2n \log \log n)^{\frac{1}{2}}/r(R^{-1}(n))} = -1,
\]
Condition (29) is the well known Von Mises sufficient condition for attraction to the extreme value distribution $\Lambda(x)$ ([3], Theorem 2.7.4, [12]). From the Duality Theorem 4.1 of [8] $X_{L_n}$ has limiting normal distribution iff the distribution $1 - \exp(-R^2(x))$ is attracted in the sense of extreme value theory to $\Lambda(x)$. The Von Mises sufficient condition for such attraction is (31). The sufficiency of (30) for (ii) is proven in [4] and (iv) is just Theorem 3'.

Since $R(x) \uparrow \infty$ as $x \to \infty$, it is clear that when $R$ has a differentiable density: (32) $\Rightarrow$ (31) $\Rightarrow$ (30) $\Rightarrow$ (29).

Under the sufficient condition (10) of Theorem 1 for $\limsup X_{L_n} / n! = e^{-c}$ a.s. and $\liminf X_{L_n} / n! = e^{-c}$ a.s. $n \to \infty$, it is quickly seen that $\lim M_n / n! = 1$ a.s. For (10) implies $\lim \log R(x) / x^2(x) = 0$ and this is the sufficient condition for a.s. stability of $\{M_n\}$ of [4]. One can also show that $\lim M_n / n! = 1$ a.s. if the sufficient conditions of Theorem 1 hold. It is enough to show that (8) implies

$$\int_0^\infty \exp(-(R(tx) - R(t))) dR(t) < \infty$$

for all $t > 1$, since (33) is necessary and sufficient for a.s. stability of $\{M_n\}$ ([1], [10]). However (8) implies that there exists $\eta > 0$ such that for $t$ greater than some $m_0$:

$$R(tx) - R(t) \geq \eta (2R(t) \log \log R(t))^1/3$$

$$> \eta R^1/3(t).$$
On the region \((m_0, \infty)\), (33) is dominated by \(\int_{m_0}^{\infty} \exp(-nR(t)^{1/2})dR(t) < \infty\) and this proves our assertion.

Next, suppose \(\lim_{x \to \infty} x^r(x)/\log R(x) = c^{-1}\) so that, from [4], we have almost surely that:

\[
\limsup_{n \to \infty} \frac{M_n}{R^{-1}(\log n)} = e^c, \quad \liminf_{n \to \infty} \frac{M_n}{R^{-1}(\log n)} = 1.
\]

The sufficient condition for this case clearly implies (15) so that it is also true almost surely that

\[
\limsup_{n \to \infty} \frac{X_n}{R^{-1}(n)} = \infty, \quad \liminf_{n \to \infty} \frac{X_n}{R^{-1}(n)} = 0.
\]

Consider again the distribution function with \(R(x) = (\log x)^2\), \(x \geq e\). As mentioned already, for this distribution it is not true that \(\frac{X_n}{R^{-1}(n)} \xrightarrow{P} 1\). Hence it is not true that \(\frac{X_n}{R^{-1}(n)} \xrightarrow{a.s.} 1\) and so by Remark 5, none of the Theorems 3, 3' or 4 hold for this distribution. (Note that \(r(x)\) is regularly varying with exponent \(-1\) which is the precluded case in Theorem 4.) Furthermore it is impossible for (20) to be true for this distribution. To see this note that

\[
\frac{R^{-1}(n+(2n \log \log n)^{1/2})-R^{-1}(n) - \exp\{(n+(2n \log \log n)^{1/2})^{1/2}\}}{\exp(n^{1/2})} \xrightarrow{a.s.} \infty.
\]

Therefore (20) is equivalent to the following two almost sure statements:
\[
\limsup \frac{X_n}{\exp\left(n + \left(2n \log \log n \right)^{1/2}\right)} = 1
\]
\[
\liminf \frac{X_n}{\exp\left(n + \left(2n \log \log n \right)^{1/2}\right)} = -1.
\]

The second statement is absurd and so (20) does not hold for the case R(x) = \((\log x)^2\).

Yet for R(x) = \((\log x)^2\) it is true almost surely that:

\[
\limsup \frac{M_n - \exp\left((\log n)^{1/2}\right)}{(\log \log n)\exp\left((\log n)^{1/2}\right)/2(\log n)^{1/2}} = 1
\]

\[
\liminf \frac{M_n - \exp\left((\log n)^{1/2}\right)}{(\log \log n)\exp\left((\log n)^{1/2}\right)/2(\log n)^{1/2}} = 0 .
\]

This follows because R(x) = \((\log x)^2\) satisfies (30).
References


