WEAK CONVERGENCE TO EXTREMAL PROCESSES

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SIDNEY I. RESNICK

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Abstract

\{X_n, n \geq 1\} are i.i.d. r.v.'s with d.f. F(x). Set

\[ M_n = \max\{X_1, \ldots, X_n\}. \]

As a basic assumption suppose normalizing constants \( a_n > 0, b_n, n \geq 1 \) exist such that

\[ \lim_{n \to \infty} P[M_n < a_n x + b_n] = G(x), \]

non-degenerate. Define the random function

\[ Y_n(t) = \frac{(M_{[nt]} - b_n)}{a_n} \text{ if } t > \frac{1}{n}, \text{ and } \frac{(X_1 - b_n)}{a_n} \text{ if } t \leq \frac{1}{n}. \]

A new proof of the original Lamperti result that \( Y_n \Rightarrow Y \) is given where \( Y \) is an extremal-G process. Other weak convergence results are shown. Let \( x(t) \) be non-decreasing and \( N_x(I) \) be the number of times \( x \) jumps in time interval \( I \). Then \( Y_n^{-1} \Rightarrow Y^{-1}, \text{ NY}_n \Rightarrow \text{ NY}, \text{ NY}_{-1} \Rightarrow \text{ NY}_{-1}. \)

From these convergences emerge a variety of limit results for record values, record value times and inter-record times.

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Weak Convergence to Extremal Processes

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1. Introduction and Preliminaries

The well-known Donsker Theorem (cf. [2]) states that a sequence of suitably normalized random functions based on partial sums of independent, identically distributed (i.i.d.) random variables with finite variances converges weakly in the uniform topology to Brownian Motion. An analogous result for maxima has long been known and was first proven by Lamperti [12] in 1964. Until recently comparatively little attention was paid to the structure of the converging processes or to that of the limiting extremal process with the result that the full potential of the basic weak convergence result was never, in our opinion, realized. In view of recent studies on the structure of maxima of i.i.d. random variables and extremal processes, another look at weak convergence questions seems justified.

Let \( \{X_n, n \geq 1\} \) be i.i.d. random variables with common distribution function (d.f.) \( F(\cdot) \) and set \( M_n = \max\{X_1, \ldots, X_n\} \). Concepts necessary to elucidate the structure of \( \{M_n\} \) are the following:

Say \( X_j \) is a record value of the sequence \( \{M_n\} \) (or \( \{X_n\} \)) if \( X_j > M_{j-1} \). By convention \( X_1 \) is a record value. The indices at which record values occur are given by the record value times \( \{L_n, n \geq 0\} \) defined by
\[ L_0 = 1, \quad L_n = \min \{ j | j > L_{n-1}, \quad X_j > X_{L_{n-1}} \} \]

and the record value sequence is \( \{X_{L_n}, \quad n \geq 0\} \). The inter-record times are the random variables \( \Delta_n \) defined by \( \Delta_n = L_n - L_{n-1}, \quad n \geq 1 \) and \( \nu_n \) is the number of record values in the sequence \( M_1, \ldots, M_n \).

For Donsker's Theorem, the appropriate limiting process is Brownian Motion. When "sum" is replaced by "max" the right processes to consider are the extremal processes defined as follows: For the given d.f. \( F(x) \) define a consistent family of finite dimensional distributions by

\[
F_{t_1, t_2, \ldots, t_k}(x_1, \ldots, x_k) = F_{t_1}(\min \{x_1, \ldots, x_k\}) F_{t_2-t_1}(\min \{x_2, \ldots, x_k\}) \cdots F_{t_k-t_{k-1}}(x_k)
\]

for \( 0 < t_1 < \ldots < t_k \) and \( x_1, \ldots, x_k \) real. There exists a process with these finite dimensional distributions called an extremal-F process. The process is denoted either as \( \{Y(t), \quad t > 0\} \), \( Y \) or if ambiguity must be precluded: \( Y-F \). \( Y \) is taken separable and measurable, is continuous in probability with right continuous, non-decreasing sample paths and is a Markov jump process. If the left and right end of \( F \) are defined respectively by \( x_0 = \inf \{x | F(x) > 0\}, \)
\( x_1 = \sup \{x | F(x) < 1\} \) then \( \lim_{t \to 0} Y(t) = x_0, \quad \lim_{t \to \infty} Y(t) = x_1 \) a.s. The random counting measure \( \nu(I) \) counts the number of jumps of \( Y \) in the time interval \( I (I \subset R^+) \) and is a non-homogeneous Poisson
process with intensity $t^{-1}$ in the case that $F$ is continuous. See [3], [15] for details.

An underlying assumption throughout this paper will be that there exist normalizing constants $a_n > 0, b_n, n \geq 1$ such that for some non-degenerate $G(x)$:

$$P[M_n \leq a_n x + b_n] = P^n(a_n x + b_n) \to G(x)$$

as $n \to \infty$. When (1) holds we write $F \in D(G)$ and say $F$ is in the domain of attraction of $G$. A basic result of extreme value theory ([7], [8]) states that $G$ can belong to the types of one of three classes denoted by $\Lambda(x), \Phi_\alpha(x), \Psi_\alpha(x)$. For convenience we will here consider only the first two as the third never offers any new challenges. Recall $\Lambda(x) = \exp(-e^{-x}), -\infty < x < \infty$ and $\Phi_\alpha(x) = \exp(-x^{-\alpha})$ for $x \geq 0$ and $= 0$ when $x < 0$. Here $\alpha$ is a positive parameter.

When (1) holds, we will be concerned with the convergence properties of the random function $Y_n(\cdot)$ defined by

$$Y_n(\cdot) = (M_{[nt]} - b_n)/a_n, \quad n^{-1} < \cdot$$

$$= (X_1 - b_n)/a_n, \quad 0 \leq \cdot \leq n^{-1}$$

In section 2 we offer a new proof of the Lamperti result that for any $0 < a < b < \infty$: $Y_n \Rightarrow Y - G$; i.e. $Y_n$ converges weakly to the extremal-$G$ process in $D[a,b]$ with respect to the usual Skorohod $J_1$ topology.
All notation and conventions are as in [2] except that "\( \Rightarrow \)" is used to denote weak convergence. Since the limiting extremal processes are stochastically continuous, weak convergence on \( D[a,b] \) for any \( a < b \) automatically extends to weak convergence on \( D[0,\infty) \) when \( G = \phi^\alpha \) and to \( D(0,\infty) \) when \( G = \Lambda \). Cf. [13]. Note for \( Y - \Lambda \) that \( Y(0) = -\infty \) and for \( Y - \phi^\alpha \) we have \( Y(0) = 0 \) (see [15] subsequent to Proposition 2).

In section 1 we also show \( Y_n^{-1} \Rightarrow Y^{-1} \). Throughout, inverses of non-decreasing functions are taken right continuous. In section 3 we consider some implications of the basic convergences.

2. The Basic Convergences

The technical difficulties of the original Lamperti proof of the following result involved showing tightness of the sequence \( \{Y_n\} \) defined by (2). In our proof, tightness presents no problem and the difficulties fall within extreme value theory rather than weak convergence theory. The tools needed to overcome the difficulties are by now rather shopworn ([8], [9], [10]).

**Theorem 1:** Suppose (1) holds and \( Y_n \) is defined by (2).

(i) If \( G = \phi^\alpha \) then \( Y_n \Rightarrow Y - \phi^\alpha \) in \( D[0,\infty) \).

(ii) If \( G = \Lambda \) then \( Y_n \Rightarrow Y - \Lambda \) in \( D(0,\infty) \).

**Proof:** The method is to prove the result for a special case and then to extend.

**Case 1:** Pick \( 0 < \alpha < 1 \) and suppose \( \{X_n, n \geq 1\} \) are i.i.d.
non-negative random variables with 

\[ P[X_1 > x] = 1 - F_1(x) = x^{-\alpha} \]

with \( L \) continuous and slowly varying. Set 

\[ S_n = X_1 + \ldots + X_n \]

and then

\[ S_{[n^*]} / F_1^{-1}(1 - n^{-1}) \Rightarrow X_{\alpha}(\ast) \]

in \( D[0, \infty) \) where \( \{ X_{\alpha}(t), \ t \geq 0 \} \) is a stable process of order \( \alpha \) (Cf. [6], p. 180-3, [18]). Define the mapping \( h:D[0, \infty) \rightarrow D[0, \infty) \) by

\[ h(x(t)) = \text{largest positive jump of } x \]  

in \([0, t]\) if there is one;

\[ = 0 \text{ otherwise.} \]

One can verify that \( h \) is continuous with respect to the \( J_1 \) topology on \( D[0, \infty) \) and hence by the continuous mapping theorem:

\[ hS_{[n^*]} / F_1^{-1}(1 - n^{-1}) \Rightarrow hX_{\alpha} \]

in \( D[0, \infty) \). Now observe that 

\[ hS_{[n^*]} / F_1^{-1}(1 - n^{-1}) = M_{[n^*]} / F_1^{-1}(1 - n^{-1}) \]

and by Proposition 3 of [15] \( hX_{\alpha} = Y - \Phi_{\alpha} \) and case 1 is disposed of.

**Case 2:** Let \( \alpha, F_1 \) be as in case 1 and suppose \( 0 < \beta < \infty \) and \( F_2 \in D(\Phi_{\beta}) \) so that \( 1 - F_2 \) is regularly varying exponent \(-\beta\) ([7], [8]). If \( X_1 \) has d.f. \( F_1 \) and \( X_2 \) has d.f. \( F_2 \) then 

\[ X_2 \overset{d}{=} \phi(X_1) \]

where \( \phi(x) = (1 - F_2)^{-1} \phi(1 - F_1)(x) \). Note \( \phi(x) \uparrow \infty \) and
\( \phi \) is regularly varying with exponent \( \alpha/\beta \) ([8], Corollary 1.21). Let \( M_n^{(1)} \) be the maximum of \( n \) i.i.d. observations from \( F_i \), \( i = 1, 2 \) so that \( M_n^{(2)} = \phi(M_n^{(1)}) \). Set \( Y_n^{(1)}(*) = M_n^{(1)}/[n^\beta/2](1-n^{-1}) \) and by case 1: 
\( Y_n^{(1)} \rightarrow Y - \phi \). At the expense of replacing \( Y_n^{(1)} \) by a random function having the same distribution, we may suppose \( Y_n^{(1)} \) converges a.s. to 
\( Y - \phi \) (Skorohod's Theorem 3.1.1 [22]). Now write

\[
M_n^{(2)}/[n^\beta/2](1-n^{-1}) = \phi(M_n^{(1)})/[\phi(F_i^{-1}(1-n^{-1}))
\]

\[
= \phi(F_i^{-1}(1-n^{-1})Y_n^{(1)}(*))/[\phi(F_i^{-1}(1-n^{-1}))]
\]

and use the definition of pointwise convergence in the Skorohod metric and the regular variation of \( \phi \) ([8], Corollary 1.2.1-4) to conclude case 2.

**Case 3:** Suppose \( F_3 \in D(A) \) and \( F_1 \) is the same as in case 1. Set

\( R_i(x) = -\log(1-F_i(x)) \) for \( i = 1, 2 \). If \( X_3 \) has d.f. \( F_3 \) and \( X_1 \) has d.f. \( F_1 \) then \( X_3 \xrightarrow{d} R_3^{-1} R_1(X_1) \). It is well known that \( F_3 \in D(A) \) iff \( R_3^{-1}\log \) is in de Haan's class \( II \); i.e.:

\[
\lim_{t \to \infty} \frac{R_3^{-1}(\log tx) - R_3^{-1}(\log t)}{R_3^{-1}(\log te) - R_3^{-1}(\log t)} = \log x
\]

(3)

for any \( x > 0 \) ([8], Theorem 2.5.1). Note the convergence is uniform on compact \( x \) sets in the foregoing. Because of the assumed regular variation of \( F_1 \) it follows easily that \( R_3^{-1} R_1 \in \Pi \) (cf. [9]).
Let $M_{n}^{(3)}$ be the maximum of $n$ i.i.d. observations from $F_{3}$ and then

$M_{n}^{(3)} = R_{3}^{-1}o_{R_{1}}(M_{n}^{(1)})$. Assume again without loss of generality that $Y_{n}^{(1)}$ converges a.s. in the Skorohod metric to $Y - \Phi_{\alpha}$. Then

$$
\frac{M_{n}^{(3)} - R_{3}^{-1}o_{R_{1}}(F_{-1}^{-1}(1-n^{-1}))}{R_{3}^{-1}o_{R_{1}}(F_{-1}^{-1}(1-(ne)^{-1})) - R_{3}^{-1}o_{R_{1}}(F_{-1}^{-1}(1-n^{-1}))}
$$

$$
= \frac{M_{n}^{(3)} - R_{3}^{-1}(\log n)}{R_{3}^{-1}(\log ne) - R_{3}^{-1}(\log n)}
$$

$$
= \frac{(R_{3}^{-1}o_{R_{1}}(F_{-1}^{-1}(1-n^{-1}))Y_{n}^{(1)}(\cdot)) - (R_{3}^{-1}o_{R_{1}}(F_{-1}^{-1}(1-n^{-1})))}{(R_{3}^{-1}o_{R_{1}}(F_{-1}^{-1}(1-n^{-1})) - (R_{3}^{-1}o_{R_{1}})(F_{-1}^{-1}(1-n^{-1})))}
$$

Now use the uniform convergence property of $(3)$ and the convergence in the Skorohod metric of $Y_{n}^{(1)}$ to $Y - \Phi_{\alpha}$ to conclude that the above converges a.s. to $\alpha \log(Y - \Phi_{\alpha})$ to $Y - \Lambda$. This concludes case 3 as well as the proof of Theorem 1.

Here are some related convergence results:

**Theorem 2:** For $n \geq 0$ $F_{n}$ are d.f.'s such that $F_{n} \Rightarrow F_{0}$. Suppose $Y^{(n)}$ is extremal-$F_{n}$. If either

(i) $F_{0}$ is continuous, strictly increasing or

(ii) $F_{n}$ is continuous, $\forall n \geq 1$

then $Y^{(n)} \Rightarrow Y^{(0)}$ in $D[0,\infty)$ if $X_{0}(F_{0}) > -\infty$ and in $D(0,\infty)$ if $X_{0}(F_{0}) = -\infty$. (Recall $X_{0}(F_{0})$ is the left end of $F_{0}$.)
Proof: (i) Set \( S_n(x) = -\log(-\log P_n(x)) \), \( n \geq 0 \), so that \( Y(n) \overset{d}{=} S_n^{-1}(Y-A) \). Further, since \( F_0 \) is continuous and strictly increasing, so is \( S_0 \) and hence \( S_n^{-1}(x) \rightarrow S_0^{-1}(x) \) uniformly on finite intervals. This is enough to show \( S_n^{-1}(Y-A) \) converges in the Skorohod metric a.s. to \( S_0^{-1}(Y-A) \) and hence the result.

(ii) From [13] it suffices to show weak convergence in \( D[a,b] \), \( 0 < a < b < \infty \). The finite dimensional d.f.'s obviously converge and the limit process is stochastically continuous so that by Theorem 15.6, p. 128 of [2] it suffices to show the existence of a non-decreasing, continuous function \( H \) on \( [a,b] \) such that for \( a \leq t_1 < t < t_2 \leq b \) and \( n \geq 1 \)

\[
(4) \quad P[|Y(n)(t) - Y(n)(t_1)| \geq \lambda |Y(n)(t_2) - Y(n)(t)| \geq \lambda] \leq (H(t_2) - H(t_1))^2.
\]

Recall that since \( P_n \) is continuous, the random measure \( \nu \) which counts the number of jumps of \( Y(n) \) is non-homogeneous Poisson, intensity \( t^{-1} \). Hence the left side of (4) is not greater than

\[
P[Y(n)(t) \neq Y(n)(t_1), Y(n)(t_2) \neq Y(n)(t_1)]
\]

\[
= P[\nu(t_1,t) = 0]^c \cap [\nu(t,t_2) = 0]^c
\]

and since \( \nu \) has independent increments the above equals
\[ P[v(t_1, t) > 0] P[v(t, t_2) > 0] = (1 - \exp\{-\log(t/t_1)\})(1 - \exp\{-\log(t_2/t)\}) \leq (\log(t/t_1))(\log(t_2/t)) \leq \{\log t/t_1 + \log(t_2/t)\}^2 = \{\log t_2 - \log t_1\}^2. \]

Set \( H(t) = \log t \) and the proof is complete.

**Remark:** Theorem 2 and the technique of embedding \( \{M_n\} \) within a suitable extremal process (cf. [16]) can be combined to yield a proof of Theorem 1.

If \( Y \) is extremal-G where \( G \) has left and right ends \( x_0, x_1 \) respectively, we may consider along with \( Y \) the inverse process \( \{Y^{-1}(x), x_0 \leq x \leq x_1\} \) defined by \( Y^{-1}(x) = \inf\{z|Y(z) > x\} \). \( Y^{-1} \) is an additive process and has been studied in [3], [17], [21]. When (1) holds and \( Y_n \) is defined by (2), Theorem 1 tells us \( Y_n \Rightarrow Y \) and we may ask if \( Y_n^{-1} \Rightarrow Y^{-1} \) in \( D[0, \infty) \) if \( G = \Phi_\alpha \) or in \( D(-\infty, \infty) \) if \( G = \Lambda \). The mapping \( x(\cdot) \Rightarrow x^{-1}(\cdot) \) is not necessarily continuous in the \( J_\perp \) topology so the conclusion that \( Y_n^{-1} \Rightarrow Y^{-1} \) does not follow immediately from Theorem 1. However \( x(\cdot) \Rightarrow x^{-1}(\cdot) \) is continuous in the \( M_\perp \) topology (cf. [22], [27]) and consequently Theorem 1 immediately guarantees convergence of the finite dimensional d.f.'s. Since both \( Y_n^{-1} \) and \( Y^{-1} \) have independent increments ([14], [17], [20], [21],) it suffices by Remark 2.2 of [23] for weak convergence in \( D[a,b] \) to
verify that

\[ \lim_{c \to 0} \lim_{n \to \infty} \sup_{|t_1 - t_2| \leq c} \sup_{a \leq t_1 < t_2 \leq b} P[Y_n^{-1}(t_2) - Y_n^{-1}(t_1) > \varepsilon] = 0 \]

for all \( \varepsilon > 0 \).

Define \( \tau(x) = \inf \{ n | M_n > x \} \) and note that \( P[\tau(x) > n] = F_n^x(x) \) so that \( E\tau(x) = \sum_{n=0}^\infty P^n(x) = 1/(1-F(x)) = (F(x))^{-1} \) where \( F = 1 - F \).

Next note that \( Y_n^{-1}(t) = \inf \{ s | Y_n(s) > t \} = \inf \{ s | M_{ns} > a_n t + b_n \} \)

\( = \tau(a_n, t + b_n)/n \). Returning to (5) we use Markov's inequality:

\[ P[Y_n^{-1}(t_2) - Y_n^{-1}(t_1) > \varepsilon] = P[\tau(a_n t_2 + b_n) - \tau(a_n t_1 + b_n) > \varepsilon n] \]

\[ \leq \frac{1}{\varepsilon n} \{ (n(F(a_n t_2 + b_n)))^{-1} \}

Equation (1) entails \( \lim_{n \to \infty} n(F(a_n x + b_n)) = -\log G(x) \) and the convergence is uniform on bounded \( x \)-intervals. This uniform convergence and the fact that \( -\log G(x) \) is uniformly continuous on bounded intervals (the extreme value distributions are continuous) combine to guarantee that (5) will hold. Thus we have proven:

**Theorem 3:** If (1) holds and \( Y_n \) is defined by (2) then
\[ y_n^{-1} \Rightarrow y^{-1} \]

in \( D(0,\infty) \) if \( G = \Phi_\alpha \) and in \( D(-\infty,\infty) \) if \( G = \Lambda \) where \( Y \) is extremal-\( G \).

3. Functions of the Basic Processes

We now derive some limit theorems for maxima and record values from Theorems 1 and 3. Knowledge of the structure of extremal processes is very helpful here as it permits explicit computation of limiting distributions.

For \( x(\cdot) \in D[1,\infty) \) let \( N_x(I) \) be the number of jumps of \( x(\cdot) \) in the time interval \( I, I \subset [1,\infty) \). For \( t > 1 \), set \( N_x(t) = N_x(1,t] \).

The left endpoint \( l \) is arbitrary and in fact we could work in \( D(0,\infty) \).

Theorem 4: Suppose (1) holds so that in the terminology of (2):

\[ y_n \Rightarrow Y - G. \]

Then \( N_y_n \Rightarrow N_y \) in \( D[1,\infty) \). In the notation of the introduction:

\[ \mu_{[n^*]} - \mu_n \Rightarrow \nu(l, \cdot) \].

Remark: The proof is quite easy if one is willing to assume that \( F \) is continuous for in that case one combines the independent increments property of \( \{\mu_{[nt]} - \mu_n, t \geq 1\} \) (see [14]) with Theorem 2.1(a) of [3] and [11], p. 37 or [24], p. 212 to conclude the result. The point of the following proof is to avoid the continuity assumption relying only on (1).
Proof: \( N \) may not be a continuous function so we cannot immediately invoke the continuous mapping theorem. However the following mapping is continuous: Define \( N(\varepsilon)x(I) \) to be the number of jumps of size \( \geq \varepsilon \) in time interval \( I, \varepsilon > 0, x \in D[l,\infty) \). Hence from Theorem 1 and continuity: \( N(\varepsilon)x_n \to N(\varepsilon)x \) in \( D[l,\infty) \) as \( n \to \infty \). Further, for any \( T > 1: \sup_{1 \leq s < T} |N_Y(s) - N(\varepsilon)x_n(s)| = N_Y(T) - N(\varepsilon)x(T) \to 0 \) a.s. as \( \varepsilon \to 0 \) since \( \forall(l,T) < \infty \) a.s. Hence in view of Theorem 4.2 of [2] and the definition of the topology on \( D[l,\infty) \), it suffices for the completion of the proof to show

\[
(6) \lim \limsup_{\varepsilon \to 0} \sup_{n \to \infty} \sup_{1 \leq s < T} |N_Y(s) - N(\varepsilon)x_n(s)| > \delta = 0
\]

for any \( 0 < \delta < 1 \).

The probability in (6) equals

\[
P[N_Y(T) - N(\varepsilon)x_n(T) > \delta] \leq P[Y_n \text{ has a jump of size } \leq \varepsilon \text{ in } [1,T]]
\]

Call this latter event \( A_\varepsilon \). For \( \eta \) arbitrarily small pick \( z_1, z_2 \) such that \( 0 < G(z_1) < \eta, 0 < 1 - G^+(z_2) \leq \eta \). The above probability is

\[
(7) P[A_\varepsilon, Y_n(T) > z_1, Y_n(T) \leq z_2] + P[Y_n(1-) \leq z_1] + P[Y_n(T) > z_2]
\]

Concentrate on the first term of (7); it can be written
\[
\Pr \left\{ \bigcup_{j=n}^{[nT]} \left[ 0 < \frac{M_j - b}{a_n} - \frac{M_{j-1} - b}{a_n} \leq \varepsilon \right] \cap \left[ \frac{M_{n-1} - b}{a_n} > z_1, \frac{M_{[nT]} - b}{a_n} < z_2 \right] \right\} \\
\leq \sum_{j=n}^{[nT]} \Pr \left[ 0 < \frac{M_j - b}{a_n} - \frac{M_{j-1} - b}{a_n} \leq \varepsilon, \ z_1 > \frac{M_{j-1} - b}{a_n} \leq z_2 \right]
\]

and remembering that \( \left\{ \frac{M_j - b}{a_n}, \ j \geq 1 \right\} \) is a Markov chain we have that

the above equals

\[
\sum_{j=n}^{[nT]} \int_{z_1}^{z_2} \left( \int_{x < \frac{M_j - b}{a_n} < x + \varepsilon} M_{j-1} = a_n x + b_n \right) d\mu^{j-1}(a_n x + b_n)
\]

\[
= \sum_{j=n}^{[nT]} \int_{z_1}^{z_2} \left( \overline{F}(a_n x + b_n) - \overline{F}(a_n (x+\varepsilon) + b_n) \right) d\mu^{j-1}(a_n x + b_n)
\]

(\text{where } \overline{F} = 1 - F)

\[
\leq \int_{z_1}^{z_2} \left( \sum_{j=n}^{[nT]} \overline{n\{F(a_n x + b_n) - F(a_n (x+\varepsilon) + b_n)\}} d \frac{\overline{F}^{n-1}(a_n x + b_n)}{nF(a_n x + b_n)} \right).
\]

As a consequence of 1 we have \( n\overline{F}(a_n x + b_n) \rightarrow -\log G(x) \equiv Q_G(x) \) as \( n \to \infty \) and hence the above integral, being taken over a finite interval, converges as \( n \to \infty \) to

\[
\int_{z_1}^{z_2} \left\{ Q_G(x) - Q_G(x+\varepsilon) \right\} d \frac{G(x)}{Q_G(x)}.
\]

Use the continuity of \( Q_G \) to infer that the limit of the integral as \( \varepsilon \to 0 \) is 0.
The last two terms of (7) are handled easily: As \( n \to \infty \) they converge to \( G(z_1) + 1 - G(z_2) \leq 2n \) (by choice of \( z_1, z_2 \)) and hence \( \lim \text{limsup } \) of (6) is \( \leq 2n \). Since \( \eta \) is arbitrarily chosen, the proof is complete.

There is a companion result dealing with \( NY_n^{-1} \). Here \( NY_n^{-1}(z_1, z_2) \) can be interpreted as the number of states visited by \( Y_n \) in the subset of the range given by \( \{z_1, z_2\} \). It is convenient to define a point process \( \xi(I) \) for intervals \( I \) relative to the original \( \{X_n\} \) sequence by

\[
\xi(I) = \# \{k \mid X_{L_k} \in I \}.
\]

In this case we have \( NY_n^{-1}(I) = \xi(a_n I + b_n) \).

**Theorem 5:** If (1) holds then

\[
NY_n^{-1} \Rightarrow NY^{-1}
\]

in \( D(0, \infty) \) or equivalently:

\[
\{I \to \xi(a_n I + b_n)\} \Rightarrow \{I \to NY^{-1}(I)\}.
\]

Here \( NY^{-1} \) is a non-homogeneous Poisson process with mean measure

\[
S(I) = -\log(-\log G(I)).
\]
Proof: The last statement is Corollary 1 of [17]. To show weak convergence in $D(0,\infty)$ of these point processes it suffices to show that the finite dimensional d.f.'s converge ([11], p. 37 or [24], p.212). Since $\xi$ has independent increments ([19], Theorem 1, [20], Theorem 1.1) all finite dimensional d.f.'s converge if the one-dimensional marginals converge and it suffices to show $\xi(a_n I + b_n) \Rightarrow NY^{-1}(I)$ or taking generating functions we must show:

$$\xi(a_n I + b_n) \rightarrow \exp((s-1)S(I))$$

(8)

$0 \leq s \leq 1$, as $n \to \infty$, the quantity on the right being the generating function of a Poisson random variable with mean $S(I)$.

The structure of $\xi$ as discussed by Shorrock [19] is as follows:

Write $F = 1 - F$ as the product of a discrete tail with a continuous tail: $F = F_d F_c$ where $F_d$ is purely discrete and $F_c$ purely continuous. Let $A$ be the atoms of $F$ in $I$ and suppose $\{y_\alpha, \alpha \in A\}$ are independent Bernoulli random variables with $P[y_\alpha = 0] = \overline{F_d}(\alpha)/\overline{F_d}(\alpha-) \equiv 1 - p_\alpha$ and $P[y_\alpha = 1] = p_\alpha$. Note that since $F \in D(G)$ $\lim_{x \to \infty} F(x)/F(x-) = 1$ [7] so in particular $p_\alpha \to 0$. Let $\xi'$ be a homogeneous rate 1 Poisson process and put $R_c(x) = -\log F_c(x)$. Then

$$\xi(I) = \xi'(R_c(I)) + \xi''(I)$$

where $\xi''(I) = \sum_{\alpha \in I} y_\alpha$ and $\xi'$ and $\xi''$ are independent.
Working from (8):
\[
\xi(a_n + b_n)_{n} \sim \exp[(s-1)R_{c}(a_n + b_n)_{n}] \prod_{\alpha \in a_{n} + b_{n}} (1 - p_{\alpha}(1-s)).
\]

Suppose \( I = (u,v) \) so that the right side becomes
\[
\left( \frac{\bar{F}_{c}(a_{n} + b_{n})}{\bar{F}_{c}(a_{n} + b_{n})} \right)^{s-1} \prod_{\alpha \in a_{n} + b_{n}} (1 - p_{\alpha}(1-s)).
\]

If I can show
\[
(9) \quad \prod_{\alpha \in a_{n} + b_{n}} (1 - p_{\alpha}(1-s)) \sim \left( \frac{\bar{F}_{c}(a_{n} + b_{n})}{\bar{F}_{d}(a_{n} + b_{n})} \right)^{s-1}
\]
as \( n \to \infty \) then as \( n \to \infty \):
\[
\xi(a_n + b_n)_{n} \sim \left( \frac{\bar{F}_{c}(a_{n} + b_{n})}{\bar{F}_{c}(a_{n} + b_{n})} \right)^{s-1} \left( \frac{\bar{F}_{d}(a_{n} + b_{n})}{\bar{F}_{d}(a_{n} + b_{n})} \right)^{s-1}
\]
\[
= \left( \frac{n \bar{F}(a_{n} + b_{n})}{n \bar{F}(a_{n} + b_{n})} \right)^{s-1} \quad \to \quad \frac{(-\log G(u)/-\log G(v))^{s-1}}
\]
as a consequence of (1) and the last expression equals
\[
\exp[(s-1)(-\log(-\log G(v) + \log(-\log G(u)))] = \exp[(s-1)S(I)]
\]
which is the desired result.
Instead of (9) it suffices to show

\[(10) \sum_{\alpha \in \mathbb{I}_n} \{ \log(1-(1-s)p_\alpha) \} - (s-1)\{R_d(a,v+b_n) - R_d(a,u+b_n)\} \rightarrow 0.\]

Since \( p_\alpha \to 0, \) \( \log(1-(1-s)p_\alpha) = -(1-s)p_\alpha + O(p_\alpha^2). \) Hence

\[\sum_{\alpha \in a_n I+b_n} \log(1-(1-s)p_\alpha) = -\sum(1-s)p_\alpha + \sum O(p_\alpha^2)\]

\[= -(1-s)\sum(1-p_\alpha) + \sum O(p_\alpha^2)\]

\[= (1-s)\sum \log(1-p_\alpha) + \sum O(p_\alpha^2)\]

\[= (1-s)\log \left\{ \prod_{a_n u+b_n < \alpha < a_n v+b_n} \overline{F}_d(\alpha)/\overline{F}_d(\alpha-) \right\} + \sum O(p_\alpha^2)\]

and since \( \overline{F}_d \) is purely discrete the above equals

\[= (1-s)\log \left\{ \overline{F}_d(a,v+b_n)/\overline{F}_d(a,u+b_n) \right\} + \sum O(p_\alpha^2)\]

\[= (s-1)(R_d(a,v+b_n) - R_d(a,u+b_n)) + \sum O(p_\alpha^2).\]

Thus the difference in (10) amounts to \( \sum_{\alpha \in a_n I+b_n} O(p_\alpha^2) \) which goes to 0 because of the following facts: (i) \( p_\alpha \to 0; \) (ii) \( a_n u + b_n \to x_1, \) the right end of \( F; \) (iii) \( \sum_{\alpha \in a_n I+b_n} p_\alpha = F(U_{M_j \text{ hits } \alpha \text{ for some } j}) \leq 1. \) The proof is complete.
Define the jump time functional \( j \) as follows: If \( x(\cdot) \) is a path of a point process \( jx = (j_1x, j_2x, \ldots) \) gives the times of jumps past \( t = 1 \); i.e.

\[
j_k(x) = \begin{cases} 
  +\infty & \text{if } \{t > 1 \mid x(t) - x(1) = k\} = \emptyset \\
  \inf \{t > 1 \mid x(t) - x(1) = k\}, & \text{otherwise}
\end{cases}
\]

for \( k = 1, 2, \ldots \). \( j \) is continuous from point process paths into \((\mathbb{R}^\infty, \mathcal{B}^\infty)\) (cf. [11], p. 57) and hence the continuous mapping theorem and Theorem 4 give \( jNY \_n \Rightarrow jNY = j\nu \) where \( \nu \) is nonhomogeneous Poisson with intensity \( t^{-1} \). A simple transformation to a homogeneous Poisson process shows that \( j\nu \) has the structure \( \{\exp[\sum_{i=1}^n Z_i, \ n \geq 1]\} \) where \( \{Z_n, \ n \geq 1\} \) are i.i.d. and \( P[Z_i > x] = e^{-x}, \ x > 0 \). The times when \( NY \_n \) or \( Y \_n \) jumps past \( t = 1 \) are \( L_k/n \) (in the terminology of section 1). The conclusion is

**Corollary 1:** If (1) holds or if \( F \) is continuous then

\[\{L_k/n \mid L_k > n\} \Rightarrow \{\exp[\sum_{i=1}^m Z_i], \ m \geq 1\} \text{ as } n \to \infty \text{ where } \{Z_n, \ n \geq 1\} \text{ are i.i.d. exponential mean } 1 \text{ random variables. Further}
\]

\[
\left\{L_{k+1}/n \mid L_k > n\right\} \Rightarrow \left\{\exp\left(\sum_{k=1}^{m+1} Z_k\right) - \exp\left(\sum_{k=1}^{m} Z_k\right), \ m \geq 1\right\}.
\]

**Proof:** The last statement is a consequence of the first via the continuous mapping theorem. The assertion concerning continuous \( F \)'s is
proven by a probability integral transformation taking the original sequence \( \{ X_n \} \) to an i.i.d. exponentially distributed sequence. \( \{ L_n \} \) is unchanged by the transformation.

The continuous mapping theorem permits other variations:

(11) \[
\left\{ \frac{L_{k+1}}{L_k} \mid L_k > n \right\} \Rightarrow \left\{ \frac{Z_m}{e^m}, \; m \geq 1 \right\}
\]

and

(12) \[
\{ \log(L_k/n) \mid L_k > n \} \Rightarrow \{ Z_m, \; m \geq 1 \}
\]

and

(13) \[
\left\{ \frac{\Delta_{k+1}/\Delta_k}{L_k > n} \right\} \Rightarrow \left\{ \begin{array}{l}
\exp\left\{ \frac{m+2}{m+1} \sum_{k=1}^m Z_k \right\} - \exp\left\{ \sum_{k=1}^{m+1} Z_k \right\} \\
\exp\left\{ \sum_{k=1}^{m+1} Z_k \right\} - \exp\left\{ \sum_{k=1}^m Z_k \right\}
\end{array} \right\}, \; m \geq 1
\]

\[
= \left\{ \begin{array}{l}
\exp\left\{ Z_{m+2} - Z_{m+1} \right\} - 1 \\
1 - \exp\left\{ -Z_{m+1} \right\}
\end{array} \right\}, \; m \geq 1
\]

The reader may be interested in comparing the results of Corollary 1 as well as (11), (12), (13) with those in section 4 of [19]. Also compare Corollary 1 with the fact that there do not exist normalizing constants \( \alpha_n > 0, \beta_n, \; n \geq 1 \) such that \( \alpha_n^{-1}(L_n - \beta_n) \) has a weak limit. Cf. [1], [19], [25].

Suppose the right end \( x_n \) of \( F \) is \( > 1 \). In this case it makes sense to apply \( j \) to \( NY_n^{-1} \) and by Theorem 5 we have \( jNY_n^{-1} \Rightarrow jNY^{-1} \) in
say $D[1, \infty)$. Note that 

$$j_{NY-1} = \{a_n^{-1}(X_{L_k} - b_n) | X_{L_k} > a_n + b_n \}$$

so that

$$\{a_n^{-1}(X_{L_k} - b_n) | X_{L_k} > a_n + b_n \} \Rightarrow j_{NY-1}$$

where $NY^{-1}$ is a Poisson process with mean measure

$$S(I) = -\log(-\log G(I)).$$

Observe that if $G = \Lambda$ then $S(x) = x$ and $NY^{-1}$ is homogeneous Poisson. If $G = \Phi_\alpha$ then $S(x) = \alpha \log x$.

This result overlaps some work of Shorrock [21].

In particular note that if $F \in D(\Phi_\alpha)$ then we can set $b_n = 0$,

$$a_n = F^{-1}(1-1/n) \quad (\text{cf. [7], [8]}) \quad \text{and (14) becomes}$$

$$\{X_{L_k} / F^{-1}(1-1/n) | X_{L_k} > F^{-1}(1-1/n) \} \Rightarrow j_{NY^{-1}}$$

or equivalently

$$\{X_{L_k} / T | X_{L_k} > T \} \Rightarrow j_{NY^{-1}}$$

as $T \to \infty$. Note $j_{NY^{-1}} = \{\exp(\alpha^{-1} \sum_{i=1}^{m} Z_i, \ m \geq 1 \}$ so that in the case that $F \in D(\Phi_\alpha)$ (equivalently: $F$ is regularly varying with exponent-$\alpha$) we have

$$\{X_{L_k} / T | X_{L_k} > T \} \Rightarrow \{\exp(\alpha^{-1} \sum_{i=1}^{m} Z_i, \ m \geq 1 \}$$

where $\{Z_i, \ i \geq 1 \}$ are i.i.d. exponential mean 1 random variables.
In the case $F \in D(\Lambda)$ we can set $b_n = F^{-1}(1 - l/n)$,
$a_n = F^{-1}(1 - l/ne) - F^{-1}(1 - l/n)$ so that $a_n + b_n = F^{-1}(1 - l/ne)$. After a
change of variable (14) becomes

\[(16) \{a(1/e\tilde{F}(T))^{-1}(X_{L_k} - b(1/e\tilde{F}(T)))|X_{L_k} > T]\Rightarrow \sum_{i=1}^{m} Z_i, \ m \geq 1 \text{ as } T \rightarrow \infty.\]

Since $a(*)$ is slowly varying we can replace $a(1/e\tilde{F}(T))$ by
$a(1/\tilde{F}(T))$. The conclusion:

**Corollary 2:** If (1) holds and $x_1(F) > 1$ then (15) and (16) hold.

In the case $F \in D(\Phi_{\alpha})$, (15) and the continuous mapping theorem
give

\[(17) \{X_{L_{k+1}}/X_{L_k} | X_{L_k} > T\} \Rightarrow \{\exp(\alpha^{-1}Z_m, \ m \geq 1)\}
\]
as $T \rightarrow \infty$ where again $\{Z_m, \ m \geq 1\}$ are i.i.d. exponential random var-
iables. Also

\[(18) \{\log(X_{L_k}/T)|X_{L_k} > T\} \Rightarrow \{\alpha^{-1}Z_m, \ m \geq 1\}.
\]

Analogous results can be deduced from (16) but the limits are not so
simple.

When $F \in D(\Lambda)$ we conclude from (16):

\[(19) \{a(1/e\tilde{F}(T))^{-1}(X_{L_k+1} - X_{L_k})|X_{L_k} > T\} \Rightarrow \{Z_m, \ m \geq 1\}\]
with an analogous result deducible from (15).

A variant of the foregoing results is obtained by looking at jump heights past time 1. The results: If \( F \in D(A) \) then

\[
\{a_n^{-1}(X_L - b_n)_n | L > n \} \Rightarrow \{Y(1) + \sum_{i=1}^m Z_i, \ m \geq 1\}
\]

and

\[
\{a_n^{-1}(X_L - \bar{X}_{L_k}) | L_k > n \} \Rightarrow \{Z_m, \ m \geq 1\}
\]

as \( n \to \infty \) where \( \{Z_m, \ m \geq 1\} \) are i.i.d. exponential random variables independent of \( Y(1) \) where \( P[Y(1) \leq x] = A(x) \). (The limit is evaluated using Theorem 3, [15].) Observe that these results entail

\[
a_n^{-1}(X_L - b_n) \Rightarrow Y(1) + Z_1
\]

\[
a_n^{-1}(X_L - \bar{X}_{L_k}) \Rightarrow Z_1 \quad \text{as} \ n \to \infty
\]

so that the size of the first jump after index \( n \) in the Markov chain \( \{M_j, \ j \geq 1\} \) is asymptotically exponentially distributed. Analogous results can be derived when \( F \in D(\Phi_{\alpha}) \).

Next we discuss the excess functional \( \gamma^+ \) defined as follows: Suppose \( x(\cdot) \) is a right continuous non-decreasing function and \( x^{-1}(\cdot) \) is its right continuous inverse. Define
\( (\gamma^+x)(t) = x(x^{-1}(t)) - t. \)

so that \( \gamma^+x(t) \) is the amount by which \( x \) surpasses height \( t \) upon first crossing level \( t \). \( \gamma^+\cdot(t) \) is continuous ([22], 2.2.13) and so from Theorem 1 \( (\gamma^+Y_n)(t) \Rightarrow (\gamma^+Y)(t) \). When \( F \in D(\Lambda) \), \( (\gamma^+Y)(t) \) is the forward recurrence time of a homogeneous Poisson process and hence \( (\gamma^+Y)(t) \overset{d}{=} Z \) an exponential, mean 1 random variable. When \( F \in D(\Phi_\alpha) \), \( (\gamma^+Y)(t) \) is the forward recurrence time of a nonhomogeneous Poisson process with mean measure \( \alpha \log I \). Hence

\[
P[(\gamma^+Y)(t) > x] = \exp(-\alpha \log(t+x)/t) = (t/t+x)^\alpha.
\]

Recall the definition of \( \tau(a) = \inf\{n| M_n > a\} \). Using \( \tau \) we can express \( (\gamma^+Y_n)(t) \) as \( M_{\tau(a_n t+b_n)} - (a_n t+b_n)/a_n \). In the case \( F \in D(\Phi_\alpha) \) we can set \( b_n = 0 \).

**Corollary 3:** (i) If \( F \in D(\Lambda) \) then for any \( t > 0 \):

\[
P[a_n^{-1}(M_{\tau(a_n t+b_n)} - (a_n t+b_n)) \leq x] \rightarrow 1 - e^{-x}
\]

for \( x \geq 0 \), as \( n \to \infty \).

(ii) If \( F \in D(\Phi_\alpha) \) then for \( t > 0, x > 0 \)

\[
P[M_{\tau(s)} - s/s \leq x] \rightarrow 1 - (1+x)^{-\alpha},
\]

as \( s \to \infty \).

Now apply \( \gamma^+ \) to \( Y_n^{-1} \) so that by Theorem 3 \( (\gamma^+Y_n^{-1})(1) \Rightarrow (\gamma^+Y^{-1})(1) \).
Observe that \((\gamma^+y^{-1})(1)\) is the forward recurrence time of the non-homogeneous Poisson process \(\nu\) with mean measure \(\log I\). Also \((\gamma^+y^{-1})_n(1) = y^{-1}_n(y_n(1)) - 1 = n^{-1}(\tau(a_n(y_n(1)) + b_n) - 1 = (\tau(M_n)_n - n)/n = L_{\mu_n+1} - n)/n\)

where \(L_{\mu_n+1}\) is the index of the first record value after the \(n\)th observation. Again the continuous mapping theorem does the trick:

**Corollary 1:** If (1) holds or if \(F\) is continuous:

\[
P [(L_{\mu_n+1} - n)/n \leq x] \to x(1+x)^{-1}
\]

for \(x \geq 0\) as \(n \to \infty\).

The functional \(\gamma^+\) corresponds to the forward recurrence time.

We can also consider backward recurrences. Define for \(x\) non-decreasing:

\[
(\gamma^-x)(t) = t - x(x^{-1}(t)-)
\]

and

\[
(\gamma x)(t) = (\gamma^+ x)(t) + (\gamma^- x)(t) .
\]

Both are continuous and from Theorem 3 we have \((\gamma^-y^{-1})_n(1) \Rightarrow (\gamma^-y^{-1})(1)\) and \((\gamma y^{-1})_n(1) \Rightarrow (\gamma y^{-1})(1)\). \(\gamma^-y^{-1}(1)\) is the backward recurrence time of the non-homogeneous Poisson process \(\nu\) from \(t = 1\) so that
\[ P[\gamma^{-1}(1) > x] = P[\nu(1-x,1) = 0] \]

\[ = \exp\{-\log 1 - \log(1-x)\} \]

\[ = 1 - x \quad \text{if} \quad x \leq 1; \]
\[ = 0 \quad \text{if} \quad x > 1. \]

Similarly \( \gamma^{-1}(1) \) is the length of the interval covering 1 which is free of the points of \( \nu \). Consequently for \( x \leq 1 \)

\[ P[\gamma^{-1}(1) \leq x] = \int_{u=1-x}^{1} e^{-\log u^{-1}} (1-e^{-\log(u+x)})u^{-1}du = x - \log(1+x). \]

In deriving this recall the intensity of \( \nu \) is \( t^{-1} \) so if we condition on the last point preceding time 1 being at \( u \), we must then require no points in \((u,1]\] and at least one point in \([1, u+x]\]. The expression for \( x > 1 \) is simpler in that the integration is from \( u = 0 \) to \( u = 1 \).

The answer for \( x > 1 \) is \( P[\gamma^{-1}(1) \leq x] = 1 + \log(x^{-1}(1+x)) \).

Finally we have \((\gamma^{-1}_n)(1) = 1 - Y^{-1}_n(1) = 1 - \tau(a_n b_n) - \tau(M_n)/n \)

\[ = 1 - \tau(M_n)/n = \left(\frac{n-L_n}{\mu_n}\right)/n \]

and \((\gamma^{-1}_n)(1) = (\gamma^{-1}_n)(1) + (\gamma^{-1}_n)(1) \)

\[ = \left(\frac{L_n + L_n}{\mu_n}\right)/n. \]

Putting the pieces together we have

**Corollary 5:** If (1) holds or if \( F \) is continuous:

\[ \lim_{n \to \infty} P[(n-L_n)/n \leq x] = x \quad \text{if} \quad 0 \leq x \leq 1 \]

\[ 1 \quad \text{if} \quad x > 1. \]
and
\[ \lim_{n \to \infty} \frac{\mu_n}{\mu_n + 1} \leq x = x - \log(1 + x) \quad \text{if } x \leq 1 \]
\[ = \log(x^{-1}(1 + x)) \quad \text{if } x > 1. \]

Next consider again the function $h$ introduced in Theorem 1:

For $x \in D(0, \infty)$ or $D(-\infty, \infty)$ let

\[ (hx)(t) = \sup_{0 < s < t} \{ x(s) - x(s-) \}. \]

$h$ is continuous so by Theorem 3: $hY_n^{-1} \Rightarrow hY^{-1}$. Since $Y_n^{-1}$ had independent increments, $hY_n^{-1}$ is a variant of the class of extremal processes considered here: $hY_n^{-1}$ is a non-homogeneous Markov process of the type studied in [26]. Cf. [15], section 2. Due to the additive structure of $Y_n^{-1}$, the Levy measure can be computed. This was done in [17], Theorem 2. Consequently

\[ P[(hY_n^{-1})(t) \leq x] = P[Y_n^{-1}(u) \text{ has no jump of size } >x, \ 0 < u \leq t] \]
\[ = \exp \left\{ - \left[ \int_{x}^{\infty} e^{-Q(t) s^{-1} ds} - \int_{x}^{\infty} e^{-Q(0) s^{-1} ds} \right] \right\} \]

where $Q(s) = -\log G(s)$.

Next we identify $(hY_n^{-1})(t)$ as the maximum holding time of
\[ Y_n(u), \ 0 < u \leq Y_n^{-1}(t) = \sup \{ \Delta_j / n \mid \tau_j \leq \tau(a_n + b_n) \}. \] Recall
\( \tau(a) = \inf \{ n \mid X_n > a \} \) and \( \tau^{-1}(t) = \frac{\tau(a, t+b_n)}{n} \). In the case that \( G = \Phi_{\alpha} \), we set \( b_n = 0, a_n = F^{-1}(1-n^{-1}) \) and \( t = 1 \) and set \( s = F^{-1}(1-n^{-1}) \).

The result:

**Corollary 6:** If (1) holds then

(i) If \( G = \Phi_{\alpha} \):

\[
\sup \{ \Delta_j F(s) \mid L_j \leq \tau(s) \} \Rightarrow \exp \left\{ - \int_0^\infty e^{-z} z^{-1} dz \right\}
\]
as \( s \to \infty \) for \( x \geq 0 \).

(ii) If \( G = \Lambda \) then

\[
\sup \{ \Delta_j/\tau(b_n) \mid L_j \leq \tau(a_n + b_n) \} \Rightarrow \exp \left\{ - \left[ \int_0^\infty e^{-z} z^{-1} dz - \int_0^\infty e^{-z} z^{-1} dz \right] \right\}
\]
as \( n \to \infty \) for \( x \geq 0 \).

Attempts to derive explicitly the d.f. of the largest jump of \( Y \) in the time interval \((1,t], t > 1\), have not been successful. Consider instead the following continuous functional: For \( x(\cdot) \) a non-decreasing function define

\[
(h'x)(t) = \max \{ x(x^{-1}(0)), \sup \{ x(s) - x(s^-) \mid 0 < x(s^-) \leq x(s) \leq t \} \}.
\]

By the continuous mapping theorem: \( (h'\bar{Y}_n)(t) \Rightarrow (h'Y)(t) \). We first compute the d.f. of \( (h'Y)(t) \). Suppose \( Y \) is extremal-\( \Lambda \) so that the point process induced by the range of \( Y \) is homogeneous Poisson, rate \( 1 \).
If then \( \{Z_j, \ j \geq 1\} \) are i.i.d. exponential mean 1 random variables and \( S_n = \sum_{j=1}^{n} Z_j \) then

\[
(24) \quad (h'Y)(t) \overset{d}{=} \max\{Z_j \mid S_j \leq t\} \equiv J(t).
\]

We show for \( y \geq 0 \):

\[
(25) \quad P[J(t) \leq y] = 1 - e^{-t} + e^{-t} \sum_{j=1}^{\infty} \frac{(-1)^j (t-jy)_+^j}{j!} e^{-u} \left( u + \frac{n-1}{(n-1)!} \right) du.
\]

where \( s_+ = 1 \) if \( s > 0 \), \( = 0 \) if \( s < 0 \). The derivation uses some facts presented in [5]. Conditioning on the last \( S_n \) before \( t \) we have:

\[
P[J(t) \leq y] = \sum_{n=1}^{\infty} \int_{0}^{t} P[J(t) \leq y \mid S_n = u] e^{-(t-u)} e^{-u} \left( u + \frac{n-1}{(n-1)!} \right) du.
\]

Note \( P[J(t) \leq y \mid S_n = u] \) is the probability that of the \( n \) subintervals induced by \( n-1 \) points chosen at random from \([0,t]\) none is longer than \( y \). This probability is computed on pages 28-29 of [5].

The rest is routine manipulation involving reversal of the order of summations.

This derivation enables one to conclude:

**Corollary 7:** If (1) holds with \( G = \Lambda \) then

\[
\max\{a_n^{-1}(M_0 - b_n), \sup\{a_n^{-1}(X_{L_k} - X_{L_k+1}) \mid 0 < X_{L_k} < X_{L_{k+1}} \leq t\}\} \Rightarrow J(t)
\]

where the d.f. of \( J(t) \) is given by (25).
References


